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Brownian continuum random trees conditioned to be large*

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Abstract

We consider a Feller diffusion $(Z_s, s \geq 0)$ (with diffusion coefficient $\sqrt{2\beta}$ and drift $\theta \in \mathbb{R}$) that we condition on $\{Z_t = a_t\}$, where a_t is a deterministic function, and we study the limit in distribution of the conditioned process and of its genealogical tree as $t \to +\infty$. When a_t does not increase too rapidly, we recover the standard size-biased process (and the associated genealogical tree given by the Kesten tree). When a_t behaves as $\alpha\beta^2 t^2$ when $\theta = 0$ or as $\alpha\, \mathrm{e}^{2\beta|\theta|t}$ when $\theta \neq 0$, we obtain a new diffusion, as already proved by Overbeck in 1994 in the case $\theta = 0$. We give a new representation of this diffusion using an elementary SDE with a Poisson immigration. The corresponding genealogical tree is described by an infinite discrete skeleton (which does not satisfy the branching property) decorated with Brownian continuum random trees given by a Poisson point measure.

As a by-product of this study, we introduce several sets of trees endowed with a Gromov-type distance which are of independent interest and which allow here to define in a formal and measurable way the decoration of a backbone with a family of continuum random trees.

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1 Introduction

1.1 The discrete case motivation

In [1], for the geometric reproduction law, and in [5], for general super-critical reproduction laws with finite mean and some special sub-critical reproduction laws, the authors consider the limit of a Galton-Watson (GW) process $(Z_n, n \in \mathbb{N})$ started at $Z_0 = 1$ conditionally on $Z_n = a_n$ as n goes to infinity, provided the event $\{Z_n = a_n\}$ has

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positive probability. They also consider more generally the local limit of the GW tree, which in particular allows to study condensation phenomenon (on this latter subject, see [29, 28, 4]). Depending on the growth rate of a_n as n goes to infinity, they observe different regimes for the limiting random tree: if $a_n = 0$ for n large, the limiting tree corresponds to the GW tree conditioned on the extinction event; if a_n is strictly positive but grows slowly (including the case a_n bounded), then the limit is the so-called Kesten tree, which consists in an infinite spine decorated with independent GW trees with the initial reproduction law; if a_n grows at a moderate speed (given in the super-critical case of finite variance by $a_n \sim \alpha m^n$ with $\alpha > 0$ and m the mean of the reproduction law), then the limit is a skeleton given by an immigration process decorated again with independent GW trees with the initial reproduction law; if a_n grows faster than m^n (that is $\lim_{n\to\infty} m^{-n} a_n = \infty$) then results are known only for the geometric reproduction law (the limit exhibits a condensation at the root, that is, the root has an infinite number of children, and then those children generate independent trees) and for bounded reproduction laws (the limit is the regular b-ary tree, with b the possible maximum number of children).

We mention that the local limit distributions of the GW tree with geometric reproduction also appear when considering the local limit of trees having n vertices with a Gibbs distribution where the energy is the height of the tree, see [21].

This work is a first step to extend those results to random real trees called Lévy trees introduced by Duquesne and Le Gall in [18, 19] which are scaling limits of (sub)critical GW trees and can be seen as genealogical trees for (sub)critical continuous state branching processes (CSBP); see also [3, 20] for the extension of this latter representation to the super-critical case. We shall only consider Feller diffusions, which correspond to CSBPs with quadratic branching mechanism and whose genealogy can be described using the Brownian continuum random tree introduced by Aldous [9]. Our results belong also to the family of works dedicated to the description of limits of conditioned random real trees, in this direction, see [34, 33, 17, 2].

1.2 Feller diffusion with Poisson immigration

We consider a (nonnegative) quadratic CSBP $Z = (Z_t, t \geq 0)$ associated with the branching mechanism:

$$\psi_\theta(\lambda) = \beta\lambda^2 + 2\beta\theta\lambda,$$

with $\beta > 0$ and $\theta \in \mathbb{R}$. The process Z is a solution to the stochastic differential equation (SDE):

$$\mathrm{d}Z_t = \sqrt{2\beta Z_t}\, \mathrm{d}B_t - 2\beta\theta Z_t \mathrm{d}t, \quad \text{for } t \geq 0,$$

where $(B_t, t \geq 0)$ is some standard Brownian motion. The CSBP is sub-critical (resp. super-critical) if $\theta > 0$ (resp. $\theta < 0$). The time scaling parameter β will be fixed, but we shall stress in the notations the size scaling parameter θ, and denote by $\mathbb{P}_x^\theta$ the distribution of Z starting at $Z_0 = x \geq 0$.

Let $a = (a_t, t \geq 0)$ be a non-negative function. We shall consider the local limit of the process Z conditionally on $\{Z_t = a_t\}$ as t goes to infinity, that is the possible limiting distribution of $Z_{[0,s]} = (Z_r, r \in [0, s])$, with s fixed, conditionally on $\{Z_t = a_t\}$ as t goes to infinity. We recall that this question is related to the description of the Martin boundary of Markov processes and extremal time-space harmonic functions, see [22]. We have for $t \geq s \geq 0$ and H_s a bounded $\sigma(Z_{[0,s]})$-measurable random variable:

$$\mathbb{E}_1^\theta \left[H_s \,|\, Z_t = a_t\right] = \mathbb{E}_1^\theta \left[H_s\, K(s, Z_s; t, a_t)\right],$$

where K is the so-called Martin kernel, see (2.22) for an explicit formula. Then, all the extremal time-space harmonic functions h appear as the limit of:

$$h(s,x) = \lim_{t\to+\infty} K(s,x;t,a_t) \quad \text{for all } s,x \in \mathbb{R}_+ \tag{1.1}$$

for some non-negative function $a = (a_t, t \geq 0)$. Overbeck [36] gives, up to a normalizing constant, all the extremal time-space harmonic functions h for the critical Feller diffusion (that is $\theta = 0$), and gives also the SDE solved by the Doob h-transform of the process Z, see Lemma 2.7 for the extremal harmonic functions and Corollary 4.2 for the SDE (1.2) below with $\theta \in \mathbb{R}$ which includes the sub-critical and super-critical cases. For keeping the introduction as simple as possible, we shall stick to the critical case $\theta = 0$ considered in [36], and choose $\beta = 1$ (the general case can be deduced using a deterministic time change or a Girsanov transformation of Z). In this case, the extremal harmonic functions h are, with B defined in (2.13) and a different normalizing constant than in [36] for $\alpha > 0$:

- **Extinction case** $a_t = 0$ for t large: $h^{\emptyset}(s,x) = 1$;
- **Low regime** $a_t > 0$ and $a_t = o(t^2)$: $h^0(s,x) = x$;
- **Moderate regime** $a_t \sim \alpha t^2$ with $\alpha \in (0,+\infty)$: $h^\alpha(s,x) = \mathrm{e}^{-\alpha s}\, \mathrm{B}(\alpha x)/\alpha$.

In particular for the three regimes, for all $x > 0$, all $s \geq 0$ and all bounded $\sigma(Z_{[0,s]})$-measurable random variable H_s, we get:

$$\lim_{t\to+\infty} \mathbb{E}_x^0 \left[H_s \,|\, Z_t = a_t\right] = \mathbb{E}_x^0 \left[H_s \,\frac{h^\alpha(s,Z_s)}{h^\alpha(0,x)}\right].$$

The case $x = 0$ is trivial as it is an absorbing state for Z.

For $\alpha \in [0,+\infty)$, the Doob h-transform of the process Z using the harmonic function h^α, denoted by $Z^\alpha = (Z_t^\alpha, t \geq 0)$, satisfies the following SDE according to [36, Theorem 3]:

$$\mathrm{d}Z_t^\alpha = \sqrt{2Z_t^\alpha}\, \mathrm{d}B_t + 2g_\alpha(Z_t^\alpha)\, \mathrm{d}t, \quad t \geq 0, \quad Z_0^\alpha \geq 0, \tag{1.2}$$

where the function g_α is, for $\alpha = 0$, constant equal to 1, and for $\alpha > 0$ with $\mathcal{H}^\alpha(s,y) = \mathrm{e}^{-\alpha s}\, \mathrm{B}(\alpha y)/\alpha$, equal to:

$$g_\alpha(y) = y\, \partial_y \log(\mathcal{H}^\alpha(\cdot,y)) = \alpha y \frac{\mathrm{B}'(\alpha y)}{\mathrm{B}(\alpha y)}\cdot$$

The process Z^α is nonnegative, and using Feller's conditions for the classification of boundaries of one-dimensional diffusion, see [30, Table 6.2 p.234], we easily get that for any $\alpha \in [0,+\infty)$, 0 is an entrance boundary point for Z^α, that is, 0 is not accessible but it is possible to start the process Z^α from 0, and also $+\infty$ is a natural boundary point for Z^α, and thus inaccessible.

Motivated by the backbone decomposition of the corresponding genealogical tree given in the next section, we provide a new representation of the process Z^α using a Poisson immigration given in Corollary 4.2 which is stated for the general case $\theta \in \mathbb{R}$.

Proposition 1.1 (Representation using a Poisson immigration, case $\theta = 0$)**.** *Let $\alpha \geq 0$ and $(S_t^\alpha, t \geq 0)$ be a Poisson process with intensity $\alpha \mathrm{d}t$, independent of the Brownian motion $(B_t, t \geq 0)$. The process Z^α starting at $Z_0^\alpha = 0$ is distributed as the solution $Y^\alpha = (Y_t^\alpha, t \geq 0)$ of:*

$$\mathrm{d}Y_t^\alpha = \sqrt{2Y_t^\alpha}\, \mathrm{d}B_t + 2\,(S_t^\alpha + 1)\, \mathrm{d}t \quad \textit{with} \quad Y_0^\alpha = 0. \tag{1.3}$$

The proof of this result, given in Section 4.2, uses a result from Rogers and Pitman [39] for a transformation of a Markov process to still be a Markov process. When the process Z^α starts at $Z_0^\alpha = x > 0$, the constant 1 in the drift term of (1.3) must be replaced by a random constant independent of B and S^α, see the beginning of the proof of Corollary 4.2 in Section 4.3.

1.3 Decomposition of the Brownian CRT with respect to n leaves taken at random at a given height

We denote by $\mathbb{N}^\theta$ the canonical σ-finite measure associated with the CSBP Z under $\mathbb{P}^\theta_\cdot$. Intuitively, under $\mathbb{N}^\theta$, the population starts with an infinitesimal individual at time $t=0$. Let $\mathcal{T}$ denote under $\mathbb{N}^\theta$ the genealogical tree of the process Z, it is the so-called Brownian continuum random tree (CRT) introduced by Aldous. In this context, the random tree $\mathcal{T}$ can be easily built from a Brownian excursion, and the measure $\mathbb{N}^\theta$ can then be identified with the excursion measure of the reflected Brownian motion. We write ϱ for the root of $\mathcal{T}$. In [19, Theorem 4.5], Duquesne and Le Gall give a decomposition of the critical or sub-critical Brownian tree $\mathcal{T}$ by taking a leaf uniformly at random at level $t \geq 0$ and decorating the branch from the root to this leaf with independent Brownian CRTs. There is no difficulty to extend this result to the supercritical case, see Corollary 5.9. We then extend this representation by giving a decomposition of the Brownian CRT when taking n leaves uniformly at random at level $t \geq 0$ and decorating the discrete tree spanned by the n leaves and the root with independent Brownian CRTs, see Theorem 5.8. This result completes the description of [20] where one chooses theses vertices at random without condition on their level.

Stating and proving this result relies on a lengthy study of various spaces of trees and the measurability of various maps defined on those sets of trees, which are detailed in Section 6. We shall present informally the mathematical objects and state the theorem in the critical case $\theta = 0$ with $\beta = 1$ for simplicity; we also write $\mathbb{N}$ for $\mathbb{N}^0$. Let Λ_t denote the local time at level t associated with the Brownian tree $\mathcal{T}$ (its total mass is equal to Z_t the size of the population at level t): this measure allows to sample random individual "uniformly" at level t; the measure Λ_t is supported by the leaves of $\mathcal{T}$ at level t. Under $\mathbb{N}[\mathrm{d}\mathcal{T}]\,\Lambda_t^{\otimes n}(\mathrm{d}\mathbf{v}^*)$ we can sample the Brownian CRT $\mathcal{T}$ with n leaves $\mathbf{v}^* = (v_1, \ldots, v_n) \in \mathcal{T}^n$ at level t. To those n distinguish vertices, we shall add the root ϱ of $\mathcal{T}$ and set $\mathbf{v} = (\varrho, \mathbf{v}^*)$, and shall see $(\mathcal{T}, \mathbf{v})$ as an element of the Polish metric space of the complete locally compact $n+1$-pointed trees, $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$, equipped with the local Gromov-Hausdorff distance (and where all $n+1$-pointed trees which are isomorphic are identified), see Section 6 for precise details. We describe the rooted tree spanned by the root and the distinguished $\mathbf{v}^*$ vertices using a combinatorial construction on growing discrete planar trees with fixed height t defined in Section 5.4.1 where starting from one branch of height t, we graft uniformly successively branches with their leaf at height t. Let us stress that we use the planar structure of the trees in this section only, and that the grafting of the new branch is uniformly done on the right or on the left. After n such steps, we obtain the random $n+1$-pointed tree $(\mathbf{T}_n^{\mathrm{unif}}, \mathbf{v}_n)$, where the distinguished vertices $\mathbf{v}_n$ are first the root, and then the leaves ranked in their arrival order (and not in the planar order). Then, very informally, on this discrete tree, for all $i \in I$ a countable set of indices, we graft at $x_i \in \mathbf{T}_n^{\mathrm{unif}}$ a subtree T_i, where $\mathcal{M}(\mathrm{d}x, \mathrm{d}\mathcal{T}) = \sum_{i\in I} \delta_{(x_i, T_i)}(\mathrm{d}x, \mathrm{d}\mathcal{T})$ is a Poisson point measure with intensity $2\,d\mathscr{L}(\mathrm{d}x)\,\mathbb{N}[\mathrm{d}\mathcal{T}]$, where $d\mathscr{L}$ is the length measure on $\mathbf{T}_n^{\mathrm{unif}}$. This grafting procedure, Graft_n, is rigorously defined in Section 7.2.2 based on the technical material from Section 6. So we are now able to state Theorem 5.10 for $\theta = 0$ and $\beta = 1$. Recall $\mathbf{v} = (\varrho, \mathbf{v}^*) \in \mathcal{T}^{n+1}$, with ϱ the root of $\mathcal{T}$.

Theorem 1.1 (Generalized n-leaves decomposition, case $\theta = 0$)**.** *Let $t > 0$ and $n \in \mathbb{N}^*$.*

For every non-negative measurable function F defined on $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$, we have:

$$\mathbb{N}\left[\int_{\mathcal{T}^n}\Lambda_t^{\otimes n}(\mathrm{d}\mathbf{v}^*)\,F(\mathcal{T},\mathbf{v})\right] = n!\,t^{n-1}\,\mathbb{E}\left[F\Big(\mathrm{Graft}_n\big((\mathbf{T}_n^{\mathrm{unif}},\mathbf{v}_n),\mathcal{M}\big)\Big)\right].$$

For $n=1$, see Theorem 5.8, this is the so-called Bismut decomposition which appears in [19] stated in the framework of the contour process of the Lévy trees. As pointed out by a thorough referee it is possible to state Theorem 1.1 using the contour process given by the Brownian excursion, providing an interesting shorter proof. We however choose to stick to the tree framework presentation in order to clarify measurability issues such as the grafting procedure for random complete locally compact trees, which is a commonly used operation on trees. We believe those topological technical results postponed to Sections 6 and 7 will be useful for future works on continuum random trees.

Let us mention here that there have been several works on skeletal/backbone decompositions for (spatial) branching processes and their corresponding genealogical trees, for example see [2, 10, 12, 20, 26, 27, 32] and the references therein. In particular, in [26], coupled systems of SDEs were established to represent the skeletal decompositions for continuous-state branching processes (conditioned on survival), where the skeletons are determined by continuous-time Galton-Watson processes. And we refer to [20] for the reconstruction of a Lévy tree from a backbone tree, which could be formed by leaves taken at random in a Poissonnian way from the Lévy tree according to the so-called mass measure; see Remark 5.4 there and [18]. For representations of branching processes (with immigration) via SDEs, we also refer to [15] and references therein.

1.4 Local limit of conditioned Brownian CRT

The Brownian CRT $\mathcal{T}$ gives the genealogical structure of the CSBP Z. We shall give a description of the genealogical structure of the CSBP associated with the Doob h-transform and prove that it appears naturally as local limit of the Brownian CRT $\mathcal{T}$ conditioned to be large. We stress that the local limits obtained here are different from the one obtained by conditioning on the non extinction at large time, see [2] in this direction. We denote by $\mathbb{T}_{\mathrm{loc-K}} = \mathbb{T}^{(0)}_{\mathrm{loc-K}}$ the set (of equivalence classes) of complete locally compact rooted real trees, see Section 6 for more details.

Recall that in the critical case $\theta = 0$ the Brownian CRT $\mathcal{T}$ is compact. In the introduction, we simply denote by $\mathbf{t}_t$ the real tree $\mathbf{t}$ truncated at level t. We denote by $\mathcal{G}_t$ the σ-field generated by $\mathcal{T}_t$ for $t \geq 0$; in particular the process Z is adapted to the filtration $(\mathcal{G}_t, t \geq 0)$. Let F be any bounded continuous function defined on $\mathbb{T}_{\mathrm{loc-K}}$.

- **Extinction case:** $a_t = 0$ for t large. We have:

  $$\lim_{t\to\infty}\mathbb{N}\big[F(\mathcal{T}_s)\,\mathbf{1}_{\{Z_t=a_t\}}\big] = \mathbb{N}\big[F(\mathcal{T}_s)\big].$$

  The result is obvious for the critical case as the tree $\mathcal{T}$ is compact $\mathbb{N}$-a.e., that is $\mathcal{T}_t = \mathcal{T}$ for t large enough. We obtain the same result in the sub-critical case $\theta > 0$. In the super-critical case $\theta < 0$, using the Girsanov transformation from [3] to define the super-critical Lévy tree, see also (5.6), we get that:

  $$\lim_{t\to\infty}\mathbb{N}^{\theta}\big[F(\mathcal{T}_s)\,\mathbf{1}_{\{Z_t=a_t\}}\big] = \mathbb{N}^{|\theta|}\big[F(\mathcal{T}_s)\big].$$

  Those results hold also in general for any compact Lévy trees.

- **Low regime:** a is positive and $a_t = o(t^2)$. We recall that the Kesten tree $\mathcal{T}^*$ is informally obtained by grafting the trees $(T_i, i \in I)$ respectively at levels $(h_i, i \in I)$ on an infinite spine, where the point measure $\sum_{i\in I}\delta_{h_i,T_i}(\mathrm{d}h,\mathrm{d}\mathcal{T})$ is a Poisson point measure with intensity measure $2\mathbf{1}_{\{h>0\}}\mathrm{d}h\,\mathbb{N}[\mathrm{d}\mathcal{T}]$. See Lemma 7.2 for a more formal definition of the Kesten tree. The Kesten tree appears already as the local limit of general compact Lévy trees when conditioning instead by $\{Z_t > 0\}$, see [2]. The next result is Theorem 5.14 restricted to the critical case $\theta = 0$ with $\mathcal{T}^{0,\theta} = \mathcal{T}^*$ (notice therein the difference of the limit between the super-critical case and the sub-critical one).

Theorem 1.2. *We have in the low regime, $a_t = o(t^2)$ and $a_t > 0$, that:*

$$\lim_{t\to\infty} \mathbb{N}\big[F(\mathcal{T}_s) \bigm| Z_t = a_t\big] = \mathbb{E}\left[F\left(\mathcal{T}_s^*\right)\right].$$

- **Moderate regime:** $a_t \sim \alpha t^2$, where $\alpha \in (0, +\infty)$. We first consider a backbone tree $\mathfrak{T}^{\alpha,0}$ representing in some sense the genealogy associated with a Poisson immigration with rate α, see Section 5.6 for a more precise description. Secondly, let the point measure $\sum_{i\in I} \delta_{x_i,T_i}(\mathrm{d}x, \mathrm{d}\mathcal{T})$ be, conditionally given $\mathfrak{T}^{\alpha,0}$, a Poisson point measure with intensity rate $2\mathscr{L}(\mathrm{d}x)\,\mathbb{N}[\mathrm{d}\mathcal{T}]$ with $\mathscr{L}(\mathrm{d}x)$ the length measure on $\mathfrak{T}^{\alpha,0}$. Then, the random tree $\mathcal{T}^{\alpha,0}$ is obtained by grafting, for $i \in I$, the tree T_i at vertex x_i on the backbone tree $\mathfrak{T}^{\alpha,0}$. (As, for $\alpha = 0$, $\mathfrak{T}^{0,0}$ can be seen as an infinite spine, the Kesten tree is indeed distributed as $\mathcal{T}^{0,0}$.) The next result is Theorem 5.13 restricted to the critical case $\theta = 0$.

Theorem 1.3. *We have in the moderate regime, $a_t \sim \alpha t^2$ with $\alpha \in (0, +\infty)$, that:*

$$\lim_{t\to\infty} \mathbb{N}\big[F(\mathcal{T}_s) \bigm| Z_t = a_t\big] = \mathbb{E}\left[F\left(\mathcal{T}_s^{\alpha,0}\right)\right].$$

Let us stress that the backbone tree does not enjoy the branching property, as already observed by [1, 5] in a discrete setting. In a forthcoming paper, we shall recover the branching structure in the backbone by considering a weighted tree.

- **High regime:** $\lim_{t\to\infty} t^{-2} a_t = +\infty$ (or $\lim_{t\to\infty} \mathrm{e}^{-2\beta|\theta|t}\, a_t = +\infty$ if $\theta \neq 0$). The description of the possible limit in this regime is still an open question. As in the discrete setting studied in [1], one could ask if there is a condensation phenomenon at the root. However, to study such local limits, which would not be locally compact (at least at or near the root), one would require a non trivial extension of the current topology developed for complete locally compact trees.

1.5 Outline of the paper

Section 2 is devoted to some notations and elementary facts for the quadratic CSBP, the transition kernel under the canonical measure $\mathbb{N}^\theta$, and the Martin boundary for the process Z (under $\mathbb{P}_x^\theta$ and the excursion measure $\mathbb{N}^\theta$). We then present families of martingales for the process Z and then the local limits of the process Z conditionally on $Z_t = a_t$ for t large and some deterministic function $(a_t, t > 0)$ (under $\mathbb{P}_x^\theta$ and the excursion measure $\mathbb{N}^\theta$) in Section 3. We prove Proposition 1.1 on the representation of the Doob h-transform of the process Z with h harmonic extremal using a Poisson immigration in the general case $\theta \in \mathbb{R}$ in Section 4, see Corollary 4.2. We provide the backbone decomposition in Section 5, with the decomposition with respect to n leaves from Theorem 1.1 in Section 5.5 and the local limit of Brownian CRT conditioned to $Z_t = a_t$ from Theorems 1.2 and 1.3 in Section 5.6. We have made the choice to use some intuitive (but abusive) definition in Section 5, in particular considering the Graft_n map in order to state the result without burdening the reader with too much technicalities; we clarify all the definitions in Section 7 using the lengthy technical Section 6.

Let us mention that the introduction is written with the time scale parameter $\beta = 1$ and for the critical case $\theta = 0$. The general cases $\beta > 0$ and $\theta \in \mathbb{R}$ could be deduced in finite time from the particular case by scaling or using the Girsanov transform on CSBP. However, if those computations are not that complicated, they are lengthy and treacherous; so we decided to treat the general cases but for the introduction.

An index of all the (numerous) relevant notations is provided at the end of the document.

2 General quadratic CSBP

2.1 Notations

We set $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_+^* = (0, +\infty)$, $\mathbb{N} = \{0, 1, 2, \cdots\}$ and $\mathbb{N}^* = \{1, 2, \cdots\}$.

For $x \in \mathbb{R}$, we set $x_+ = \max(0, x)$ and $x_- = \max(0, -x)$, so that $x = x_+ - x_-$. We write δ_x for the Dirac mass at x.

Let $(E, \mathscr{E})$ be a measured space. For a (nonnegative) measure μ on E and $A \in \mathscr{E}$, we denote by $\mu_{|A}(\mathrm{d}x)$ the measure $\mathbf{1}_A(x)\mu(\mathrm{d}x)$. We write $\mu(f) = (f, \mu) = \int f \, \mathrm{d}\mu = \langle f, \mu \rangle$ for the integral of the measurable real-valued function f with respect to the measure μ, whenever it is meaningful.

We say that a function from a measurable space to a measurable space is bi-measurable if it is measurable and the image of any measurable set is a measurable set (when the function is one-to-one this is equivalent to the function and its inverse being measurable).

2.2 Quadratic CSBP

Most results in this section can be found in [18, 16, 3, 20]. Let $\beta > 0$ be fixed. Let $\theta \in \mathbb{R}$. We consider the quadratic branching mechanism ψ_θ given for $\lambda \in \mathbb{R}$ by:

$$\psi_\theta(\lambda) = \beta\lambda^2 + 2\beta\theta\lambda. \tag{2.1}$$

The corresponding CSBP $Z = (Z_t, t \geq 0)$ is the unique strong solution to the following stochastic differential equation (SDE):

$$\mathrm{d}Z_t = \sqrt{2\beta Z_t}\, \mathrm{d}B_t - 2\beta\theta Z_t \mathrm{d}t \quad \text{for } t \geq 0, \tag{2.2}$$

where $B = (B_t, t \geq 0)$ is a standard Brownian motion and $Z_0 = x \geq 0$. For $t \geq 0$, let $\mathcal{F}_t$ be the σ-field generated by $(Z_s, t \in [0, t])$. We write $\mathbb{P}_x^\theta$ to stress the value of the parameter θ, and the initial value of the process Z, $Z_0 = x$. We denote by $\mathbb{N}^\theta$ the canonical measure of the process Z with $Z_0 = 0$, normalized in such a way that for $\lambda \geq 0$:

$$\mathbb{N}^\theta \left[1 - \mathrm{e}^{-\lambda\sigma}\right] = \psi_\theta^{-1}(\lambda),$$

where $\sigma = \int_0^\infty Z_t \, \mathrm{d}t$ is the total size of the population under the canonical measure $\mathbb{N}^\theta$ and $\psi_\theta^{-1}(\lambda)$ is the only solution t to $\psi_\theta(t) = \lambda$ such that $t \geq 2\theta_-$. In particular, the process $(Z_t, t \geq 0)$ under $\mathbb{P}_x^\theta$ is distributed as the process $\left(\sum_{i \in I} Z_t^{(i)}, t \geq 0\right)$ where $\sum_{i \in I} \delta_{Z^{(i)}}$ is a Poisson point measure with intensity $x\mathbb{N}^\theta(dZ)$. We refer to [18] for $\theta \geq 0$ (critical and sub-critical case) and to [16, 3, 20] for $\theta < 0$ (super-critical case) for a detailed presentation of the CSBP process Z and the corresponding continuum Brownian random tree $\mathcal{T}$.

With a slight abuse, we say that two random variables or functionals Y' and Y'' have the same distribution under $\mathbb{N}^\theta$ if the pushed forward measures of $\mathbb{N}^\theta$ by Y' and Y'' are equal.

We introduce the following positive functions c^θ and $\tilde{c}^\theta$ defined for $t \in (0, +\infty)$ by $c_t^0 = \tilde{c}_t^0 = 1/\beta t$ and for $\theta \neq 0$:

$$c_t^\theta = \frac{2\theta}{\mathrm{e}^{2\beta\theta t} - 1} \quad \text{and} \quad \tilde{c}_t^\theta = c_t^{-\theta} = \frac{2\theta}{1 - \mathrm{e}^{-2\beta\theta t}}. \tag{2.3}$$

Remark 2.1 (Scaling property of Z). *In this remark, we write $Z^{[\beta,\theta]}$ for Z under $\mathbb{N}^\theta$ or $\mathbb{P}^\theta_x$ in order to stress the dependence in $\beta > 0$ and $\theta \in \mathbb{R}$. Let $Y = (Y_s, s \geq 0)$ be a Feller diffusion that is, $Y = Z^{[\beta=1,\theta=0]}$. Under $\mathbb{P}_x$, it is given as the unique strong solution to the SDE, with initial condition $Y_0 = x$:*

$$\mathrm{d}Y_s = \sqrt{2Y_s}\, \mathrm{d}B_s, \quad \text{for } s \geq 0. \tag{2.4}$$

We denote by $(Q_t, t \geq 0)$ the semi-group of the diffusion Y and recall that it is a Feller semi-group with the so-called following branching property:

$$Q_t(x + x', \cdot) = Q_t(x, \cdot) * Q_t(x', \cdot) \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad x, x' \in \mathbb{R}_+. \tag{2.5}$$

We shall denote by $\mathbb{N}$ the canonical measure of Y.

For $\beta > 0$ and $\theta \in \mathbb{R}$, the process $Z^{[\beta,\theta]}$ under $\mathbb{N}^\theta$ (resp. $\mathbb{P}^\theta_x$) is distributed as the process:

$$\left(\mathrm{e}^{-2\beta\theta t}\, Y_{1/c^\theta_t},\, t \geq 0\right) \tag{2.6}$$

under $\mathbb{N}$ (resp. $\mathbb{P}_x$), with the convention that $1/c^\theta_t = 0$ for $t = 0$. Notice that the range of $1/c^\theta_t$ as t runs in $\mathbb{R}_+$ is $[0, 1/(2\theta_-))$. Even though, using this scaling and time change, it is (almost) enough to state the forthcoming results for the particular case $\beta = 1$ and $\theta = 0$, we shall keep general values for the parameters in order to better understand their role. This ends the remark.

The functions c^θ and $\tilde{c}^\theta$ are decreasing with:

$$\lim_{t\to 0+} c^\theta_t = \lim_{t\to 0+} \tilde{c}^\theta_t = +\infty, \quad \lim_{t\to +\infty} c^\theta_t = 2\theta_- \quad \text{and} \quad \lim_{t\to +\infty} \tilde{c}^\theta_t = 2\theta_+. \tag{2.7}$$

We also have for $t > 0$:

$$\tilde{c}^\theta_t = c^{-\theta}_t = c^\theta_t + 2\theta. \tag{2.8}$$

We define for $t > 0$ and $\lambda > -\tilde{c}^\theta_t$:

$$u^\theta(\lambda, t) = \frac{\lambda c^\theta_t}{\tilde{c}^\theta_t + \lambda} = c^\theta_t - \frac{c^\theta_t\, \tilde{c}^\theta_t}{\tilde{c}^\theta_t + \lambda}, \tag{2.9}$$

and set $u^\theta(\lambda, 0) = \lambda$ for $t = 0$. This gives that for $t > 0$ and $\lambda > -\tilde{c}^\theta_t$:

$$u^\theta(\lambda, t) = \begin{cases} \dfrac{2\theta\lambda}{(2\theta + \lambda)\, \mathrm{e}^{2\beta\theta t} - \lambda}, & \text{if } \theta \neq 0, \\ \lambda/(1 + \lambda\beta t), & \text{if } \theta = 0. \end{cases}$$

For $r > 0$ and $t \geq 0$, we have that:

$$u^\theta(c^\theta_r, t) = c^\theta_{t+r}.$$

We recall from the above mentioned references ([18, 16, 3, 20]) for $\lambda \geq 0$ and by analytic continuation for $\lambda < 0$, that for $t > 0$ and $x \geq 0$:

$$\mathbb{N}^\theta\left[1 - \mathrm{e}^{-\lambda Z_t}\right] = u^\theta(\lambda, t) \quad \text{and} \quad \mathbb{E}^\theta_x\left[\mathrm{e}^{-\lambda Z_t}\right] = \mathrm{e}^{-x u^\theta(\lambda,t)} \quad \text{for all } \lambda > -\tilde{c}^\theta_t. \tag{2.10}$$

We denote by $\zeta = \inf\{t > 0;\, Z_t = 0\}$ the lifetime of the process Z. We recall that for all $t > 0$:

$$\mathbb{N}^\theta[\zeta > t] = \lim_{\lambda\to\infty} u^\theta(\lambda, t) = c^\theta(t).$$

By considering the series in λ in (2.9) and (2.10), we deduce that for all $t > 0$ and $n \in \mathbb{N}^*$:

$$\mathbb{N}^\theta\left[\left(\tilde{c}^\theta_t Z_t\right)^n\right] = n! c^\theta_t. \tag{2.11}$$

We now give a martingales related to the CSBP Z. Since $u^\theta(\lambda,t) = u^{-\theta}(\lambda+2\theta,t) - 2\theta$ for $\lambda \geq c_t^\theta - 2\theta$, we deduce that the process $\left(\mathrm{e}^{2\theta Z_t},\, t\in I\right)$ is a martingale with respect to the filtration $(\mathcal{F}_t, t\geq 0)$ under $\mathbb{N}^\theta$ with $I=\mathbb{R}_+^*$ and under $\mathbb{P}_x^\theta$ with $I=\mathbb{R}_+$. Furthermore, according to [3, Section 4], we have that for $\theta\in\mathbb{R}$, $t>0$ and $x\geq 0$:

$$\mathbb{N}^{-\theta}[\mathrm{d}Z]_{|\mathcal{F}_t} = \mathrm{e}^{2\theta Z_t}\,\mathbb{N}^\theta[\mathrm{d}Z]_{|\mathcal{F}_t} \quad \text{and} \quad \mathbb{E}_x^{-\theta}[\mathrm{d}Z]_{|\mathcal{F}_t} = \mathrm{e}^{2\theta(Z_t - x)}\,\mathbb{E}_x^\theta[\mathrm{d}Z]_{|\mathcal{F}_t}. \tag{2.12}$$

Recall that $\tilde{c}_t^\theta$ is decreasing in t, and thus $-\tilde{c}_{t+r}^\theta > -\tilde{c}_r^\theta$. The next lemma is an easy consequence of (2.10) and the following elementary equality:

$$u(-\tilde{c}_{t+r}^\theta, t) = -\tilde{c}_r^\theta \quad \text{for all } t\geq 0 \text{ and } r>0.$$

Lemma 2.2. *Let $\theta\in\mathbb{R}$, $x\in\mathbb{R}_+$, $r>0$ and the quadratic CSBP $(Z_t, t\geq 0)$ solution of* (2.2)*. The process $\left(\mathrm{e}^{\tilde{c}_{t+r}^\theta Z_t},\, t\in I\right)$ is a martingale with respect to the filtration $(\mathcal{F}_t, t\geq 0)$ under $\mathbb{N}^\theta$ with $I=\mathbb{R}_+^*$ and under $\mathbb{P}_x^\theta$ with $I=\mathbb{R}_+$.*

2.3 Transition densities and Martin Kernel

We first provide the densities of the entrance law $\mathbb{N}^\theta[Z_t\in \mathrm{d}x,\, \zeta>t]$ and the transition kernel $\mathbb{N}^\theta[Z_{t+s}\in\mathrm{d}y|\, Z_s = x]$ of the (homogeneous) CSBP Z under its excursion measure for $t>0$, $x>0$ and $y\geq 0$. We shall consider the function B_0 and B on $\mathbb{R}_+$ defined by:

$$\mathrm{B}_0(x) = \sum_{k\in\mathbb{N}} \frac{x^k}{k!(k+1)!} \quad \text{and} \quad \mathrm{B}(x) = x\mathrm{B}_0(x) = \sqrt{x}\, I_1\big(2\sqrt{x}\big), \tag{2.13}$$

where $I_1(x) = \sum_{i\in\mathbb{N}} (x/2)^{2i+1}/i!(i+1)!$ is the Bessel function. Notice that $\mathrm{B}_0(0)=1$.

Lemma 2.3 (Entrance law and transition densities of Z)**.** *Let $\theta\in\mathbb{R}$. Let $s,t>0$ and $y\geq 0$. We have $\mathbb{N}^\theta[Z_{t+s}\in\mathrm{d}y|\, Z_s=0] = \delta_0(\mathrm{d}y)$ and for $x>0$:*

$$\begin{aligned} \mathbb{N}^\theta[Z_t\in\mathrm{d}x,\, \zeta>t] &= q_t^\theta(x)\,\mathrm{d}x,\\ \mathbb{N}^\theta[Z_{t+s}\in\mathrm{d}y|\, Z_s = x] = \mathbb{P}_x^\theta(Z_t\in\mathrm{d}y) &= \mathrm{e}^{-xc_t^\theta}\,\delta_0(\mathrm{d}y) + \mathfrak{q}_t^\theta(x,y)\,\mathrm{d}y, \end{aligned}$$

where:

$$q_t^\theta(x) = c_t^\theta \tilde{c}_t^\theta\, \mathrm{e}^{-\tilde{c}_t^\theta x}, \tag{2.14}$$

$$\mathfrak{q}_t^\theta(x,y) = x c_t^\theta \tilde{c}_t^\theta\; \mathrm{e}^{-(x+y)c_t^\theta - 2\theta y}\, \mathrm{B}_0\left(xy c_t^\theta \tilde{c}_t^\theta\right). \tag{2.15}$$

Notice that for $x=0$ the transition kernel of Z is a Dirac mass at 0; this amount to take $\mathfrak{q}_t^\theta(0,y)=0$ for all $t>0$ and $y\geq 0$, which is consistent with (2.15). We also mention that $\int_{(0,\infty)} \mathfrak{q}_t^\theta(x,y)\,\mathrm{d}y = 1-\mathrm{e}^{xc_t^\theta}$.

The case $\theta=0$ and $\beta=1$ is in [36]. We provide a short proof for any parameters for the reader convenience.

Proof. We omit the parameter θ in the proof. On one hand, we get that for $\lambda\geq 0$:

$$\mathbb{N}\left[\mathrm{e}^{-\lambda Z_t}\,\mathbf{1}_{\{\zeta>t\}}\right] = -\mathbb{N}\left[1-\mathrm{e}^{-\lambda Z_t}\right] + \mathbb{N}\left[\zeta>t\right] = c(t) - u(\lambda,t).$$

On the other hand, using (2.9), we get:

$$\int_0^\infty c_t\tilde{c}_t\,\mathrm{e}^{-(\tilde{c}_t+\lambda)x}\,\mathrm{d}x = c(t) - u(\lambda,t).$$

Then use that finite positive measures on $\mathbb{R}_+$ are characterized by their Laplace transform to obtain that $\mathbb{N}^\theta[Z_t\in\mathrm{d}x,\,\zeta>t] = q_t(x)\,\mathrm{d}x$ with q_t given by (2.14).

For the transition kernel, we get that for $\lambda \geq 0$:

$$\mathbb{N}\left[\mathrm{e}^{-\lambda Z_{t+s}} \;\middle|\; Z_s = x\right] = \mathrm{e}^{-xu(\lambda,t)} = \mathrm{e}^{-\frac{a}{b}+\frac{a}{b+\lambda}},$$

where, thanks to (2.9), $a = xc_t\tilde{c}_t$ and $b = \tilde{c}_t$. Notice that:

$$\mathrm{e}^{\frac{a}{b+\lambda}} = 1 + \sum_{k\in\mathbb{N}} \frac{1}{(k+1)!}\left(\frac{a}{b+\lambda}\right)^{k+1} = 1 + a\sum_{k\in\mathbb{N}} \int_0^{+\infty} \frac{(ay)^k}{k!(k+1)!}\,\mathrm{e}^{-by-\lambda y}\,\mathrm{d}y.$$

Using (2.3), we deduce that $\mathbb{N}^\theta[Z_{t+s} \in \mathrm{d}y |\, Z_s = x] = \mathrm{e}^{-xc_t}\,\delta_0(\mathrm{d}y) + \mathfrak{q}_t(x,y)\,\mathrm{d}y$, with $\mathfrak{q}_t(x,y)$ given by (2.15). □

Let us notice that $\mathfrak{q}_t^\theta(x,y)$ is also the transition density of the CSBP Z under $\mathbb{P}_{x_0}^\theta$ for every $x_0 \geq 0$. As for $t > s \geq 0$ and $x, y \in \mathbb{R}_+$, the probability measure $\mathbb{P}_x^\theta(Z_{t-s} \in \mathrm{d}y)$ is absolutely continuous w.r.t. $\mathbb{P}_1^\theta(Z_t \in \mathrm{d}y)$, the Martin kernel is defined by the Radon-Nikodym derivative:

$$K(s,x;t,y) = \frac{\mathbb{P}_x^\theta(Z_{t-s} \in \mathrm{d}y)}{\mathbb{P}_1^\theta(Z_t \in \mathrm{d}y)}. \tag{2.16}$$

2.4 Martin boundary

According to Overbeck [36], see also [22, Section 10], all extremal (non-negative) time-space harmonic functions for the CSBP Z appear as the limit of the Martin kernel $K(s,x;t,a_t)$, see (2.16), as t goes to infinity and where $(a_t, t \geq 0)$ is a non-negative function. To study the possible limits:

$$\lim_{t\to\infty} K(s,x;t,a_t), \tag{2.17}$$

we shall consider the functions on $\mathbb{R}_+^2$:

$$H^{0,\theta}(s,x) = x\;\mathrm{e}^{2\beta\theta s} \quad \text{and} \quad H^{\alpha,\theta}(s,x) = \mathrm{e}^{-\alpha/c_s^\theta}\;\frac{\mathrm{B}\left(\alpha x\,\mathrm{e}^{2\beta\theta s}\right)}{\alpha}, \tag{2.18}$$

for $\alpha > 0$ and with B defined in (2.13). Notice that $\lim_{\alpha\to 0} H^{\alpha,\theta} = H^{0,\theta}$.

We consider the following intermediary result with the functions q and $\mathfrak{q}$ given in (2.14) and (2.15).

Lemma 2.4. *Let $s \geq 0$ and $x \geq 0$. If $(a_t, t \geq 0)$ is positive and $\lim_{t\to+\infty} a_t\, c_t^\theta \tilde{c}_t^\theta = \alpha \in [0,+\infty)$, then we have:*

$$\lim_{t\to\infty} \frac{\mathfrak{q}_{t-s}^\theta(x,a_t)}{q_t^\theta(a_t)} = \mathrm{e}^{-2\theta_- x}\;H^{\alpha,|\theta|}(s,x). \tag{2.19}$$

Proof. We omit the superscript θ in the proof. We get from (2.15) and (2.8) that for $t \geq s > 0$ and $y > 0$:

$$\frac{\mathfrak{q}_{t-s}^\theta(x,y)}{q_t^\theta(y)} = x\,\mathrm{e}^{-xc_{t-s}}\;\mathrm{e}^{-y(c_{t-s}-c_t)}\;\frac{c_{t-s}\tilde{c}_{t-s}}{c_t\tilde{c}_t}\,\mathrm{B}_0\left(xyc_{t-s}\tilde{c}_{t-s}\right).$$

Recall from (2.7) that $\lim_{t\to+\infty} c_t = 2\theta_-$ so that:

$$\lim_{t\to\infty} \mathrm{e}^{-xc_{t-s}} = \mathrm{e}^{-2\theta_- x}. \tag{2.20}$$

It is elementary to check that:

$$\lim_{t\to\infty} c_{t-s}\tilde{c}_{t-s}/c_t\tilde{c}_t = \mathrm{e}^{2\beta|\theta|s} \quad \text{and} \quad \lim_{t\to\infty}(c_{t-s} - c_t)/c_t\tilde{c}_t = \frac{\mathrm{e}^{2\beta|\theta|s}-1}{2|\theta|} = \frac{1}{c_s^{|\theta|}},$$

where the latter limit is simply βs if $\theta = 0$. The result is then immediate. □

The result below for $\theta = 0$ appears in [36, Section 5], and the proof for general θ, based on (2.16) and Lemma 2.4 is similar.

Lemma 2.5 (Martin boundary)**.** *Let $s \geq 0$ and $x \geq 0$.*

(i) Extinction case. If $a_t = 0$ for t large enough, then the limit (2.17) *exists and is equal to:*

$$h^{\emptyset,\theta}(s,x) = h^{\emptyset,\theta}(x) = \mathrm{e}^{-2\theta_-(x-1)}.$$

(ii) Low and moderate regimes. If the sequence $(a_t, t \geq 0)$ is positive and $\lim_{t\to+\infty} a_t\, c_t^\theta \tilde{c}_t^\theta = \alpha \in [0, +\infty)$, then the limit (2.17) *exists and is equal to:*

$$h^{\alpha,\theta}(s,x) = h^{\emptyset,\theta}(x)\ \frac{H^{\alpha,|\theta|}(s,x)}{H^{\alpha,|\theta|}(0,1)}.$$

(iii) High regime. If $\lim_{t\to+\infty} a_t\, c_t^\theta \tilde{c}_t^\theta = +\infty$, then the limit (2.17) *exists and is equal to:*

$$h^\infty(s,x) = \mathbf{1}_{\{s=0,\, x=1\}}.$$

By considering the accumulation points of the sequence $(a_t\, c_t^\theta \tilde{c}_t^\theta, t > 0)$, we deduce from Lemma 2.5 that there is no other possible case for the existence of the limit (2.17).

Remark 2.6 (Equivalent condition for the moderate regime)**.** *The moderate regime condition $\lim_{t\to+\infty} a_t\, c_t^\theta \tilde{c}_t^\theta = \alpha \in (0, +\infty)$, which appears in Lemma 2.5 (ii), is in fact equivalent to:*

$$a_t \sim \begin{cases} \alpha\beta^2 t^2 & \text{if } \theta = 0, \\ \alpha(2\theta)^{-2}\,\mathrm{e}^{2\beta|\theta|t} & \text{if } \theta \neq 0. \end{cases} \tag{2.21}$$

Proof. We omit the superscript θ in the proof. The low and moderate regimes are a direct consequence of Lemma 2.4. In the extinction case, use that $K(s,x;t,0) = \mathrm{e}^{-xc_{t-s}+c_t}$ and (2.20) to get the result. For the high regime, using that for $y > 0$:

$$K(s,x;t,y) = \mathrm{e}^{-xc_{t-s}+c_t} \times \begin{cases} 0 & \text{if } x = 0, \\ \dfrac{\mathrm{B}(xy\, c_{t-s}\tilde{c}_{t-s})}{\mathrm{B}(y\, c_t\tilde{c}_t)}\ \mathrm{e}^{-y(c_{t-s}-c_t)} & \text{if } x > 0. \end{cases} \tag{2.22}$$

Equation (2.13), the asymptotics of the Bessel function $I_1(z) \sim \mathrm{e}^z/\sqrt{2\pi z}$ as z goes to infinity and (2.20), we deduce that $\lim_{t\to\infty} K(s,x;t,a_t) = \mathbf{1}_{\{s=0,\, x=1\}}$. □

Using similar arguments as in [36, Section 5] stated for $\theta = 0$, the Girsanov transform (2.12) to reduce the cases $\theta < 0$ to $\theta > 0$ and then Remark 2.1 to reduce those latter cases to the case $\theta = 0$, we get the following result.

Lemma 2.7 (Extremal harmonic functions)**.** *Let $\beta > 0$ be fixed. Let $\theta \in \mathbb{R}$. The extremal time-space harmonic functions of $Z^{[\beta,\theta]}$ are the functions $h^{\alpha,\theta}$ for $\alpha \in \mathcal{A}$ where $\mathcal{A} = \{\emptyset\} \bigcup [0, +\infty)$.*

Proof. Notice that h^∞ is not an harmonic function. According to [22, Section 10], the functions $h^{\alpha,\theta}$ with $\alpha \in \mathcal{A}$ are the only possible extremal harmonic functions. Thanks to (2.12) it is enough to consider the case $\theta \geq 0$. Thanks to Remark 2.1, for $\theta \geq 0$, we have that:

$$\begin{aligned} \mathbb{E}^\theta\left[F(Z_{[0,t]}^{[\beta,\theta]})\, h^{\alpha,\theta}(t, Z_t^{[\beta,\theta]})\right] &= \mathbb{E}\left[F(\mathrm{e}^{-2\beta\theta s}\, Y_{1/c_s^\theta}, s \in [0,t])\, h^{\alpha,\theta}(t, \mathrm{e}^{-2\beta\theta t}\, Y_{1/c_t^\theta})\right] \\ &= C_\alpha\, \mathbb{E}\left[F(\mathrm{e}^{-2\beta\theta s}\, Y_{1/c_s^\theta}, s \in [0,t])\ \mathrm{e}^{-\alpha/c_t^\theta}\ \mathrm{B}(\alpha Y_{1/c_t^\theta})\right], \end{aligned}$$

where for $\alpha = 0$, $C_\alpha \mathrm{B}(\alpha x)$ is simply replaced by $C_0\, x$, and C_α is a finite positive constant. Then use that 1, $(Y_s, s \geq 0)$ and $(\mathrm{e}^{-\alpha s}\,\mathrm{B}(\alpha Y_s), s \geq 0)$ for $\alpha \in (0, +\infty)$ are martingales, see [36], to conclude when $\theta \geq 0$. □

3 Local limits for the process Z

3.1 Some martingales

We present in this section two martingales which will naturally appear in the local limits for the Brownian continuum random tree (CRT). Let $\alpha \geq 0$. Define:

$$\mathcal{H}^\alpha(s, y) = \mathrm{e}^{-\alpha s}\, y \mathrm{B}_0(\alpha y), \quad s \geq 0,\, y \geq 0, \tag{3.1}$$

where B_0 is defined in (2.13). Recall $\theta \in \mathbb{R}$. Let $M^{\alpha,\theta} = (M_t^{\alpha,\theta}, t > 0)$ be the process defined by:

$$M_t^{\alpha,\theta} = \mathcal{H}^\alpha(1/c_t^\theta,\, \mathrm{e}^{2\beta\theta t}\, Z_t) = H^{\alpha,\theta}(t, Z_t), \tag{3.2}$$

where the last equality is a direct consequence of (2.18) on the definition of $H^{\alpha,\theta}$. For $\theta = 0$, this formula corresponds to [36, Eq. (19)] up to a normalizing constant.

Using Theorem 3 of [36] and Remark 2.1, we get the following result (where a martingale under the σ-finite excursion measure is understood to be a process with integrable marginals).

Proposition 3.1. *Let $\theta \in \mathbb{R}$, $\alpha \geq 0$ and $x \in \mathbb{R}_+$. The process $\left(M_t^{\alpha,\theta},\, t \in I\right)$ is a non-negative martingale under $\mathbb{N}^\theta$ with $I = (0, +\infty)$ and under $\mathbb{P}_x^\theta$ with $I = \mathbb{R}_+$.*

Proof. The case $\theta = 0$ under $\mathbb{P}_x$ is in [36, Section 5]. For $\theta \neq 0$, use Remark 2.1 to get the result under $\mathbb{P}_x$ for all $\theta \in \mathbb{R}$.

Moreover, for all $t > 0$, we have, using (2.11) and $\tilde{c}_t^\theta / c_t^\theta = \mathrm{e}^{2\beta\theta t}$, that:

$$\mathbb{N}^\theta \left[M_t^{\alpha,\theta}\right] = 1. \tag{3.3}$$

Then use the Markov property under the excursion measure to conclude the result also holds under $\mathbb{N}^\theta$. □

We introduce an other family of related martingales. For $\theta \in \mathbb{R}$ and $\alpha \geq 0$, we set $\tilde{M}^{\alpha,\theta} = (\tilde{M}_t^{\alpha,\theta}, t > 0)$ with:

$$\tilde{M}_t^{\alpha,\theta} = \mathrm{e}^{2\theta Z_t}\, M_t^{\alpha,-\theta} = \mathcal{H}^\alpha(1/\tilde{c}_t^\theta,\, \mathrm{e}^{-2\beta\theta t}\, Z_t)\, \mathrm{e}^{2\theta Z_t}, \tag{3.4}$$

using (3.2) and $c_t^{-\theta} = \tilde{c}_t^\theta$ for the second equality. We then deduce from Proposition 3.1 the following corollary.

Corollary 3.2. *Let $\theta \in \mathbb{R}$, $\alpha \geq 0$. The process $\tilde{M}^{\alpha,\theta}$ is a martingale under $\mathbb{N}^\theta$, and for $t > 0$ and any non-negative $\mathcal{F}_t$-measurable random variable H_t, we have:*

$$\mathbb{N}^\theta[H_t\, \tilde{M}_t^{\alpha,\theta}] = \mathbb{N}^{-\theta}\left[H_t\, M_t^{\alpha,-\theta}\right]. \tag{3.5}$$

Remark 3.3 (The case $\theta = 0$ and $\alpha = 0$)**.** *Let $t > 0$. For $\theta = 0$, we have:*

$$\tilde{M}_t^{\alpha,0} = M_t^{\alpha,0} = \mathcal{H}^\alpha(\beta t,\, Z_t)\cdot$$

For $\alpha = 0$, we have:

$$M_t^{0,\theta} = Z_t\, \mathrm{e}^{2\beta\theta t} \quad \text{and} \quad \tilde{M}_t^{0,\theta} = Z_t\, \mathrm{e}^{2\theta(Z_t - \beta t)}. \tag{3.6}$$

Then for $\alpha = \theta = 0$, we have:

$$\tilde{M}_t^{0,0} = M_t^{0,0} = Z_t.$$

Remark 3.4 (The case $\alpha < 0$)**.** *We consider the process $M^{\alpha,\theta}$ defined by* (3.2) *and* (3.1) *with α negative. Using $|M_t^{\alpha,\theta}| \leq \mathrm{e}^{2|\alpha|/c_t^\theta}\, M_t^{|\alpha|,\theta}$, we get the integrability of the process $M^{\alpha,\theta}$. Using that B_0 is a converging series with positive terms, we deduce that the martingale property of the process $M^{\alpha,\theta}$ holds as in Proposition 3.1 for α negative.*

3.2 Local limit

We first consider the Poisson regime, whose name is inherited from the representation given in Proposition 4.1 based on a Poisson immigration. Let $a = (a_t, t > 0)$ be a positive function.

Proposition 3.5 (Poisson regime). *Let $\theta \in \mathbb{R}$, $s > 0$ and H_s be a bounded $\mathcal{F}_s$-measurable random variable. Let $\alpha \in (0, +\infty)$. Assume the function a is such that as $t \to \infty$ large:*

$$a_t \sim \begin{cases} \alpha\beta^2 t^2 & \text{if } \theta = 0, \\ \alpha(2\theta)^{-2}\, \mathrm{e}^{2\beta|\theta|t} & \text{if } \theta \neq 0. \end{cases}$$

Then we have:

$$\lim_{t\to\infty} \mathbb{N}^\theta[H_s |\, Z_t = a_t] = \mathbb{N}^{|\theta|}\left[H_s\, M_s^{\alpha,|\theta|}\right] = \begin{cases} \mathbb{N}^\theta[H_s\, M_s^{\alpha,\theta}] & \text{if } \theta \geq 0, \\ \mathbb{N}^\theta[H_s\, \tilde{M}_s^{\alpha,\theta}] & \text{if } \theta \leq 0. \end{cases} \tag{3.7}$$

Proof. Let $s > 0$ and H_s be fixed. For $t > 0$, thanks to (2.14) on the entrance law density and (2.15) on the transition kernel density, we have:

$$\mathbb{N}^\theta[H_s |\, Z_{t+s} = a_{t+s}] = \frac{\mathbb{N}^\theta\left[H_s \mathfrak{q}_t(Z_s, a_{t+s})\right]}{q_{t+s}(a_{t+s})}.$$

Then use Lemma 2.4 to get the existence of the limit:

$$\lim_{t\to+\infty} \frac{\mathfrak{q}_t(Z_s, a_{t+s})}{q_{t+s}(a_{t+s})} = \begin{cases} M_s^{\alpha,\theta} & \text{if } \theta \geq 0, \\ \tilde{M}_s^{\alpha,\theta} & \text{if } \theta \leq 0. \end{cases} \tag{3.8}$$

The Markov property, Proposition 3.1 (see (3.3) in its proof) and Corollary 3.2 (with $H_s = 1$) give that the $\mathbb{N}^\theta[\mathfrak{q}_t(Z_s, a_{t+s})/q_{t+s}(a_{t+s})]$, $\mathbb{N}^\theta[M_s^{\alpha,\theta}]$ and $\mathbb{N}^\theta[\tilde{M}_s^{\alpha,\theta}]$ are all equal to 1. To conclude use that, for nonnegative functions, the a.e. convergence and the convergence of the integrals imply the L^1 convergence in (3.8). This concludes the proof. □

The same proof can be used for the Kesten regime.

Proposition 3.6 (Kesten regime). *Let $\theta \in \mathbb{R}$, $s > 0$ and H_s be a bounded non-negative $\mathcal{F}_s$-measurable random variable. Assume the function a is positive ($a_t > 0$) and such that as $t \to \infty$:*

$$a_t = \begin{cases} o(t^2) & \text{if } \theta = 0, \\ o(\mathrm{e}^{2\beta|\theta|t}) & \text{if } \theta \neq 0. \end{cases}$$

Then we have:

$$\lim_{t\to\infty} \mathbb{N}^\theta[H_s |\, Z_t = a_t] = \mathbb{N}^{|\theta|}\left[H_s\, Z_s\, \mathrm{e}^{2\beta|\theta|s}\right] = \begin{cases} \mathbb{N}^\theta[H_s\, M_s^{0,\theta}] & \text{if } \theta \geq 0, \\ \mathbb{N}^\theta[H_s\, \tilde{M}_s^{0,\theta}] & \text{if } \theta \leq 0. \end{cases} \tag{3.9}$$

For completeness, we add the well known extinction case, that is the function $a_t = 0$ for large t, which is a direct consequence of (2.12). Since the event $\{Z_t = 0\}$ has infinite measure under $\mathbb{N}^\theta$, we consider the restriction instead of the conditioning.

Proposition 3.7 (Extinction regime). *Let $\theta \in \mathbb{R}$, $s > 0$ and H_s be a bounded non-negative $\mathcal{F}_s$-measurable random variable. Then we have:*

$$\lim_{t\to\infty} \mathbb{N}^\theta\left[H_s\, \mathbf{1}_{\{Z_t = 0\}}\right] = \mathbb{N}^{|\theta|}[H_s] = \begin{cases} \mathbb{N}^\theta[H_s] & \text{if } \theta \geq 0, \\ \mathbb{N}^{-\theta}[H_s] & \text{if } \theta \leq 0. \end{cases} \tag{3.10}$$

4 Change of measure

We give a representation of the distribution of the process Z under the change of measure given by the martingale $M^{\alpha,\theta}$ using a Poisson immigration; and we identify it with the solution of the SDE from [36, Theorem 3]. Even if Proposition 4.1 and Corollary 4.2 below are a direct consequence of Proposition 5.11 and Theorem 5.13 (see Remarks 5.12 and 5.15), we provide an independent proof in Sections 4.2 and 4.3 which is interesting by itself. The proof will be done for $\beta = 1$ and $\theta = 0$, and then use a time-change, see Remark 2.1, to get $\theta \in \mathbb{R}$.

4.1 SDE representation

Let $\beta > 0$ and $\theta \in \mathbb{R}$. Let $B = (B_t, t \geq 0)$ be a standard Brownian motion. Let $\alpha > 0$ and $S^{\alpha,\theta}(\mathrm{d}t)$ be a Poisson point measure on $\mathbb{R}_+$ with intensity $\alpha\beta\, \mathrm{e}^{2\beta\theta t}\, \mathrm{d}t$, independent of the Brownian motion B. We set $S_t^{\alpha,\theta} = S^{\alpha,\theta}([0,t])$ for $t \in \mathbb{R}_+$. We define the process $Y^{\alpha,\theta} = (Y_t^{\alpha,\theta}, t \geq 0)$ under $\mathbb{P}^\theta$ as the unique strong solution (conditionally on S) of the following SDE:

$$\mathrm{d}Y_t^{\alpha,\theta} = \sqrt{2\beta Y_t^{\alpha,\theta}}\, \mathrm{d}B_t - 2\beta\theta Y_t^{\alpha,\theta} \mathrm{d}t + 2\beta\, (S_t^{\alpha,\theta} + 1)\, \mathrm{d}t \quad \text{for } t \geq 0, \quad \text{and } Y_0^{\alpha,\theta} = 0. \tag{4.1}$$

Notice that by definition, we have $S_0^{\alpha,\theta} = 0$.

Proposition 4.1 (An SDE with Poisson drift). *Let $\alpha > 0$, $\theta \in \mathbb{R}$ and $t_0 > 0$. The process $(Z_t, t \in [0, t_0])$ under $\mathbb{N}^\theta \left[\bullet\, M_{t_0}^{\alpha,\theta}\right]$ is distributed as the process $(Y_t^{\alpha,\theta}, t \in [0, t_0])$.*

The proof of this proposition is detailed in Section 4.2. Notice that Corollary 3.2 provides a Girsanov transform between the law of $Y^{\alpha,\theta}$ and $Y^{\alpha,-\theta}$ starting from 0.

As $\partial_y \log(\mathcal{H}^\alpha(t, y))$ does not depend on t, we simply write $\partial_y \log(\mathcal{H}^\alpha(\cdot, y))$.

Corollary 4.2 (The SDE with Poisson drift is a diffusion). *Let $\alpha > 0$ and $\theta \geq 0$. The process $(Y_t^{\alpha,\theta}, t \geq 0)$ satisfies the stochastic differential equation with initial condition $Y_0^{\alpha,\theta} = 0$:*

$$\mathrm{d}Y_t^{\alpha,\theta} = \sqrt{2\beta Y_t^{\alpha,\theta}}\, \mathrm{d}B_t - 2\beta\theta Y_t^{\alpha,\theta}\, \mathrm{d}t + 2\beta\, \mathrm{e}^{2\beta\theta t}\, Y_t^{\alpha,\theta}\, \partial_y \log(\mathcal{H}^\alpha(\cdot, \mathrm{e}^{2\beta\theta t}\, Y_t^{\alpha,\theta}))\, \mathrm{d}t, \quad t \geq 0. \tag{4.2}$$

The proof of this corollary is detailed in Section 4.3. We mention that the solution to the SDE (4.2) is path-wise unique, see Theorem IX.3.5.(iii) in [38], and thus the SDE (4.2) has a unique strong solution.

In [36, Theorem 3] when $\beta = 1$ and $\theta = 0$ (the function h therein is given by $y^{-1}\, \mathcal{H}^\alpha(s, y)$ up to a multiplicative constant), it is already proven that the solution of (4.2) is the Doob h-transform of the Feller diffusion. Proposition 4.1 and Corollary 4.2 give an alternative proof of this result. Equation (4.1) gives a more useful description of this process which will be translated in terms of genealogical trees in the next sections.

4.2 Proof of Proposition 4.1

Following Remark 2.1, we first use a scaling argument to remove the parameters β and θ.

Let $\alpha > 0$. We simply write (Y^α, S^α) for $(Y^{\alpha,\theta}, S^{\alpha,\theta})$ when $\theta = 0$ and $\beta = 1$, that is, $S^\alpha = (S_t^\alpha, t \geq 0)$ is a Poisson process with parameter α independent of the Brownian motion B and $Y^\alpha = (Y_t^\alpha, t \geq 0)$ is the unique strong solution (conditionally on S) of the following SDE:

$$\mathrm{d}Y_t^\alpha = \sqrt{2Y_t^\alpha}\, \mathrm{d}B_t + 2\, (S_t^\alpha + 1)\, \mathrm{d}t \quad \text{for } t \geq 0, \quad \text{and} \quad Y_0^\alpha = 0. \tag{4.3}$$

Let $\beta, \alpha > 0$ and $\theta \in \mathbb{R}$. Recall that $(Y^{\alpha,\theta}, S^{\alpha,\theta})$ depends also on β. Define the process $(Y'^{\alpha}, S'^{\alpha}) = ((Y_s'^{\alpha}, S_s'^{\alpha}), s \in [0, 1/(2\theta_-)))$ by:

$$Y_s'^{\alpha} = \mathrm{e}^{2\beta\theta t}\, Y_t^{\alpha,\theta} \quad \text{and} \quad S_s'^{\alpha} = S_t^{\alpha,\theta}, \quad \text{with} \quad s = \frac{1}{c_t^{\theta}} \cdot \tag{4.4}$$

Then, it is elementary that this deterministic time change yields the following result.

Lemma 4.3. *Let $\beta, \alpha > 0$ and $\theta \in \mathbb{R}$. The process $(Y'^{\alpha}, S'^{\alpha})$ under $\mathbb{P}^{\theta}$ (whose law depends on (β, θ) and α) is distributed as $\Big((Y_s^{\alpha}, S_s^{\alpha}), s \in \big[0, 1/(2\theta_-)\big)\Big)$.*

Let $(P_t, t \in \mathbb{R}_+)$ be the transition semi-group on $\mathbb{R}_+ \times \mathbb{N}$ of the Markov process (Y^{α}, S^{α}). The next result is necessary when checking conditions from [39] to ensure that a projection of a Markov process is still a Markov process. Let C_b (resp. C_0) denotes the set of continuous real-valued functions defined on $\mathbb{R}_+ \times \mathbb{N}$ which are bounded (resp. which vanish at infinity, with $\mathbb{R}_+ \times \mathbb{N}$ endowed with the L^{∞} norm).

Lemma 4.4. *The semi-group $(P_t, t \in \mathbb{R}_+)$ has the Feller property, that is $P_t C_b \subset C_b$ for all $t \geq 0$. The semi-group $(P_t, t \in \mathbb{R}_+)$ is also a Feller semi-group, that is, $P_t C_0 \subset C_0$ for all $t \geq 0$ and $\lim_{t \to 0+} P_t f(y, k) = f(y, k)$ for all $f \in C_0$ and $(y, k) \in \mathbb{R}_+ \times \mathbb{N}$.*

Proof. Let $\Big((Y_t^{\alpha,(y,k)}, S_t^{\alpha,(y,k)}), t \geq 0\Big)$ denote the solution of the SDE (4.3) starting from $(y, k) \in \mathbb{R}_+ \times \mathbb{N}$, that is, $S_t^{\alpha,(y,k)} = S_t^{\alpha} + k$ and (conditionally on $S^{\alpha,(y,k)}$) the process $Y^{\alpha,(y,k)}$ is a strong solution to the SDE:

$$\mathrm{d}Y_t^{\alpha,(y,k)} = \sqrt{2Y_t^{\alpha,(y,k)}}\, \mathrm{d}B_t + 2\,(S_t^{\alpha,(y,k)} + 1)\, \mathrm{d}t \quad \text{for } t \geq 0, \quad \text{and } Y_0^{\alpha,(y,k)} = y.$$

Let $(X_t^y, t \geq 0)$ be a Feller diffusion starting from y (it is distributed as a solution to the SDE (2.4)), independent of the $(Y_t^{\alpha,(y,k)}, S_t^{\alpha,(y,k)})_{t \geq 0}$. By the branching property, see (2.5), we have the equality in distribution for the processes:

$$\Big((Y_t^{\alpha,(y,k)}, S_t^{\alpha,(y,k)}), t \geq 0\Big) \overset{(d)}{=} \Big((Y_t^{\alpha,(0,k)} + X_t^y, S_t^{\alpha,(0,k)}), t \geq 0\Big).$$

Recall Q_t denote the semi-group of the process X^x, see Remark 2.1. Then for every $t \geq 0$, $x, y \in \mathbb{R}_+$, $k \in \mathbb{N}$ and every bounded continuous function f defined on $\mathbb{R}_+ \times \mathbb{N}$, we have:

$$\begin{aligned} P_t f(x, k) - P_t f(y, k) &= \mathbb{E}\left[f\left(Y_t^{\alpha,(x,k)}, S_t^{\alpha,(x,k)}\right) - f\left(Y_t^{\alpha,(y,k)}, S_t^{\alpha,(y,k)}\right)\right] \\ &= \mathbb{E}\left[f\left(Y_t^{\alpha,(0,k)} + X_t^x, S_t^{\alpha,(0,k)}\right) - f\left(Y_t^{\alpha,(0,k)} + X_t^y, S_t^{\alpha,(0,k)}\right)\right] \\ &= \mathbb{E}\left[Q_t f_{\left(Y_t^{\alpha,(0,k)}, S_t^{\alpha,(0,k)}\right)}(x) - Q_t f_{\left(Y_t^{\alpha,(0,k)}, S_t^{\alpha,(0,k)}\right)}(y)\right], \end{aligned}$$

where $f_{(y,k)}$ is the continuous map $x \mapsto f\,(y + x, k)$. By the Feller property of the semi-group Q_t and the dominated convergence theorem, we deduce that $\lim_{y \to x} P_t f(x, k) - P_t f(y, k) = 0$. This gives the Feller property of the kernel P_t.

The continuity $\lim_{t \to 0+} P_t f(y, k) = f(y, k)$, for $f \in C_b$, is a direct consequence of the path right-continuity at time $t = 0$ of the process $(Y^{\alpha,(y,k)}, S^{\alpha,(y,k)})$. Since $P_t C_0 \subset P_t C_b \subset C_b$, to conclude that the semi-group $(P_t, t \in \mathbb{R}_+)$ is Feller, it is enough to prove that $P_t f$ vanish at infinity for $f \in C_0$. Let $f \in C_0$. Notice that $S_t^{\alpha,(y,k)} = S_t^{\alpha} + k$ and S_t^{α} is nonnegative. It is well known that on can define the family of Feller diffusions $(X_t^y, (t, y) \in \mathbb{R}_+^2)$ in such a way that a.s. $\lim_{y \to \infty} X_t^y = +\infty$. Using that $Y_t^{\alpha,(0,k)}$ is also nonnegative, we deduce that $\lim_{\max(y,k) \to \infty} f\left(Y_t^{\alpha,(y,k)}, S_t^{\alpha,(y,k)}\right) = 0$ a.s. and we get $\lim_{\max(y,k) \to \infty} P_t f(y, k) = 0$ by dominated convergence. This finishes the proof. ☐

We now give the density of (Y_t^α, S_t^α). Recall that $Y_0^\alpha = S_0^\alpha = 0$. Let N be the counting measure on $\mathbb{N}$.

Lemma 4.5. *Let $t > 0$. The random variable (Y_t^α, S_t^α) has a density f on $\mathbb{R}_+ \times \mathbb{N}$ with respect to $\mathrm{d}y \otimes \mathrm{N}(\mathrm{d}k)$ given by:*

$$f(y,k) = \frac{1}{t^2}\,\frac{\alpha^k\, y^{k+1}}{k!(k+1)!}\,\mathrm{e}^{-(\alpha t + t^{-1} y)}, \quad y \geq 0,\, k \in \mathbb{N}. \tag{4.5}$$

Proof. Conditionally on S, by (4.3), we can see Y^α as a quadratic CSBP (with $\beta = 1$) with immigration whose rate is $2(S_t^\alpha + 1)\mathrm{d}t$. This implies that, conditionally on S^α, the process Y^α is distributed as $\left(\sum_{i\in I} \mathbf{1}_{\{h_i \leq t\}}\, Y_{t-h_i}^{(i)}, t \geq 0\right)$, where $\sum_{i \in I} \delta_{(h_i, Y^{(i)})}(\mathrm{d}t, \mathrm{d}Y)$ is a Poisson point measure on $\mathbb{R}_+ \times \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ with intensity $2(S_t^\alpha + 1)\mathrm{d}t\, \mathbb{N}[\mathrm{d}Y]$ and $\mathbb{N}$ is the excursion measure of a CSBP with branching mechanism $\psi(\lambda) = \lambda^2$.

We deduce that for $\lambda, \mu \geq 0$:

$$\mathbb{E}\left[\mathrm{e}^{-\lambda Y_t^\alpha - \mu S_t^\alpha}\right] = \mathbb{E}\left[\mathrm{e}^{-\mu S_t^\alpha - \int_0^t 2(S_r^\alpha+1)\mathbb{N}\left[1-\mathrm{e}^{-\lambda Y_{t-r}}\right]\mathrm{d}r}\right] = \mathbb{E}\left[\mathrm{e}^{-\mu S_t^\alpha - 2\int_0^t (S_r^\alpha+1)\frac{\lambda}{1+(t-r)\lambda}\,\mathrm{d}r}\right],$$

where we used (2.10) for the last equality (with $\beta = 1$ and $\theta = 0$). Denote by $(\xi_i, i \in \mathbb{N}^*)$ the increasing sequence of the jumping times of the Poisson process S^α, and set $\xi_0 = 0$. Then, we have on $\{S_t^\alpha = k\}$:

$$\begin{aligned}\int_0^t (S_r^\alpha + 1)\frac{\lambda}{1+(t-r)\lambda}\,\mathrm{d}r &= \sum_{i=0}^k (i+1)\int_{\xi_i}^{\xi_{i+1}\wedge t} \frac{\lambda}{1+(t-r)\lambda}\,\mathrm{d}r \\ &= -\sum_{i=0}^k (i+1)\log(1+(t-r)\lambda)\Big|_{\xi_i}^{\xi_{i+1}\wedge t} \\ &= \sum_{i=0}^k \log(1+(t-\xi_i)\lambda).\end{aligned}$$

Conditionally on $\{S_t^\alpha = k\}$, the random set $\{\xi_1, \ldots, \xi_k\}$ is distributed as $\{tU_1, \ldots, tU_k\}$ (notice the order is unimportant and is not preserved), where $U_1, \ldots, U_k$ are independent random variables uniformly distributed on $[0,1]$. We deduce that:

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{-\lambda Y_t^\alpha - \mu S_t^\alpha}\right] &= \sum_{k\in\mathbb{N}} \frac{(\alpha t)^k\,\mathrm{e}^{-\alpha t - \mu k}}{k!}\,\mathbb{E}\left[\prod_{i=1}^k \big(1+t(1-U_i)\lambda\big)^{-2}\right](1+t\lambda)^{-2} \\ &= \sum_{k\in\mathbb{N}} \frac{(\alpha t)^k\,\mathrm{e}^{-\alpha t-\mu k}}{k!}\,(1+t\lambda)^{-k-2} \\ &= \sum_{k\in\mathbb{N}} \int_{\mathbb{R}_+} \mathrm{d}y\; f(y,k)\,\mathrm{e}^{-\lambda y - \mu k},\end{aligned}$$

where for the last equality, we used the definition of f given in (4.5). This finishes the proof. □

Recall that $Y_0^\alpha = S_0^\alpha = 0$. Let p_t be the distribution density of Y_t^α for $t \in \mathbb{R}_+$. We have $p_0 = \delta_0$ the Dirac mass at 0, and for $t > 0$, we deduce from Lemma 4.5 that $p_t(\mathrm{d}y)$ has a density on $\mathbb{R}_+$ with respect to the Lebesgue measure given by:

$$p_t(dy) = t^{-2}\,\mathrm{e}^{-y/t}\,\mathcal{H}^\alpha(t,\, y)\mathrm{d}y, \quad t > 0,\, y \geq 0, \tag{4.6}$$

where $\mathcal{H}^\alpha$ is defined in (3.1). We now give some properties of the conditional law of S_t given Y_t. Recall B_0 defined in (2.13).

Lemma 4.6. *Let $y \in \mathbb{R}_+$. The law of S_t^α conditionally on $\{Y_t^\alpha = y\}$ does not depend on t. More precisely, we get for all $t \geq 0$, $k \in \mathbb{N}$ and $y \geq 0$:*

$$\mathbb{P}(S_t^\alpha = k |\, Y_t^\alpha = y) = \frac{1}{\mathrm{B}_0(\alpha y)} \, \frac{(\alpha y)^k}{k!(k+1)!}. \tag{4.7}$$

Proof. Using Lemma 4.5, we directly have (4.7) for $t > 0$. Notice that for $y = 0$, we have $\mathrm{B}_0(0) = 1$ and

$$\frac{1}{\mathrm{B}_0(\alpha y)} \, \frac{(\alpha y)^k}{k!(k+1)!} = \mathbf{1}_{\{k=0\}}.$$

As $(Y_0^\alpha, S_0^\alpha) = (0, 0)$, we deduce that (4.7) also holds for $t = 0$. □

We can now prove the Markov property of the process $Y^\alpha = (Y_t^\alpha, t \geq 0)$.

Lemma 4.7. *The process Y^α from* (3.2) *with initial condition $Y_0^\alpha = S_0^\alpha = 0$ is a Markov process on $\mathbb{R}_+$, and its transition semi-group $(Q_t, t \in \mathbb{R}_+)$ is the unique semi-group on $\mathcal{C}_b(\mathbb{R}_+, \mathbb{R}_+)$ with the Feller property and such that $p_t = p_0 Q_t$ for $t \in \mathbb{R}_+$, where the probability measure p_t is defined in* (4.6).

Proof. We say that a probability kernel K is continuous if for all continuous and bounded function f, Kf is also continuous (and bounded). We say that a collection of probabilities $\mathcal{L}$ on a metric space is determining if for two bounded continuous functions f and g, $\mu(f) = \mu(g)$ for all $\mu \in \mathcal{L}$ implies $f = g$. The proof is base on [39, Lemma 1], which we recall:

Let E and E' be metric spaces endowed with their Borel σ-fields, $X = (X_t, t \geq 0)$ a E-valued continuous time Markov process with fixed initial distribution λ and transition semigroup $(P_t, t \geq 0)$ that we suppose to be Feller (that is P_t is continuous for every $t \geq 0$), ϕ a continuous function from E to E', Λ a continuous probability kernel from E' to E (that is $\Lambda : (y, A) \mapsto \Lambda(y, A)$, $y \in E'$ and $A \in \mathcal{B}(E)$), and p_t the distribution of $\phi \circ X_t$. Suppose further that:

(i) for each $t \geq 0$ a conditional distribution for X_t given $\phi \circ X_t = y$ is $\Lambda(y, \cdot)$, $y \in E'$;

(ii) the collection $\{p_t : \ t \geq 0\}$ is determining.

Then $\phi \circ X_t$ is Markov with initial law p_0 and transition semigroup $(Q_t : \ t \geq 0)$, which is the unique semigroup on E' with the Feller property such that $p_t = p_0 Q_t$, $t \geq 0$.

We now check hypothesis from this lemma with the notation $E = \mathbb{R}_+ \times \mathbb{N}$, $E' = \mathbb{R}_+$, $X = (Y^\alpha, S^\alpha)$, $\lambda = \delta_{(0,0)}$ and $\phi(y, s) = y$. The semi-group $(P_t, t \geq 0)$ is Feller, see Lemma 4.4. The probability kernel $\Lambda(y; \mathrm{d}z, \mathrm{d}k) = \mathbb{P}(S_t^\alpha = k |\, Y_t^\alpha = y) \, \delta_y(\mathrm{d}z) \, \mathrm{N}(\mathrm{d}k)$ does not depend on t by Lemma 4.6 and is clearly continuous. This gives condition (i). We now check condition (ii), that is the one-dimensional marginal distributions of Y^α, $(p_t, t \in \mathbb{R}_+)$, are determining. To prove this, notice that:

$$t^2 \, \mathrm{e}^{\alpha t} \, \mathbb{E}\big[h(Y_t^\alpha)\big] = \int_{\mathbb{R}_+} \mathrm{e}^{-t^{-1} y} \, H(y) \mathrm{d}y,$$

where $H(y) = h(y) y \, \mathrm{B}_0(\alpha y)$. As the Laplace transform characterizes bounded continuous functions, we deduce that if $\mathbb{E}[h(Y_t^\alpha)] = \mathbb{E}[g(Y_t^\alpha)]$ for all $t \in \mathbb{R}_+$, then $H = G$ (with $G(y) = g(y) y \, \mathrm{B}_0(\alpha y)$) and thus $h = g$ on $(0, +\infty)$ and by continuity on $\mathbb{R}_+$.

As the assumption of [39, Lemma 1] are satisfied, we deduce that Y^α is a Markov process, and that its transition semi-group $(Q_t, t \in \mathbb{R}_+)$ is the unique Feller semi-group such that $p_t = p_0 Q_t$ for $t \in \mathbb{R}_+$, with p_t the distribution of Y_t^α. □

Remark 4.8. *Let us stress that if we consider the Markov process* $(Y^{\alpha,(y,k)}, S^{\alpha,(y,k)})$*, the solution of the SDE* (4.3) *starting from* $(y, k) \in \mathbb{R}_+ \times \mathbb{N}$*, then there is no guarantee that* $Y^{\alpha,(y,k)}$ *is Markov if* $y > 0$*. However, for* $y > 0$*, if* ξ_y *is a* $\mathbb{N}$*-valued random variable with distribution given by* (4.7)*, and we consider the process* $(Y^{\alpha,(y,\xi_y)}, S^{\alpha,(y,\xi_y)})$ *with random initial condition* (y, ξ_y)*, according to Lemma 4.6 and the Markov property, we deduce that* $Y^{\alpha,(y,\xi_y)}$ *is indeed Markov with transition semigroup* $(Q_t, t \in \mathbb{R}_+)$ *and initial condition* y*.*

We now compare the distribution of Y^α and the distribution of the Feller diffusion Y defined in Remark 2.1, which is a CSBP with parameter $\beta = 1$ and $\theta = 0$. Following (3.2), we set for $t > 0$:

$$M_t^\alpha = \mathcal{H}^\alpha(t, Y_t) = \mathrm{e}^{-\alpha t}\, Y_t\, \mathrm{B}_0(\alpha Y_t).$$

Let $\mathbb{N}$ denote the canonical measure of Y.

Lemma 4.9. *Let* $\alpha > 0$*. Let* $t_0 > 0$*. The process* $(Y_t^\alpha, t \in [0, t_0])$ *has the same distribution as the process* $(Y_t, t \in [0, t_0])$ *under* $\mathbb{N}\left[\bullet\, M_{t_0}^\alpha\right]$*.*

Proof. Notice that $Y_0 = 0$ and that the process $(Y_t, t \in [0, t_0])$ is continuous $\mathbb{N}$-a.e. and thus $\mathbb{N}\left[\bullet\, M_{t_0}^\alpha\right]$-a.e. We first check that the two processes have the same one-dimensional marginals. Clearly $Y_0^\alpha = Y_0 = 0$. Let $t > 0$. According to Lemma 2.3, the entrance law of Y_t under $\mathbb{N}$ has density $y \mapsto t^{-2}\, \mathrm{e}^{-y/t}$. Recall the density p_t defined in (4.6). We deduce that for $\lambda \geq 0$:

$$\mathbb{N}\left[\mathrm{e}^{-\lambda Y_t}\, M_t^\alpha\right] = \int_{\mathbb{R}_+} \mathrm{e}^{-\lambda y}\ \mathcal{H}^\alpha(t, y)\, t^{-2}\, \mathrm{e}^{-y/t}\ \mathrm{d}y = \int_{\mathbb{R}_+} \mathrm{e}^{-\lambda y}\ p_t(\mathrm{d}y) = \mathbb{E}\left[\mathrm{e}^{-\lambda Y_t^\alpha}\right].$$

Since the Laplace transform characterizes the probability distribution on $\mathbb{R}_+$, we obtain that Y_t^α has the same distribution as Y_t under $\mathbb{N}\left[\bullet\, M_t^\alpha\right]$.

Using Doob's h-transform, we get that the process $(Y_t, t \in [0, t_0])$ under $\mathbb{N}\left[\bullet\, M_{t_0}^\alpha\right]$ is Markov. Using that M^α is a martingale under $\mathbb{N}$ (see Proposition 3.1 and use that Y is distributed as Z when $\beta = 1$, $\theta = 0$), that M_t^α is a function of Y_t, and that Y is Markov under $\mathbb{N}$, we get that $(Y_t, t \in [0, t_0])$ under $\mathbb{N}\left[\bullet\, M_{t_0}^\alpha\right]$ is also Feller. We deduce from the uniqueness property of Lemma 4.7 and the identification of the one-dimensional marginals from the first step of the proof, that $(Y_t^\alpha, t \in [0, t_0])$ has the same distribution as $(Y_t, t \in [0, t_0])$ under $\mathbb{N}\left[\bullet\, M_{t_0}^\alpha\right]$. ☐

We can now give the proof of Proposition 4.1. Let $\beta, \alpha > 0$, $\theta \in \mathbb{R}$ and $t_0 > 0$. Using the time changes given by Remark 2.1 and (4.4), we deduce that the process $(Y_t^{\alpha,\theta}, t \in [0, t_0])$ is distributed as the process $(Z_t, t \in [0, t_0])$ under $\mathbb{N}^\theta\left[\bullet\, M_{t_0}^{\alpha,\theta}\right]$.

4.3 Proof of Corollary 4.2

We first consider the case $\theta = 0$. Recall that $Y_0^\alpha = S_0^\alpha = 0$. Since Y^α satisfies (4.3), then by Lemma 4.6 and Remark 4.8, the Markov process $(\tilde{Y}_t^{\alpha,y}, t \geq 0)$ starting from $y \geq 0$ with the same semigroup as Y^α solves the following equation:

$$\mathrm{d}\tilde{Y}_t^{\alpha,y} = \sqrt{2\tilde{Y}_t^{\alpha,y}}\, \mathrm{d}B_t + 2(S_t^\alpha + \xi^y)\, \mathrm{d}t, \quad t \geq 0, \quad \text{and} \quad \tilde{Y}_0^{\alpha,y} = y,$$

where S^α and ξ^y are independent and (since $S_0^\alpha = 0$) for $k \in \mathbb{N}$:

$$\mathbb{P}(\xi^y = k + 1) = \frac{1}{\mathrm{B}_0(\alpha y)}\ \frac{(\alpha y)^k}{k!(k+1)!}.$$

We deduce that $\mathbb{E}[\xi^y] = g_\alpha(y)$ with:

$$g_\alpha(y) = \frac{1}{\mathrm{B}_0(\alpha y)} \sum_{k=0}^{\infty} \frac{(\alpha y)^k}{(k!)^2} = y\, \partial_y \log(\mathcal{H}^\alpha(\cdot, y)),$$

where $\partial_y \log(\mathcal{H}^\alpha(t, y))$ does not depend on t.

As g_α is finite, we deduce that $\mathbb{E}[\tilde{Y}_t^{\alpha,y}] = 2g_\alpha(y)t + \alpha t^2$ is also finite. Since Y^α is a Feller process by Lemma 4.7, we can consider its semi-group, say $(Q_t, t \in \mathbb{R}_+)$, which is also the semi-group of $\tilde{Y}^{\alpha,y}$, and its corresponding infinitesimal generator, say $\mathcal{A}$, defined on the set C_0 of real-valued continuous function defined on $\mathbb{R}_+$ and vanishing at infinity (possibly on a subspace). Notice the process $\tilde{Y}^{\alpha,y}$ is continuous. We consider the Dynkin's characteristic operator $\mathcal{U}$ defined by, for $y \in \mathbb{R}_+$:

$$\mathcal{U}f(y) = \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}_y[f(\tilde{Y}^{\alpha,y}_{\tau_{y,\varepsilon}})] - f(y)}{\mathbb{E}_y[\tau_{y,\varepsilon}]},$$

when the limit exists, and $\tau_{y,\varepsilon} = \inf\{t > 0 : |\tilde{Y}^{\alpha,y} - y| > \varepsilon\}$. The domain $\mathcal{D}_\mathcal{U}$ is defined to be the set of those $f \in C_0$ for which $\mathcal{U}f(y)$ exists for all $y \in \mathbb{R}_+$ and $\mathcal{U}f \in C_0$, see [40, Section III.12]. According to Dynkin's theorem, see Theorem III.12.2 therein, we also have that $\mathcal{A} = \mathcal{U}$ (with the same domain).

Let C_K^∞ be the set of all infinitely differentiable functions f with compact support in $\mathbb{R}_+$. By conditioning on (S^α, ξ^y) and applying Itô's formula, one easily get that for any $f \in C_K^\infty$ and $y \geq 0$:

$$\mathcal{A}f(y) = y\, f''(y) + 2g_\alpha(y)\, f'(y).$$

Now for the second order differential operator $y\, \partial_y^2 + 2g_\alpha(y)\, \partial_y$, using Feller's conditions for the classification of boundaries of one-dimensional diffusion, see [30, Table 6.2 p.234], we easily get that 0 is an entrance boundary point (that is, 0 is not accessible but it is possible to start the process from 0) and that $+\infty$ is a natural boundary point; thus both 0 and $+\infty$ are inaccessible.

The drift coefficient g_α has a continuous derivative; but it is not bounded, so one can not directly use [23], see Theorem 1.1 in Section 8.1 and Theorem 2.1, to completely identify the generator. However, stopping the process $\tilde{Y}^{\alpha,y}$ when it reaches a large level, say M, one can show that up to this stopping time the process $\tilde{Y}^{\alpha,y}$, with $y < M$, is a diffusion which is distributed as the solution of the SDE $\mathrm{d}\hat{Y}_t^{\alpha,y} = \sqrt{2\hat{Y}_t^{\alpha,y}}\, \mathrm{d}B_t + 2g_\alpha(\hat{Y}_t^{\alpha,y})\, \mathrm{d}t$ starting from $\hat{Y}_0^{\alpha,y} = 0$. We leave this details to the interested reader.

We then deduce that Y^α is a diffusion starting from $Y_0^\alpha = 0$ and for $t \geq 0$:

$$\mathrm{d}Y_t^\alpha = \sqrt{2Y_t^\alpha}\, \mathrm{d}B_t + 2g_\alpha(Y_t^\alpha)\, \mathrm{d}t.$$

That is, the proces Y^α is a solution to the SDE (4.2) with $\theta = 0$. Recall from (4.4) that $Y_t^{\alpha,\theta} = \mathrm{e}^{-2\beta\theta t}\, Y_s^\alpha$ with $s = 1/c_t^\theta = (\mathrm{e}^{2\beta\theta t} - 1)/2\theta$. With this deterministic time-change, we deduce that the process $Y^{\alpha,\theta}$ satisfies (4.2).

5 Backbone decomposition

We introduce basic facts on the space of real trees in Section 5.1. We recall some properties of the Brownian CRT in Section 5.2. We give in Section 5.4 a recursive construction of some discrete random trees using a grafting procedure defined in Section 5.3. Let us stress that the measurable and topological properties of the grafting procedure, as well as its formal definition, are discussed in detail in Section 6.3. In Section 5.5, we provide a decomposition of a (sub)critical Brownian CRT according to n leaves at a given distance from the root and uniformly chosen at random, this is a generalization of the case $n = 1$ from [19, Theorem 4.5]. We prove our main results in Section 5.6 on the local convergence of the Brownian CRT conditioned to have a large population at time t, as t goes to infinity.

5.1 Notations for trees

5.1.1 Real trees

We use the framework of real trees to encode the genealogy of a continuous state branching process. We refer to [24] for a detailed introduction to real trees.

A real tree (or simply a tree in the rest of the text) is a metric space (T, d) that satisfies the two following properties for every $u, v \in T$:

(i) There is a unique isometric map $f_{u,v}$ from $[0, d(u, v)]$ into T such that:

$$f_{u,v}(0) = u \quad \text{and} \quad f_{u,v}\big(d(u, v)\big) = v.$$

(ii) If φ is a continuous injective map from $[0, 1]$ into T such that $\varphi(0) = u$ and $\varphi(1) = v$, then the range of φ is also the range of $f_{u,v}$.

The range of the map $f_{u,v}$ is denoted by $[\![u, v]\!]$. It is the unique continuous path that links u to v in the tree. We will write $[\![u, v[\![$ (resp. $]\!]u, v]\!]$, $]\!]u, v[\![$) for $[\![u, v]\!] \setminus \{v\}$ (resp. $[\![u, v]\!] \setminus \{u\}$, $[\![u, v]\!] \setminus \{u, v\}$).

A rooted tree is a tree (T, d) with a distinguished vertex denoted by ϱ and called the root. We always consider rooted trees in this work. For an element x of a rooted tree (T, d, ϱ), we denote by $H(x) = d(\varrho, x)$ its height, and we set $H(T) = \sup_{x \in T} H(x)$ the height of the tree T.

An element x of $T \setminus \{\varrho\}$ is a leaf if $T \setminus \{x\}$ has only one connected component; by convention the root is a leaf if and only if T is reduced to the root. We denote by $\mathrm{Lf}(T)$ the (non-empty) set of leaves of T. The skeleton of the tree is the set $\mathrm{Sk}(T) = T \setminus \mathrm{Lf}(T)$. The set of branching points (or vertices) $\mathrm{Br}(T)$ is the set of $x \in T$ such that $T \setminus \{x\}$ has at least 3 connected components if $x \neq \varrho$ or at least 2 components if $x = \varrho$.

For a vertex $x \in T$, we define the subtree T_x "above" x as:

$$T_x = \big\{y \in T : \ x \in [\![\varrho, y]\!]\big\}.$$

The real tree T_x is endowed with the distance induced by T and will be rooted at x. If $u, v \in T$, we denote by $u \wedge v$ the most recent common ancestor of u and v, *i.e.* the unique vertex of T such that:

$$[\![\varrho, u]\!] \cap [\![\varrho, v]\!] = [\![\varrho, u \wedge v]\!].$$

If (T, d, ϱ) is a rooted real tree and a is a positive real number, we define the scaled tree aT as:

$$aT = (T, ad, \varrho) \tag{5.1}$$

where all the distances in the tree T are multiplied by the factor a.

The trace of the Borel σ-field of T on $\mathrm{Sk}(T)$ is generated by the sets $[\![s, s']\!]$, $s, s' \in \mathrm{Sk}(T)$ (see [25]). Hence, there exists a σ-finite Borel measure $\mathscr{L}^T$ on T, such that:

$$\mathscr{L}^T\big(\mathrm{Lf}(T)\big) = 0 \quad \text{and} \quad \mathscr{L}^T\big([\![s, s']\!]\big) = d(s, s').$$

This measure $\mathscr{L}^T$ is called the length measure on T. When there is no ambiguity, we simply write $\mathscr{L}$ for $\mathscr{L}^T$.

5.1.2 Gromov-Hausdorff distance and sets of trees

We endow the set of (isometry equivalence classes) of rooted real trees with the classical Gromov-Hausdorff distance whose definition (with the notion of correspondences) is described below. We refer to [31] for general results on Gromov-Hausdorff metrics.

Let (T, d, ϱ) and (T', d', ϱ') be two rooted compact real trees. A correspondence $\mathcal{R}$ between T and T' is a subset of $T \times T'$ such that:

(i) for all $x \in T$, there exists $x' \in T'$ such that $(x, x') \in \mathcal{R}$,

(ii) for all $x' \in T'$, there exists $x \in T$ such that $(x, x') \in \mathcal{R}$,

(iii) $(\varrho, \varrho') \in \mathcal{R}$.

The distortion of such a correspondence $\mathcal{R}$ is defined as:

$$\mathrm{dist}(\mathcal{R}) = \sup \left\{ \left| d(x,y) - d'(x',y') \right|;\ (x,x'),(y,y') \in \mathcal{R} \right\}.$$

For two compact rooted trees (T, d, ϱ) and (T', d', ϱ') we set:

$$d_{GH}(T, T') = \inf \frac{1}{2} \mathrm{dist}(\mathcal{R}),$$

where the infimum is taken over all the correspondences between (T, d, ϱ) and (T', d', ϱ'). The function d_{GH} is the so-called Gromov-Hausdorff pseudo-distance, see [35]. Furthermore, we have that $d_{\mathrm{GH}}(T, T') = 0$ if and only if there exists an isometric bijection from (T, d) to (T', d') which preserves the root. The relation $d_{\mathrm{GH}}(T, T') = 0$ defines an equivalence relation between compact rooted trees. The set $\mathbb{T}_{\mathrm{K}}$ of equivalence classes of compact rooted trees endowed with d_{GH} is then a metric Polish space, see [35, Proposition 9]. We shall consider below the trivial tree $\mathrm{T}_0 \in \mathbb{T}_{\mathrm{K}}$ reduced to its root.

We can generalize this definition to compact n-pointed rooted trees where a n-pointed rooted tree is a triplet $(T, d, \mathbf{v})$ where (T, d) is a rooted real tree and $\mathbf{v} = (v_0, v_1, \dots, v_n)$ with that $v_0 = \varrho$ is the root of T and $v_1, \dots, v_n$ are n distinguished (possibly equal) vertices. A correspondence between two n-pointed rooted trees $(T, d, \mathbf{v})$ and $(T', d', \mathbf{v}')$ is a correspondence between (T, d, ϱ) and (T', d', ϱ') which satisfies moreover $(v_i, v_i') \in \mathcal{R}$ for all $i \in \{1, \dots, n\}$, where $\mathbf{v}' = (v_0', v_1', \dots, v_n')$ with $v_0' = \varrho'$, the root of T'. The distance $d_{\mathrm{GH}}^{(n)}$ on the space $\mathbb{T}_{\mathrm{K}}^{(n)}$ of equivalence classes of compact n-pointed rooted trees is then defined in the same way as d_{GH} on $\mathbb{T}_{\mathrm{K}}$, and following the arguments of [35] one get that the metric space $(\mathbb{T}_{\mathrm{K}}^{(n)}, d_{\mathrm{GH}}^{(n)})$ is Polish; and notice that $(\mathbb{T}_{\mathrm{K}}, d_{\mathrm{GH}}) = (\mathbb{T}_{\mathrm{K}}^{(0)}, d_{\mathrm{GH}}^{(0)})$.

For a rooted n-pointed tree $(T, d, \mathbf{v})$ and $t \geq t_T = \max_{i \in \{0, \dots, n\}} d(\varrho, v_i)$, we define the rooted n-pointed tree T truncated at level t as $(r_t(T, \mathbf{v}), d, \mathbf{v})$ with:

$$r_t(T, \mathbf{v}) = \left\{ x \in T :\ H(x) \leq t \right\}, \tag{5.2}$$

and the distance on $r_t(T, \mathbf{v})$ is given by the restriction of the distance d. We shall simply write $r_t(T, \mathbf{v})$ for $(r_t(T, \mathbf{v}), d, \mathbf{v})$. By the Hopf-Rinow theorem, the tree $r_t(T, \mathbf{v})$ is a compact n-pointed rooted tree for all $t \geq t_T$ if the rooted n-pointed tree $(T, d, \mathbf{v})$ is complete and locally compact, see [6] and references therein. Two complete locally compact trees $(T, d, \mathbf{v})$ and $(T', d', \mathbf{v}')$ are equivalent if and only if there exists an isometric one-to-one map from (T, d) to (T', d') which preserves the distinguished vertices. This defines indeed an equivalence relation. The set $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ of equivalence classes of complete locally compact rooted trees is then endowed with a distance $d_{\mathrm{LGH}}^{(n)}$ in the spirit of [6], see Section 6.2 below and more precisely Proposition 6.4, so that it is a metric Polish space and $\mathbb{T}_{\mathrm{K}}^{(n)}$ is an open dense subset of $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$. For $n = 0$, we simply write $\mathbb{T}_{\mathrm{loc-K}}$ and d_{LGH} for $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ and $d_{\mathrm{LGH}}^{(n)}$. We shall consider below the infinite spine tree $\mathrm{T}_1 = (\mathbb{R}_+, |\cdot|, 0) \in \mathbb{T}_{\mathrm{loc-K}}$, where $|\cdot|$ is the usual Euclidean distance.

5.1.3 Grafting operation

We recall the grafting operation of [2]. Let $(T, d, (\varrho, x))$ be a complete locally compact rooted 1-pointed tree and (T', d', ϱ') be complete locally compact rooted trees. We define the tree $T \circledast_x T'$ as the tree obtained by grafting T' on the tree T at vertex x. We set:

$$T \circledast_x T' = T \sqcup \big(T' \setminus \{\varrho'\}\big), \tag{5.3}$$

$$\forall y, y' \in T \circledast_x T',\ d^{\circledast}(y, y') = \begin{cases} d(y, y') & \text{if } y, y' \in T, \\ d'(y, y') & \text{if } y, y' \in T', \\ d(y, x) + d'(\varrho', y') & \text{if } y \in T,\ y' \in T', \end{cases} \tag{5.4}$$

where $\sqcup$ denotes the disjoint union of two sets. By construction $(T \circledast_x T', d^{\circledast}, \varrho)$ is a complete locally compact rooted tree. It is easy to see that the equivalence class of $T \circledast_x T'$ does not depend of the choice of the representatives in the equivalence classes of T and T' and hence the grafting operation is well-defined on $\mathbb{T}_{\mathrm{loc-K}}$; it is even continuous, see Lemma 7.1. We also refer to Section 6.3 for a more general grafting procedure and its topological properties.

Let $(T, \mathbf{v}) \in \mathbb{T}^{(n)}_{\mathrm{loc-K}}$ be either the infinite spine tree T_1 (and $n = 0$) or a discrete tree (and $n \in \mathbb{N}^*$), that is, a compact rooted real tree with all its leaves being distinguished, see (6.11) for a more formal definition. Let $\mathcal{M} = \sum_{i \in I} \delta_{(x_i, T_i)}$ be a point measure on the $T \times \mathbb{T}_{\mathrm{loc-K}}$. We define intuitively the tree $\mathrm{Graft}_n((T, \mathbf{v}), \mathcal{M})$ as the tree:

$$\mathrm{Graft}_n((T, \mathbf{v}), \mathcal{M}) = \big(T \circledast_{x_i, i \in I} (T_i, i \in I), \mathbf{v}\big) \tag{5.5}$$

obtained by grafting each complete locally compact rooted tree T_i on T at point x_i (and keeping the n distinguished elements $\mathbf{v}$ of T). It is not clear that the resulting tree belongs to $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$ (some assumptions must be added to $\mathcal{M}$) nor that this infinite grafting procedure can be proceeded in a measurable way (so that we get indeed a random tree when the tree T and the point measure $\mathcal{M}$ are random). We give a formal definition of this procedure in Section 7.2 and check that it is well defined (after some lengthy topological preliminaries) with good measurable property in $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$, where $\mathcal{M}$ is a particular Poisson point measure considered in the context of the backbone decomposition from Section 5.5. Even if the presentation (5.5) is abusive, we stick to this informal definition for simplicity.

5.2 Brownian CRTs and Kesten trees

Brownian CRTs are random trees in $\mathbb{T}_{\mathrm{loc-K}}$ that encode the genealogy of continuous-state branching processes.

Before recalling the definition of such trees, we give some additional notation. For a complete locally compact rooted tree $\mathbf{t}$, we define the population at level a as the subset:

$$\mathcal{Z}_{\mathbf{t}}(a) = \{u \in \mathbf{t},\ H(u) = a\}.$$

We denote by $(\mathbf{t}^{(i),*}, i \in I)$ the connected components of the open set $\mathbf{t} \setminus r_a(\mathbf{t})$. For every $i \in I$, let ϱ_i be the MRCA of $\mathbf{t}^{(i),*}$, which is equivalently characterized by $[\![\varrho, \varrho_i]\!] = \bigcap_{u \in \mathbf{t}^{(i),*}} [\![\varrho, u]\!]$; notice that $\varrho_i \in \mathcal{Z}_{\mathbf{t}}(a)$. We then set $\mathbf{t}^{(i)} = \mathbf{t}^{(i),*} \cup \{\varrho_i\}$ so that $\mathbf{t}^{(i)}$ is a complete locally compact rooted tree with root ϱ_i, and we consider the point measure on $\mathcal{Z}_{\mathbf{t}}(a) \times \mathbb{T}_{\mathrm{loc-K}}$:

$$\mathcal{N}^{\mathbf{t}}_a = \sum_{i \in I} \delta_{(\varrho_i, \mathbf{t}^{(i)})}.$$

We then recall the definition of the excursion measure $\mathbb{N}^\theta$ for $\beta > 0$ and $\theta \geq 0$ associated with a Brownian CRT from [19]. The underlying parameter β is fixed, and will be omitted from the notation. There exists a measure $\mathbb{N}^\theta$ on $\mathbb{T}_{\mathrm{K}}$ (and hence on $\mathbb{T}_{\mathrm{loc-K}}$) such that:

(i) **Existence of a local time.** For every $a \geq 0$ and for $\mathbb{N}^\theta[\mathrm{d}\mathcal{T}]$-a.e. $\mathcal{T}$, there exists a finite measure Λ_a on $\mathcal{T}$ such that

  (a) $\Lambda_0 = 0$ and, for every $a > 0$, Λ_a is supported on $\mathcal{Z}_\mathcal{T}(a)$.

  (b) For every $a > 0$, $\mathbb{N}^\theta[\mathrm{d}\mathcal{T}]$-a.e., we have $\{\Lambda_a \neq 0\} = \{H(\mathcal{T}) > a\}$.

  (c) For every $a > 0$, $\mathbb{N}^\theta[\mathrm{d}\mathcal{T}]$-a.e., we have for every continuous function φ on $\mathcal{T}$:

$$\begin{aligned}\langle \Lambda_a, \varphi \rangle &= \lim_{\varepsilon \to 0+} \frac{1}{c_\varepsilon^\theta} \int \mathcal{N}_a^\mathcal{T}(\mathrm{d}u, \mathrm{d}\mathcal{T}')\varphi(u)\mathbf{1}_{\{H(\mathcal{T}') \geq \varepsilon\}} \\ &= \lim_{\varepsilon \to 0+} \frac{1}{c_\varepsilon^\theta} \int \mathcal{N}_{a-\varepsilon}^\mathcal{T}(\mathrm{d}u, \mathrm{d}\mathcal{T}')\varphi(u)\mathbf{1}_{\{H(\mathcal{T}') \geq \varepsilon\}}.\end{aligned}$$

(ii) **Branching property.** For every $a > 0$, the conditional distribution of the point measure $\mathcal{N}_a^\mathcal{T}(\mathrm{d}u, \mathrm{d}\mathcal{T}')$, under the probability measure $\mathbb{N}^\theta[\mathrm{d}\mathcal{T} \mid H(\mathcal{T}) > a]$ and given $r_a(\mathcal{T})$, is that of a Poisson point measure on $\mathcal{Z}_\mathcal{T}(a) \times \mathbb{T}_{\mathrm{loc-K}}$ with intensity $\Lambda_a(\mathrm{d}u)\mathbb{N}^\theta[\mathrm{d}\mathcal{T}']$.

(iii) **Regularity of the local time process.** We can choose a modification of the process $(\Lambda_a, a \geq 0)$ in such a way that the map $a \longmapsto \Lambda_a$ is $\mathbb{N}^\theta[\mathrm{d}\mathcal{T}]$-a.e. continuous for the weak topology of finite measures on $\mathcal{T}$.

(iv) **Link with CSBP.** Under $\mathbb{N}^\theta[\mathrm{d}\mathcal{T}]$, the process $(\langle \Lambda_a, 1 \rangle, a \geq 0)$ is distributed as a CSBP under its canonical measure with branching mechanism:

$$\psi(\lambda) = \beta\lambda^2 + 2\beta\theta\lambda, \quad \lambda \geq 0.$$

We now extend the definition of the measure $\mathbb{N}^\theta$ (only on $\mathbb{T}_{\mathrm{loc-K}}$) for $\theta < 0$ by a Girsanov transform, following [7]. For $t \geq 0$, set $\mathcal{G}_t = \sigma(r_t(\mathcal{T}))$ and $Z_t = \Lambda_t(\mathcal{T})$, the latter notation is consistent with Section 2.2. The CSBP process $Z = (Z_t, t \geq 0)$ is Markov with respect to the filtration $(\mathcal{G}_t, t \geq 0)$. For $\theta < 0$ and $t > 0$, we set:

$$\mathbb{N}^{-\theta}[\mathrm{d}\mathcal{T}]_{|\mathcal{G}_t} = \mathrm{e}^{2\theta Z_t}\, \mathbb{N}^\theta[\mathrm{d}\mathcal{T}]_{|\mathcal{G}_t}. \tag{5.6}$$

Then properties (i) to (iv) still hold for every $\theta \in \mathbb{R}$. This Girsanov transform is consistent with the Girsanov transform of CSBPs given by (2.12). Let us stress that the measure $\mathbb{N}^\theta$ on $\mathbb{T}_{\mathrm{loc-K}}$ depends also on the parameter $\beta > 0$.

The so-called Kesten tree with parameters $(\beta, \theta) \in \mathbb{R}_+^* \times \mathbb{R}_+$, which appears first in [8] for $\theta = 0$ and $\beta = 1/2$ as the self-similar continuum random tree, can also be obtained as the genealogical tree associated with the continuous-state branching process with the same parameters, conditioned on non-extinction (see for instance [33]). This latter process can also be constructed by adding to the initial process a particular immigration. We use this second approach to extend the definition of the Kesten tree for $\theta < 0$.

Using our framework, the Kesten tree with parameters $(\beta, \theta) \in \mathbb{R}_+^* \times \mathbb{R}$ is built as a countable family of trees defined by a Poisson point measure grafted on the infinite spine tree T_1:

$$\mathcal{T}^* = \mathrm{Graft}_0(\mathrm{T}_1, \mathcal{M}), \tag{5.7}$$

with $\mathcal{M}(\mathrm{d}h, \mathrm{d}T)$ a Poisson point measure on $\mathbb{R}_+ \times \mathbb{T}_{\mathrm{loc-K}}$ with intensity $2\beta\mathbf{1}_{\{h>0\}}\mathrm{d}h\, \mathbb{N}^\theta[\mathrm{d}T]$. We refer to Section 7.2 for a more formal presentation which in particular implies that the Kesten tree is a $\mathbb{T}_{\mathrm{loc-K}}$-valued random variable, see Lemma 7.2.

5.3 The set of (planar) discrete trees

A discrete tree is a compact rooted tree with a finite number of leaves. We denote by $\mathbb{T}_{\text{dis}}^{(n)}$ the subset of n-pointed discrete tree whose leaves are distinguished (see (6.11) for a formal definition): for $(\mathbf{t}, \mathbf{v}) \in \mathbb{T}_{\text{dis}}^{(n)}$ with $\mathbf{v} = (v_0 = \varrho, \ldots, v_n)$, we have that $\text{Lf}(\mathbf{t}) \subset \{v_0, \ldots, v_n\}$. According to Lemma 6.11, the set $\mathbb{T}_{\text{dis}}^{(n)}$ is closed. We can consider a discrete tree with a planar structure by enumerating its leaves, or more precisely its distinguished vertices, "from the left to the right". This will allow us to define oriented grafting; this will be used in the next section. Intuitively a discrete tree $(\mathbf{t}, \mathbf{v}) \in \mathbb{T}_{\text{dis}}^{(n)}$ is a planar tree if for all $x \in \mathbf{t}$, there exists $0 \leq i_{\text{g}} \leq i_{\text{d}} \leq n$ such that $v_i \in \mathbf{t}_x$ if and only if $i \in \{i_{\text{g}}, \ldots, i_{\text{d}}\}$; we check in Section 7.3 that the set of (equivalence classes of) n-pointed planar tree $\mathbb{T}_{\text{plan}}^{(n)} \subset \mathbb{T}_{\text{dis}}^{(n)}$ is also closed.

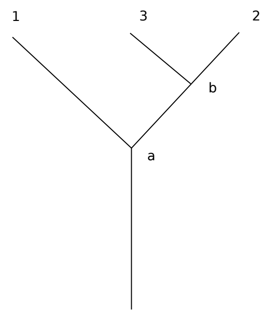

Figure 1: A tree $(\mathbf{t}, \mathbf{v}) \in \mathbb{T}_{\text{plan}}^{(3)}$ with $\mathbf{v} = (\varrho, 1, 3, 2)$

We now define an oriented grafting of a discrete tree $\mathbf{t}'$ on a discrete tree $\mathbf{t}$ at point $x \in \mathbf{t}$; we shall use later on this construction for planar trees; this is similar to the first grafting defined in Section 5.1.3 but for the ordering of the distinguished vertices. Formally, if $(\mathbf{t}, \mathbf{v})$ be an n-pointed discrete tree with $\mathbf{v} = (v_0 = \varrho, v_1, \ldots, v_n)$, $(\mathbf{t}', \mathbf{v}')$ an m-pointed discrete tree with $\mathbf{v}' = (v'_0 = \varrho', v'_1, \ldots, v'_m)$ and $x \in \mathbf{t}$, we define for $\varepsilon \in \{\text{g}, \text{d}\}$:

$$(\mathbf{t}, \mathbf{v}) \circledast_x^{\varepsilon} (\mathbf{t}', \mathbf{v}') = (\mathbf{t} \circledast_x \mathbf{t}', \mathbf{v} \circledast^{\varepsilon} \mathbf{v}') \in \mathbb{T}_{\text{dis}}^{(n+m)} \tag{5.8}$$

with $\mathbf{t} \circledast_x \mathbf{t}'$ defined in (5.3) and:

$$\mathbf{v} \circledast^{\text{g}} \mathbf{v}' = (v_0, \ldots, v_{i_{\text{g}}-1}, v'_1, \ldots, v'_m, v_{i_{\text{g}}}, \ldots, v_n), \tag{5.9}$$

$$\mathbf{v} \circledast^{\text{d}} \mathbf{v}' = (v_0, \ldots, v_{i_{\text{d}}}, v'_1, \ldots, v'_m, v_{i_{\text{d}}+1}, \ldots, v_n), \tag{5.10}$$

where:

$$i_{\text{g}} = \min\{i \in \{0, \ldots, n\} : \; v_i \in \mathbf{t}_x\} \quad \text{and} \quad i_{\text{d}} = \max\{i \in \{0, \ldots, n\} : \; v_i \in \mathbf{t}_x\}, \tag{5.11}$$

and the convention that if $i_{\text{g}} = 0$ (that is, $x = \varrho$), then $\mathbf{v} \circledast^{\text{g}} \mathbf{v}' = (v_0, v'_1, \ldots, v'_m, v_1, \ldots, v_n)$, and if $i_{\text{d}} = n$, then $\mathbf{v} \circledast^{\text{d}} \mathbf{v}' = (v_0, \ldots, v_n, v'_1, \ldots, v'_m)$. Let us stress that i_{g} and i_{d} are well defined as all the leaf are distinguished. Notice also that if $(\mathbf{t}, \mathbf{v})$ and $(\mathbf{t}', \mathbf{v}')$ are planar, so is $(\mathbf{t}, \mathbf{v}) \circledast_x^{\varepsilon} (\mathbf{t}', \mathbf{v}')$.

Furthermore, for $i \in \{1, \ldots, n\}$ and $h \leq H(v_i)$, we shall consider the grafting of $\mathbf{t}'$ at $x_{i,h} \in \mathbf{t}$ the point of $[\![\varrho, v_i]\!]$ at height h:

$$(\mathbf{t}, \mathbf{v}) \circledast_{i,h}^{\varepsilon} (\mathbf{t}', \mathbf{v}') = (\mathbf{t}, \mathbf{v}) \circledast_{x_{i,h}}^{\varepsilon} (\mathbf{t}', \mathbf{v}'). \tag{5.12}$$

Notice this latter grafting is well defined on the equivalent classes of discrete trees, and it is measurable thanks to Lemma 7.3.

5.4 A discrete random tree constructed by successive grafts

5.4.1 A random tree

In this section, for $a \geq 0$, we denote by $\big([0,a],(0,a)\big) \in \mathbb{T}^{(1)}_{\text{dis}}$ the (equivalent class of the) tree $[0,a]$ endowed with the usual distance on $\mathbb{R}$, rooted at $\varrho = 0$ and pointed at a; and when there is no possible confusion we simply denote it by $[0,a]$.

Let $t > 0$ and let ν be a probability measure on $[0,t]$. Let $\xi = (\xi_k,\, k \in \mathbb{N}^*)$ be a sequence of independent random variables with distribution ν and let $\big((K_k,\varepsilon_k),\, k \in \mathbb{N}^*\big)$ be a sequence of independent random variables independent of the sequence ξ, with K_k uniformly distributed on $\{1,\dots,k\}$ independent of ε_k uniformly distributed on $\{\mathrm{g},\mathrm{d}\}$. For every integer $n \geq 2$, we set $(\xi^{(n)}_1,\dots,\xi^{(n)}_{n-1})$ the increasing order statistic of $(\xi_1,\dots,\xi_{n-1})$. Then we define the family of pointed planar trees $\big((\mathbf{T}^{(n)}_1,\mathbf{v}^{(n)}_1),\dots,(\mathbf{T}^{(n)}_n,\mathbf{v}^{(n)}_n)\big)$, with $(\mathbf{T}^{(n)}_k,\mathbf{v}^{(n)}_k) \in \mathbb{T}^{(k)}_{\text{plan}}$, recursively by:

- $\mathbf{T}^{(n)}_1 = [0,t]$, that is, $(\mathbf{T}^{(n)}_1,\mathbf{v}^{(n)}_1) = \big([0,t],(0,t)\big) \in \mathbb{T}^{(1)}_{\text{plan}}$.
- For every $k \in \{1,\dots,n-1\}$, conditionally given the random variable $(\mathbf{T}^{(n)}_k,\mathbf{v}^{(n)}_k)$ in $\mathbb{T}^{(k)}_{\text{plan}}$, we define the $\mathbb{T}^{(k+1)}_{\text{loc}-\text{K}}$-valued random variable $(\mathbf{T}^{(n)}_{k+1},\mathbf{v}^{(n)}_{k+1})$ by grafting a branch of length $t-\xi^{(n)}_k$ uniformly on the left or on the right of a uniformly chosen vertex among the k vertices of $\mathbf{T}^{(n)}_k$ at level $\xi^{(n)}_k$, and the new leaf (which is, as all the other leaves, at level t) is added to the vector recording the distinguished vertices. Formally, using the grafting procedure (5.12) we set:

$$(\mathbf{T}^{(n)}_{k+1},\mathbf{v}^{(n)}_{k+1}) = (\mathbf{T}^{(n)}_k,\mathbf{v}^{(n)}_k) \circledast^{\varepsilon_k}_{K_k,\,\xi^{(n)}_k} \left[0,t-\xi^{(n)}_k\right]. \tag{5.13}$$

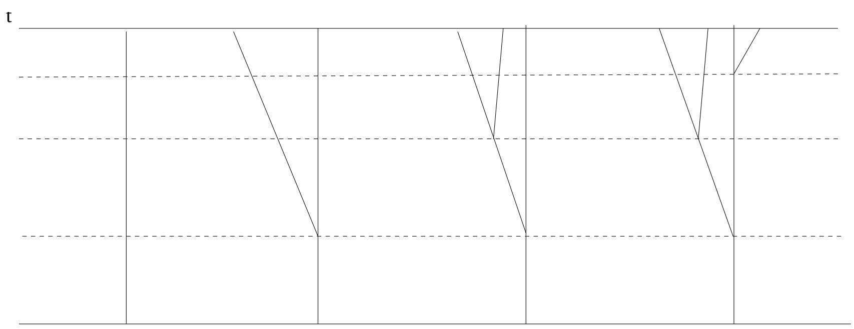

Figure 2: The trees $\mathbf{T}^{(4)}_1$, $\mathbf{T}^{(4)}_2$, $\mathbf{T}^{(4)}_3$ and $\mathbf{T}^{(4)}_4$ obtained from the sequences $(K_1 = 1, K_2 = 1, K_3 = 2)$ and $(\varepsilon_1 = \mathrm{g}, \varepsilon_2 = \mathrm{d}, \varepsilon_3 = \mathrm{d})$. The dashed lines represent the levels $\xi^{(4)}_1, \xi^{(4)}_2, \xi^{(4)}_3$.

By construction, we get that $(\mathbf{T}^{(n)}_k,\mathbf{v}^{(n)}_k)$ belongs to $\mathbb{T}^{(k)}_{\text{plan}}$ for all $k \in \{1,\dots,n\}$. To simplify the notations, we set $\mathbf{T}_n = (\mathbf{T}^{(n)}_n,\mathbf{v}^{(n)}_n)$.

Recall that for a rooted tree T, $\mathscr{L}^T$ denotes its length measure; and we simply write $\mathscr{L}$ when there is no ambiguity. The next lemma relates the distributions of $\mathbf{T}_n$ and of $\mathbf{T}_{n+1}$; its proof is given in the next section. Recall that $\mathbb{T}^{(n)}_{\text{plan}}$ is a subset of $\mathbb{T}^{(n)}_{\text{dis}}$.

Lemma 5.1. *Let $t \geq 0$. Assume that the probability distribution ν has a positive density f_{dens} with respect to the Lebesgue measure on $[0,t]$. For $n \in \mathbb{N}^*$, G a measurable non-negative function defined on $\mathbb{T}^{(n+1)}_{\text{dis}}$, and ε a random variable uniformly distributed on*

$\{\mathrm{g},\mathrm{d}\}$ *and independent of* $\mathbf{T}_n$*, we have:*

$$\mathbb{E}\left[\int_{\mathbf{T}_n} \mathscr{L}(\mathrm{d}x)\, f_{\mathrm{dens}}\big(H(x)\big)\, G\Big(\mathbf{T}_n \circledast_x^{\varepsilon} \big[0, t - H(x)\big]\Big)\right] = \frac{n+1}{2}\, \mathbb{E}\left[G\big(\mathbf{T}_{n+1}\big)\right]. \tag{5.14}$$

Remark 5.2. *We comment on the left-hand side of* (5.14)*. First notice the grafting on* $\mathbf{T}_n$ *is oriented, which is consistent with the fact that the discrete tree* $\mathbf{T}_n$ *is a planar tree. Second, we check that the integral*

$$\mathcal{I} = \int_{\mathbf{T}_n} \mathscr{L}(\mathrm{d}x)\, f_{\mathrm{dens}}(H(x))\, G\big(\mathbf{T}_n \circledast_x^{\varepsilon} [0, t - H(x)]\big)$$

is a non-negative random variable. Recall that $\mathbf{T}_n = (\mathbf{T}_n^{(n)}, \mathbf{v}_n^{(n)})$*. Then, we can write* $\mathcal{I}$ *as follows:*

$$\mathcal{I} = \sum_{k=1}^{n} \int_{\xi_{k-1}^{(n)}}^{t} \mathrm{d}h\, f_{\mathrm{dens}}(h)\, G\big(\mathbf{T}_n \circledast_{k,h}^{\varepsilon} [0, t-h]\big),$$

with the convention that $\xi_0^{(n)} = 0$*. Therefore, using the continuity of the grafting function, see Lemma 7.3, we obtain that* $\mathcal{I}$ *is a non-negative real-valued random variable, and thus its expectation is well defined.*

5.4.2 Proof of Lemma 5.1

The proof is based on two technical lemmas. We first consider the case $t = 1$ and ν the uniform distribution on $[0,1]$. Let us denote by $\mathbf{T}_n^{\mathrm{unif}}$ for $\mathbf{T}_n$ when ν is the uniform distribution on $[0,1]$.

Lemma 5.3. *For* $n \in \mathbb{N}^*$*,* G *a measurable non-negative functional defined on* $\mathbb{T}_{\mathrm{dis}}^{(n+1)}$*, and* ε *a* $\{\mathrm{g},\mathrm{d}\}$*-valued uniform random variable independent of* $\mathbf{T}_n^{\mathrm{unif}}$*, we have:*

$$\mathbb{E}\left[\int_{\mathbf{T}_n^{\mathrm{unif}}} \mathscr{L}(\mathrm{d}x)\, G\Big(\mathbf{T}_n^{\mathrm{unif}} \circledast_x^{\varepsilon} \big[0, t - H(x)\big]\Big)\right] = \frac{n+1}{2}\, \mathbb{E}\left[G\big(\mathbf{T}_{n+1}^{\mathrm{unif}}\big)\right]. \tag{5.15}$$

Remark 5.4. *From* (5.15)*, we see* $\frac{n+1}{2}$ *is just the mean length of* $\mathbf{T}_n^{\mathrm{unif}}$*.*

Proof. To simplify notation, we write $\mathbf{T}_n$ for $\mathbf{T}_n^{\mathrm{unif}}$. We give a proof by induction. For $n = 1$, this is a direct consequence of the construction of $\mathbf{T}_2 = \mathbf{T}_2^{(2)}$ from $\mathbf{T}_1^{(2)} = \mathbf{T}_1 = [0,1]$ given by (5.13).

Let $n \in \mathbb{N}^*$ and assume that (5.15) holds for n replaced by any $k \in \{1, \ldots, n-1\}$. We will use for the proof a special representation of planar binary trees. Let T be a compact planar binary tree rooted at ϱ, with all leaves at height 1; in particular the tree T has a finite number of leaves. If T has at least two leaves, since it is compact with a finite number of leaves, there exists a lowest branching vertex, say x. We set $h = H(x)$ and $\tilde{T}^{\mathrm{g}}$ (resp. $\tilde{T}^{\mathrm{d}}$) the left (resp. right) subtree above x. In our settings, we have:

$$h = H(x) \quad \text{and} \quad T = \big(⟦\varrho, x⟧ \circledast_x \tilde{T}^{\mathrm{g}}\big) \circledast_x^{\mathrm{d}} \tilde{T}^{\mathrm{d}} = \big(⟦\varrho, x⟧ \circledast_x \tilde{T}^{\mathrm{d}}\big) \circledast_x^{\mathrm{g}} \tilde{T}^{\mathrm{g}},$$

where, with a slight abuse of language (see Lemma 6.6 for formal justification), one has removed the vertex x from the distinguished vertices after the graftings. For convenience, we consider the scaled left and right trees $T^{\mathrm{g}} = (1-h)^{-1}\tilde{T}^{\mathrm{g}}$ and $T^{\mathrm{d}} = (1-h)^{-1}\tilde{T}^{\mathrm{d}}$ (recall (5.1) for the definition of a scaled tree), so that T^{g} and T^{d} are rooted bounded binary planar trees with all their leaves at height 1. We call $(h, T^{\mathrm{g}}, T^{\mathrm{d}})$ the decomposition of T according to its lowest branching vertex.

Let $(\xi_1^{(n+1)}, \mathbf{T}_{n+1}^{\mathrm{g}}, \mathbf{T}_{n+1}^{\mathrm{d}})$ be the decomposition of $\mathbf{T}_{n+1}$ according to its lowest branching vertex (which is indeed at height $\xi_1^{(n+1)}$ by construction). Denote by I_{n+1} the number of leaves of $\mathbf{T}_{n+1}^{\mathrm{g}}$. Using a Pólya urn starting with two balls of color g and d, we get that, by construction, I_{n+1} is the number of balls of color g in the urn after n draws. Thus I_{n+1} is uniform on $\{1, \dots, n\}$ and independent of $\xi_1^{(n+1)}$. Notice that if U is a uniform random variable on $[0,1]$, for every $h \in (0,1)$, conditionally given $\{U \geq h\}$, the random variable $(1-h)^{-1}(U-h)$ is still uniformly distributed on $[0,1]$. This gives that, conditionally on $\{\xi_1^{(n+1)} = h\}$ and $\{I_{n+1} = i\}$, the two trees $\mathbf{T}_{n+1}^{\mathrm{g}}$ and $\mathbf{T}_{n+1}^{\mathrm{d}}$ are independent and distributed respectively as $\mathbf{T}_i$ and $\mathbf{T}_{n+1-i}$.

We consider a measurable non-negative functional G defined on the space of rooted compact binary planar trees with a finite number of leaves, all of them at height 1 of the form:

$$G(T) = g_1(h)\, g_2(T^{\mathrm{g}})\, g_3(T^{\mathrm{d}}), \tag{5.16}$$

where the g_i's are measurable non-negative functionals and $(h, T^{\mathrm{g}}, T^{\mathrm{d}})$ is the decomposition of T according to its lowest branching vertex. Setting $f_j(i) = \mathbb{E}\left[g_j(\mathbf{T}_i)\right]$ for $j \in \{2,3\}$, we have since $\xi_1^{(n+1)}$ is distributed according to a $\beta(1,n)$ distribution (as the minimum of n independent uniform random variables):

$$\begin{aligned} \mathbb{E}\left[G(\mathbf{T}_{n+1})\right] &= \left(\int_0^1 g_1(h)\, n(1-h)^{n-1}\, \mathrm{d}h\right) \frac{1}{n} \sum_{i=1}^{n} f_2(i) f_3(n+1-i) \\ &= \left(\int_0^1 g_1(h)\, (1-h)^{n-1}\, \mathrm{d}h\right) \sum_{i=1}^{n} f_2(i) f_3(n+1-i). \end{aligned} \tag{5.17}$$

On the other hand, let $(\xi_1^{(n)}, \mathbf{T}_n^{\mathrm{g}}, \mathbf{T}_n^{\mathrm{d}})$ be the decomposition of $\mathbf{T}_n$ according to its lowest branching vertex. Let $x \in \mathbf{T}_n$ and set $h = H(x)$.

- If $h < \xi_1^{(n)}$, the decomposition of $\mathbf{T}_n \circledast_x^{\mathrm{g}} [0, 1-h]$ according to its lowest branching vertex is given by $(h, [0,1], (1-h)^{-1}\mathbf{T}_n')$ where $\mathbf{T}_n'$ is as the tree $\mathbf{T}_n$ but for its lowest branch whose length is $\xi_1^{(n)} - h$ instead of $\xi_1^{(n)}$. Notice that the shapes of the tree $\mathbf{T}_n'$ and $\mathbf{T}_n$ are the same. Then using again the property of conditioned uniform random variables, we deduce that conditionally on $\{\xi_1^{(n)} \geq h\}$, the tree $(1-h)^{-1}\mathbf{T}_n'$ is distributed as $\mathbf{T}_n$. Thus, we get:

$$\mathbb{E}\left[\int_{\mathbf{T}_n} \mathbf{1}_{\{H(x)<\xi_1^{(n)}\}}\, \mathscr{L}(\mathrm{d}x)\, G\big(\mathbf{T}_n \circledast_x^{\mathrm{g}} [0,1-h]\big)\right] = \mathbb{E}\left[\int_0^{\xi_1^{(n)}} g_1(h)\, \mathrm{d}h\right] f_2(1) f_3(n). \tag{5.18}$$

  By symmetry, we have:

$$\mathbb{E}\left[\int_{\mathbf{T}_n} \mathbf{1}_{\{H(x)<\xi_1^{(n)}\}}\, \mathscr{L}(\mathrm{d}x)\, G\big(\mathbf{T}_n \circledast_x^{\mathrm{d}} [0,1-h]\big)\right] = \mathbb{E}\left[\int_0^{\xi_1^{(n)}} g_1(h)\, \mathrm{d}h\right] f_2(n) f_3(1). \tag{5.19}$$

- For $x \in \tilde{\mathbf{T}}_n^{\mathrm{g}}$, the decomposition of $\mathbf{T}_n \circledast_x^{\varepsilon} [0, 1-h]$ according to its lowest branching vertex is given by $(\xi_1^{(n)}, (1-h)^{-1}\mathcal{T}', \mathbf{T}_n^{\mathrm{d}})$, where $\mathcal{T}' = \tilde{\mathbf{T}}_n^{\mathrm{g}} \circledast_x^{\varepsilon} [0, 1-h]$. Notice that the length measure on the tree $\mathbf{T}_n^{\mathrm{g}}$ is obtained by scaling by $(1-\xi_1^{(n)})^{-1}$ the length measure on $\mathbf{T}_n$ restricted to $\tilde{\mathbf{T}}_n^{\mathrm{g}}$. We deduce that:

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbf{T}_n} \mathbf{1}_{\{x\in\tilde{\mathbf{T}}_n^{\mathrm{g}}\}}\, \mathscr{L}^{\mathbf{T}_n}(\mathrm{d}x)\, G\big(\mathbf{T}_n \circledast_x^{\varepsilon} [0,1-h]\big)\right]\\
&= \mathbb{E}\left[g_1(\xi_1^{(n)}) g_3(\mathbf{T}_n^{\mathrm{d}}) \int_{\mathbf{T}_n} \mathbf{1}_{\{x\in\tilde{\mathbf{T}}_n^{\mathrm{g}}\}}\, \mathscr{L}^{\mathbf{T}_n}(\mathrm{d}x)\, g_2\Big((1-\xi_1^{(n)})^{-1}\big(\tilde{\mathbf{T}}_n^{\mathrm{g}} \circledast_x^{\varepsilon} [0,1-H(x)]\big)\Big)\right]\\
&= \mathbb{E}\left[g_1(\xi_1^{(n)}) g_3(\mathbf{T}_n^{\mathrm{d}}) \int_{\mathbf{T}_n^{\mathrm{g}}} (1-\xi_1^{(n)}) \mathscr{L}^{\mathbf{T}_n^{\mathrm{g}}}(\mathrm{d}y)\, g_2\Big(\mathbf{T}_n^{\mathrm{g}} \circledast_y^{\varepsilon} \big[0,1-H(y)\big]\Big)\right]\\
&= \mathbb{E}\left[(1-\xi_1^{(n)}) g_1(\xi_1^{(n)})\right] \frac{1}{n-1}\sum_{i=1}^{n-1} f_3(n-i) \mathbb{E}\left[\int_{\mathbf{T}_i} \mathscr{L}^{\mathbf{T}_i}(\mathrm{d}y)\, g_2\Big(\mathbf{T}_i \circledast_y^{\varepsilon} \big[0,1-H(y)\big]\Big)\right],
\end{aligned}$$

where we used the distribution of $(\mathbf{T}_n^{\mathrm{g}}, \mathbf{T}_n^{\mathrm{d}})$ conditionally on $\xi_1^{(n)}$ and I_n for the last equality. Using that, by induction, (5.15) holds for $n=i$, we get:

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbf{T}_n} \mathbf{1}_{\{x\in\tilde{\mathbf{T}}_n^{\mathrm{g}}\}}\, \mathscr{L}(\mathrm{d}x)\, G\big(\mathbf{T}_n \circledast_x^{\varepsilon} [0,1-h]\big)\right]\\
&\qquad= \mathbb{E}\left[(1-\xi_1^{(n)}) g_1(\xi_1^{(n)})\right] \frac{1}{n-1}\sum_{i=1}^{n-1} \frac{i+1}{2}\, f_2(i+1)\, f_3(n-i)\\
&\qquad= \mathbb{E}\left[(1-\xi_1^{(n)}) g_1(\xi_1^{(n)})\right] \frac{1}{n-1}\sum_{i=2}^{n} \frac{i}{2}\, f_2(i)\, f_3(n-i+1).
\end{aligned} \tag{5.20}$$

- By symmetry, for $x\in\tilde{\mathbf{T}}_n^{\mathrm{d}}$, we get:

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbf{T}_n} \mathbf{1}_{\{x\in\tilde{\mathbf{T}}_n^{\mathrm{d}}\}}\, \mathscr{L}(\mathrm{d}x)\, G(\mathbf{T}_n \circledast_x^{\varepsilon} [0,1-h])\right]\\
&\qquad= \mathbb{E}\left[(1-\xi_1^{(n)}) g_1(\xi_1^{(n)})\right] \frac{1}{n-1}\sum_{i=2}^{n} \frac{i}{2}\, f_3(i)\, f_2(n-i+1)\\
&\qquad= \mathbb{E}\left[(1-\xi_1^{(n)}) g_1(\xi_1^{(n)})\right] \frac{1}{n-1}\sum_{i=1}^{n-1} \frac{n-i+1}{2}\, f_2(i)\, f_3(n-i+1).
\end{aligned} \tag{5.21}$$

Summing (5.18) times $\mathbb{P}(\varepsilon=\mathrm{g})=1/2$, (5.19) times $\mathbb{P}(\varepsilon=\mathrm{d})=1/2$, (5.20) and (5.21), and using that $\xi_1^{(n)}$ has distribution $\beta(1,n-1)$ so that:

$$\mathbb{E}\left[\int_0^{\xi_1^{(n)}} g_1(h)\,\mathrm{d}h\right] = \frac{1}{n-1}\,\mathbb{E}\left[(1-\xi_1^{(n)}) g_1(\xi_1^{(n)})\right] = \int_0^1 g_1(h)(1-h)^{n-1}\,\mathrm{d}h,$$

we deduce that:

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbf{T}_n} \mathscr{L}(\mathrm{d}x)\, G\Big(\mathbf{T}_n \circledast_x^{\varepsilon} \big[0,1-H(x)\big]\Big)\right]\\
&\qquad\qquad= \left(\int_0^1 g_1(h)(1-h)^{n-1}\,\mathrm{d}h\right)\sum_{i=1}^{n} \frac{n+1}{2}\, f_2(i) f_3(n+1-i).
\end{aligned}$$

Thanks to (5.17), we deduce that (5.15) holds for G given by (5.16). Then use a monotone class argument to conclude that (5.15) holds for any measurable non-negative G. This concludes the proof by induction. □

We now consider $t \geq 0$ and assume that the probability distribution ν has a positive density f_{dens} with respect to the Lebesgue measure on $[0,t]$. Let F denote the cumulative distribution function of ν. By the assumptions on f_{dens}, F is a bijection from $[0,t]$ onto $[0,1]$ and its inverse F^{-1} is continuous. For a compact rooted real tree (T,d,ϱ), we define:

$$\forall x \in T,\, H^{f_{\text{dens}}}(x) = F^{-1}(H(x)),$$
$$\forall x,y \in T,\, d^{f_{\text{dens}}}(x,y) = H^{f_{\text{dens}}}(x) + H^{f_{\text{dens}}}(y) - 2H^{f_{\text{dens}}}(x \wedge y).$$

The scaling map $R^{f_{\text{dens}}} : (T,d,\varrho) \longmapsto (T, d^{f_{\text{dens}}}, \varrho)$ is then well-defined from $\{T \in \mathbb{T}_{\text{K}} : H(T) \leq 1\}$ to $\mathbb{T}_{\text{K}}$. We shall now prove it is continuous.

Lemma 5.5. *The map $R^{f_{\text{dens}}}$ from $\{T \in \mathbb{T}_{\text{K}},\ H(T) \leq 1\}$ to $\mathbb{T}_{\text{K}}$ is continuous.*

Proof. Let $\varepsilon > 0$. As F^{-1} is uniformly continuous with our assumptions, there exists $\delta > 0$ such that, for every $x,y \in [0,1]$:

$$|x-y| < \delta \Longrightarrow \left|F^{-1}(x) - F^{-1}(y)\right| \leq \frac{\varepsilon}{2}.$$

Let $T, T' \in \mathbb{T}_{\text{K}}$ with $H(T) \leq 1$ and $H(T') \leq 1$ such that $d_{\text{GH}}(T,T') < \delta/8$. Then, there exists a correspondence $\mathcal{R}$ between (elements in the equivalence classes) T and T' such that $\text{dist}\ (\mathcal{R}) \leq 2d_{\text{GH}}(T,T') + \delta/4 < \delta/2$.

For every $(x,x'),(y,y') \in \mathcal{R}$, we have:

$$\begin{aligned}
&\left|d^{f_{\text{dens}}}(x,y) - d'^{f_{\text{dens}}}(x',y')\right| \\
&\quad = \Big|H^{f_{\text{dens}}}(x) + H^{f_{\text{dens}}}(y) - 2H^{f_{\text{dens}}}(x \wedge y) - H^{f_{\text{dens}}}(x') \\
&\qquad\qquad - H^{f_{\text{dens}}}(y') + 2H^{f_{\text{dens}}}(x' \wedge y')\Big| \\
&\quad \leq \Big|F^{-1}\big(H(x)\big) - F^{-1}\big(H(x')\big)\Big| + \Big|F^{-1}\big(H(y)\big) - F^{-1}\big(H(y')\big)\Big| \\
&\qquad\qquad + 2\Big|F^{-1}\big(H(x \wedge y)\big) - F^{-1}\big(H(x' \wedge y')\big)\Big|.
\end{aligned}$$

As $(x,x') \in \mathcal{R}$, we have $\left|H(x) - H(x')\right| \leq \text{dist}\ (\mathcal{R}) < \delta$ and consequently,

$$\Big|F^{-1}\big(H(x)\big) - F^{-1}\big(H(x')\big)\Big| < \varepsilon/2.$$

Similarly, we have

$$p\Big|F^{-1}\big(H(y)\big) - F^{-1}\big(H(y')\big)\Big| < \varepsilon/2.$$

We also have:

$$\begin{aligned}
\left|H(x \wedge y) - H(x \wedge y)\right| &= \frac{1}{2}\left|H(x) + H(y) - d(x,y) - H(x') - H(y') + d'(x',y')\right| \\
&\leq \frac{1}{2}\left|H(x) - H(x')\right| + \frac{1}{2}\left|H(y) - H(y')\right| + \frac{1}{2}\left|d(x,y) - d'(x',y')\right| \\
&\leq \frac{3}{2}\text{dist}\ (\mathcal{R}) \\
&< \delta.
\end{aligned}$$

This gives $\Big|F^{-1}\big(H(x \wedge y)\big) - F^{-1}\big(H(x' \wedge y')\big)\Big| < \varepsilon/2$.

To conclude, we have $\text{dist}^{f_{\text{dens}}}(\mathcal{R}) < 2\varepsilon$ which implies that $d_{\text{GH}}^{f_{\text{dens}}}(T,T') < \varepsilon$. This gives the continuity of the map $R^{f_{\text{dens}}}$. □

We now prove Lemma 5.1. Recall that $\mathbf{T}_n$ denotes the trees constructed with the probability measure $\nu(\mathrm{d}x) = f_{\mathrm{dens}}(x)\,\mathrm{d}x$ and $\mathbf{T}_n^{\mathrm{unif}}$ the trees constructed with the uniform distribution on $[0,1]$ as studied in the first step. By construction, for all $n \in \mathbb{N}^*$, the random variables $R^{f_{\mathrm{dens}}}(\mathbf{T}_n^{\mathrm{unif}})$ and $\mathbf{T}_n$ have the same distribution. Notice also that, for every $T \in \mathbb{T}_{\mathrm{K}}$ and every non-negative measurable function g on $\mathbb{R}_+ \times \mathbb{T}_{\mathrm{loc-K}}$, we have:

$$\int_T \mathscr{L}^T(\mathrm{d}y)\, g\big(H(y), T\big) = \int_{R^{f_{\mathrm{dens}}}(T)} \mathscr{L}^{R^{f_{\mathrm{dens}}}(T)}(\mathrm{d}x)\, f_{\mathrm{dens}}\big(H^{f_{\mathrm{dens}}}(x)\big)\, g\big(H^{f_{\mathrm{dens}}}(x), R^{f_{\mathrm{dens}}}(T)\big).$$

Let G be a measurable non-negative functional defined on the space of rooted compact binary planar trees with a finite number of leaves, all of them at height t. We first have:

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbf{T}_n} \mathscr{L}^{\mathbf{T}_n}(\mathrm{d}x)\, f_{\mathrm{dens}}\big(H(x)\big)\, G\Big(\mathbf{T}_n \circledast_x^{\varepsilon} \big[0, t - H(x)\big]\Big)\right] \\
&= \mathbb{E}\Bigg[\int_{R^{f_{\mathrm{dens}}}(\mathbf{T}_n^{\mathrm{unif}})} \mathscr{L}^{R^{f_{\mathrm{dens}}}(\mathbf{T}_n^{\mathrm{unif}})}(\mathrm{d}x)\, f_{\mathrm{dens}}\big(H^{f_{\mathrm{dens}}}(x)\big) \\
&\qquad\qquad G\Big(R^{f_{\mathrm{dens}}}(\mathbf{T}_n^{\mathrm{unif}}) \circledast_x^{\varepsilon} \big[0, t - H^{f_{\mathrm{dens}}}(x)\big]\Big)\Bigg] \\
&= \mathbb{E}\left[\int_{\mathbf{T}_n^{\mathrm{unif}}} \mathscr{L}^{\mathbf{T}_n^{\mathrm{unif}}}(\mathrm{d}y)\; G \circ R^{f_{\mathrm{dens}}}\Big(\mathbf{T}_n^{\mathrm{unif}} \circledast_y^{\varepsilon} \big[0, 1 - H(y)\big]\Big)\right].
\end{aligned}$$

Applying Lemma 5.1, and then that $R^{f_{\mathrm{dens}}}(\mathbf{T}_{n+1}^{\mathrm{unif}})$ and $\mathbf{T}_{n+1}$ have the same distribution, we get the result.

5.4.3 An infinite tree with no leaves

Let f_{int} be a positive locally integrable function on $[0, +\infty)$. Let S be a Poisson point measure on $\mathbb{R}_+$ with intensity $f_{\mathrm{int}}(t)\,\mathrm{d}t$. We denote by $(\xi_i, i \geq 1)$ the increasing sequence of the atoms of S and by N the process $\big(N_t = S\big([0,t]\big),\ t \geq 0\big)$.

Let $(\varepsilon_n, n \geq 1)$ be independent random variables uniformly distributed on $\{\mathrm{g}, \mathrm{d}\}$ and let $(K_n, n \geq 1)$ be independent random variables uniformly distributed on $\{1, 2, \dots, n\}$ respectively, all these variables being independent and independent of S.

We define a tree-valued process $(\mathfrak{T}_t, t \geq 0)$ where, for every $t \geq 0$, the random tree $\mathfrak{T}_t$ is compact, has height t and $N_t + 1$ leaves, all of them at height t. Before going into this construction, we first define a growing procedure on rooted n-pointed trees for $n \in \mathbb{N}^*$:

$$\mathrm{Growth}_n((T, \mathbf{v}), h) \in \mathbb{T}_{\mathrm{dis}}^{(n)} \quad \text{with} \quad (T, \mathbf{v}) \in \mathbb{T}_{\mathrm{dis}}^{(n)} \quad \text{and} \quad h \in \mathbb{R}_+, \tag{5.22}$$

as the tree obtained by grafting on all the distinguished vertices of T, but the root (that is, on $\mathbf{v}^* = (v_1, \dots, v_n)$) a branch of length h, distinguishing the new leaves with the order naturally induced by $\mathbf{v}^*$ and removing the vertices $\mathbf{v}^*$ from the list of distinguished vertices. This function is formally defined in Section 7.5, see also Lemma 7.4 for its measurability.

We can now construct the process $(\mathfrak{T}_t, t \geq 0)$ inductively. For $0 \leq t \leq \xi_1$, we set $\mathfrak{T}_t = \big([0,t], (0,t)\big)$ and $N_t = 0$.

Let $n \in \mathbb{N}^*$ and assume that $(\mathfrak{T}_{\xi_n}, \mathbf{v}_n)$ is a tree of height ξ_n with n leaves, all of them at height ξ_n and distinguished (*i.e.* the vector $\mathbf{v}_n$ is composed of the root of $\mathfrak{T}_{\xi_n}$ and all its leaves). Then, we define the process on $(\xi_n, \xi_{n+1}]$ by setting, for every $t \in (\xi_n, \xi_{n+1}]$:

$$\mathfrak{T}_t = \mathrm{Growth}_n(\mathfrak{T}_{\xi_n}, t - \xi_n) \circledast_{K_n, \xi_n}^{\varepsilon_n} [0, t - \xi_n] \quad \text{and} \quad N_t = n.$$

Standard properties of Poisson processes give the following result.

Lemma 5.6. *For every $n \geq 1$ and every $t > 0$, conditionally given $N_t = n-1$, the tree $\mathfrak{T}_t$ is distributed as the tree $\mathbf{T}_n$ of Section 5.4.1 associated with the density f_{dens} on $[0,t]$ given by:*

$$f_{\text{dens}}(u) = \frac{f_{\text{int}}(u)}{F(t)} \mathbf{1}_{[0,t]}(u) \quad \textit{with} \quad F(t) = \int_0^t f_{\text{int}}(u)\, \mathrm{d}u. \tag{5.23}$$

We now view the tree $\mathfrak{T}_t$ as a real-tree of $\mathbb{T}_{\text{loc}-\text{K}}$ (we forget about the distinguished leaves which is a continuous operation thanks to Lemma 6.6). It is easy to see that the process $(\mathfrak{T}_t, t \geq 0)$ satisfies the Cauchy property in $\mathbb{T}_{\text{loc}-\text{K}}$ as $r_s(\mathfrak{T}_t) = r_s(\mathfrak{T}_{t'})$ for every $s \leq t \leq t'$. Thus this sequence converges a.s. in $\mathbb{T}_{\text{loc}-\text{K}}$, and we write:

$$\mathfrak{T}^{\text{ske}} = \lim_{t \to +\infty} \mathfrak{T}_t. \tag{5.24}$$

The tree $\mathfrak{T}^{\text{ske}}$ is a $\mathbb{T}_{\text{loc}-\text{K}}$-valued random variable which has no leaves. The tree $\mathfrak{T}^{\text{ske}}$ will serve as a backbone for the description of the genealogical tree of the conditioned CSBP.

We present now an ancillary result which is interesting by itself; it is a consequence of Lemma 5.1 on two tree-valued processes that have the same one-dimensional marginal.

We first consider the process $(\mathfrak{T}_t, t \geq 0)$ associated with the intensity $f_{\text{int}} \equiv 1$, that is, $f_{\text{int}}(t) = 1$ for all $t \geq 0$. Then we construct a sequence $\mathbf{t} = (\mathbf{t}_n, n \geq 1)$ of increasing real trees, with $\mathbf{t}_n \in \mathbb{T}_{\text{K}}^{(n)}$ for every $n \geq 1$, all of them of height 1. Let $(\varepsilon_k, k \geq 1)$ be independent random variables uniformly distributed on $\{\text{g}, \text{d}\}$. We define the sequence $\mathbf{t}$ by induction by setting first $\mathbf{t}_1 = \big([0,1], (0,1)\big)$. Let $n \geq 1$ and assume that $(\mathbf{t}_n, \mathbf{v}_n)$ is a tree of $\mathbb{T}_{\text{K}}^{(n)}$ with height 1 and with n leaves all of them at height 1. Conditionally given $\mathbf{t}_n$, let V_{n+1} be a random element on $\mathbf{t}_n$ uniformly chosen according to the length measure; that is V_{n+1} is distributed according to the measure $c_n \mathscr{L}$, with $\mathscr{L}$ the length measure on $\mathbf{t}_n$ and the normalization $c_n = 1/\mathscr{L}(\mathbf{t}_n)$. Notice that V_{n+1} is a.s. not a leaf nor the root of $\mathbf{t}_n$. Then we set:

$$\mathbf{t}_{n+1} = \mathbf{t}_n \circledast_{V_{n+1}}^{\varepsilon_{n+1}} \big[0, 1 - H(V_{n+1})\big].$$

In particular, for every measurable nonnegative function G, we have:

$$\mathbb{E}\big[G(\mathbf{t}_{n+1}) |\, \mathbf{t}_1, \ldots, \mathbf{t}_n, \varepsilon_{n+1}\big] = \int_{\mathbf{t}_n} \frac{\mathscr{L}(\mathrm{d}x)}{\mathscr{L}(\mathbf{t}_n)}\, G\Big(\mathbf{t}_n \circledast_x^{\varepsilon_{n+1}} \big[0, 1 - H(x)\big]\Big). \tag{5.25}$$

From Definition (6.12), we see that the measurable function $\tilde{N}_t$ records the number of vertices at level t of a tree without leaves: $\tilde{N}_t(T) = \text{Card}\,\Big(\big\{x \in T :\ H(x) = t\big\}\Big)$. Let us consider the continuous (see Lemma 6.6) canonical projection $\Pi_n^\circ : \mathbb{T}_{\text{K}}^{(n)} \longrightarrow \mathbb{T}_{\text{K}}$ defined by $\Pi_n^\circ(\mathbf{t}, \mathbf{v}) = \mathbf{t}$. We write $\mathfrak{T}_t^{\text{ske}} = r_t(\mathfrak{T}^{\text{ske}})$ for $t \geq 0$.

Proposition 5.7. *Let $n \geq 1$ and $f_{\text{int}} \equiv 1$. For all measurable non-negative functional G defined on $\mathbb{T}_{\text{dis}}^{(n)}$, we have, with $\mathscr{L}$ the length measure on $\mathbf{t}_n$:*

$$\mathbb{E}\left[G(\mathfrak{T}_1) \,\Big|\, N_1 = n-1\right] = \frac{2^{n-1}}{n!} \mathbb{E}\left[G(\mathbf{t}_n) \prod_{k=1}^{n-1} \mathscr{L}(\mathbf{t}_k)\right], \tag{5.26}$$

and for all measurable non-negative functional G defined on $\mathbb{T}_{\text{K}}$ (or on $\mathbb{T}_{\text{loc}-\text{K}}$):

$$\mathbb{E}\left[G(\mathfrak{T}_1^{\text{ske}}) \,\Big|\, \tilde{N}_1(\mathfrak{T}^{\text{ske}}) = n\right] = \mathbb{E}\left[G \circ \Pi_n^\circ(\mathfrak{T}_1) \,\Big|\, N_1 = n-1\right]. \tag{5.27}$$

Proof. By construction, we have that the process $\Big(\big(\mathfrak{T}_t^{\text{ske}}, \tilde{N}_t(\mathfrak{T}^{\text{ske}})\big), t \geq 0\Big)$ is distributed as the process $\Big(\big(\Pi_{N_t+1}^\circ(\mathfrak{T}_t), N_t + 1\big), t \geq 0\Big)$. This gives (5.27).

We now prove (5.26) by induction. Thanks to Lemma 5.6, conditionally given $N_1 = n-1$, the tree $\mathfrak{T}_1$ is distributed as $\mathbf{T}_n^{\text{unif}}$. For $n=1$, we have $\mathbf{T}_n^{\text{unif}} = \mathbf{t}_1 = ([0,1],(0,1))$ hence Equation (5.26) holds. Let us suppose that (5.26) holds for some $n \geq 1$. Applying Lemma 5.1, one gets:

$$\mathbb{E}\big[G(\mathbf{T}_{n+1}^{\text{unif}})\big] = \frac{2}{n+1}\mathbb{E}\left[\mathscr{L}(\mathbf{T}_n^{\text{unif}})\int_{\mathbf{T}_n^{\text{unif}}} \frac{\mathscr{L}^{\mathbf{T}_n^{\text{unif}}}(\mathrm{d}x)}{\mathscr{L}^{\mathbf{T}_n^{\text{unif}}}(\mathbf{T}_n^{\text{unif}})} G\Big(\mathbf{T}_n^{\text{unif}} \circledast_x^{\varepsilon} \big[0,1-H(x)\big]\Big)\right].$$

Now we apply the induction assumption for the right-hand side of the previous equation to get:

$$\begin{aligned}
&\mathbb{E}\big[G(\mathbf{T}_{n+1}^{\text{unif}})\big] \\
&= \frac{2}{n+1}\frac{2^{n-1}}{n!}\left[\mathscr{L}^{\mathbf{t}_n}(\mathbf{t}_n)\int_{\mathbf{t}_n} \frac{\mathscr{L}^{\mathbf{t}_n}(\mathrm{d}x)}{\mathscr{L}^{\mathbf{t}_n}(\mathbf{t}_n)} G\Big(\mathbf{t}_n \circledast_x^{\varepsilon_{n+1}} \big[0,1-H(x)\big]\Big)\prod_{k=1}^{n-1}\mathscr{L}^{\mathbf{t}_n}(\mathbf{t}_k)\right] \\
&= \frac{2^n}{(n+1)!}\mathbb{E}\left[G(\mathbf{t}_{n+1})\prod_{k=1}^{n}\mathscr{L}^{\mathbf{t}_{n+1}}(\mathbf{t}_k)\right],
\end{aligned}$$

by definition of the tree $\mathbf{t}_{n+1}$ and by (5.25). This gives that (5.26) holds with n replaced by $n+1$. This concludes the proof by induction. $\square$

5.5 The n-leaves decomposition of the Brownian CRT

The decomposition of a (sub)critical Brownian CRT $\mathcal{T}$ according to a spine $[\![\emptyset, x]\!]$, where $x \in \mathcal{T}$ is a leaf picked at random at level $t>0$, that is according to the local time $\Lambda_t(\mathrm{d}x)$, is given in Theorem 4.5 in [19]. In our setting, it can be rephrased in the next theorem. Notice that, for $t>0$, the (planar discrete) 1-pointed tree $[0,t] \in \mathbb{T}_{\text{loc}-\text{K}}^{(1)}$ denotes the segment $[0,t]$ endowed with the Euclidean distance, with the root 0 and the distinguished vertex t. Recall that the grafting operation Graft_n on a n-pointed discrete tree of trees formalized by atoms of a Poisson point measure $\mathcal{M}$ has been intuitively presented in (5.5) (or (7.3)) and formally defined in Section 7.2.2, see (7.4) therein, using the theoretical background of Section 6.8, so that $\text{Graft}_1([0,t],\mathcal{M})$ in (5.28) below is a well defined $\mathbb{T}_{\text{loc}-\text{K}}^{(1)}$-valued random variable.

Theorem 5.8 ([19])**.** *Let $\beta>0$, $\theta \geq 0$ and $t>0$. Let $\mathcal{M}$ be under $\mathbb{E}$ a Poisson measure with intensity $2\beta\,\mathbf{1}_{[0,t]}(s)\mathrm{d}s\mathbb{N}^\theta[\mathrm{d}\mathcal{T}]$. For every non-negative measurable functional F on $\mathbb{T}_{\text{loc}-\text{K}}^{(1)}$ (or $\mathbb{T}_{\text{K}}^{(1)}$), we have, with ϱ the root of $\mathcal{T}$:*

$$\mathbb{N}^\theta\left[\int_{\mathcal{T}} \Lambda_t(\mathrm{d}v)\, F\big(\mathcal{T},(\varrho,v)\big)\right] = \mathrm{e}^{-2\beta\theta t}\,\mathbb{E}\left[F\big(\text{Graft}_1([0,t],\mathcal{M})\big)\right]. \tag{5.28}$$

We extend this result to the super-critical case $\theta<0$.

Corollary 5.9 (One-leaf decomposition)**.** *Let $\beta>0$, $\theta \in \mathbb{R}$ and $t>0$. Let $\mathcal{M}$ be under $\mathbb{E}$ a Poisson measure with intensity $2\beta\,\mathbf{1}_{[0,t]}(s)\mathrm{d}s\mathbb{N}^\theta[\mathrm{d}\mathcal{T}]$. For every non-negative measurable functional F on $\mathbb{T}_{\text{loc}-\text{K}}^{(1)}$, Equation* (5.28) *holds.*

Proof. Let $(T,\mathbf{v}) \in \mathbb{T}_{\text{loc}-\text{K}}^{(1)}$ with $\mathbf{v} = (\varrho, v)$. We denote by $(T_i^\circ, i \in I)$ the connected components of the set $T \setminus [\![\varrho, v]\!]$. For every $x \in T$, there exists a unique $x_i \in T$ such that $\cap_{x\in T_i^\circ}[\![\varrho, x]\!] = [\![\varrho, x_i]\!]$ and we set $T_i = T_i^\circ \cup \{x_i\}$ viewed as a real tree rooted at x_i. Then we define the point measure $\boldsymbol{\mathcal{M}}(T,\mathbf{v})$ on $\mathbb{R}_+ \times \mathbb{T}_{\text{loc}-\text{K}}$ by:

$$\boldsymbol{\mathcal{M}}(T,\mathbf{v}) = \sum_{i\in I}\delta_{H(x_i),T_i}.$$

This map is well defined according to Corollary 6.34. Even if we shall not use it as such, let us mention that a.s. $\boldsymbol{\mathcal{M}}(\mathrm{Graft}_1([0,t],\mathcal{M}))=\mathcal{M}$; this can be easily deduced from Proposition 6.33.

We first prove (5.28) for functionals F of the form:

$$F(T,\mathbf{v})=\mathrm{e}^{-\langle \Phi, \boldsymbol{\mathcal{M}}(T,\mathbf{v})\rangle}, \tag{5.29}$$

where $(T,\mathbf{v})\in\mathbb{T}^{(1)}_{\mathrm{loc-K}}$, Φ is a continuous non-negative function with bounded support defined on $\mathbb{R}_+\times\mathbb{T}^*_{\mathrm{loc-K}}$ (with $\mathbb{T}^*_{\mathrm{loc-K}}=\mathbb{T}_{\mathrm{loc-K}}\setminus\{\mathrm{T}_0\}$ where $\mathrm{T}_0\in\mathbb{T}_{\mathrm{loc-K}}$ is the tree reduced to its root, see Section 6.9).

For simplicity, we write $(\mathcal{T},v)$ for the 1-pointed tree $(\mathcal{T},(\varrho,v))$. Let $\theta>0$. Using (5.6), we have for every $s>t$ that:

$$\mathbb{N}^{-\theta}\left[\int_{\mathcal{T}}\Lambda_t(\mathrm{d}v)\,\mathrm{e}^{-\langle\Phi,\,\boldsymbol{\mathcal{M}}(r_s(\mathcal{T},v))\rangle}\right]=\mathbb{N}^{\theta}\left[\int_{\mathcal{T}}\Lambda_t(\mathrm{d}v)\,\mathrm{e}^{2\theta Z_s-\langle\Phi,\,\boldsymbol{\mathcal{M}}(r_s(\mathcal{T},v))\rangle}\right].$$

We apply then (5.28) to get:

$$\begin{aligned}\mathbb{N}^{-\theta}&\left[\int_{\mathcal{T}}\Lambda_t(\mathrm{d}v)\,\mathrm{e}^{-\langle\Phi,\,\boldsymbol{\mathcal{M}}(r_s(\mathcal{T},v))\rangle}\right]\\
&=\mathrm{e}^{-2\beta\theta t}\,\mathbb{E}\left[\mathrm{e}^{2\theta Z_s}\,F(r_s(\mathrm{Graft}_1([0,t],\mathcal{M}))\right]\\
&=\exp\left\{-2\beta\theta t-2\beta\int_0^t\mathrm{d}a\,\mathbb{N}^{\theta}\left[1-\mathrm{e}^{-\Phi(a,r_{s-a}(\mathcal{T}))+2\theta Z_{s-a}}\right]\right\}\\
&=\exp\left\{-2\beta\theta t-2\beta\int_0^t\mathrm{d}a\,\left(\mathbb{N}^{-\theta}\left[1-\mathrm{e}^{-\Phi(a,r_{s-a}(\mathcal{T}))}\right]+\mathbb{N}^{\theta}\left[1-\mathrm{e}^{2\theta Z_{s-a}}\right]\right)\right\}\\
&=\exp\left\{2\beta\theta t-2\beta\int_0^t\mathrm{d}a\,\mathbb{N}^{-\theta}\left[1-\mathrm{e}^{-\Phi(a,r_{s-a}(\mathcal{T}))}\right]\right\},\end{aligned}$$

where we used standard property of Poisson point measures for the second equality, (5.6) again for the third one, and that $\mathbb{N}^{\theta}\left[1-\mathrm{e}^{2\theta Z_a}\right]=u(-2\theta,a)=-2\theta$, see (2.8) and (2.9), for the last one. As Φ has bounded support, we get taking s large enough:

$$\mathbb{N}^{-\theta}\left[\int_{\mathcal{T}}\Lambda_t(\mathrm{d}v)\,\mathrm{e}^{-\langle\Phi,\,\boldsymbol{\mathcal{M}}(\mathcal{T},v)\rangle}\right]=\exp\left\{2\beta\theta t-2\beta\int_0^t\mathrm{d}a\,\mathbb{N}^{-\theta}\left[1-\mathrm{e}^{-\Phi(a,\mathcal{T})}\right]\right\}.$$

Then the result follows from the definition of $\mathrm{Graft}_1([0,t],\mathcal{M})$, that is (5.28) holds for F given by (5.29).

As $(\mathcal{T},v)$ is a measurable function of $\boldsymbol{\mathcal{M}}(\mathcal{T},v)$, see Section 7.6, we then conclude by the monotone class theorem that Equation (5.28) holds for any non-negative measurable function F defined on $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$. □

Let $\beta>0$, $\theta\in\mathbb{R}$ and $t>0$. Recall $\tilde{c}^{\theta}_t=(2\theta)/(1-\mathrm{e}^{-2\beta\theta t})$ defined in (2.3). We consider the probability measure on $[0,t]$:

$$\nu(ds)=\frac{2\beta\theta\,\mathrm{e}^{2\beta\theta s}}{\mathrm{e}^{2\beta\theta t}-1}\,\mathbf{1}_{[0,t]}(s)\,\mathrm{d}s=\beta\,\tilde{c}_t(\theta)\;\mathrm{e}^{-2\beta\theta(t-s)}\;\mathbf{1}_{[0,t]}(s)\,\mathrm{d}s. \tag{5.30}$$

Let $(\mathbf{T}_n,\mathbf{v}_n)$ be, under $\mathbb{P}^{\theta,t}$, the planar tree, element of $\mathbb{T}^{(n)}_{\mathrm{dis}}$, defined in Section 5.4.1 associated with the measure ν and $t>0$ (recall that all the distinguished vertices from $\mathbf{v}_n$ but the root are at distance t from the root). The following theorem is a generalization of Theorem 5.8 when picking n leaves uniformly at random at level t.

Theorem 5.10 (Generalized n-leaves decomposition)**.** *Let $\beta > 0$, $\theta \in \mathbb{R}$, $t > 0$ and $n \in \mathbb{N}^*$. For every non-negative measurable function F defined on $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$, we have:*

$$\mathbb{N}^\theta \left[\int_{\mathcal{T}^n} \Lambda_t^{\otimes n}(\mathrm{d}\mathbf{v}^*)\, F(\mathcal{T}, \mathbf{v}) \right] = n!\, \left(\tilde{c}_t^\theta\right)^{1-n} \mathrm{e}^{-2\beta\theta t}\, \mathbb{E}^{\theta,t} \left[F\Big(\mathrm{Graft}_n\big((\mathbf{T}_n, \mathbf{v}_n), \mathcal{M}\big)\Big)\right], \quad (5.31)$$

where $\mathbf{v} = (\varrho, \mathbf{v}^*) \in \mathcal{T}^{n+1}$, with ϱ the root of $\mathcal{T}$, and, under $\mathbb{E}^{\theta,t}$, conditionally given $(\mathbf{T}_n, \mathbf{v}_n)$, $\mathcal{M}(\mathrm{d}x, \mathrm{d}\mathcal{T})$ is a Poisson point measure on $\mathbf{T}_n \times \mathbb{T}_{\mathrm{loc-K}}$ with intensity $2\beta\, d\mathscr{L}^{\mathbf{T}_n}(\mathrm{d}x)\mathbb{N}^\theta[\mathrm{d}\mathcal{T}]$.

We stress again that measurability of the grafting map Graft_n on a discrete tree is formally stated in Section 7.2.2, so that $\mathrm{Graft}_n\big((\mathbf{T}_n, \mathbf{v}_n), \mathcal{M}\big)$ is indeed a $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$-valued random variable (see in particular (7.3) and (7.4) therein). The proof of this theorem is postponed to Section 8 as it heavily relies on the topological setting developed in Section 6.

5.6 Local limit of conditioned Brownian CRT

Let $\beta > 0$, $\theta, \alpha \in \mathbb{R}_+$ and let $S^{\alpha,\theta}$ be a Poisson point measure on $[0, \infty)$ with intensity measure $f_{\mathrm{int}}(t)\, \mathrm{d}t$, where:

$$f_{\mathrm{int}}(t) = \alpha\beta\, \mathrm{e}^{2\beta\theta t}, \quad t \geq 0. \quad (5.32)$$

We first consider the case $\alpha > 0$. Denote by $(\xi_i, i \in \mathbb{N}^*)$ the increasing sequence of jumping times of the inhomogeneous Poisson process $(N_t^{\alpha,\theta} = S^{\alpha,\theta}([0,t]), t \geq 0)$. We consider the $\mathbb{T}^{(n)}_{\mathrm{dis}}$-valued random variable $\mathfrak{T}_{\xi_n}$ of Section 5.4.3 for $n \geq 1$ associated with f_{int}. In particular, recall that, for every $n \geq 1$, $\mathfrak{T}_{\xi_n}$ is a discrete tree with n distinguished leaves, where all of them are at height ξ_n. Recall the construction of the infinite backbone $\mathfrak{T}^{\mathrm{ske}}$ in Section 5.4.3 from the sequence of trees $\mathfrak{T}_{\xi_n}$. Notice its distribution depends on α and θ (and also β which is fixed). We informally define $\mathcal{T}^{\alpha,\theta}$ as the tree obtained by grafting on $\mathfrak{T}^{\mathrm{ske}}$ (whose distribution depends on α and θ) a tree $\mathcal{T}_i$ at point x_i where, conditionally given $\mathfrak{T}^{\mathrm{ske}}$, the family $((x_i, \mathcal{T}_i), i \in I)$ is the atoms of a Poisson point measure on $\mathfrak{T}^{\mathrm{ske}} \times \mathbb{T}_{\mathrm{loc-K}}$ with intensity $2\beta\, \mathscr{L}^{\mathfrak{T}^{\mathrm{ske}}}(\mathrm{d}x)\mathbb{N}^\theta(\mathrm{d}\mathcal{T})$.

For $\alpha = 0$, the infinite backbone rooted tree $\mathfrak{T}^{\mathrm{ske}}$ has only one branch and is identified with $(\mathbb{R}_+, 0)$, and the tree $\mathcal{T}^{0,\theta}$ is then identified with the Kesten tree with parameter (β, θ) defined in Section 5.2 and formally in Section 7.2.1.

Since we are considering equivalence class of trees, it is ambiguous to present $\mathfrak{T}^{\mathrm{ske}}$ as a subtree of $\mathcal{T}^{\alpha,\theta}$. This motivates the introduction of marked trees in Section 6.4; and to avoid confusion, we shall denote $\mathfrak{T}^{\alpha,\theta}$ the subtree of $\mathcal{T}^{\alpha,\theta}$; it is in the same equivalence class as $\mathfrak{T}^{\mathrm{ske}}$ in $\mathbb{T}_{\mathrm{loc-K}}$. We refer to Section 7.7 for a formal and more rigorous definition of the trees $(\mathcal{T}^{\alpha,\theta}, \mathfrak{T}^{\alpha,\theta})$. We then define the random process $(\mathcal{T}_t^{\alpha,\theta}, t \geq 0)$ by setting:

$$\mathcal{T}_t^{\alpha,\theta} = r_t(\mathcal{T}^{\alpha,\theta}).$$

Recall that the $\mathbb{T}_{\mathrm{loc-K}}$-valued function $\mathcal{T}$ is under $\mathbb{N}^\theta$ a Brownian tree; and we write $\mathcal{T}_t = r_t(\mathcal{T})$. We now give the main result of this section.

Proposition 5.11 (Representation of an h-transform of the CRT)**.** *Let $\beta \in \mathbb{R}_+^*$, $\theta, \alpha \in \mathbb{R}_+$ and $t > 0$. For every non-negative measurable functional F on $\mathbb{T}_{\mathrm{loc-K}}$ (or $\mathbb{T}_{\mathrm{K}}$), we have:*

$$\mathbb{E}\left[F\left(\mathcal{T}_t^{\alpha,\theta}\right)\right] = \mathbb{N}^\theta \left[F\left(\mathcal{T}_t\right)\, M_t^{\alpha,\theta}\right].$$

Remark 5.12 (On the h-transform of Z)**.** *By considering the size population at level t of $\mathcal{T}^{\alpha,\theta}$, the above proposition gives a representation of the process $(Z_t, t \geq 0)$ under $\mathbb{N}^\theta[\cdot\, M^{\alpha,\theta}]$ as a quadratic CSBP with a Poisson immigration given by $\mathfrak{T}^{\mathrm{ske}}$ and the grafting intensity $2\beta\, \mathscr{L}^{\mathfrak{T}^{\mathrm{ske}}}(\mathrm{d}x)\mathbb{N}^\theta(\mathrm{d}\mathcal{T})$. As $(\tilde{N}_t(\mathfrak{T}^{\mathrm{ske}}), t \geq 0)$ is distributed as $(N_t^{\alpha,\theta} + 1, t \geq 0)$, that is, as $(S_t^{\alpha,\theta} + 1, t \geq 0)$, this provides another proof of Proposition 4.1.*

Proof of Proposition 5.11. We first consider the case $\alpha > 0$. Let us fix $t > 0$, and write N_t for $N_t^{\alpha,\theta}$. Recall $\tilde{N}_t(T)$ is the number of vertices of T at level t. Since $\mathfrak{T}^{\text{ske}} = \mathfrak{T}^{\alpha,\theta}$ in $\mathbb{T}_{\text{loc-K}}$, we get that $\tilde{N}_t(\mathfrak{T}^{\alpha,\theta}) = \tilde{N}_t(\mathfrak{T}^{\text{ske}})$ is distributed as $N_t + 1$. The fact that $(\mathcal{T}_t^{\alpha,\theta}, \tilde{N}_t(\mathfrak{T}^{\alpha,\theta}))$ is a well defined random variable is detailed at the end of Section 7.7. We shall also consider the truncated backbone $\mathfrak{T}_t^{\alpha,\theta} = r_t(\mathfrak{T}^{\alpha,\theta})$ for $t \geq 0$, and see $\mathfrak{T}_t^{\alpha,\theta}$ as a subtree of $\mathcal{T}_t^{\alpha,\theta}$.

Let $(\mathbf{T}_n, n \geq 0)$ be the sequence of trees defined in Section 5.4.1 associated with the function:

$$f_{\text{dens}}(s) = \beta\, \tilde{c}_t(\theta)\ \mathrm{e}^{-2\beta\theta(t-s)}\ \mathbf{1}_{[0,t]}(s). \tag{5.33}$$

Recall the continuous canonical projection $\Pi_n^\circ : \mathbb{T}_{\text{K}}^{(n)} \longrightarrow \mathbb{T}_{\text{K}}$ defined by $\Pi_n^\circ(\mathbf{t}, \mathbf{v}) = \mathbf{t}$. Set $\text{Graft}_k^\circ = \Pi_k^\circ \circ \text{Graft}_k$. Then we have:

$$\begin{aligned}
\mathbb{E}\left[F\left(\mathcal{T}_t^{\alpha,\theta}\right)\right] &= \sum_{n\in\mathbb{N}} \mathbb{E}\left[F\left(\mathcal{T}_t^{\alpha,\theta}\right)\ \Big|\ \tilde{N}_t(\mathfrak{T}^{\alpha,\theta}) = n+1\right] \mathbb{P}(N_t^{\alpha,\theta} = n)\\
&= \sum_{n\in\mathbb{N}} \mathbb{E}\left[F\Big(r_t\big(\text{Graft}_{n+1}^\circ(\mathfrak{T}_t^{\alpha,\theta}, \mathcal{M}_t)\big)\Big)\ \Big|\ N_t^{\alpha,\theta} = n\right] \frac{(\alpha/c_t^\theta)^n\, \mathrm{e}^{-\alpha/c_t^\theta}}{n!}\\
&= \sum_{n\in\mathbb{N}} \mathbb{E}\left[F\Big(r_t\big(\text{Graft}_{n+1}^\circ(\mathbf{T}_{n+1}, \tilde{\mathcal{M}}_t)\big)\Big)\right] \frac{(\alpha/c_t^\theta)^n\, \mathrm{e}^{-\alpha/c_t^\theta}}{n!},
\end{aligned}$$

where we used that $\tilde{N}_t(\mathfrak{T}^{\alpha,\theta})$ is distributed as $N_t^{\alpha,\theta}+1$ for the first equality, that conditionally on $\tilde{N}_t(\mathfrak{T}^{\alpha,\theta}) = n+1$, the random tree $\mathcal{T}_t^{\alpha,\theta}$ is distributed as $r_t\left(\text{Graft}_{n+1}^\circ(\mathfrak{T}_t^{\alpha,\theta}, \mathcal{M}_t)\right)$ conditionally on $N_t^{\alpha,\theta} = n$ where, conditionally given $\mathfrak{T}_t$, $\mathcal{M}_t$ (resp. $\tilde{\mathcal{M}}_t$) is a Poisson point measure on $\mathfrak{T}_t^{\alpha,\theta} \times \mathbb{T}_{\text{K}}$ (resp. $\mathbf{T}_{n+1} \times \mathbb{T}_{\text{K}}$) with intensity $2\beta\, \mathscr{L}^{\mathfrak{T}_t^{\alpha,\theta}}(\mathrm{d}x)\mathbb{N}^\theta(\mathrm{d}\theta)$ (resp. $2\beta\, \mathscr{L}^{\mathbf{T}_{n+1}}(\mathrm{d}x)\mathbb{N}^\theta(\mathrm{d}\theta)$), and that $N_t^{\alpha,\theta}$ is distributed as a Poisson process with intensity α at time $1/c_t^\theta$ (see Lemma 4.3) for the second one, and that $\mathfrak{T}_t^{\alpha,\theta}$ conditionally on $N_t^{\alpha,\theta} = n$ is distributed as $\mathbf{T}_{n+1}$ with f_{int} and f_{dens} in (5.23) given by (5.32) and (5.33) (see Lemma 5.6) for the last one. Using Theorem 5.10 and that $\nu(\mathrm{d}s)$ in (5.30) is exactly $f_{\text{dens}}(s)\,\mathrm{d}s$ with f_{dens} given by (5.33), we have:

$$\begin{aligned}
\mathbb{E}\left[F\left(r_t\left(\text{Graft}_{n+1}^\circ(\mathbf{T}_{n+1}, \tilde{\mathcal{M}}_t)\right)\right)\right] &= \frac{\left(\tilde{c}_t^\theta\right)^n \mathrm{e}^{2\beta\theta t}}{(n+1)!}\, \mathbb{N}^\theta\left[\int_{\mathcal{T}^{n+1}} \Lambda_t^{\otimes(n+1)}(\mathrm{d}\mathbf{v}^*)\, F\big(r_t(\mathcal{T})\big)\right]\\
&= \frac{\left(\tilde{c}_t^\theta\right)^n \mathrm{e}^{2\beta\theta t}}{(n+1)!}\, \mathbb{N}^\theta\left[\int_{\mathcal{T}^{n+1}} \Lambda_t^{\otimes(n+1)}(\mathrm{d}\mathbf{v}^*)\, F(\mathcal{T}_t)\right]\\
&= \frac{\left(\tilde{c}_t^\theta\right)^n \mathrm{e}^{2\beta\theta t}}{(n+1)!}\, \mathbb{N}^\theta\left[Z_t^{n+1} F(\mathcal{T}_t)\right],
\end{aligned}$$

as $Z_t = \Lambda_t(\mathbf{1})$ is the total local time of $\mathcal{T}$ at level t. Thus, using the definition of $M_t^{\alpha,\theta}$ in (3.2), we obtain:

$$\mathbb{E}\left[F\left(\mathcal{T}_t^{\alpha,\theta}\right)\right] = \sum_{n\in\mathbb{N}} \frac{\left(\tilde{c}_t^\theta\right)^n \mathrm{e}^{2\beta\theta t}}{(n+1)!}\, \mathbb{N}^\theta\left[Z_t^{n+1} F(\mathcal{T}_t)\right] \frac{(\alpha/c_t^\theta)^n\, \mathrm{e}^{-\alpha/c_t^\theta}}{n!} = \mathbb{N}^\theta\left[F\left(\mathcal{T}_t\right)\ M_t^{\alpha,\theta}\right].$$

The simpler case $\alpha = 0$, which is left to the reader, can also be handled in a similar way. □

As a conclusion, we deduce the following result for $\alpha > 0$.

Theorem 5.13 (Local limit of CRT in the Poisson regime). *Let $\alpha, \beta > 0$, $\theta \in \mathbb{R}$. Assume that the function a is such that as $t \to \infty$:*

$$a_t \sim \begin{cases} \alpha\beta^2 t^2, & \text{if } \theta = 0, \\ \alpha(2\theta)^{-2}\,\mathrm{e}^{2\beta|\theta|t}, & \text{if } \theta \neq 0. \end{cases}$$

For every non-negative measurable function F on $\mathbb{T}_{\mathrm{K}}$ and $s > 0$, we have:

$$\lim_{t\to\infty} \mathbb{N}^\theta \left[F\left(\mathcal{T}_s\right) \mid Z_t = a_t\right] = \mathbb{E}\left[F\left(\mathcal{T}_s^{\alpha,|\theta|}\right)\right].$$

Proof. Clearly, Proposition 3.5 still holds if H_s is $\mathcal{G}_s = \sigma(r_s(\mathcal{T}))$ measurable, that is $H_s = F(\mathcal{T}_s)$ with F non-negative defined on $\mathbb{T}_{\mathrm{loc-K}}$, and Z_t is the total local time of $\mathcal{T}$ at level t, see Section 5.2. We deduce that:

$$\lim_{t\to\infty} \mathbb{N}^\theta [F(\mathcal{T}_s)|\, Z_t = a_t] = \mathbb{N}^{|\theta|}\left[F(\mathcal{T}_s)\, M_s^{\alpha,|\theta|}\right] = \mathbb{E}\left[F(\mathcal{T}_s^{\alpha,|\theta|})\right],$$

where we used Proposition 5.11 for the last equality. □

Similarly, we also get the following result for $\alpha = 0$. Recall that $\mathcal{T}^{0,\theta}$ is a Kesten tree with parameter (β, θ).

Theorem 5.14 (Local limit of CRT in the Kesten regime). *Let $\beta > 0$, $\theta \in \mathbb{R}$. Assume that the function a is positive such that as $t \to \infty$:*

$$a_t = \begin{cases} o(t^2), & \text{if } \theta = 0, \\ o(\mathrm{e}^{2\beta|\theta|t}) & \text{if } \theta \neq 0. \end{cases}$$

For every non-negative measurable function F on $\mathbb{T}_{\mathrm{K}}$ and $s > 0$, we have:

$$\lim_{t\to\infty} \mathbb{N}^\theta \left[F\left(\mathcal{T}_s\right) \mid Z_t = a_t\right] = \mathbb{E}\left[F\left(\mathcal{T}_s^{0,|\theta|}\right)\right].$$

Remark 5.15. *Using [36], Corollary 4.2 on the SDE for the size-population process Z, is a direct consequence of Theorems 5.13 (for $\alpha \in (0, +\infty)$) and 5.14 (for $\alpha = 0$) and Remark 5.12.*

6 Set of trees, topology and measurability

In a nutshell, the main objective of this section is to define the grafting and splitting functions, as well as the decorating and de-decorating functions in a measurable way on the set of complete locally compact rooted real trees, so that we can properly define in Section 7 the random variables used the in the previous sections. An index of all the (numerous) relevant notations of this section is provided at the end of the document.

We keep the basic definitions and notations for rooted real trees from Section 5.1. In Section 6.1 we consider the regularity of the spanning of subtrees. In Section 6.2, we study the Polish spaces of equivalent classes of compact (resp. complete locally compact) rooted trees with distinguished vertices endowed with the Gromov-Hausdorff distance. We define various grafting measurable operations (denoted by $\circledast_*^*$) of a tree on an another tree in Section 6.3. Motivated by the fact that some random trees are obtained as decorated backbone trees, we introduce in Section 6.4 the space of marked trees, that is of trees with a distinguished subtree (or backbone tree). We also establish in this section the measurability of various truncation maps. The short Section 6.5 is devoted to special case of the backbone tree being reduced to an infinite spine (this is the case for the Kesten tree). In Section 6.6, we consider specifically discrete trees which are spanned by n distinguished vertices, and describe them as a set of branches indexed by all the possible subsets of the n distinguished vertices. This description is then used in Section 6.7 to split (with a function Split_n) a complete locally compact tree with n distinguished vertices as subtrees supported by the different branches of the discrete tree spanned by the distinguished vertices. Then, we provide in a sense the inverse construction in Section 6.8 where (with a function Graft_n) we decorate the branches of a discrete trees with subtrees. In Section 6.9, we describe a measurable procedure to decorate a branch with a family of subtrees given by the atoms of a point measure on the set of trees (the function Tree) and a measurable procedure to describe the decoration of a distinguished branch of a tree (the function $\boldsymbol{\mathcal{M}}$) through a point measure on the set of trees.

We shall use many times Lusin's theorem from [37] or [11, Exercise 6.10.54 p.60] which states that, if f is a measurable function defined on a Borel subset A of a Polish space to a Polish space, then $f(B)$ is a Borel set for all Borel subsets $B \subset A$ if and only if the set of all values y, such that $f^{-1}(\{y\})$ is uncountable, is at most countable.

6.1 Continuity of the map Span

Recall the definition of the set $\mathbb{T}_{\mathrm{K}}^{(n)}$ of n-pointed compact rooted tree in Subsection 5.1.2, endowed with the distance $d_{\mathrm{GH}}^{(n)}$. Recall also the definition of the tree spanned by n vertices. For a rooted n-pointed tree $(T, d, \mathbf{v})$, with $\mathbf{v} = (\varrho, v_1 \dots, v_n)$, we denote the corresponding spanned tree $\mathrm{Span}^\circ(T, \mathbf{v})$ as:

$$\mathrm{Span}^\circ(T, \mathbf{v}) = \bigcup_{k=1}^{n} [\![\varrho, v_k]\!]. \tag{6.1}$$

The tree $(\mathrm{Span}^\circ(T, \mathbf{v}), d, \varrho)$ will be simply denoted by $\mathrm{Span}^\circ(T, \mathbf{v})$, whereas we will denote by $\mathrm{Span}(T, \mathbf{v})$ the rooted n-pointed tree $(\mathrm{Span}^\circ(T, \mathbf{v}), d, \mathbf{v})$. For $y \in T$, we also define $p_{\mathbf{v}}(y)$, the projection of y on $\mathrm{Span}^\circ(T, \mathbf{v})$, as the only point of $\mathrm{Span}^\circ(T, \mathbf{v})$ such that:

$$[\![\varrho, y]\!] \cap \mathrm{Span}^\circ(T, \mathbf{v}) = [\![\varrho, p_{\mathbf{v}}(y)]\!]. \tag{6.2}$$

Let us state a technical result which will be used several times in what follows.

Lemma 6.1. *Let $n \in \mathbb{N}$. Let $(T, d, \mathbf{v})$ and $(T', d', \mathbf{v}')$ be two compact rooted n-pointed trees and let $\mathcal{R}$ be a correspondence between them. For every $(x, x') \in \mathcal{R}$ with $x' \in \mathrm{Span}^\circ(T', \mathbf{v}')$, we have:*

$$d(x, p_{\mathbf{v}}(x)) \leq \frac{3}{2} \mathrm{dist}\ (\mathcal{R}).$$

Proof. Let $(x, x') \in \mathcal{R}$ with $x' \in \mathrm{Span}^\circ(T', \mathbf{v}')$. First remark that there exist $k, \ell \in \{0, \dots, n\}$ such that $p_{\mathbf{v}}(x) \in [\![v_k, v_\ell]\!]$ and $x' \in [\![v'_k, v'_\ell]\!]$. Indeed, let us set:

$$A = \big\{v_k \colon\ p_{\mathbf{v}}(x) \in [\![\varrho, v_k]\!]\big\} \quad \text{and} \quad A' = \big\{v'_k \colon\ x' \in [\![\varrho', v'_k]\!]\big\}.$$

Notice that $A \neq \emptyset$ and $A' \neq \emptyset$. If there exists $k \geq 1$ such that $v_k \in A$ and $v'_k \in A'$, then one can take $\ell = 0$ so that $v_\ell = \varrho$ and $v'_\ell = \varrho'$. Otherwise, take k and ℓ with $k \neq \ell$ such that $v_k \in A$ and $v'_\ell \in A'$. In this case, we get $v_\ell \notin A$. Clearly we have $p_{\mathbf{v}}(x) \in [\![v_k, v_\ell]\!]$ and by a similar argument, $x' \in [\![v'_k, v'_\ell]\!]$. Therefore, we have:

$$2d\big(x, p_{\mathbf{v}}(x)\big) = d(x, v_k) + d(x, v_\ell) - d(v_k, v_\ell) \leq d'(x', v'_k) + d'(x', v'_\ell) - d'(v'_k, v'_\ell) + 3\, \mathrm{dist}\ (\mathcal{R}).$$

Then, use that $d'(x', v'_k) + d'(x', v'_\ell) - d'(v'_k, v'_\ell) = 0$, as $x' \in [\![v'_k, v'_\ell]\!]$, to conclude. □

If $(T, \mathbf{v})$ and $(T', \mathbf{v}')$ belong to the same equivalence class in $\mathbb{T}_{\mathrm{K}}^{(n)}$, then so do $\mathrm{Span}(T, \mathbf{v})$ and $\mathrm{Span}(T', \mathbf{v}')$ in $\mathbb{T}_{\mathrm{K}}^{(n)}$. Therefore, the function $(T, \mathbf{v}) \mapsto \mathrm{Span}(T, \mathbf{v})$ is well defined from $\mathbb{T}_{\mathrm{K}}^{(n)}$ to $\mathbb{T}_{\mathrm{K}}^{(n)}$. A first consequence of Lemma 6.1 is that this function is Lipschitz continuous; this result will be completed in Lemma 6.7.

Lemma 6.2 (Continuity of the map Span)**.** *Let $n \in \mathbb{N}$. The map $(T, \mathbf{v}) \mapsto \mathrm{Span}(T, \mathbf{v})$ is 4-Lipschitz continuous from $\mathbb{T}_{\mathrm{K}}^{(n)}$ to $\mathbb{T}_{\mathrm{K}}^{(n)}$.*

Proof. Let $(T, \mathbf{v}), (T', \mathbf{v}')$ be two compact rooted n-pointed trees and let $\mathcal{R}$ be a correspondence between them. Let us set with obvious notations:

$$\tilde{\mathcal{R}} = \Big\{ \big(x, p'_{\mathbf{v}'}(x')\big) :\ (x, x') \in \mathcal{R},\ x \in \mathrm{Span}^\circ(T, \mathbf{v}) \Big\} \cup \Big\{ \big(p_{\mathbf{v}}(x), x'\big) :\ (x, x') \in \mathcal{R},\ x' \in \mathrm{Span}^\circ(T', \mathbf{v}') \Big\}. \quad (6.3)$$

Clearly, $\tilde{\mathcal{R}}$ is a correspondence between $\mathrm{Span}(T, \mathbf{v})$ and $\mathrm{Span}(T', \mathbf{v}')$. We now compute its distortion. We consider the case $x \in \mathrm{Span}(T, \mathbf{v})$, $y' \in \mathrm{Span}(T', \mathbf{v}')$ and $(x, x'), (y, y') \in \mathcal{R}$, so that $(x, p'_{\mathbf{v}'}(x'))$ and $(p_{\mathbf{v}}(y), y')$ belong to $\tilde{\mathcal{R}}$. We have:

$$\begin{aligned} \Big| d\big(x, p_{\mathbf{v}}(y)\big) - d'\big(p'_{\mathbf{v}'}(x'), y'\big) \Big| &= \Big| d(x, y) - d\big(y, p_{\mathbf{v}}(y)\big) - d'(x', y') + d'\big(x', p_{\mathbf{v}'}(x')\big) \Big| \\ &\leq \big| d(x, y) - d'(x', y') \big| + d\big(y, p_{\mathbf{v}}(y)\big) + d'\big(x', p_{\mathbf{v}'}(x')\big) \\ &\leq 4 \,\mathrm{dist}\ (\mathcal{R}), \end{aligned}$$

where we used Lemma 6.1 for the last inequality. The other cases can be treated similarly. This implies that $\mathrm{dist}\ (\tilde{\mathcal{R}}) \leq 4 \,\mathrm{dist}\ (\mathcal{R})$ and thus, by definition of $d_{\mathrm{GH}}^{(n)}$:

$$d_{\mathrm{GH}}^{(n)}\big(\mathrm{Span}(T, \mathbf{v}), \mathrm{Span}(T', \mathbf{v}')\big) \leq 4 \, d_{\mathrm{GH}}^{(n)}\big((T, \mathbf{v}), (T', \mathbf{v}')\big).$$

□

6.2 Set of (equivalence classes of) rooted n-pointed complete locally compact trees

Recall the definition of the height $H(x) = d(\varrho, x)$ of a vertex x in a rooted tree (T, d, ρ). For a rooted n-pointed tree $(T, d, \mathbf{v})$ and $t \geq 0$, we define the rooted n-pointed tree T truncated at level t as $(r_t(T, \mathbf{v}), d, \mathbf{v})$ with:

$$r_t(T, \mathbf{v}) = \big\{ x \in T :\ H(x) \leq t \big\} \cup \big\{ \mathrm{Span}^\circ(T, \mathbf{v}) \big\}, \quad (6.4)$$

and the distance on $r_t(T, \mathbf{v})$ is given by the restriction of the distance d. We shall simply write $r_t(T, \mathbf{v})$ for $(r_t(T, \mathbf{v}), d, \mathbf{v})$. (Notice that for $t \geq t_T = \max_{i \in \{0, \ldots, n\}} d(\varrho, v_i)$ the truncated operations defined by (6.4) and (5.2) coincide.)

If $(T, \mathbf{v})$ and $(T', \mathbf{v}')$ are in the same equivalence class of $\mathbb{T}_{\mathrm{K}}^{(n)}$, so are $r_t(T, \mathbf{v})$ and $r_t(T', \mathbf{v}')$. Thus the function r_t can be seen as a map from $\mathbb{T}_{\mathrm{K}}^{(n)}$ to itself. When $n = 0$, we shall simply write $r_t(T)$ for $r_t(T, \varrho)$. The next lemma is about the continuity of r_t.

Lemma 6.3 (Continuity of r_t)**.** *Let $n \in \mathbb{N}$. For $s, t \geq 0$ and $(T, \mathbf{v}), (T', \mathbf{v}') \in \mathbb{T}_{\mathrm{K}}^{(n)}$, we have:*

$$d_{\mathrm{GH}}^{(n)}\left(r_t(T, \mathbf{v}), r_{t+s}(T', \mathbf{v}')\right) \leq 4 \, d_{\mathrm{GH}}^{(n)}\big((T, \mathbf{v}), (T', \mathbf{v}')\big) + s. \quad (6.5)$$

The map $\big(t, (T, \mathbf{v})\big) \mapsto r_t(T, \mathbf{v})$ is continuous from $\mathbb{R}_+ \times \mathbb{T}_{\mathrm{K}}^{(n)}$ to $\mathbb{T}_{\mathrm{K}}^{(n)}$.

Proof. Let $(T,d,\mathbf{v}),(T',d',\mathbf{v}')$ be two compact rooted n-pointed trees. Firstly, notice that $d_{\mathrm{GH}}^{(n)}\big(r_{t+s}(T,\mathbf{v}),r_t(T,\mathbf{v})\big)\leq s$. Secondly, recall Definition (6.2) of the projection $p_{\mathbf{v}}$ on $\mathrm{Span}^\circ(T,\mathbf{v})$. For $y\in T$, we also define the projection $p_t(y)$ of y on $r_t(T,\mathbf{v})$ as the only point of $r_t(T,\mathbf{v})$ such that:

$$[\![\varrho,y]\!]\cap r_t(T,\mathbf{v})=[\![\varrho,p_t(y)]\!].$$

We first prove the analogue of Lemma 6.1. Let $\mathcal{R}$ be a correspondence between $(T,\mathbf{v})$ and $(T',\mathbf{v}')$. Let $(x,x')\in\mathcal{R}$ with $x'\in r_t(T',\mathbf{v}')$. By construction, we have $p_t(x)\in[\![p_{\mathbf{v}}(x),x]\!]$. If $x'\in\mathrm{Span}(T',\mathbf{v}')$, then we deduce from Lemma 6.1 that $d\big(x,p_t(x)\big)\leq d\big(x,p_{\mathbf{v}}(x)\big)\leq\frac{3}{2}\mathrm{dist}\ (\mathcal{R})$. If $x'\in r_t(T',\mathbf{v}')\setminus\mathrm{Span}(T',\mathbf{v}')$, then we have $H(x')\leq t$ and thus $H(x)=d(\varrho,x)\leq d'(\varrho',x')+\mathrm{dist}\ (\mathcal{R})\leq t+\mathrm{dist}\ (\mathcal{R})$, which implies that $d\big(x,p_t(x)\big)\leq\mathrm{dist}\ (\mathcal{R})$. In conclusion, we get $d\big(x,p_t(x)\big)\leq\frac{3}{2}\mathrm{dist}\ (\mathcal{R})$. Now, arguing as in the proof of Lemma 6.2, we deduce that $d_{\mathrm{GH}}^{(n)}\big(r_t(T,\mathbf{v}),r_t(T',\mathbf{v})\big)\leq 4\,d_{\mathrm{GH}}^{(n)}\big((T,\mathbf{v}),(T',\mathbf{v}')\big)$. This gives the result. □

If a rooted n-pointed tree $(T,d,\mathbf{v})$ is complete and locally compact, then the rooted tree $r_t(T,\mathbf{v})$ is compact for all $t\geq 0$. Following [6], we set for two complete locally compact rooted n-pointed trees $(T,\mathbf{v})$ and $(T',\mathbf{v}')$:

$$d_{\mathrm{LGH}}^{(n)}((T,\mathbf{v}),(T',\mathbf{v}'))=\int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}t\ \left(1\wedge d_{\mathrm{GH}}^{(n)}\big(r_t(T,\mathbf{v}),r_t(T',\mathbf{v}')\big)\right).$$

Furthermore, we have that $d_{\mathrm{LGH}}^{(n)}\big((T,\mathbf{v}),(T',\mathbf{v}')\big)=0$ if and only if there exists an isometric bijection from (T,d) to (T',d') which preserves the distinguished vertices (this can easily be proved with similar arguments as for [6, Proposition 5.3]). The relation $d_{\mathrm{LGH}}^{(n)}\big((T,\mathbf{v}),(T',\mathbf{v}')\big)=0$ defines an equivalence relation. Arguing as in [6] where $n=0$, we get the following result. Following the notations in [6], for $n=0$, we simply write $\mathbb{T}_{\mathrm{loc-K}}$ and d_{LGH} for $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ and $d_{\mathrm{LGH}}^{(n)}$.

Proposition 6.4 ($\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ is Polish)**.** *The set $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ of equivalence classes of complete locally compact rooted n-pointed trees endowed with $d_{\mathrm{LGH}}^{(n)}$ is a metric Polish space. Furthermore, the set $\mathbb{T}_{\mathrm{K}}^{(n)}$ of equivalence classes of compact rooted n-pointed trees is an open dense subset of $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$.*

We first provide a short proof for the following inequalities.

Lemma 6.5 (Inequalities for $d_{\mathrm{GH}}^{(n)}$ and $d_{\mathrm{LGH}}^{(n)}$)**.** *Let $n\in\mathbb{N}$. For $(T,\mathbf{v}),(T',\mathbf{v}')\in\mathbb{T}_{\mathrm{K}}^{(n)}$, we have:*

$$d_{\mathrm{LGH}}^{(n)}\big((T,\mathbf{v}),(T',\mathbf{v}')\big)\leq 1\wedge 4\,d_{\mathrm{GH}}^{(n)}\big((T,\mathbf{v}),(T',\mathbf{v})\big). \tag{6.6}$$

For $(T,\mathbf{v}),(T',\mathbf{v}')\in\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ and $s,t\geq 0$, we have:

$$d_{\mathrm{LGH}}^{(n)}\big(r_t(T,\mathbf{v}),r_{t+s}(T',\mathbf{v}')\big)\leq 4\,d_{\mathrm{LGH}}^{(n)}\big((T,\mathbf{v}),(T',\mathbf{v}')\big)+s, \tag{6.7}$$

$$d_{\mathrm{GH}}^{(n)}\big(r_t(T,\mathbf{v}),r_t(T',\mathbf{v}')\big)\leq 4\,\mathrm{e}^t\ d_{\mathrm{LGH}}^{(n)}\big((T,\mathbf{v}),(T',\mathbf{v}')\big). \tag{6.8}$$

The map $\big(t,(T,\mathbf{v})\big)\mapsto r_t(T,\mathbf{v})$ is continuous from $\mathbb{R}_+\times\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ to $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ (and to $\mathbb{T}_{\mathrm{K}}^{(n)}$).

Proof. Equation (6.6) is a direct consequence of (6.5) with $s=0$ and the definition of $d_{\mathrm{LGH}}^{(n)}$. Equation (6.7) follows from similar arguments, using also that $r_{t'}\circ r_u=r_u\circ r_{t'}=r_{t'\wedge u}$. For $t\leq s$, we have $4^{-1}\,d_{\mathrm{GH}}^{(n)}\big(r_t(T,\mathbf{v}),r_t(T',\mathbf{v}')\big)\leq d_{\mathrm{GH}}^{(n)}\big(r_s(T,\mathbf{v}),r_s(T',\mathbf{v}')\big)$. Integrating with respect to $\mathrm{e}^{-s}\ \mathrm{d}s$ gives (6.8). The continuity of the map $(t,(T,\mathbf{v}))\mapsto r_t(T)$ is a direct consequence of (6.7). □

We deduce from (6.6) and (6.8) that all the measurable sets of $(\mathbb{T}_{\mathrm{K}}^{(n)}, d_{\mathrm{GH}}^{(n)})$ are measurable sets of $(\mathbb{T}_{\mathrm{loc-K}}^{(n)}, d_{\mathrm{LGH}}^{(n)})$, and that a converging sequence in $(\mathbb{T}_{\mathrm{K}}^{(n)}, d_{\mathrm{GH}}^{(n)})$ is also converging in $(\mathbb{T}_{\mathrm{loc-K}}^{(n)}, d_{\mathrm{LGH}}^{(n)})$. We also we deduce from (6.6) that the restriction to $\mathbb{T}_{\mathrm{K}}^{(n)}$ of a continuous function defined on $(\mathbb{T}_{\mathrm{loc-K}}^{(n)}, d_{\mathrm{LGH}}^{(n)})$ is also continuous on $(\mathbb{T}_{\mathrm{K}}^{(n)}, d_{\mathrm{GH}}^{(n)})$.

Removing from $\mathbf{v}$ some of the distinguished vertices (but the root) is continuous, see the next lemma. For $(T, \mathbf{v} = (v_0 = \varrho, \dots, v_n)) \in \mathbb{T}_{\mathrm{loc-K}}^{(n)}$ and $0 \in A \subset \{0, \dots, n\}$, we set:

$$\Pi_n^{\circ, A}(T, \mathbf{v}) = (T, \mathbf{v}_A) \quad \text{with} \quad \mathbf{v}_A = (v_i, i \in A). \tag{6.9}$$

For simplicity, we shall write Π_n° for $\Pi_n^{\circ, A}$ when A is reduced to $\{0\}$, so that Π_n° corresponds to removing all the distinguished vertices but the root.

Lemma 6.6 (Removing some distinguished vertices is continuous). *Let* $n \in \mathbb{N}$ *and* $0 \in A \subset \{0, \dots, n\}$. *The map* $\Pi_n^{\circ, A}$ *from* $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ *to* $\mathbb{T}_{\mathrm{loc-K}}^{(k)}$*, with* k *the cardinal of* A*, is* 1*-Lipschitz continuous.*

Proof. First, notice that the equivalence class of $(T, \mathbf{v}_A)$ in $\mathbb{T}_{\mathrm{loc-K}}^{(k)}$ does not depend of the choice of $(T, \mathbf{v})$ in its equivalence class in $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$. Thus the map $\Pi_n^{\circ, A}$ is well defined from $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ to $\mathbb{T}_{\mathrm{loc-K}}^{(k)}$. It is clearly 1-Lipschitz continuous since a correspondence between the trees $(T, \mathbf{v})$ and $(T', \mathbf{v}')$ is also a correspondence between $(T, \mathbf{v}_A)$ and $(T, \mathbf{v}'_A)$. ☐

We give an immediate consequence on the continuity of the maps Span and Span°.

Lemma 6.7 (Continuity of the maps Span and Span°). *Let* $n \in \mathbb{N}$. *The map* $(T, \mathbf{v}) \mapsto \mathrm{Span}(T, \mathbf{v})$ *and* $(T, \mathbf{v}) \mapsto \mathrm{Span}^\circ(T, \mathbf{v})$ *are* 4*-Lipschitz continuous from* $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ *to* $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ *and to* $\mathbb{T}_{\mathrm{loc-K}}$ *respectively.*

Proof. Notice that $d_{\mathrm{LGH}}^{(n)}\big(\mathrm{Span}(T, \mathbf{v}), \mathrm{Span}(T', \mathbf{v})\big) = d_{\mathrm{GH}}^{(n)}\big(\mathrm{Span}(T, \mathbf{v}), \mathrm{Span}(T', \mathbf{v})\big)$, and thus the map Span from $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ to $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ is 4-Lipschitz continuous, thanks to Lemma 6.2. Then use Lemma 6.6 on the continuity of Π_n° and the fact that $\mathrm{Span}^\circ = \Pi_n^\circ \circ \mathrm{Span}$ to conclude. ☐

Next, we check that rerooting or reordering the distinguished vertices is a continuous operation. For a vector $\mathbf{v} = (v_0, \dots, v_n)$ and a permutation π of $\{0, \dots, n\}$, we set $\mathbf{v}^\pi = (v_{\pi(0)}, \dots, v_{\pi(n)})$.

Remark 6.8. *One can see that the map* $(T, \mathbf{v}) \mapsto (T, \mathbf{v}^\pi)$ *is an isometry on* $\mathbb{T}_{\mathrm{K}}^{(n)}$. *The next lemma is an extension to complete locally trees.*

Lemma 6.9 (Permuting the distinguished vertices is continuous). *Let* $n \in \mathbb{N}$ *and let* π *be a permutation on* $\{0, \dots, n\}$. *The map* $(T, \mathbf{v}) \mapsto (T, \mathbf{v}^\pi)$ *defined on* $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ *is continuous.*

Proof. First notice that if $(T, \mathbf{v})$ and $(T', \mathbf{v}')$ are rooted n-pointed trees belonging to the same equivalence class of $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$, so do $(T, \mathbf{v}^\pi)$ and $(T', \mathbf{v}'^\pi)$. Thus, the map $(T, \mathbf{v}) \mapsto (T, \mathbf{v}^\pi)$ is indeed well-defined on $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$. We shall use the following notation: we denote by r_t° the truncation r_t when one forgets about the distinguished vertices (but the root): $r_t^\circ = \Pi_n^\circ \circ r_t$. (Take care that $\Pi_n^\circ \circ r_t \neq r_t \circ \Pi_n^\circ$.) To prove the continuity of the map, we consider two cases.

1st case: No rerooting, $\pi(0) = 0$. In that case, for every $t \geq 0$ and every $(T, \mathbf{v}) \in \mathbb{T}_{\mathrm{loc-K}}^{(n)}$, we have that $r_t^\circ(T, \mathbf{v}) = r_t^\circ(T, \mathbf{v}^\pi)$ and thus we get that:

$$d_{\mathrm{LGH}}^{(n)}\big((T, \mathbf{v}^\pi), (T', \mathbf{v}'^\pi)\big) = d_{\mathrm{LGH}}^{(n)}\big((T, \mathbf{v}), (T', \mathbf{v}')\big).$$

This trivially implies the continuity of the map.

2nd case: With rerooting, $\pi(k_0)=0$ for some $k_0\neq 0$. Let $(T,\mathbf{v}),\ (T',\mathbf{v}')\in \mathbb{T}^{(n)}_{\mathrm{loc-K}}$, with $\mathbf{v}=(v_0=\varrho,\dots,v_n)$ and $\mathbf{v}'=(v'_0=\varrho',\dots,v'_n)$, such that $d^{(n)}_{\mathrm{LGH}}\big((T,\mathbf{v}),(T',\mathbf{v}')\big)<1/2$. As v_{k_0} and v'_{k_0} are always in correspondence as well as ϱ and ϱ', we have, for every $t\geq 0$ that:

$$\big|H(v_{k_0})-H(v'_{k_0})\big|\leq 2d^{(n)}_{\mathrm{GH}}\big(r_t(T,\mathbf{v}),r_t(T',\mathbf{v}')\big).$$

Multiplying by e^{-t} and integrating yields:

$$1\wedge\big|H(v_{k_0})-H(v'_{k_0})\big|\leq 2d^{(n)}_{\mathrm{LGH}}\big((T,\mathbf{v}),(T',\mathbf{v}')\big)<1,$$

and hence:

$$H(v'_{k_0})\leq H(v_{k_0})+1.$$

We set $h_0=H(v_{k_0})+1$. Then, for every $t\geq 0$, we have:

$$r^\circ_t(T,\mathbf{v}^\pi)\subset r^\circ_{t+h_0}(T,\mathbf{v})\quad\text{and thus}\quad r_t(T,\mathbf{v}^\pi)=r_t\Big(r^\circ_{t+h_0}(T,\mathbf{v}),\mathbf{v}^\pi\Big),$$

and the same holds for T'. Consequently, applying Lemma 6.3, we have:

$$\begin{aligned}
d^{(n)}_{\mathrm{LGH}}\big((T,\mathbf{v}^\pi),(T',\mathbf{v}'^\pi)\big)&\leq 4\int_0^{+\infty}\mathrm{d}t\,\mathrm{e}^{-t}\,\Big(1\wedge d^{(n)}_{\mathrm{GH}}\Big(\big(r^\circ_{t+h_0}(T,\mathbf{v}),\mathbf{v}^\pi\big),\big(r^\circ_{t+h_0}(T',\mathbf{v}'),\mathbf{v}'^\pi\big)\Big)\Big)\\
&=4\int_0^{+\infty}\mathrm{d}t\,\mathrm{e}^{-t}\,\Big(1\wedge d^{(n)}_{\mathrm{GH}}\big(r_{t+h_0}(T,\mathbf{v}),r_{t+h_0}(T',\mathbf{v}')\big)\Big)\\
&\leq 4\,\mathrm{e}^{h_0}\,d^{(n)}_{\mathrm{LGH}}\big((T,\mathbf{v}),(T',\mathbf{v}')\big),
\end{aligned}$$

where we used for the second inequality that $d^{(n)}_{\mathrm{GH}}\big((\tilde T,\mathbf{v}^\pi),(\tilde T',\mathbf{v}'^\pi)\big)=d^{(n)}_{\mathrm{GH}}\big((\tilde T,\mathbf{v}),(\tilde T,\mathbf{v}')\big)$ for $(\tilde T,\mathbf{v}),(\tilde T',\mathbf{v}')\in\mathbb{T}^{(n)}_{\mathrm{K}}$. The continuity of the map follows. □

We shall also consider the set of trees whose root is not a branching vertex:

$$\mathbb{T}^{(n),0}_{\mathrm{loc-K}}=\big\{(T,\mathbf{v})\in\mathbb{T}^{(n)}_{\mathrm{loc-K}}:\ \varrho\not\in\mathrm{Br}(T)\big\}. \tag{6.10}$$

We shall simply write $\mathbb{T}^0_{\mathrm{loc-K}}$ for $\mathbb{T}^{(n),0}_{\mathrm{loc-K}}$ when $n=0$.

Lemma 6.10. *The set $\mathbb{T}^{(n),0}_{\mathrm{loc-K}}$ is a Borel subset of $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$.*

Proof. For a rooted tree T, we define its diameter by $\mathrm{diam}\,(T)=\sup\{d(x,y):\ x,y\in T\}$. Notice that $H(T)\leq\mathrm{diam}\,(T)\leq 2H(T)$. Clearly the function diam is constant on all equivalent classes of $\mathbb{T}^{(n)}_{\mathrm{K}}$ and thus of $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$. If $\mathrm{diam}\,(T)=2H(T)<+\infty$, then we deduce that the root is a branching vertex. Recall Π°_n for (6.9). More generally, we get that:

$$\mathbb{T}^{(n),0}_{\mathrm{loc-K}}=\bigcup_{n\in\mathbb{N}^*}D_{1/n}\quad\text{with}\quad D_t=\Big\{T\in\mathbb{T}^{(n)}_{\mathrm{loc-K}}:\ \mathrm{diam}\,\big(r_t\circ\Pi^\circ_n(T)\big)=2t\Big\}.$$

Since the functions diam , r_t and Π°_n are continuous, we deduce that D_t is closed, and hence $\mathbb{T}^{(n),0}_{\mathrm{loc-K}}$ is a Borel subset of $\mathbb{T}_{\mathrm{loc-K}}$. □

We now define the set of discrete trees. We say that a rooted n-pointed tree $(T,d,\mathbf{v})$ is a discrete tree if T is equal to the tree spanned by its distinguished vertices: $T=\mathrm{Span}^\circ(T,\mathbf{v})$. We define the set of (equivalence classes of) discrete trees with at most n leaves as:

$$\mathbb{T}^{(n)}_{\mathrm{dis}}=\big\{(T,\mathbf{v})\in\mathbb{T}^{(n)}_{\mathrm{loc-K}}:\ (T,\mathbf{v})=\mathrm{Span}(T,\mathbf{v})\big\}. \tag{6.11}$$

As a direct consequence of the continuity of the map Span we get the following result.

Lemma 6.11. *Let $n \in \mathbb{N}$. The set of discrete trees $\mathbb{T}_{\mathrm{dis}}^{(n)}$ is a closed subset of $\mathbb{T}_{\mathrm{K}}^{(n)}$ and of $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$.*

We end this section with partial measurability result on the number of vertices at a given height of a tree.

Remark 6.12. *It is immediate to check that the map $(T, \mathbf{v}) \mapsto \big(d(v_i, v_j),\ i, j \in \{0, \dots, n\}\big)$ is injective $1/2$-Lipschitz continuous from $(\mathbb{T}_{\mathrm{loc-K}}^{(n)}, d_{\mathrm{LGH}}^{(n)})$ to $\mathbb{R}^{(n+1)\times(n+1)}$ endowed with the supremum norm (*i.e.* the maximum of the distances between coordinates). It is also bi-measurable thanks to Lusin's theorem.*

Let $\mathbb{T}_{\mathrm{loc-K}}^{\mathrm{no\,leaf}}$ be the set of trees with no leaves:

$$\mathbb{T}_{\mathrm{loc-K}}^{\mathrm{no\,leaf}} = \big\{T \in \mathbb{T}_{\mathrm{loc-K}} :\ \mathrm{Lf}(T) = \emptyset\big\}.$$

For $T \in \mathbb{T}_{\mathrm{loc-K}}^{\mathrm{no\,leaf}}$ and $t \geq 0$, let $\tilde{N}_t(T)$ denotes the number of vertices at height t of T:

$$\tilde{N}_t(T) = \mathrm{Card}\ \Big(\big\{x \in T :\ H(x) = t\big\}\Big). \tag{6.12}$$

It is easy to prove (and left as an exercise to the reader) that $\tilde{N}_t(T)$ is finite using that T is complete locally compact without leaves. We have the following result.

Lemma 6.13 (Measurability of $\tilde{N}_t$)**.** *The set $\mathbb{T}_{\mathrm{loc-K}}^{\mathrm{no\,leaf}}$ is a Borel subset of $\mathbb{T}_{\mathrm{loc-K}}$ and the map $(t, T) \mapsto \tilde{N}_t(T)$ is measurable from $\mathbb{R}_+ \times \mathbb{T}_{\mathrm{loc-K}}^{\mathrm{no\,leaf}}$ to $\mathbb{N}$.*

Proof. Let $t \geq 0$ and let $\Theta_n(t)$ be the set of discrete trees such that all the distinguished vertices (but the root) are leaves at height t:

$$\Theta_n(t) = \big\{T \in \mathbb{T}_{\mathrm{dis}}^{(n)} :\ d(\varrho, v_i) = t \text{ and } d(v_i, v_j) > 0 \text{ for all } i, j \in \{1, \dots, n\}\big\}.$$

Thanks to Remark 6.12, $\Theta_n(t)$ is a Borel set of $\mathbb{T}_{\mathrm{dis}}^{(n)} \subset \mathbb{T}_{\mathrm{K}}^{(n)} \subset \mathbb{T}_{\mathrm{loc-K}}^{(n)}$. For $T \in \mathbb{T}_{\mathrm{loc-K}}$, we get that $\big\{T' \in \mathbb{T}_{\mathrm{dis}}^{(n)} :\ \Pi_n^\circ(T') = T\big\}$ is finite. We deduce from Lusin's theorem that Π_n° restricted to $\mathbb{T}_{\mathrm{dis}}^{(n)}$ is bi-measurable. This implies that the set $\Pi_n^\circ(\Theta_n(t))$ is a Borel subset of $\mathbb{T}_{\mathrm{loc-K}}$. We deduce that the set of trees with no leaves, $\mathbb{T}_{\mathrm{loc-K}}^{\mathrm{no\,leaf}}$, which is formally defined by:

$$\mathbb{T}_{\mathrm{loc-K}}^{\mathrm{no\,leaf}} = \bigcap_{k \in \mathbb{N}^*} \bigcup_{n \in \mathbb{N}} r_k^{-1}\Big(\Pi_n^\circ\big(\Theta_n(k)\big)\Big),$$

is a Borel subset of $\mathbb{T}_{\mathrm{loc-K}}$. We also get that $\big\{T \in \mathbb{T}_{\mathrm{loc-K}}^{\mathrm{no\,leaf}} :\ \tilde{N}_t(T) = n\big\} = r_t^{-1}\Big(\Pi_n^\circ\big(\Theta_n(t)\big)\Big)$; this implies that the map $\tilde{N}_t$ is measurable. Since $t \mapsto \tilde{N}_t(T)$ is non-decreasing and left-continuous, we deduce that the map $(t, T) \mapsto \tilde{N}_t(T)$ is measurable from $\mathbb{R}_+ \times \mathbb{T}_{\mathrm{loc-K}}^{\mathrm{no\,leaf}}$ to $\mathbb{N}$. ☐

6.3 Grafting a discrete tree on another one

We define, in a slightly more general context than Section 5.1.3, the grafting of a complete locally compact rooted tree at a distinguished vertex of an another compact locally compact rooted tree. For $(T, \mathbf{v}) \in \mathbb{T}_{\mathrm{loc-K}}^{(n)}$ and $(T', \mathbf{v}') \in \mathbb{T}_{\mathrm{loc-K}}^{(k)}$ and $i \in \{0, \dots, n\}$, with $n, k \geq 0$, $\mathbf{v} = (v_0 = \varrho, \dots, v_n)$ and $\mathbf{v}' = (v'_0 = \varrho', \dots, v'_k)$, we define the tree $T \circledast_i T'$ by (5.3) and the distance $d^\circledast$ by (5.4) with x replaced by v_i, and consider the distinguished vertices $\mathbf{v} \circledast \mathbf{v}' = (v_0 = \varrho, \dots, v_n, v'_1, \dots, v'_k)$.

Lemma 6.14 (Continuity of the grafting map)**.** *Let $n, k \in \mathbb{N}$ and $i \in \{0, \dots, n\}$. The map $\big((T, \mathbf{v}), (T', \mathbf{v}')\big) \mapsto (T \circledast_i T', \mathbf{v} \circledast \mathbf{v}')$, is continuous from $\mathbb{T}_{\mathrm{loc-K}}^{(n)} \times \mathbb{T}_{\mathrm{loc-K}}^{(k)}$ to $\mathbb{T}_{\mathrm{loc-K}}^{(n+k)}$.*

Proof. Let $(T_1, \mathbf{v}_1), (T_1', \mathbf{v}_1') \in \mathbb{T}^{(n)}_{\mathrm{loc-K}}$ and $(T_2, \mathbf{v}_2), (T_2', \mathbf{v}_2') \in \mathbb{T}^{(k)}_{\mathrm{loc-K}}$. Set $T = T_1 \circledast_i T_2$, $T' = T_1' \circledast_i T_2'$, $\mathbf{v} = \mathbf{v}_1 \circledast \mathbf{v}_2$, and $\mathbf{v}' = \mathbf{v}_1' \circledast \mathbf{v}_2'$.

First suppose that the trees are compact, that is $(T_1, \mathbf{v}_1), (T_1', \mathbf{v}_1') \in \mathbb{T}^{(n)}_{\mathrm{K}}$ and $(T_2, \mathbf{v}_2)$, $(T_2', \mathbf{v}_2') \in \mathbb{T}^{(k)}_{\mathrm{K}}$. Let $\mathcal{R}_1$ be a correspondence between (elements of the classes) $(T_1, \mathbf{v}_1)$ and $(T_1', \mathbf{v}_1')$ and let $\mathcal{R}_2$ be a correspondence between (elements of the classes) $(T_2, \mathbf{v}_2)$ and $(T_2', \mathbf{v}_2')$. We set $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ with ϱ_2 and ϱ_2' replaced respectively by v_i and v_i'. It defines a correspondence between $(T, \mathbf{v})$ and $(T', \mathbf{v}')$. For every $(x, x'), (y, y') \in \mathcal{R}$, we have:

$$\left|d^{\circledast}(x,y) - d'^{\circledast}(x',y')\right| = \begin{cases} \left|d_1(x,y) - d_1'(x',y')\right| \leq \mathrm{dist}\,(\mathcal{R}_1) & \text{if } (x,x'), (y,y') \in \mathcal{R}_1, \\ \left|d_2(x,y) - d_2'(x',y')\right| \leq \mathrm{dist}\,(\mathcal{R}_2) & \text{if } (x,x'), (y,y') \in \mathcal{R}_2, \end{cases}$$

and if $(x, x') \in \mathcal{R}_1$ and $(y, y') \in \mathcal{R}_2$, we have:

$$\begin{aligned} \left|d^{\circledast}(x,y) - d'^{\circledast}(x',y')\right| &= \left|d_1(x, v_i) + d_2(\rho_2, y) - d_2'(\rho_2', y') - d_1'(x', v_i')\right| \\ &\leq \left|d_1(x, v_i) - d_1(x', v_i')\right| + \left|d_2(\varrho_2, y) - d_2'(\varrho_2', y')\right| \\ &\leq \mathrm{dist}\,(\mathcal{R}_1) + \mathrm{dist}\,(\mathcal{R}_2). \end{aligned}$$

This gives:

$$d_{\mathrm{GH}}^{(n+k)}\big((T, \mathbf{v}), (T', \mathbf{v}')\big) \leq d_{\mathrm{GH}}^{(n)}\big((T_1, \mathbf{v}_1), (T_1', \mathbf{v}_1')\big) + d_{\mathrm{GH}}^{(k)}\big((T_2, \mathbf{v}_2), (T_2', \mathbf{v}_2')\big). \tag{6.13}$$

Now consider $(T_1, \mathbf{v}_1), (T_1', \mathbf{v}_1') \in \mathbb{T}^{(n)}_{\mathrm{loc-K}}$ and $(T_2, \mathbf{v}_2), (T_2', \mathbf{v}_2') \in \mathbb{T}^{(k)}_{\mathrm{loc-K}}$. Without loss of generality we assume that $H(v_i') \geq H(v_i)$. Remark that, for every $t \geq 0$, we have, with $a_+ = \max(a, 0)$:

$$r_t(T, \mathbf{v}) = r_t(T_1, \mathbf{v}_1) \circledast_i r_{(t-H(v_i))_+}(T_2, \mathbf{v}_2)$$

and hence

$$\begin{aligned} &d_{\mathrm{GH}}^{(n+k)}\big(r_t(T, \mathbf{v}), r_t(T', \mathbf{v}')\big) \\ &\quad = d_{\mathrm{GH}}^{(n+k)}\left(r_t(T_1, \mathbf{v}_1) \circledast_i r_{(t-H(v_i))_+}(T_2, \mathbf{v}_2), r_t(T_1', \mathbf{v}_1') \circledast_i r_{(t-H(v_i'))_+}(T_2', \mathbf{v}_2')\right). \end{aligned}$$

Therefore, we have:

$$\begin{aligned} &d_{\mathrm{LGH}}^{(n+k)}\big((T, \mathbf{v}), (T', \mathbf{v}')\big) \\ &\quad = \int_0^{+\infty} \mathrm{d}t\, \mathrm{e}^{-t} \left(1 \wedge d_{\mathrm{GH}}^{(n+k)}\big(r_t(T, \mathbf{v}), r_t(T', \mathbf{v}')\big)\right) \\ &\quad \leq \int_0^{+\infty} \mathrm{d}t\, \mathrm{e}^{-t} \left(1 \wedge d_{\mathrm{GH}}^{(n)}\left(r_t(T_1, \mathbf{v}_1), r_t(T_1', \mathbf{v}_1')\right)\right) \\ &\qquad\qquad + \int_0^{+\infty} \mathrm{d}t\, \mathrm{e}^{-t} \left(1 \wedge d_{\mathrm{GH}}^{(k)}\left(r_{(t-H(v_i))_+}(T_2, \mathbf{v}_2), r_{(t-H(v_i'))_+}(T_2', \mathbf{v}_2')\right)\right) \\ &\quad \leq d_{\mathrm{LGH}}^{(n)}\big((T_1, \mathbf{v}_1), (T_1', \mathbf{v}_1')\big) + 4\, \mathrm{e}^{-H(v_i')}\, d_{\mathrm{LGH}}^{(k)}\big((T_2, \mathbf{v}_2), (T_2', \mathbf{v}_2')\big) + H(v_i') - H(v_i) \\ &\quad \leq 3\, d_{\mathrm{LGH}}^{(n)}\big((T_1, \mathbf{v}_1), (T_1', \mathbf{v}_1')\big) + 4 d_{\mathrm{LGH}}^{(k)}\big((T_2, \mathbf{v}_2), (T_2', \mathbf{v}_2')\big), \end{aligned}$$

where we used Equation (6.13) for the first inequality and Lemma 6.3 for the second one. This completes the proof. □

We shall use a version of the grafting procedure where, instead of grafting on v_i, we shall graft on the branch $[\![\varrho, v_i]\!]$ at height h provided that $H(v_i) \geq h$. Let $n \in \mathbb{N}$ and $i \in \{0, \ldots, n\}$ be given. For $h \in \mathbb{R}_+$ and $(T, \mathbf{v}) \in \mathbb{T}^{(n)}_{\mathrm{K}}$, we denote by $x_{i,h}$ the unique vertex of T that satisfies $x_{i,h} \in [\![\varrho, v_i]\!]$ and $H(x_{i,h}) = H(v_i) \wedge h$. Then, the map

$\big(h, (T, \mathbf{v})\big) \mapsto \big(T, (\mathbf{v}, x_{i,h})\big)$ is clearly continuous from $\mathbb{R}_+ \times \mathbb{T}^{(n)}_{\mathrm{loc-K}}$ to $\mathbb{T}^{(n+1)}_{\mathrm{loc-K}}$. We then define the grafting map $\circledast_{i,h}$ by:

$$\big(h, (T, \mathbf{v}), (T', \mathbf{v}')\big) \mapsto T \circledast_{i,h} T' = (T \circledast_{i,h} T', \mathbf{v} \circledast \mathbf{v}'), \tag{6.14}$$

as the composition of

[adding the vertex $x_{i,h}$]: $\big(h, (T, \mathbf{v})\big) \mapsto (T, \tilde{\mathbf{v}})$ with $\mathbf{v} = (v_0 = \varrho, \dots, v_n)$ and $\tilde{\mathbf{v}} = (\mathbf{v}, x_{i,h}) = (\tilde{v}_0 = \varrho, \dots, \tilde{v}_n = v_n, \tilde{v}_{n+1} = x_{i,h})$,

[grafting]: $\big((T, \tilde{\mathbf{v}}), (T', \mathbf{v}')\big) \mapsto (T \circledast_{n+1} T', \tilde{\mathbf{v}} \circledast \mathbf{v}')$ and

[removing the $(n+1)$-th distinguished vertex]: $(T'' = T \circledast_{n+1} T', \tilde{\mathbf{v}} \circledast \mathbf{v}') \mapsto (T'', \mathbf{v} \circledast \mathbf{v}')$.

Since all those maps are continuous, we get the following result.

Lemma 6.15 (Continuity of the grafting map $\circledast_{i,h}$). *Let $n, k \in \mathbb{N}$, $i \in \{0, \dots, n\}$. The map $\big(h, (T, \mathbf{v}), (T', \mathbf{v}')\big) \mapsto T \circledast_{i,h} T'$ is continuous from $\mathbb{R}_+ \times \mathbb{T}^{(n)}_{\mathrm{loc-K}} \times \mathbb{T}^{(k)}_{\mathrm{loc-K}}$ to $\mathbb{T}^{(n+k)}_{\mathrm{loc-K}}$.*

6.4 Set of (equivalence classes of) marked trees

We shall consider trees with a marked infinite branch; for this reason we introduce the notion of marked trees. In this part, we do not record an order on the marked vertices as in the n-pointed trees.

We say that (T, S, d, ϱ) is a marked rooted tree if (T, d, ϱ) is a rooted tree and the set of marks S is a subtree of T with the same root (that is $\varrho \in S$) endowed with the restriction of the distance d. A correspondence between two compact marked rooted trees (T, S, d, ϱ) and (T', S', d', ϱ') is a set $\mathcal{R} \subset T \times T'$ such that $\mathcal{R}$ is a correspondence between (T, d, ϱ) and (T', d', ϱ') and $\mathcal{R} \cap (S \times S')$ is also a correspondence between (S, d, ϱ) and (S', d', ϱ'). Then, we set:

$$d^{[2]}_{\mathrm{GH}}\big((T, S), (T', S')\big) = \inf \frac{1}{2} \operatorname{dist}(\mathcal{R}),$$

where the infimum is taken over all the correspondences $\mathcal{R}$ between (T, S, d, ϱ) and (T', d', S', ϱ'). An easy extension of [6] gives that $d^{[2]}_{\mathrm{GH}}$ is a pseudo-distance, and that $d^{[2]}_{\mathrm{GH}}(T, T') = 0$ if and only if there exists an isometric one-to-one map φ from (T, d) to (T', d') which preserves the root and which is also one-to-one from S to S'. The relation $d^{[2]}_{\mathrm{GH}}((T, S), (T', S')) = 0$ defines an equivalence relation. The set $\mathbb{T}^{[2]}_{\mathrm{K}}$ of equivalence classes of compact marked rooted trees (T, S, d, ϱ) endowed with $d^{[2]}_{\mathrm{GH}}$ is then a metric Polish space. We simply write (T, S) for (T, S, d, ϱ), and unless specified otherwise, we shall denote also by (T, S) its equivalence class. Since

$$d_{\mathrm{GH}}(T, T') \vee d_{\mathrm{GH}}(S, S') \leq d^{[2]}_{\mathrm{GH}}((T, S), (T', S')), \tag{6.15}$$

we deduce that the map $(T, S) \mapsto (T, S)$ from $\mathbb{T}^{[2]}_{\mathrm{K}}$ to $(\mathbb{T}_{\mathrm{K}})^2$ (endowed with the maximum distance on the coordinates) is continuous. For $t \geq 0$, we define the truncation function $r^{[2]}_t$ of a marked rooted tree (T, S, d, ϱ) as the marked rooted tree $r^{[2]}_t(T, S) = (r_t(T), r_t(S), d, \varrho)$, where we recall that $r_t(T) = \{x \in T : \, H(x) \leq t\}$. If (T, S) and (T', S') are in the same equivalence class of $\mathbb{T}^{[2]}_{\mathrm{K}}$, so are $r^{[2]}_t(T, S)$ and $r^{[2]}_t(T', S')$; thus the function $r^{[2]}_t$ can be seen as a map from $\mathbb{T}^{[2]}_{\mathrm{K}}$ to itself. Similarly to (6.5), we have for $t, s \geq 0$ and $(T, S), (T', S') \in \mathbb{T}^{[2]}_{\mathrm{K}}$:

$$d^{[2]}_{\mathrm{GH}}\left(r^{[2]}_t(T, S), r^{[2]}_{t+s}(T', S')\right) \leq 4\, d^{[2]}_{\mathrm{GH}}\big((T, S), (T', S')\big) + s. \tag{6.16}$$

This implies that the map $(t,(T,S)) \mapsto r_t^{[2]}(T,S)$ is continuous from $\mathbb{R}_+ \times \mathbb{T}_{\mathrm{K}}^{[2]}$ to $\mathbb{T}_{\mathrm{K}}^{[2]}$.

If a complete marked rooted tree (T,S,d,ϱ) is locally compact, then the marked rooted tree $r_t^{[2]}(T,S)$ is compact for all $t \geq 0$. Following [6], we consider for two complete locally compact marked rooted trees (T,S) and (T',S'):

$$d_{\mathrm{LGH}}^{[2]}((T,S),(T',S')) = \int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}t\left(1 \wedge d_{\mathrm{GH}}^{[2]}\left(r_t^{[2]}(T,S), r_t^{[2]}(T',S')\right)\right). \tag{6.17}$$

Furthermore, we have that $d_{\mathrm{LGH}}^{[2]}((T,S),(T',S')) = 0$ if and only if there exists an isometric one-to-one map φ from (T,d) to (T',d') which is one-to-one from S to S' and preserves the roots. Thus the relation $d_{\mathrm{LGH}}^{[2]}((T,S),(T',S')) = 0$ defines an equivalence relation, see [7, Proposition 5.3]. The set $\mathbb{T}_{\mathrm{loc-K}}^{[2]}$ of equivalence classes of complete locally compact marked rooted trees (T,S,d,ϱ) endowed with $d_{\mathrm{LGH}}^{[2]}$ is then a metric Polish space. Furthermore, $\mathbb{T}_{\mathrm{K}}^{[2]}$ is an open dense subset of $\mathbb{T}_{\mathrm{loc-K}}^{[2]}$. Combining (6.15) and the definition of $d_{\mathrm{LGH}}^{[2]}$, we get the elementary following result.

Lemma 6.16 (Regularity of the projection)**.** *The map* $\mathrm{P}:\ (T,S) \mapsto T$ *from* $\mathbb{T}_{\mathrm{loc-K}}^{[2]}$ *to* $\mathbb{T}_{\mathrm{loc-K}}$ *is 1-Lipschitz.*

Similar equations to (6.6), (6.7) and (6.8) holds with $d_{\mathrm{LGH}}^{(n)}$ and $d_{\mathrm{GH}}^{(n)}$ replaced by $d_{\mathrm{LGH}}^{[2]}$ and $d_{\mathrm{GH}}^{[2]}$. For future use, let us give the equations corresponding to (6.7) and (6.8). For $(T,S),(T',S') \in \mathbb{T}_{\mathrm{loc-K}}^{[2]}$ and $s,t \geq 0$, we have:

$$d_{\mathrm{LGH}}^{[2]}\left(r_t^{[2]}(T,S), r_{t+s}^{[2]}(T',S')\right) \leq 4\, d_{\mathrm{LGH}}^{[2]}\big((T,S),(T',S')\big) + s, \tag{6.18}$$

$$d_{\mathrm{GH}}^{[2]}\left(r_t^{[2]}(T,S), r_t^{[2]}(T',S')\right) \leq 4\,\mathrm{e}^t\ d_{\mathrm{LGH}}^{[2]}\big((T,S),(T',S')\big). \tag{6.19}$$

We also we have the following result consequences of (6.16) and (6.18).

Lemma 6.17 (Continuity of the truncation map)**.** *Let* $n \in \mathbb{N}$*. The map*

$$\big(t,(T,S)\big) \mapsto r_t^{[2]}(T,S)$$

is continuous from $\mathbb{R}_+ \times \mathbb{T}_{\mathrm{K}}^{[2]}$ *to* $\mathbb{T}_{\mathrm{K}}^{[2]}$ *and from* $\mathbb{R}_+ \times \mathbb{T}_{\mathrm{loc-K}}^{[2]}$ *to* $\mathbb{T}_{\mathrm{loc-K}}^{[2]}$ *(and to* $\mathbb{T}_{\mathrm{K}}^{[2]}$*).*

We give in the next lemma an example of a $\mathbb{T}_{\mathrm{K}}^{[2]}$ and $\mathbb{T}_{\mathrm{loc-K}}^{[2]}$ valued function.

Lemma 6.18 (Continuity of Span°)**.** *Let* $n \in \mathbb{N}$*. The map*

$$(T,d,\mathbf{v}) \mapsto \big(\Pi_n^\circ(T), \mathrm{Span}^\circ(T,\mathbf{v}), d, \varrho\big)$$

from $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ *to* $\mathbb{T}_{\mathrm{loc-K}}^{[2]}$ *(resp. from* $\mathbb{T}_{\mathrm{K}}^{(n)}$ *to* $\mathbb{T}_{\mathrm{K}}^{[2]}$*) is injective, bi-measurable and* 16*-Lipschitz (resp.* 4*-Lipschitz) continuous.*

Proof. We first consider the compact case. Let $(T,\mathbf{v})$ and $(T',\mathbf{v}')$ be rooted n-pointed compact trees and let $\mathcal{R}$ be a correspondence between them. Recall the definition of $p_{\mathbf{v}}$ in (6.2) as the projection on $\mathrm{Span}^\circ(T,\mathbf{v})$ and the correspondence $\tilde{\mathcal{R}}$ from (6.3). We set $\mathcal{R}^{[2]} = \mathcal{R} \cup \tilde{\mathcal{R}}$. By construction $\mathcal{R}^{[2]}$ is a correspondence between $\big(T, \mathrm{Span}^\circ(T,\mathbf{v})\big)$ and $\big(T', \mathrm{Span}^\circ(T',\mathbf{v}')\big)$. From the proof of Lemma 6.2, we get that $\mathrm{dist}\,(\mathcal{R}^{[2]}) \leq 4\,\mathrm{dist}\,(\mathcal{R})$. This directly implies that:

$$d_{\mathrm{GH}}^{[2]}\Big(\big(T, \mathrm{Span}^\circ(T,\mathbf{v})\big), \big(T', \mathrm{Span}^\circ(T',\mathbf{v}')\big)\Big) \leq 4\, d_{\mathrm{GH}}^{(n)}\big((T,\mathbf{v}),(T',\mathbf{v}')\big). \tag{6.20}$$

This gives that the map $(T,d,\mathbf{v}) \mapsto \big(T, \mathrm{Span}^\circ(T,\mathbf{v}), d, \varrho\big)$ from $\mathbb{T}_{\mathrm{K}}^{(n)}$ to $\mathbb{T}_{\mathrm{K}}^{[2]}$ is 4-Lipschitz continuous.

We now consider complete locally compact trees. Let $(T,\mathbf{v})$ and $(T',\mathbf{v}')$ belong to $\mathbb{T}^{[2]}_{\mathrm{loc-K}}$. We have:

$$\begin{aligned}
d_{\mathrm{LGH}}^{[2]}&\Big(\big(T,\mathrm{Span}^\circ(T,\mathbf{v})\big),\big(T',\mathrm{Span}^\circ(T',\mathbf{v}')\big)\Big)\\
&= \int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}t\,\Big(1\wedge d_{\mathrm{GH}}^{[2]}\Big(r_t^{[2]}\big((T,\mathrm{Span}^\circ(T,\mathbf{v})\big), r_t^{[2]}\big(T',\mathrm{Span}^\circ(T',\mathbf{v}')\big)\Big)\Big)\\
&\leq 4\int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}t\,\Big(1\wedge d_{\mathrm{GH}}^{[2]}\Big(\big(r_t(T,\mathbf{v}),\mathrm{Span}^\circ(T,\mathbf{v})\big),\big(r_t(T',\mathbf{v}'),\mathrm{Span}^\circ(T',\mathbf{v}')\big)\Big)\Big)\\
&\leq 16\int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}t\,\Big(1\wedge d_{\mathrm{GH}}^{(n)}\left(r_t(T,\mathbf{v}), r_t(T',\mathbf{v}')\right)\Big)\\
&= 16\, d_{\mathrm{LGH}}^{(n)}\left((T,\mathbf{v}),(T',\mathbf{v}')\right),
\end{aligned}$$

where we used (6.16) (with T and S replaced respectively by $r_t(T,\mathbf{v})$ and $\mathrm{Span}^\circ(T,\mathbf{v})$ and similarly for T' and S') for the first inequality, and (6.20) (with $(T,\mathbf{v})$ replaced by $r_t(T,\mathbf{v})$) as well as the relation $\mathrm{Span}^\circ(r_t(T,\mathbf{v})) = \mathrm{Span}^\circ(T,\mathbf{v})$ for the second. This gives that the map $(T,d,\mathbf{v}) \mapsto (T,\mathrm{Span}^\circ(T,\mathbf{v}),d,\varrho)$ from $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$ to $\mathbb{T}^{[2]}_{\mathrm{loc-K}}$ is 16-Lipschitz continuous.

Clearly those maps are injective and thus bi-measurable thanks to Lusin's theorem. □

Remark 6.19. *Let us stress that for $(T,\mathbf{v})$ a rooted n-pointed compact tree, the rooted tree*

$$r_t^{[2]}\big(T,\mathrm{Span}^\circ(T,\mathbf{v})\big) = \Big(r_t(T), r_t\big(\mathrm{Span}^\circ(T,\mathbf{v})\big)\Big)$$

and the rooted tree

$$\Big(r_t(T),\mathrm{Span}^\circ\big(r_t(T,\mathbf{v})\big)\Big) = \big(r_t(T),\mathrm{Span}^\circ(T,\mathbf{v})\big)$$

differ if and only if t is smaller than the height of $\mathrm{Span}^\circ(T,\mathbf{v})$.

Let (T,S,d,ϱ) be a marked complete locally compact rooted tree. To simplify, we shall only write (T,S) for (T,S,d,ϱ). We define the projection of $z\in T$ on S, $p_S(z)\in S$, as the element of S uniquely defined by:

$$[\![\varrho, p_S(z)]\!] = [\![\varrho, z]\!]\cap S.$$

Now, we consider the truncation of a marked tree at a given height, say t, of the marked subtree. For $t\geq 0$ and $\varepsilon\in\{-,+\}$, we set:

$$r_t^{[2],\varepsilon}(T,S) = \Big(r_{t,1}^{[2],\varepsilon}(T,S), r_t(S)\Big) \tag{6.21}$$

with:

$$\begin{aligned}
r_{t,1}^{[2],+}(T,S) &= \Big\{x\in T:\ H\big(p_S(x)\big)\leq t\Big\},\\
r_{t,1}^{[2],-}(T,S) &= \Big\{x\in T:\ H\big(p_S(x)\big)< t\Big\}\cup\big\{x\in S:\ H(x)=t\big\}.
\end{aligned}$$

See Figure 3 for an instance of $r_t^{[2],\varepsilon}(T,S)$, where S is an infinite branch. For $\varepsilon\in\{+,-\}$, we also denote by $r_t^{[2],\varepsilon}(T,S)$ the marked rooted tree $\big(r_t^{[2],\varepsilon}(T,S), d, \varrho\big)$ endowed with the restriction of the distance d and the root ϱ. Furthermore, if (T,S) and

(T', S') belong to the same equivalence class of $\mathbb{T}^{[2]}_{\mathrm{loc-K}}$ or $\mathbb{T}^{[2]}_{\mathrm{K}}$, then so do $r_t^{[2],\varepsilon}(T,S)$ and $r_t^{[2],\varepsilon}(T',S')$. Thus the map $\big(t,(T,S)\big) \mapsto r_t^{[2],\varepsilon}(T,S)$ is a well defined map from $\mathbb{R}_+ \times \mathbb{T}^{[2]}_{\mathrm{loc-K}}$ to $\mathbb{T}^{[2]}_{\mathrm{loc-K}}$ for $\varepsilon \in \{+,-\}$.

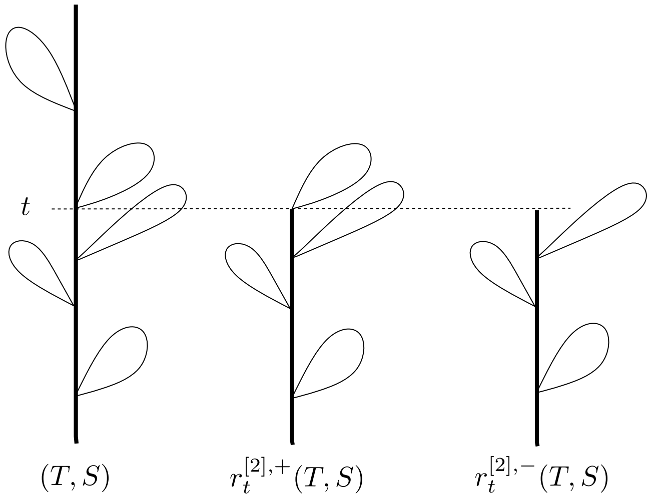

Figure 3: Example of restrictions of a tree T with a marked spine S (in bold).

Remark 6.20 (Examples). *We give elementary examples. For* $\varepsilon \in \{+,-\}$ *and* $t > 0$*, we have that* $r_t^{[2],\varepsilon}\big(T,\{\varrho\}\big) = \big(T,\{\varrho\}\big)$ *and* $r_0^{[2],-}\big(T,\{\varrho\}\big) = \big(\{\varrho\},\{\varrho\}\big)$ *as well as* $r_0^{[2],+}\big(T,\{\varrho\}\big) = \big(T,\{\varrho\}\big)$*. We also have for* $t \in \mathbb{R}_+$ *that* $r_t^{[2],\varepsilon}(T,T) = \big(r_t(T), r_t(T)\big)$*.*

Remark 6.21 (The map $r_t^{[2],\varepsilon}$ is not continuous). *Let* $\varepsilon \in \{+,-\}$ *and* $t > 0$*. The function* $r_t^{[2],\varepsilon}$ *is not continuous from* $\mathbb{T}^{[2]}_{\mathrm{loc-K}}$ *to itself. Indeed take* $t = 1$ *without loss of generality and consider* $T = [0,2]$ *and* $S_\delta = [0,\delta]$*, with* $\delta \in [0,2]$*,* $\varrho = 0$ *and the Euclidean distance. Notice that* $\big([0,1],[0,1]\big) = (S_1,S_1) \neq (T,S_1)$*. Then we have that* $\lim_{\delta\to 1} d^{[2]}_{\mathrm{GH}}\big((T,S_\delta),(T,S_1)\big) = 0$*,* $r_1^{[2],\varepsilon}(T,S_\delta) = (T,S_\delta)$ *for* $\delta < 1$*,* $r_1^{[2],\varepsilon}(T,S_\delta) = (S_1,S_1)$ *for* $\delta > 1$*,* $r_1^{[2],-}(T,S_1) = (S_1,S_1)$ *and* $r_1^{[2],+}(T,S_1) = (T,S_1)$*.*

We have the following measurability result.

Lemma 6.22 (Measurability of some truncation maps). *Let* $\varepsilon \in \{+,-\}$*. The map* $\big(t,(T,S)\big) \mapsto r_t^{[2],\varepsilon}(T,S)$ *is measurable from* $\mathbb{R}_+ \times \mathbb{T}^{[2]}_{\mathrm{loc-K}}$ *to* $\mathbb{T}^{[2]}_{\mathrm{loc-K}}$*.*

Proof. Let $a > 0$. For a marked tree $(T,S) = (T,S,d,\varrho)$, we define its partial dilatation $R_a(T,S) = (T,S,d_a,\varrho)$ as the marked tree with $d_a(x,y) = ad\big(x,p_S(x)\big) + d\big(p_S(x),p_S(y)\big) + ad\big(y,p_S(y)\big)$ if $p_S(x) \neq p_S(y)$ and $d_a(x,y) = ad(x,y)$ if $p_S(x) = p_S(y)$. Intuitively the distances on T are multiplied by a outside S. The equivalence class of $R_a(T,S)$ in $\mathbb{T}^{[2]}_{\mathrm{loc-K}}$ does not depend of the choice of (T,S) in its equivalence class in $\mathbb{T}^{[2]}_{\mathrm{loc-K}}$; so the map R_a is well defined on $\mathbb{T}^{[2]}_{\mathrm{loc-K}}$ to itself. Notice that the map R_a is continuous and one-to-one with inverse $R_{1/a}$. It is immediate to check that, for $t \geq 0$:

$$r_t^{[2],-} = \lim_{a\to 0+} R_{1/a} \circ r_t^{[2]} \circ R_a.$$

This and Lemma 6.17 imply the measurability of the map $\big(t,(T,S)\big) \mapsto r_t^{[2],-}(T,S)$. Then, notice that $\lim_{s\downarrow t} r_s^{[2],-} = r_t^{[2],+}$ to get the measurability of the map $(t,(T,S)) \mapsto r_t^{[2],+}(T,S)$. $\square$

We end this section by proving (in a very similar way) that the map $r_*^{[2]}$ below, which consists in cleaning the root, that is, in erasing the bushes at the root of a marked tree is measurable. For $(T,S)=(T,S,d,\varrho)$ a marked complete locally compact rooted tree, we set:

$$r_*^{[2]}(T,S)=\Big(r_{*,1}^{[2]}(T,S),S\Big)\quad\text{with}\quad r_{*,1}^{[2]}(T,S)=\big\{x\in T:\, p_S(x)\neq\varrho\big\}\cup\{\varrho\}. \tag{6.22}$$

We also denote by $r_*^{[2]}(T,S)$ the marked rooted tree $\big(r_*^{[2]}(T,S),d,\varrho\big)$ endowed with the restriction of the distance d and the root ϱ. Furthermore, if (T,S) and (T',S') belong to the same equivalence class of $\mathbb{T}_{\text{loc}-\text{K}}^{[2]}$, then so do $r_*^{[2]}(T,S)$ and $r_*^{[2]}(T',S')$. Thus the map $r_*^{[2]}$ is well-defined from $\mathbb{T}_{\text{loc}-\text{K}}^{[2]}$ to $\mathbb{T}_{\text{loc}-\text{K}}^{[2]}$.

Lemma 6.23 (Measurability of the root cleaning map)**.** *The map $r_*^{[2]}$ is measurable from $\mathbb{T}_{\text{loc}-\text{K}}^{[2]}$ to $\mathbb{T}_{\text{loc}-\text{K}}^{[2]}$.*

Proof. Let $a>0$. For a marked tree $(T,S)=(T,S,d,\varrho)$, we define its partial dilatation $R'_a(T,S)=(T,S,d'_a,\varrho)$ as the marked tree with $d'_a(x,y)=F_a(t)d(x,y)$ if $p_S(x)=p_S(y)$ with $t=H\big(p_S(x)\big)$, and otherwise $d'_a(x,y)=F_a(t)d\big(x,p_S(x)\big)+ad\big(p_S(x),p_S(y)\big)+F_a(s)d\big(y,p_S(y)\big)$ with $t=H\big(p_S(x)\big)$, $s=H\big(p_S(y)\big)$, and the function F_a defined for $t\geq 0$ by $F_a(t)=t\wedge a+a^{-2}(a-t)_+$ if $a\leq 1$, and $F_a(t)=1/F_{1/a}(at)$ if $a>1$. Notice that for $x\in T\setminus\{\varrho\}$, we have, as a goes down to 0, that: $d'_a(x,\varrho)\sim ad(x,\rho)$ as well as $d'_{1/a}(x,\varrho)\sim a^{-1}d(x,\rho)$ if $p_S(x)\neq\varrho$; and $d'_a(x,\varrho)\sim a^{-1}d(x,\rho)$ as well as $d'_a(x,\varrho)\sim ad(x,\rho)$ if $p_S(x)=\varrho$.

The equivalence class of $R'_a(T,S)$ in $\mathbb{T}_{\text{loc}-\text{K}}^{[2]}$ does not depend of the choice of (T,S) in its equivalence class in $\mathbb{T}_{\text{loc}-\text{K}}^{[2]}$; so the map R'_a is well defined on $\mathbb{T}_{\text{loc}-\text{K}}^{[2]}$ to itself. Notice that the map R'_a is continuous and one-to-one with inverse $R'_{1/a}$. It is immediate to check that for $t>0$:

$$r_*^{[2]}=\lim_{a\to 0+}R_{1/a}\circ r_t^{[2]}\circ R_a.$$

This and Lemma 6.17 imply the measurability of the map $r_*^{[2]}$. □

6.5 Set of (equivalence classes of) trees with one infinite marked branch

Let us denote by $\mathrm{T}_0=(\varrho,\{\varrho\})$ the rooted tree reduced to its root. Notice that $r_0^{[2],+}(T,S)=\{(\mathrm{T}_0,\mathrm{T}_0)\}$ if and only if $[\![\varrho,x]\!]\cap S=\{\varrho\}$ implies $x=\varrho$. Let $T_1=([0,\infty),d,0)$ be the tree consisting of only one infinite branch. We consider the set (of equivalence classes) of complete locally compact rooted trees with one infinite marked branch and its subset of trees whose root is not a branching vertex:

$$\mathbb{T}_{\text{loc}-\text{K}}^{\text{spine}}=\Big\{(T,S)\in\mathbb{T}_{\text{loc}-\text{K}}^{[2]}:\; S=T_1\text{ in }\mathbb{T}_{\text{loc-K}}\Big\}, \tag{6.23}$$

$$\mathbb{T}_{\text{loc}-\text{K}}^{\text{spine},0}=\Big\{(T,S)\in\mathbb{T}_{\text{loc}-\text{K}}^{\text{spine}}:\; \varrho\not\in\mathrm{Br}(T)\Big\}. \tag{6.24}$$

Lemma 6.24. *The sets $\mathbb{T}_{\text{loc}-\text{K}}^{\text{spine}}$ and $\mathbb{T}_{\text{loc}-\text{K}}^{\text{spine},0}$ are Borel subsets of $\mathbb{T}_{\text{loc}-\text{K}}^{[2]}$.*

Proof. Consider the projection $\tilde{\Pi}:(T,S)\mapsto S$ from $\mathbb{T}_{\text{loc}-\text{K}}^{[2]}$ to $\mathbb{T}_{\text{loc}-\text{K}}$, which is by construction 1-Lipschitz and thus continuous. As $\mathbb{T}_{\text{loc}-\text{K}}^{\text{spine}}=\tilde{\Pi}^{-1}\big(\{T_1\}\big)$, we get that $\mathbb{T}_{\text{loc}-\text{K}}^{\text{spine}}$ is Borel.

Notice that for $(T,S)\in\mathbb{T}_{\text{loc}-\text{K}}^{\text{spine}}$, then, by definition of $r_t^{[2],+}$, we get that the root is not a branching vertex of (T,S) if and only if $r_0^{[2],+}(T,S)=(\mathrm{T}_0,\mathrm{T}_0)$. Then, the set $\mathbb{T}_{\text{loc}-\text{K}}^{\text{spine},0}=\mathbb{T}_{\text{loc}-\text{K}}^{\text{spine}}\cap(r_0^{[2],+})^{-1}\Big(\big\{(\mathrm{T}_0,\mathrm{T}_0)\big\}\Big)$ is Borel as the map $r_0^{[2],+}$ is measurable according to Lemma 6.22. □

We shall be mainly consider elements of $\mathbb{T}_{\text{loc}-\text{K}}^{\text{spine},0}$ in what follows. For simplicity, we shall write $T^* = (T, S)$ for an element of $\mathbb{T}_{\text{loc}-\text{K}}^{\text{spine},0}$. For $t \geq 0$ and $T^* = (T, S)$ in $\mathbb{T}_{\text{loc}-\text{K}}^{\text{spine},0}$, we have $r_t^{[2],+}(T^*) = \big(r_{t,1}^{[2],+}(T), r_t(S)\big)$ where the rooted tree $r_t(S)$ is given by $\big([\![\varrho, x]\!], \varrho\big)$ with $x \in S$ uniquely characterized by $d(\varrho, x) = t$. We shall consider a slight modification of $r_t^{[2],+}$ on $\mathbb{T}_{\text{loc}-\text{K}}^{\text{spine},0}$, say $\tilde{r}_t^{[2],+}$, where one keeps track only of (ϱ, x) instead of $r_t(S)$:

$$\tilde{r}_t^{[2],+}(T^*) = \big(r_{t,1}^{[2],+}(T), (\varrho, x)\big). \tag{6.25}$$

It is left to the reader to check that $\tilde{r}_t^{[2],\varepsilon}$ is defined on $\mathbb{T}_{\text{loc}-\text{K}}^{\text{spine},0}$ and $\mathbb{T}_{\text{loc}-\text{K}}^{(1)}$-valued. Similarly to Lemma 6.22, we get the following result.

Lemma 6.25. *The function $(t, T^*) \mapsto \tilde{r}_t^{[2],+}(T^*)$ from $\mathbb{R}_+ \times \mathbb{T}_{\text{loc}-\text{K}}^{\text{spine},0}$ to $\mathbb{T}_{\text{loc}-\text{K}}^{(1)}$ is measurable.*

6.6 Another representation for discrete trees

Let $n \in \mathbb{N}$ be fixed. Let $(T, \mathbf{v})$, with $\mathbf{v} = (v_0 = \varrho, \ldots, v_n)$, be a complete locally compact rooted n-pointed tree. We will decompose the tree $\text{Span}(T, \mathbf{v})$ as a sequence of edges. To do so, we introduce some notations. Let $A \subset \{0, \ldots, n\}$ be non-empty. We set $\mathbf{v}_A = (v_i, i \in A)$. We denote by v_A the most recent common ancestor of $\mathbf{v}_A$, which is the only element of T such that:

$$[\![\varrho, v_A]\!] = \bigcap_{k \in A} [\![\varrho, v_k]\!]. \tag{6.26}$$

Notice that $v_{\{i\}} = v_i$. Recall that for $x \in T$, T_x is the subtree of T above x and rooted at x. Let $\mathcal{P}_n^+$ be the set of all subsets $A \subset \{1, \ldots, n\}$ such that $A \neq \emptyset$. For $A \in \mathcal{P}_n^+$, if $T_{v_A} \cap \text{Span}^\circ(T, \mathbf{v}_{A^c}) \neq \emptyset$ with $A^c = \{0, 1, 2, \cdots, n\} \setminus A$, we set $w_A = v_A$, otherwise we define $w_A \in [\![\varrho, v_A]\!]$ as the only element of T such that:

$$[\![\varrho, w_A]\!] = \text{Span}^\circ(T, \mathbf{v}_{A^c}) \cap \text{Span}^\circ\big(T, (\varrho, \mathbf{v}_A)\big). \tag{6.27}$$

Equivalently w_A is the only element in $[\![\varrho, v_A]\!]$ such that $w_A = v_{A \cup \{k_0\}}$ for some $k_0 \in A^c$ and for all $k \in A^c$, we have $v_{A \cup \{k\}} \in [\![\varrho, w_A]\!]$. Notice that $w_{\{1, \ldots, n\}} = \varrho$. We also record the lengths of all the branches $[\![w_A, v_A]\!]$:

$$\mathbf{L}_n(T, \mathbf{v}) = \big(\ell_A(T, \mathbf{v}), A \in \mathcal{P}_n^+\big) \quad \text{with} \quad \ell_A(T, \mathbf{v}) = d(w_A, v_A). \tag{6.28}$$

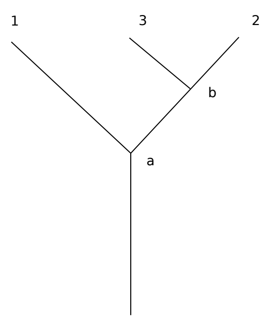

Figure 4: A discrete trees spanned by the leaves $\{1, 2, 3\}$.

For instance, we record the quantity of interest in Table 1 for the discrete tree spanned by the leaves $\{1, 2, 3\}$ from Figure 4. We can see that each branch of the discrete tree appears (through their length) once and only once in $\mathbf{L}_3(T, \mathbf{v})$.

Set $\hat{\mathbf{v}} = \big(\hat{\mathbf{v}}_0 = \varrho, (v_A, A \in \mathcal{P}_n^+)\big) \in T^{2^n}$, so that $(T, \hat{\mathbf{v}})$ is a complete locally compact rooted $(2^n - 1)$-pointed tree with the same root ϱ as T. Notice that all the vertices in

Table 1: Quantities of interest for the discrete tree from Figure 4.

$A \subset \mathcal{P}_3^+$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
v_A	1	2	3	a	a	b	a
w_A	a	b	b	a	a	a	ϱ
ℓ_A	$d(a,1)$	$d(b,2)$	$d(b,3)$	0	0	$d(a,b)$	$d(\varrho,a)$

$\mathbf{v}$ appear in $\hat{\mathbf{v}}$ (possibly more than once), and that w_A also appears in $\hat{\mathbf{v}}$ for all $A \in \mathcal{P}_n^+$. Recall the set of discrete trees defined at the end of Section 6.2. The next lemma states that $\mathbf{L}_n$ encodes discrete trees continuously. Set $\mathrm{Im}\,(\mathbf{L}_n) \subset \mathbb{R}_+^{\mathcal{P}_n^+}$ (with $\mathbb{R}_+^{\mathcal{P}_n^+} = \mathbb{R}_+^{2^n-1}$) for the image of $\mathbf{L}_n$.

Lemma 6.26 (Regularity of the branch lengths as a function of the tree)**.** *Let $n \in \mathbb{N}^*$. The map $(T, \mathbf{v}) \mapsto (T, \hat{\mathbf{v}})$ is well defined from $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ to $\mathbb{T}_{\mathrm{loc-K}}^{(2^n-1)}$, and it is continuous. The function $\mathbf{L}_n$ is well defined from $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ to $\mathrm{Im}\,(\mathbf{L}_n) \subset \mathbb{R}_+^{\mathcal{P}_n^+}$ and is continuous. Furthermore, $\mathrm{Im}\,(\mathbf{L}_n)$ is closed and $\mathbf{L}_n$ is a one-to-one bi-measurable map from $\mathbb{T}_{\mathrm{dis}}^{(n)}$ to $\mathrm{Im}\,(\mathbf{L}_n)$.*

Proof. If $(T, \mathbf{v})$ and $(T', \mathbf{v}')$ belong to the same equivalence class in $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$, then we deduce from (6.26) and (6.27) that $(T, \hat{\mathbf{v}})$ and $(T', \hat{\mathbf{v}}')$ belong also to the same equivalence class. This implies that the function $(T, \mathbf{v}) \mapsto (T, \hat{\mathbf{v}})$ is well defined from $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ to $\mathbb{T}_{\mathrm{loc-K}}^{(2^n-1)}$. We deduce from (6.26) and (6.27) that this function is in fact continuous on $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$. We also get that the function $\mathbf{L}_n$ is well defined from $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ to $\mathbb{R}_+^{\mathcal{P}_n^+}$.

We shall now precise the image of the function $\mathbf{L}_n$ and prove its continuity. Recall $x_+ = \max(x, 0)$ denotes the positive part of $x \in \mathbb{R}$. We define the function L from $\mathbb{R}_+^{(n+1)\times(n+1)}$ to $\mathbb{R}_+^{\mathcal{P}_n^+}$ by, for $\mathrm{d} = (\mathrm{d}_{ij}, 0 \leq i, j \leq n)$ and $A \in \mathcal{P}_n^+$:

$$L_A(\mathrm{d}) = \frac{1}{4} \inf \Big\{ (\mathrm{d}_{ii'} + \mathrm{d}_{ij'} + \mathrm{d}_{ji'} + d_{jj'} - 2\mathrm{d}_{ij} - 2\mathrm{d}_{i'j'})_+ \;:\; i, j \in A \text{ and } i', j' \in A^c \Big\},$$

where $A^c = \{0, \ldots, n\} \setminus A$. We also define the function D from $\mathbb{R}_+^{\mathcal{P}_n^+}$ to $\mathbb{R}_+^{(n+1)\times(n+1)}$ by, for $\ell = (\ell_A, A \in \mathcal{P}_n^+)$ and $i, j \in \{0, \ldots, n\}$:

$$D_{ij}(\ell) = \sum_{A \in \mathcal{P}_n^+} \ell_A \left(\mathbf{1}_{\{i \in A, j \notin A\}} + \mathbf{1}_{\{i \notin A, j \in A\}} \right). \tag{6.29}$$

The functions L and D are continuous. Consider the closed subset $\mathcal{Q}^{(n)}$ of $\mathbb{R}_+^{(n+1)\times(n+1)}$ satisfying the so-called four-point condition, that is the set of all $(\mathrm{d}_{ij}, 0 \leq i, j \leq n) \in \mathbb{R}_+^{(n+1)\times(n+1)}$ such that:

$$\mathrm{d}_{ij} + \mathrm{d}_{i'j'} \leq \max(\mathrm{d}_{ii'} + \mathrm{d}_{jj'}, \mathrm{d}_{ij'} + \mathrm{d}_{ji'}) \quad \text{for all} \quad i, j, i', j' \in \{0, \ldots, n\}.$$

Notice that the four-point condition is also used to characterize metric spaces which are real trees, see [24]. Then, one can check that the function L is one-to-one from $\mathcal{Q}^{(n)}$ to $L(\mathcal{Q}^{(n)})$ with inverse D. We also get that $L(\mathcal{Q}^{(n)})$ is closed (indeed if $(\ell^k = L(\mathrm{d}^k), k \in \mathbb{N})$ is a sequence of elements of $L(\mathcal{Q}^{(n)})$ converging to a limit, say ℓ, then it is bounded and thus the sequence $(\mathrm{d}^k, k \in \mathbb{N})$ is also bounded. Hence there is a converging subsequence, and denote by d its limit which belongs to $\mathcal{Q}^{(n)}$ as this set is closed. Since L is continuous, we get that $L(\mathrm{d}) = \ell$ and thus ℓ belongs to $L(\mathcal{Q}^{(n)})$, which gives that $L(\mathcal{Q}^{(n)})$ is closed). Since for $(T, \mathbf{v}) \in \mathbb{T}_{\mathrm{loc-K}}^{(n)}$, we have that $\mathbf{L}_n(T, \mathbf{v}) = L\big(d(v_i, v_j), 0 \leq i, j \leq n\big)$, we deduce that the function $\mathbf{L}_n$ is continuous from $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ to $L(\mathcal{Q}^{(n)})$.

We now prove that $\mathrm{Im}\,(\mathbf{L}_n) = L(\mathcal{Q}^{(n)})$ and that $\mathbf{L}_n$ is one-to-one from $\mathbb{T}_{\mathrm{dis}}^{(n)}$ to $L(\mathcal{Q}^{(n)})$. Let $\ell = (\ell_A, A \in \mathcal{P}_n^+) \in L(\mathcal{Q}^{(n)})$. Thus, there exists a sequence $\mathrm{d} = (\mathrm{d}_{ij}, 0 \le i, j \le n) \in \mathcal{Q}^{(n)}$ which satisfies the four-point condition and such that $L(\mathrm{d}) = \ell$. Since d satisfies the four-point condition, we get that there exists a discrete tree $(T, d, \mathbf{v}) \in \mathbb{T}_{\mathrm{dis}}^{(n)}$ such that $d(v_i, v_j) = \mathrm{d}_{ij}$ for all $i, j \in \{0, \ldots, n\}$. This proves that $\mathrm{Im}\,(\mathbf{L}_n) = L(\mathcal{Q}^{(n)})$. Then use that L is one-to-one from $\mathcal{Q}^{(n)}$ to $L(\mathcal{Q}^{(n)})$ with inverse D and that two discrete trees $(T, d, \mathbf{v})$ and $(T', d', \mathbf{v}')$ are equal in $\mathbb{T}_{\mathrm{dis}}^{(n)}$ if and only if $d(v_i, v_j) = d'(v'_i, v'_j)$ for all $i, j \in \{0, \ldots, n\}$ to deduce that $\mathbf{L}_n$ is one-to-one from $\mathbb{T}_{\mathrm{dis}}^{(n)}$ to $L(\mathcal{Q}^{(n)})$ and thus bi-measurable thanks to Lusin's theorem. □

6.7 The splitting operator for a pointed tree

We want now to decompose the pointed tree $(T, \mathbf{v})$ along the branches of $\mathrm{Span}^\circ(T, \mathbf{v})$. We keep notations from Section 6.6.

Let $(T, \mathbf{v})$, with $\mathbf{v} = (v_0 = \varrho, \ldots, v_n)$, be a complete locally compact rooted n-pointed tree. Recall Definition (6.2) of the projection $p_{\mathbf{v}}$ on $\mathrm{Span}(T, \mathbf{v})$. For $A \in \mathcal{P}_n^+$, consider the rooted 1-pointed tree:

$$\hat{T}_A(T, \mathbf{v}) = \big(T_A(T, \mathbf{v}), (\varrho_A, v_A)\big) \in \mathbb{T}_{\mathrm{loc-K}}^{(1)}, \tag{6.30}$$

with root $\rho_A = w_A$ and

$$T_A(T, \mathbf{v}) = \big\{x \in T:\ p_{\mathbf{v}}(x) \in]\!]w_A, v_A]\!]\big\} \cup \{w_A\}.$$

By construction, we have that $\ell_A(T, \mathbf{v}) = d(\varrho_A, v_A)$.

Notice that $\ell_A(T, \mathbf{v}) = 0$ if and only if $\hat{T}_A(T, \mathbf{v})$ is reduced to its root, that is, $\big(\{\varrho_A\}, (\varrho_A, \varrho_A)\big)$. Notice also that $\ell_A(T, \mathbf{v}) > 0$ implies that $\hat{T}_A$ belongs to $\mathbb{T}_{\mathrm{loc-K}}^{(1),0}$, the set of trees in $\mathbb{T}_{\mathrm{loc-K}}^{(1)}$ such that the root is not a branching point (see Definition (6.10)). We also define the rooted 1-pointed tree $\hat{T}_{\{0\}}(T, \mathbf{v}) \in \mathbb{T}_{\mathrm{loc-K}}^{(1)} = \big(T_{\{0\}}(T, \mathbf{v}), (\varrho, \varrho)\big)$ by:

$$T_{\{0\}}(T, \mathbf{v}) = \big\{x \in T:\ ]\!]\varrho, x]\!] \cap \mathrm{Span}^\circ(T, \mathbf{v}) = \emptyset\big\},$$

with root ϱ and distinguished vertex also ϱ. If $(T, \mathbf{v})$ and $(T', \mathbf{v}')$ belong to the same equivalence class in $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$, then we get that $\hat{T}_A(T, \mathbf{v})$ and $\hat{T}_A(T', \mathbf{v}')$ belong also to the same equivalent class in $\mathbb{T}_{\mathrm{loc-K}}^{(1)}$ for $A \in \mathcal{P}_n = \mathcal{P}_n^+ \cup \{\{0\}\}$. Thus, the map Split_n defined on $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ by:

$$\mathrm{Split}_n(T, \mathbf{v}) = \Big(\hat{T}_A(T, \mathbf{v}), A \in \mathcal{P}_n\Big) \tag{6.31}$$

takes values in $\left(\mathbb{T}_{\mathrm{loc-K}}^{(1)}\right)^{2^n}$. We give an instance of the function Split_n in Figure 5.

Lemma 6.27 (Measurability of the splitting map)**.** *Let $n \in \mathbb{N}^*$. The map Split_n from $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ to $\left(\mathbb{T}_{\mathrm{loc-K}}^{(1)}\right)^{2^n}$ is measurable.*

Proof. The proof is divided into three steps.

Step 1: The map $\hat{T}_{\{0\}}$ is measurable. Let $(T, \mathbf{v}) \in \mathbb{T}_{\mathrm{loc-K}}^{(n)}$. By construction, we have that $r_0^{[2],+}\big(T, \mathrm{Span}^\circ(T, \mathbf{v})\big) = \big(T_{\{0\}}(T, \mathbf{v}), \mathrm{T}_0\big)$. We deduce from Lemma 6.22 on the measurability of $r_t^{[2],\varepsilon}$, that the map $(T, \mathbf{v}) \mapsto \hat{T}_{\{0\}} = \big(T_{\{0\}}(T, \mathbf{v}), (\varrho, \varrho)\big)$ is measurable.

Step 2: A measurable truncation function. Let $n \ge 1$. Let $(T, \mathbf{v})$ be a rooted n-pointed tree. Recall the definition of $\hat{T}_A(T, \mathbf{v})$ from (6.30). We set $q(T, \mathbf{v}) = \hat{T}_{\{1,2,\ldots,n\}}(T, \mathbf{v})$ so that q is a map from $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ to $\mathbb{T}_{\mathrm{loc-K}}^{(1)}$. Recall the measurable truncation functions $r_t^{[2],+}$ and $r_*^{[2]}$ from (6.25) and (6.22), respectively.

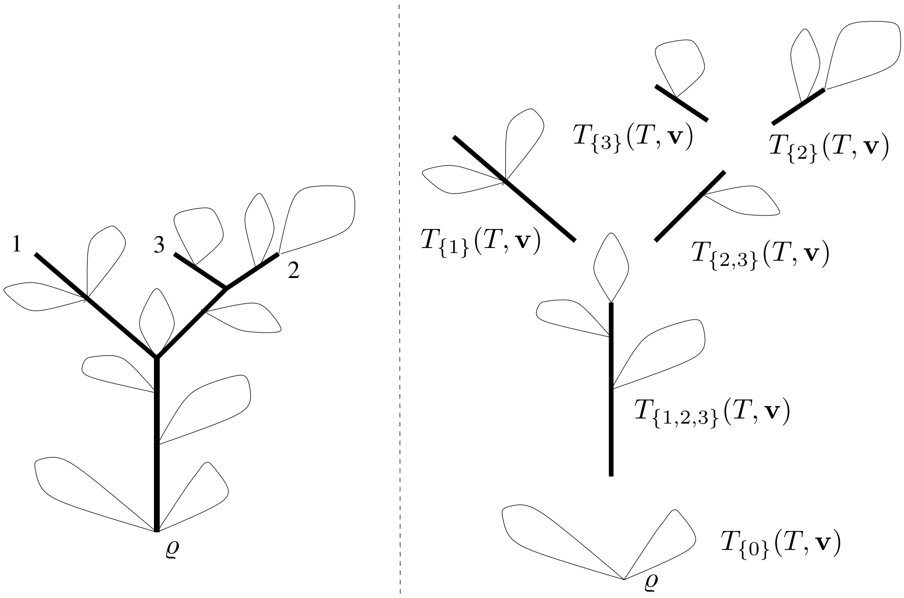

Figure 5: The splitting of the left hand tree with respect to $\mathbf{v} = \{\varrho, 1, 2, 3\}$. In this instance, $T_{\{1,2\}}$ and $T_{\{1,3\}}$ are reduced to their own root.

We set:

$$q'(T, \mathbf{v}) = r_*^{[2]} \circ r_{d(\varrho, w_{\{1,\dots,b\}})}^{[2],+} \big(T, \mathrm{Span}^\circ(T, \mathbf{v})\big).$$

Thanks to Lemma 6.18, the map $(T, \mathbf{v}) \mapsto \big(T, \mathrm{Span}^\circ(T, \mathbf{v})\big)$ is continuous from $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ to $\mathbb{T}_{\mathrm{loc-K}}^{[2]}$. Thanks to Lemma 6.26 and Remark 6.12, we get that the map $(T, \mathbf{v}) \mapsto d(\varrho, w_{\{1,\dots,b\}})$ is continuous from $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ to $\mathbb{R}_+$. Then, use Lemmas 6.22 and 6.23 on the measurability of $r_t^{[2],\varepsilon}$ and $r_*^{[2]}$ to conclude that the map q' from $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ to $\mathbb{T}_{\mathrm{loc-K}}^{[2]}$ is measurable and it has the same image as the map $\big(T, (\varrho, v)\big) \mapsto \big(T, [\![\varrho, v]\!]\big)$ from $\mathbb{T}_{\mathrm{loc-K}}^{(1)}$ to $\mathbb{T}_{\mathrm{loc-K}}^{[2]}$. According to Lemma 6.18 (with $n = 1$), this latter map is injective and measurable. Hence the map q, which is the composition of q' and this latter map, is measurable.

Step 3: Conclusion. Let $A \subset \{1, \dots, n\}$ be non-empty. Notice that $\hat{T}_A$ is the image of $(T, \mathbf{v})$ by: the expansion procedure $(T, \mathbf{v}) \mapsto (T, \hat{\mathbf{v}})$ from the first part of Lemma 6.26, the rerooting at w_A from Lemma 6.9, the reducing procedure from Lemma 6.6 where one forgets about all $w_{A'}$ and $v_{A'}$ for $A' \subset A^c$, and then the function q from Step 2. This implies that the function $(T, \mathbf{v}) \mapsto \hat{T}_A(T, \mathbf{v})$ is measurable from $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ to $\mathbb{T}_{\mathrm{loc-K}}^{(1)}$. □

6.8 The grafting procedure

Let $n \in \mathbb{N}^*$. Let $\ell = (\ell_A, A \in \mathcal{P}_n^+) \in \mathrm{Im}\,(\mathbf{L}_n)$. According to Lemma 6.26, there exists a unique (up to the equivalence in $\mathbb{T}_{\mathrm{K}}^{(n)}$) rooted n-pointed discrete tree $(S, \mathbf{v})$ (that is $S = \mathrm{Span}^\circ(S, \mathbf{v})$) such that $\mathbf{L}_n(S, \mathbf{v}) = \ell$. Recall v_A and w_A defined in Section 6.6 for $A \in \mathcal{P}_n^+$ so that:

$$S = \bigcup_{A \in \mathcal{P}_n^+} [\![w_A, v_A]\!], \tag{6.32}$$

where the sets $(]\!]w_A, v_A[\![, A \in \mathcal{P}_n^+)$ are pairwise disjoint.

Recall that $\mathbb{T}_{\mathrm{loc-K}}^{\mathrm{spine},0}$ denotes the set (of equivalence classes) of complete locally compact rooted trees with one infinite marked branch such that the root is not a branching vertex. Let $T^* = (T^*_A, A \in \mathcal{P}^+_n)$ be a family of elements of equivalence classes in $\mathbb{T}^{\text{spine},0}_{\text{loc}-\text{K}}$. Then, we define the tree $(T, \mathbf{v}) = \text{Graft}_n(\ell, T^*)$, where T is the tree S with that the branches $]\!]w_A, v_A]\!]$ are replaced by the trees given by the first component of $r^{[2],+}_{\ell_A}(T^*_A)$ (where the second component has been identified to $[\![w_A, v_A]\!]$).

We now provide a more formal construction of $\text{Graft}_n(\ell, T^*)$. Let $\ell \in \text{Im}\,(\mathbf{L}_n)$, and consider the rooted n-pointed discrete tree $(S, \mathbf{v}) = \mathbf{L}_n^{-1}(\ell) \in \mathbb{T}^{(n)}_{\text{dis}}$ and $\mathbf{v} = (v_0 = \varrho, \dots, v_n)$. Set $\hat{\mathbf{v}} = (\hat{\mathbf{v}}_0 = \varrho, (v_A, A \in \mathcal{P}^+_n)) \in T^{2^n}$, with v_A the most recent common ancestor of $(v_i, i \in A)$ defined in (6.26). Thus, we get that $(S, \hat{\mathbf{v}}) \in \mathbb{T}^{(2^n-1)}_{\text{dis}}$ is a rooted (2^n-1)-pointed discrete tree with the same root ϱ as S.

In a first step, we build by a backward induction an "increasing" sequence of discrete trees $\big((S_k, \mathbf{v}_k), k \in \{0, \dots, 2^n-1\}\big)$ such that $(S_k, \mathbf{v}_k) \in \mathbb{T}^{(k)}_{\text{dis}}$ with root ϱ. We set $(S_{2^n-1}, \mathbf{v}_{2^n-1}) = (S, \hat{\mathbf{v}})$. Recall that x is a leaf of a tree T with root ϱ if $x \in [\![\varrho, y]\!] \subset T$ implies $y = x$. Assume that $(S_{k+1}, \mathbf{v}_{k+1})$ is defined for some $k \geq 0$. We consider the lexicographical order on the non-empty sets of $\mathbb{N}$ defined recursively as follow: for $A, B \subset \mathbb{N}$ non empty, we write $A < B$: if $\min A < \min B$; or if $\min A = \min B$ and A is a singleton but not B; or if $\min A = \min B$, A and B are not singletons and $A' < B'$ where $A' = A \setminus \{\min A\}$ and similarly for B'. Notice this order is total. We set:

$$A_{k+1} = \max\big\{A \in \mathcal{P}^+_n, v_A \in \mathbf{v}_{k+1} \text{ and } v_A \text{ is a leaf of } (S_{k+1}, \mathbf{v}_{k+1})\big\}.$$

Then, we define $\mathbf{v}_k$ as the sequence $\mathbf{v}_{k+1}$ where $v_{A_{k+1}}$ has been removed (notice that the first element of $\mathbf{v}_k$ is still the root ϱ), and we set $(S_k, \mathbf{v}_k) = \text{Span}(S, \mathbf{v}_k) \in \mathbb{T}^{(k)}_{\text{dis}}$. We also set $B_k = \max\{B \in \mathcal{P}_n : v_B = w_{A_{k+1}}\}$. By construction, $v_{B_k} = w_{A_{k+1}}$ belongs to the sequence $\mathbf{v}_k$ and is therefore an element of $\mathbf{v}$ for some index, and, with a slight abuse of notation, we simply denote this index by B_k. We have, using the grafting operation from Section 6.3 that:

$$(S_{k+1}, \mathbf{v}_{k+1}) = (S_k, \mathbf{v}_k) \circledast_{B_k} [0, \ell_{A_{k+1}}], \tag{6.33}$$

where the equality holds in $\mathbb{T}^{(k+1)}_{\text{loc}-\text{K}}$ (and in $\mathbb{T}^{(k+1)}_{\text{dis}}$) and by convention $[0, t]$ denotes the discrete 1-pointed tree $\big([0, t], (0, t)\big)$ with root 0. Notice that $\ell_{A_{k+1}} = 0$ if and only if $\text{Span}^\circ(S, \mathbf{v}_k) = \text{Span}^\circ(S, \mathbf{v}_{k+1})$. Eventually, notice that $(S_0, \mathbf{v}_0) = \big(\{\varrho\}, \varrho\big)$ is the rooted tree reduced to its root $\varrho = v_{\{0\}}$ and $B_0 = \{0\}$. Let us stress, that in Section 6.3, the vector $\mathbf{v}_{k+1}$ is obtained by adding the distinguished vertex $\ell_{A_{k+1}}$ of $[0, \ell_{A_{k+1}}]$ to $\mathbf{v}_k$. However here we identify $[0, \ell_{A_{k+1}}]$ with $[\![v_{B_k} = w_{A_{k+1}}, v_{A_{k+1}}]\!]$ and add the distinguished vertex $v_{A_{k+1}}$ to $\mathbf{v}_k$ in order to obtain $\mathbf{v}_{k+1}$.

For instance, we give in Table 2 the sequences $(A_k, 1 \leq k \leq 2^n-1)$ and $(B_k, 0 \leq k \leq 2^n-2)$ for the tree of Figure 4.

Table 2: The sequences $(A_{k+1}, 0 \leq k \leq 6)$, $(B_k, 0 \leq k \leq 6)$ and $(\ell_{A_{k+1}}, 0 \leq k \leq 6)$ for the tree of Figure 4.

k	0	1	2	3	4	5	6
A_{k+1}	$\{1,2\}$	$\{1,2,3\}$	$\{1,3\}$	$\{1\}$	$\{2,3\}$	$\{2\}$	$\{3\}$
B_k	$\{0\}$	$\{1,3\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{2,3\}$	$\{2,3\}$
$\ell_{A_{k+1}}$	$d(\varrho, a)$	0	0	$d(1,a)$	$d(a,b)$	$d(2,b)$	$d(3,b)$

Remark 6.28. *The family $\big\{A_k, k \in \{1, 2^n-1\}\big\}$ is exactly equal to $\mathcal{P}^+_n$. Furthermore the sequence $\ell \in \text{Im}\,(\mathbf{L}_n) \subset \mathbb{R}_+^{2^n-1}$ provides implicitly two unique ordered sequences $\mathcal{A}(\ell) = \big(A_k, k \in \{1, 2^n-1\}\big)$ (of all elements of $\mathcal{P}^+_n$) and $\mathcal{B}(\ell) = (B_k, k \in \{0, 2^n-2\})$ (of elements of $\mathcal{P}^n = \mathcal{P}^+_n \cup \{\{0\}\}$), and an "increasing" way to built $\mathbf{L}_n^{-1}(\ell)$ recursively by*

adding at step $k \in \{0, 2^n - 2\}$ a branch of length $\ell_{A_{k+1}}$ (and graft it on v_{B_k} chosen among $\mathbf{v}_k$). It is obvious from the construction that if ℓ and ℓ' are two sequences in $\mathrm{Im}\,(\mathbf{L}_n)$ with the same zeros (that is, $\ell_A = 0$ if and only if $\ell'_A = 0$), then we have $\mathcal{A}(\ell) = \mathcal{A}(\ell')$ and $\mathcal{B}(\ell) = \mathcal{B}(\ell')$. Thus, the sets $\mathcal{A}(\ell)$ and $\mathcal{B}(\ell)$ are implicitly coded by the zeros of ℓ.

In a second step, given $\mathcal{A}(\ell)$ and $\mathcal{B}(\ell)$ from Remark 6.28 and a sequence $T^* = (T^*_A, A \in \mathcal{P}^n_+)$ in $\mathbb{T}^{\mathrm{spine},0}_{\mathrm{loc-K}}$, we build by a forward induction an "increasing" sequence of marked complete locally compact trees $\big((T_k, \mathbf{v}_k), k \in \{0, \ldots, 2^n - 1\}\big)$ such that $(T_k, \mathbf{v}_k)$ belongs to $\mathbb{T}^{(k)}_{\mathrm{loc-K}}$, has root ϱ, and the components of the vector $\mathbf{v}_k$ can be ranked as the root $\varrho = v_{\{0\}}$ and $(v_{A_i}, 1 \leq i \leq k)$. Recall also the truncation function $\tilde{r}^{[2],+}_t$ given in (6.25). We set $(\mathrm{T}_0, \mathbf{v}_0) = \big(\{\varrho\}, \varrho\big)$ and for $k \in \{0, 2^n - 2\}$:

$$(T_{k+1}, \mathbf{v}_{k+1}) = (T_k, \mathbf{v}_k) \circledast_{B_k} \tilde{r}^{[2],+}_{\ell_{A_{k+1}}}(T^*_{A_{k+1}}), \tag{6.34}$$

where the distinguished vertex of $\tilde{r}^{[2],+}_{\ell_{A_{k+1}}}(T^*_{A_{k+1}})$ is identified with $v_{A_{k+1}}$ (and its root with v_{B_k}). Then, we set:

$$\mathrm{Graft}_n(\ell, T^*) = (T_{2^n-1}, \mathbf{v}) \quad \text{with} \quad \mathbf{v} = (v_{\{k\}}, 0 \leq k \leq n). \tag{6.35}$$

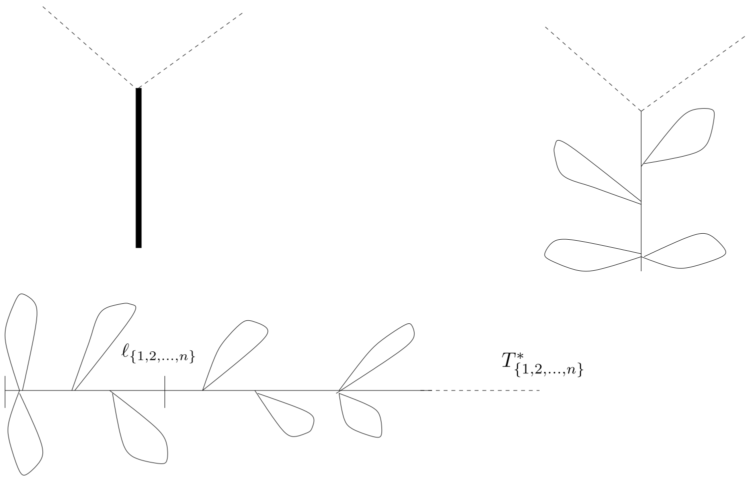

Figure 6: Example of a replacement of the branch $]\!]w_{\{1,\ldots,n\}}, v_{\{1,\ldots,n\}}]\!]$.
Upper left: The tree S with the branch $]\!]w_{\{1,\ldots,n\}}, v_{\{1,\ldots,n\}}]\!]$ in bold.
Upper right: The branch $]\!]w_{\{1,\ldots,n\}}, v_{\{1,\ldots,n\}}]\!]$ replaced by the first component of the marked tree $r^{[2],+}_{\ell_{\{1,\ldots,n\}}}(T^*_{\{1,\ldots,n\}})$.
Lower: The tree $T^*_{\{1,\ldots,n\}}$ with its marked infinite branch.

It is easy to check that the equivalence class of $(T_{2^n-1}, \mathbf{v})$ in $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$ does not depend on the choice of $T^* = (T^*_A, A \in \mathcal{P}^+_n)$ in their own equivalence class. Thus, the map Graft_n defined by:

$$(\ell, T^*) \mapsto \mathrm{Graft}_n(\ell, T^*)$$

is well defined from $\mathrm{Im}\,(\mathbf{L}_n) \times \left(\mathbb{T}^{\mathrm{spine},0}_{\mathrm{loc-K}}\right)^{\mathcal{P}^+_n}$ to $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$. The main result of this section is the measurability of the map Graft_n.

Lemma 6.29 (Measurability of the grafting map)**.** *Let $n \in \mathbb{N}^*$. The map Graft_n from $\mathrm{Im}\,(\mathbf{L}_n) \times \left(\mathbb{T}^{\mathrm{spine},0}_{\mathrm{loc-K}}\right)^{\mathcal{P}^+_n}$ to $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$ is measurable.*

Proof. For $J \subset \mathcal{P}_+^n$, we write $I_J = \big\{\ell \in \mathrm{Im}\,(\mathbf{L}_n) :\ \ell_A = 0 \text{ if and only if } A \in J\big\}$. Thus, the closed set $\mathrm{Im}\,(\mathbf{L}_n)$ of $\mathbb{R}_+^{\mathcal{P}_+^n}$ can be written as the union of I_J over all the subsets J of $\mathcal{P}_+^n$. Furthermore, the sets $(I_J, J \subset \mathcal{P}_+^n)$ are Borel sets (as $\mathrm{Im}\,(\mathbf{L}_n)$ is a Borel set), and they are pairwise disjoint. Thanks to Remark 6.28, the maps $\ell \mapsto \mathcal{A}(\ell)$ and $\ell \mapsto \mathcal{B}(\ell)$ are constant over I_J. We deduce from Equation (6.35) and recursion (6.34), Lemma 6.14 on the continuity of the grafting procedure and Lemma 6.25 on the measurability of $(t, T) \mapsto \tilde{r}_t^{[2],+}(T)$ that the function Graft_n from $I_J \times \left(\mathbb{T}_{\mathrm{loc-K}}^{\mathrm{spine},0}\right)^{\mathcal{P}_n^+}$ to $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ is measurable (as long as I_J is not empty). Since there is a finite number of such sets I_J, we deduce that the function Graft_n from $\mathrm{Im}\,(\mathbf{L}_n) \times \left(\mathbb{T}_{\mathrm{loc-K}}^{\mathrm{spine},0}\right)^{\mathcal{P}_n^+}$ to $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ is measurable. □

Remark 6.30. *Since the map $\mathbf{L}_n$ is continuous one-to-one from $\mathbb{T}_{\mathrm{dis}}^{(n)}$ to $\mathrm{Im}\,(\mathbf{L}_n)$, we deduce that the map:*

$$(T, T^*) \mapsto \mathrm{Graft}_n\big(\mathbf{L}_n(T), T^*\big)$$

from $\mathbb{T}_{\mathrm{dis}}^{(n)} \times \left(\mathbb{T}_{\mathrm{loc-K}}^{\mathrm{spine},0}\right)^{\mathcal{P}_n^+}$ to $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ is measurable. Without ambiguity, we shall simply write $\mathrm{Graft}_n(T, T^*)$ for $\mathrm{Graft}_n\big(\mathbf{L}_n(T), T^*\big)$.

Remark 6.31. *Intuitively, the maps Graft_n and Split_n should be the inverse one of the other. More precisely, we have the following result. For every $(T, (\varrho, v)) \in \mathbb{T}_{\mathrm{loc-K}}^{(1)}$, we define the tree $\mathrm{Sp}(T) = (T', S') \in \mathbb{T}_{\mathrm{loc-K}}^{\mathrm{spine},0}$ by $T' = \Pi_1^\circ\,(T \circledast_1 [0,\infty))$ with the marked spine $S = \Pi_1^\circ\,([\![\varrho, v]\!] \circledast_1 [0,\infty))$. Then then we have, for every $(T, \mathbf{v}) \in \mathbb{T}_{\mathrm{loc-K}}^{(n),0}$ (that is, the root of T is not a branching vertex, see Definition* (6.10)*), that the following equality hold in $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$:*

$$\mathrm{Graft}_n\Big(\mathrm{Span}_n(T, \mathbf{v}), \mathrm{Sp}\big(\mathrm{Split}_n(T, \mathbf{v})\big)\Big) = (T, \mathbf{v}), \tag{6.36}$$

where $\mathrm{Sp}(T_A, A \in \mathcal{P}_n) = (\mathrm{Sp}(T_A), A \in \mathcal{P}_n^+)$.

6.9 A measure associated with trees in $\mathbb{T}_{\mathrm{loc-K}}^{\mathrm{spine},0}$ or $\mathbb{T}_{\mathrm{loc-K}}^{(1)}$

Recall $\mathrm{T}_0 = (\{\varrho\}, \varrho) \in \mathbb{T}_{\mathrm{loc-K}}$ is the tree reduced to its root. We define

$$\mathbb{T}_{\mathrm{loc-K}}^* = \mathbb{T}_{\mathrm{loc-K}} \setminus \{\mathrm{T}_0\} \tag{6.37}$$

endowed with the distance:

$$d_{\mathrm{LGH}}^*(T, T') = d_{\mathrm{LGH}}(T, T') + \left|H(T)^{-1} - H(T')^{-1}\right|.$$

Clearly $(\mathbb{T}_{\mathrm{loc-K}}^*, d_{\mathrm{LGH}}^*)$ is Polish with the topology induced by the topology on $\mathbb{T}_{\mathrm{loc-K}}$ (as H is continuous on $\mathbb{T}_{\mathrm{loc-K}}$), and for all $\varepsilon > 0$, the sets $B_{\mathbb{T}_{\mathrm{loc-K}}^*}(\varepsilon) = \{T \in \mathbb{T}_{\mathrm{loc-K}}^* :\ H(T) \geq \varepsilon\}$ are closed and bounded. Furthermore, every bounded set is a subset of $B_{\mathbb{T}_{\mathrm{loc-K}}^*}(\varepsilon)$ for $\varepsilon > 0$ small enough. Set $E = \mathbb{R}_+ \times \mathbb{T}_{\mathrm{loc-K}}^*$ endowed with the distance $d_E((u, T), (u', T')) = |u - u'| + d_{\mathrm{LGH}}^*(T, T')$, so that (E, d_E) is a Polish space. Every bounded set of E is a subset of $B_E(\varepsilon) = [0, \varepsilon^{-1}] \times B_{\mathbb{T}_{\mathrm{loc-K}}^*}(\varepsilon)$ for $\varepsilon > 0$ small enough. We define $\mathbb{M}(E)$, the set of point measures on E which are bounded on bounded sets, that is finite on $B_E(\varepsilon)$ for all $\varepsilon > 0$. We say that a sequence $(\mathcal{M}_n, n \in \mathbb{N})$ of elements of $\mathbb{M}(E)$ converges to a limit $\mathcal{M}$, if $\lim_{n\to\infty} \mathcal{M}_n(f) = \mathcal{M}(f)$ for all continuous functions on E with bounded support. According to [14, Proposition 9.1.IV] the space $\mathbb{M}(E)$ is Polish and the Borel σ-field is the smallest σ-field such that the application $\mathcal{M} \mapsto \mathcal{M}(A)$ is measurable for every Borel set A of E.

We build a tree from a point measure $\mathcal{M} = \sum_{i\in I} \delta_{(h_i, T_i)} \in \mathbb{M}(E)$ by grafting T_i at height h_i on an infinite spine. Recall the infinite spine $\mathrm{T}_1 = (\mathbb{R}_+, 0)$ endowed with the

Euclidean distance is an element of $\mathbb{T}^{\text{spine},0}_{\text{loc-K}} \subset \mathbb{T}_{\text{loc-K}}$. For $T \in \mathbb{T}_{\text{loc-K}}$, let $(\tilde{T}, d, \varrho)$ denote a rooted complete locally compact tree in the equivalent class T. With obvious notation, we define the tree T' as follow:

$$T' = \tilde{\mathrm{T}}_1 \sqcup_{i\in I} (\tilde{T}_i \setminus \{\varrho_i\})$$

and $\forall x, x' \in T'$,

$$d(x,x') = \begin{cases} d_i(x,x') & \text{if } x,x' \in \tilde{T}_i, i \in I, \\ |x-x'| & \text{if } x,x' \in \tilde{\mathrm{T}}_1, \\ d_i(x,\varrho_i) + |h_i - x| & \text{if } x \in \tilde{T}_i,\ x' \in \tilde{\mathrm{T}}_1, i \in I, \\ d_i(x,\varrho_i) + d_j(x',\varrho_j) + |h_i - h_j| & \text{if } x \in \tilde{T}_i,\ x' \in \tilde{T}_j \text{ with } i \neq j,\ i,j \in I, \end{cases}$$

where $\sqcup$ denotes the disjoint union. By construction T' is a tree rooted at $\varrho = \varrho_1$, the root of $\tilde{\mathrm{T}}_1$. Because $\mathcal{M}$ is finite on bounded sets of E, it is not difficult to check that T' is complete and locally compact. It is easy to see that the equivalence class of $\mathrm{Tree}(\mathcal{M}) = (T', \tilde{\mathrm{T}}_1)$ in $\mathbb{T}^{[2]}_{\text{loc-K}}$ does not depend of the choice of the representatives in the equivalence classes of T_1 and T_i for $i \in I$. Hence, identifying $\mathrm{Tree}(\mathcal{M})$ with its equivalence class, we get that the map Tree is well defined from $\mathbb{M}(E)$ into $\mathbb{T}^{[2]}_{\text{loc-K}}$.

Lemma 6.32 (Regularity of the map Tree)**.** *The map* Tree *from* $\mathbb{M}(E)$ *to* $\mathbb{T}^{[2]}_{\text{loc-K}}$ *(or* $\mathbb{T}^{\text{spine}}_{\text{loc-K}}$*) is continuous.*

Proof. We only give the principal arguments of the proof. Let $(\mathcal{M}_n, n \in \mathbb{N})$ a sequence of point measures, elements of $\mathbb{M}(E)$, which converges to $\mathcal{M}$. Let $\varepsilon > 0$ be fixed such that $\mathcal{M}(\partial B_E(\varepsilon)) = 0$. For n large enough, we have $\mathcal{M}_n(B_E(\varepsilon)) = \mathcal{M}(B_E(\varepsilon))$ and the atoms of $\mathcal{M}_n$ in $B_E(\varepsilon)$ converge to the atoms of $\mathcal{M}$ in $B_E(\varepsilon)$. Using correspondence between the representations of the atoms, and similar arguments as in the proof of Lemma 6.14, we deduce that the distance between $\mathrm{Tree}(\mathcal{M}_n)$ and $\mathrm{Tree}(\mathcal{M})$ (in $\mathbb{T}^{[2]}_{\text{loc-K}}$) is small if $\varepsilon > 0$ is small (to prove this statement in detail, one can use the distance on $\mathbb{M}(E)$ given in [13, Equation (A2.6.1)]). This means that $\lim_{n\to\infty} d^{[2]}_{\text{LGH}}(\mathrm{Tree}(\mathcal{M}_n), \mathrm{Tree}(\mathcal{M})) = 0$, and thus the map Tree is continuous on $\mathbb{T}^{[2]}_{\text{loc-K}}$. □

We shall now prove that the restriction of the map Tree to a subset of $\mathbb{M}(E)$ is injective and bi-measurable. For this reason, we consider the subset of $\mathbb{T}_{\text{loc-K}}$ of (equivalence classes of) trees not reduced to their root and such that the root is not a branching vertex (recall Definitions (6.37) and (6.10) with $n = 0$):

$$\mathbb{T}^{0,*}_{\text{loc-K}} = \mathbb{T}^{*}_{\text{loc-K}} \cap \mathbb{T}^{0}_{\text{loc-K}}. \tag{6.38}$$

As a direct consequence of Lemma 6.10, $\mathbb{T}^{0,*}_{\text{loc-K}}$ is a Borel subset of $\mathbb{T}_{\text{loc-K}}$ and thus of $\mathbb{T}^{*}_{\text{loc-K}}$. In particular, the following subset of $\mathbb{M}(E)$ is a Borel set (recall $E = \mathbb{R}_+ \times \mathbb{T}^{*}_{\text{loc-K}}$):

$$\tilde{\mathbb{M}}(E) = \Big\{\mathcal{M} \in \mathbb{M}(E) :\ \mathcal{M}\big(\mathbb{R}_+ \times (\mathbb{T}^{0,*}_{\text{loc-K}})^c\big) = 0\Big\}. \tag{6.39}$$

We now introduce a map $\boldsymbol{\mathcal{M}}$ from $\mathbb{T}^{\text{spine}}_{\text{loc-K}}$ to $\mathbb{M}(E)$ as follow. Let $T^* = (T, \mathrm{T}_1)$ be a rooted complete locally compact tree with an infinite marked spine. In particular, we have $\mathrm{T}_1 \subset T$ and T_1 is equivalent to $(\mathbb{R}_+, d, 0)$. Let $(T_i^\circ, i \in I)$ be the family of the connected components of $T \setminus \mathrm{T}_1$. For every $i \in I$, let us denote by x_i the MRCA of T_i°, that is, the unique point of T_1 such that for every $x \in T_i^\circ$, $[\![\varrho, x]\!] \cap \mathrm{T}_1 = [\![\varrho, x_i]\!]$. We then set $T_i = T_i^\circ \cup \{x_i\}$ viewed as a complete locally compact tree rooted at x_i. Then, we define the point measure $\boldsymbol{\mathcal{M}}(T^*)$ on $\mathbb{R}_+ \times \mathbb{T}^*_{\mathrm{loc-K}} \subset \mathbb{R}_+ \times \mathbb{T}_{\mathrm{loc-K}}$ by:

$$\boldsymbol{\mathcal{M}}(T^*) = \sum_{i\in I} \delta_{(H(x_i),T_i)}. \tag{6.40}$$

As $\boldsymbol{\mathcal{M}}(T^*)$ does not depends on the representatives chosen in the equivalence class of T^* in $\mathbb{T}^{\mathrm{spine}}_{\mathrm{loc-K}}$, we deduce that $\boldsymbol{\mathcal{M}} : T^* \mapsto \boldsymbol{\mathcal{M}}(T^*)$ is a map from $\mathbb{T}^{\mathrm{spine}}_{\mathrm{loc-K}}$ to $\mathbb{M}(E)$. We now give the main result of this section.

Proposition 6.33 (Regularity of the maps Tree and $\boldsymbol{\mathcal{M}}$)**.** *The map $\boldsymbol{\mathcal{M}}$ is bi-measurable from $\mathbb{T}^{\mathrm{spine}}_{\mathrm{loc-K}}$ to $\tilde{\mathbb{M}}(E)$ with $\tilde{\mathbb{M}}(E) = \mathrm{Im}(\boldsymbol{\mathcal{M}})$. The map Tree is bi-measurable from $\tilde{\mathbb{M}}(E)$ to $\mathbb{T}^{\mathrm{spine}}_{\mathrm{loc-K}}$. Furthermore, the map $\mathrm{Tree} \circ \boldsymbol{\mathcal{M}}$ is the identity map on $\mathbb{T}^{\mathrm{spine}}_{\mathrm{loc-K}}$ and $\boldsymbol{\mathcal{M}} \circ \mathrm{Tree}$ is the identity map on $\tilde{\mathbb{M}}(E)$.*

Proof. By construction, the roots of all the trees T_i in the point measure $\boldsymbol{\mathcal{M}}(T^*)$ are not branching vertices, so that $\boldsymbol{\mathcal{M}}(T^*)$ belongs to $\tilde{\mathbb{M}}(E) \subset \mathbb{M}(E)$. We also get by construction that $\mathrm{Tree}(\boldsymbol{\mathcal{M}}(T^*)) = T^*$. This implies that $\boldsymbol{\mathcal{M}}$ is injective and thus bi-measurable thanks to Lusin's theorem.

We also have by construction that $\boldsymbol{\mathcal{M}} \circ \mathrm{Tree}(\mathcal{M}) = \mathcal{M}$ for $\mathcal{M} \in \tilde{\mathbb{M}}(E)$. This implies that $\mathrm{Im}(\boldsymbol{\mathcal{M}}) = \tilde{\mathbb{M}}(E)$ and also that Tree restricted to $\tilde{\mathbb{M}}(E)$ is injective and thus bi-measurable thanks to Lusin's theorem. □

We extend the map $T^* \mapsto \boldsymbol{\mathcal{M}}(T^*)$ to $\mathbb{T}^{(1)}_{\mathrm{loc-K}}$ in the following way. For $\big(T, \mathbf{v} = (\varrho, v_1)\big) \in \mathbb{T}^{(1)}_{\mathrm{loc-K}}$, we graft the infinite spine T_1 on v_1 and consider the rooted complete locally compact tree with an infinite marked spine $\mathrm{Sp}(T) \in \mathbb{T}^{\mathrm{spine}}_{\mathrm{loc-K}}$ defined in Remark 6.31. Then, we define $\boldsymbol{\mathcal{M}}(T, \mathbf{v})$ as $\boldsymbol{\mathcal{M}}(\mathrm{Sp}(T))$. From the continuity of the grafting procedure, see Lemma 6.14 and the continuity of Π_1°, see Lemma 6.6, and the measurability of the map $\boldsymbol{\mathcal{M}}$, we deduce that the map $(T, \mathbf{v}) \mapsto \boldsymbol{\mathcal{M}}(T, \mathbf{v})$, which we still denote by $\boldsymbol{\mathcal{M}}$ is measurable. In fact, we have the stronger following result. Consider the set of (equivalent classes of) n-pointed rooted complete locally compact tree such that the root is not a branching vertex and the distinguished vertices are not equal to the root:

$$\mathbb{T}^{(n),0,*}_{\mathrm{loc-K}} = \big\{(T, \mathbf{v}) \in \mathbb{T}^{(n),0}_{\mathrm{loc-K}} :\ d(\varrho, v_i) > 0 \text{ for all } i \in \{1, \ldots, n\}\big\}, \tag{6.41}$$

where $\mathbf{v} = (\varrho, v_1, \ldots, v_n)$. According to Lemma 6.10 and Remark 6.12, the set $\mathbb{T}^{(n),0,*}_{\mathrm{loc-K}}$ is a Borel subset of $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$. Recall from (6.38) that the Borel set $\mathbb{T}^{0,*}_{\mathrm{loc-K}}$ is the set of (equivalence class of) 1-pointed rooted complete locally compact trees such that the root is not a branching vertex and the distinguished vertex is not equal to the root.

Corollary 6.34 (Recovering $(T, \mathbf{v})$ from $\boldsymbol{\mathcal{M}}(T, \mathbf{v})$)**.** *The following map from $\mathbb{T}^{(1)}_{\mathrm{loc-K}}$ to $\mathbb{R}_+ \times \mathbb{M}(E)$ defined by:*

$$(T, \mathbf{v}) \mapsto \big(d(\varrho, v), \boldsymbol{\mathcal{M}}(T, \mathbf{v})\big)$$

is measurable and its restriction to $\mathbb{T}^{(1),0,*}_{\mathrm{loc-K}}$ is injective and bi-measurable.

Proof. Set $\mathbb{M}^*(E) = \big\{\mathcal{M} \in \mathbb{M}(E) :\ \mathcal{M}\big(\{0\} \times \mathbb{T}^*_{\mathrm{loc-K}}\big) = 0\big\}$. For $\mathcal{M} \in \mathbb{M}^*(E)$, we get that $\mathrm{Tree}(\mathcal{M})$ belongs to $\mathbb{T}^{\mathrm{spine},0}_{\mathrm{loc-K}}$. Write $[0, a] \in \mathbb{T}^{(1)}_{\mathrm{loc-K}}$ for the tree $[0, a]$ with root 0 and distinguished vertex $a \geq 0$. We define a map g on $\mathbb{R}_+ \times \mathbb{M}^*(E)$ by $g(a, \mathcal{M}) = \mathrm{Graft}_1\big([0, a], \mathrm{Tree}(\mathcal{M})\big)$. Thanks to the continuity of the grafting procedure, see Lemma 6.29 and of the function Tree, see Lemma 6.32, we deduce that g is continuous.

Let $(T, \mathbf{v}) \in \mathbb{T}^{(1),0,*}_{\mathrm{loc-K}}$. As the root of T is not a branching vertex, we get that $\boldsymbol{\mathcal{M}}(T, \mathbf{v})$ belongs to $\mathbb{M}^*(E)$, and thus $g\big(d(\varrho, v), \boldsymbol{\mathcal{M}}(T, \mathbf{v})\big)$, where $\mathbf{v} = (\varrho, v)$, is well defined and in fact equal to $(T, \mathbf{v})$ thanks to (6.36) with $n = 1$. This implies that the map $(T, \mathbf{v}) \mapsto \big(d(\varrho, v), \boldsymbol{\mathcal{M}}(T, \mathbf{v})\big)$ defined on $\mathbb{T}^{(1),0,*}_{\mathrm{loc-K}}$ is injective, and thus bi-measurable by Lusin's theorem. □

We extend this result to n-pointed trees. Recall from (6.31) that, for $(T, \mathbf{v}) \in \mathbb{T}^{(n)}_{\mathrm{loc-K}}$, we have $\mathrm{Split}_n(T, \mathbf{v}) = \Big(\hat{T}_A(T, \mathbf{v}), A \in \mathcal{P}_n\Big)$ and set $\mathcal{M}_A[T, \mathbf{v}] = \boldsymbol{\mathcal{M}}\big(\hat{T}_A(T, \mathbf{v})\big)$ for $A \in \mathcal{P}_n^+$.

Corollary 6.35 (Recovering $(T, \mathbf{v})$ from the $\mathcal{M}_A[T, \mathbf{v}]$)**.** *The following map from* $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$ *to* $\mathbb{T}^{(n)}_{\mathrm{dis}} \times \mathbb{M}(E)^{\mathcal{P}_n^+}$ *defined by:*

$$(T, \mathbf{v}) \mapsto \Big(\mathrm{Span}_n(T, \mathbf{v}), \big(\mathcal{M}_A[T, \mathbf{v}], A \in \mathcal{P}_n^+\big)\Big)$$

is measurable and its restriction to $\mathbb{T}^{(n),0,*}_{\mathrm{loc-K}}$ *is injective and bi-measurable.*

Proof. Using the measurability of the functions Span from $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$ to $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$ (see Lemma 6.7), $\mathbf{L}_n$ from $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$ to $\mathbb{R}_+^{\mathcal{P}_n^+}$ (see Lemma 6.26), Split_n from $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$ to $\Big(\mathbb{T}^{(1)}_{\mathrm{loc-K}}\Big)^{2^n}$ (see Lemma 6.27) and the map $(T, \mathbf{v}) \mapsto \boldsymbol{\mathcal{M}}(T, \mathbf{v})$ from $\mathbb{T}^{(1)}_{\mathrm{loc-K}}$ to $\mathbb{M}(E)$ (see Corollary 6.34), we deduce that the following map, say g_1, from $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$ to $\mathbb{T}^{(n)}_{\mathrm{loc-K}} \times \big(\mathbb{R}_+ \times \mathbb{M}(E)\big)^{\mathcal{P}_n^+}$ is measurable:

$$g_1 : (T, \mathbf{v}) \mapsto \Big(\mathrm{Span}(T, \mathbf{v}), \big((\ell_A(T, \mathbf{v}), \mathcal{M}_A[T, \mathbf{v}]), A \in \mathcal{P}_n^+\big)\Big).$$

Notice that $(T, \mathbf{v}) \in \mathbb{T}^{(n),0,*}_{\mathrm{loc-K}}$ implies that $\hat{T}_{\{0\}}$ is reduced to its root. Using the measurable functions Graft_n and the map defined in Corollary 6.34, we easily deduce that g_1 restricted to $\mathbb{T}^{(n),0,*}_{\mathrm{loc-K}}$ is injective and thus bi-measurable by Lusin's theorem. Since $\mathbf{L}_n(T, \mathbf{v})$ is also equal to $\mathbf{L}_n\big(\mathrm{Span}(T, \mathbf{v})\big)$, we deduce that the following map g_2, from $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$ to $\mathbb{T}^{(n)}_{\mathrm{loc-K}} \times \mathbb{M}(E)^{\mathcal{P}_n^+}$ is measurable:

$$g_2 : (T, \mathbf{v}) \mapsto \Big(\mathrm{Span}(T, \mathbf{v}), \big(\mathcal{M}_A[T, \mathbf{v}], A \in \mathcal{P}_n^+\big)\Big).$$

Furthermore, its restriction to $\mathbb{T}^{(n),0,*}_{\mathrm{loc-K}}$ is also injective and thus bi-measurable. □

7 Formal definitions of the objects informally introduced in Section 5

In this section we check that the topological and measurability results obtained in the previous section allows to precisely define the objects which are introduced in Section 5.

7.1 The elementary grafting operation

In Section 5.1.3, we considered the map:

$$\big((T, (\varrho, x)), (T', \varrho')\big) \mapsto (T \circledast_x T', \rho). \tag{7.1}$$

Lemma 7.1. *The map* (7.1) *from* $\mathbb{T}^{(1)}_{\mathrm{loc-K}} \times \mathbb{T}_{\mathrm{loc-K}}$ *to* $\mathbb{T}_{\mathrm{loc-K}}$ *is continuous.*

Proof. The map (7.1) is the composition of the continuous grafting the map from Lemma 6.14 (with $n = i = 1$, $k = 0$ and $v_i = x$) with the map Π_1° defined in (6.9) which removes x from the distinguished vertices, as this latter map is also continuous by Lemma 6.6. □

7.2 The grafting operation (5.5)

In this section we give a precise definition of the grafting procedure given in (5.5). Recall T_0 is the tree reduced to its root and the infinite spine tree $\mathrm{T}_1 \in \mathbb{T}_{\mathrm{loc-K}}$ is identified as the set $\mathbb{R}_+$ with the usual Euclidean distance and root $\varrho = 0$. We also recall that $\mathbb{T}^*_{\mathrm{loc-K}} = \mathbb{T}_{\mathrm{loc-K}} \setminus \{\mathrm{T}_0\}$, see (6.37).

Unfortunately, it is not possible to prove in general the regularity property of the grafting procedure Graft_n defined informally by (5.5). To stay close to this informal presentation, we consider the case where $n = 0$ and $(T, \mathbf{v}) = \mathrm{T}_1$ is just the infinite spine and the case where $(T, \mathbf{v})$ is a discrete tree, element of $\mathbb{T}^{(n)}_{\mathrm{dis}}$.

7.2.1 The spine case: $(T, \mathbf{v}) = \mathrm{T}_1$

This case appears in the definition of the Kesten tree in (5.7). Let $\mathcal{M}$ be a point measure on $E = \mathbb{R}_+ \times \mathbb{T}^*_{\mathrm{loc-K}}$ (or equivalently on $\mathrm{T}_1 \times \mathbb{T}^*_{\mathrm{loc-K}}$) with the restriction that $\mathcal{M}$ belongs to $\mathbb{M}(E)$, the set of point measures on E which are bounded on bounded sets introduced in Section 6.9. Then the grafting procedure $\mathrm{Graft}_0(\mathrm{T}_1, \mathcal{M})$ is precisely defined by:

$$\mathrm{Graft}_0(\mathrm{T}_1, \mathcal{M}) = \mathrm{P} \circ \mathrm{Tree}(\mathcal{M}),$$

where the reconstruction map Tree is continuous, see Lemma 6.32 and the projection map P is also continuous, see Lemma 6.16. More precisely, seeing T_1 as a distinguished spine of $\mathrm{Graft}_0(\mathrm{T}_1, \mathcal{M})$, we also have:

$$(\mathrm{Graft}_0(\mathrm{T}_1, \mathcal{M}), \mathrm{T}_1) = \mathrm{Tree}(\mathcal{M}) \quad \text{in} \quad \mathbb{T}^{[2]}_{\mathrm{loc-K}}.$$

It is then elementary to check that the Kesten tree is well defined.

Lemma 7.2 (The Kesten tree is well defined). *Let $\mathcal{M}(\mathrm{d}h, \mathrm{d}T)$ be a Poisson point measure on $\mathbb{R}_+ \times \mathbb{T}_{\mathrm{loc-K}}$ with intensity $2\beta \mathbf{1}_{\{h>0\}} \mathrm{d}h\, \mathbb{N}^\theta[\mathrm{d}T]$. Then the Kesten tree $\mathcal{T}^* = \mathrm{Graft}_0(\mathrm{T}_1, \mathcal{M})$ is a $\mathbb{T}_{\mathrm{loc-K}}$-valued random variable.*

Proof. Since $\mathrm{P} \circ \mathrm{Tree}$ is continuous, it is enough to check that a.s. the random variable $\mathcal{M}$ belongs to $\mathbb{M}(E)$. Keeping the notations from Section 6.9, we get:

$$\mathbb{E}\left[\mathcal{M}(B_E(\varepsilon))\right] = 2\beta\varepsilon^{-1}\mathbb{N}^\theta[H(T) \geq \varepsilon] = 2\beta\varepsilon^{-1} c q(\varepsilon) < +\infty.$$

Thus the point measure $\mathcal{M}$ is a.s. bounded on bounded sets of E. □

Let us notice that $(\mathcal{T}^*, \mathrm{T}_1) = \mathrm{Tree}(\mathcal{M})$ is a $\mathbb{T}^{[2]}_{\mathrm{loc-K}}$-valued random variable, which we call the Kesten tree with its distinguished spine; by definition (6.23) and (6.24), it is also a $\mathbb{T}^{\mathrm{spine}}_{\mathrm{loc-K}}$-valued and a $\mathbb{T}^{\mathrm{spine},0}_{\mathrm{loc-K}}$-valued random variable. Let us stress that the Kesten tree has a unique spine (which is then distinguished) if $\theta > 0$ and a countable number of spines if $\theta < 0$ with only one of them being distinguished.

7.2.2 The discrete case: $(T, \mathbf{v}) \in \mathbb{T}^{(n)}_{\mathrm{dis}}$

For $n \geq 1$, the construction is much more technical (even though the case $n = 1$ could be still handled by hand), and we shall only consider grafting on a discrete tree, using the theoretical background of Section 6.8. First recall the measurable map $\boldsymbol{\mathcal{M}}$ defined in (6.40) which intuitively from a complete locally compact rooted tree with a marked infinite spine $(T, \tilde{\mathrm{T}}_1)$ (in the sense of Section 6.4, with $\tilde{\mathrm{T}}_1$ equivalent to T_1 and seen as a subset of T) gives a point measure recording the heights h_i and the complete locally compact trees $T_i \neq \mathrm{T}_0$ such that $(T, \tilde{\mathrm{T}}_1)$ is in the same equivalence class as the infinite spine tree T_1 on which the T_i are grafted at h_i. See Proposition 6.33 for the measurable property of the application $\boldsymbol{\mathcal{M}}$. From the proof of Lemma 7.2, we deduce from Proposition 6.33 that, if $\mathcal{M}(\mathrm{d}h, \mathrm{d}T)$ is a Poisson point measure on $\mathbb{R}_+ \times \mathbb{T}_{\mathrm{loc-K}}$ with intensity $2\beta \mathbf{1}_{\{h>0\}} \mathrm{d}h\, \mathbb{N}^\theta[\mathrm{d}T]$, then:

$$\boldsymbol{\mathcal{M}}(\mathrm{Tree}(\mathcal{M})) = \mathcal{M}.$$

For this reason, it is natural to identify $\mathcal{M}$ with the $\mathbb{T}^{\mathrm{spine},0}_{\mathrm{loc-K}}$-valued random variable $(\mathcal{T}^*, \mathrm{T}_1) = \mathrm{Tree}(\mathcal{M})$.

From Lemma 6.29 and Remark 6.30, we get that the map:

$$(T, T^*) \mapsto \mathrm{Graft}_n\big(T, T^*\big) \quad \text{with} \quad \mathrm{Graft}_n\big(T, T^*\big) = \mathrm{Graft}_n\big(\mathbf{L}_n(T), T^*\big)$$

from $\mathbb{T}^{(n)}_{\mathrm{dis}} \times \left(\mathbb{T}^{\mathrm{spine},0}_{\mathrm{loc-K}}\right)^{\mathcal{P}_n^+}$ to $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$, which consists in replacing the branches of the discrete tree T with the truncated part of the complete locally compact tree with a distinguished spine, is measurable. Now for $A \in \mathcal{P}_n^+$, identifying the complete locally compact tree with a distinguished spine T^*_A with the point measure $\boldsymbol{\mathcal{M}}_A = \boldsymbol{\mathcal{M}}(T^*_A)$ allow the following identification:

$$\mathrm{Graft}_n\left(T, (\boldsymbol{\mathcal{M}}_A)_{A\in\mathcal{P}_n^+}\right) = \mathrm{Graft}_n\big(T, T^*\big).$$

We shall consider the case where the random variables $(\boldsymbol{\mathcal{M}}_A)_{A\in\mathcal{P}_n^+}$ are independent Poisson point measure on E with the same intensity $2\beta \mathbf{1}_{\{h>0\}} \mathrm{d}h\, \mathbb{N}^\theta[\mathrm{d}T]$. In this case, the complete locally compact n-pointed random tree $\mathrm{Graft}_n\left(T, (\boldsymbol{\mathcal{M}}_A)_{A\in\mathcal{P}_n^+}\right)$ is informally obtained by grafting, for all $i \in I$, on $x_i \in T$ the tree $T_i \in \mathbb{T}_{\mathrm{loc-K}}$, where $\mathcal{M}'(\mathrm{d}x, \mathrm{d}\mathcal{T}) = \sum_{i\in I} \delta_{(x_i, T_i)}(\mathrm{d}x, \mathrm{d}\mathcal{T})$ is, conditionally on T, a Poisson point measure on $T \times \mathbb{T}_{\mathrm{loc-K}}$ with intensity $2\beta\, d\mathscr{L}^T(\mathrm{d}x) \mathbb{N}^\theta[\mathrm{d}\mathcal{T}]$; and we shall write:

$$\mathrm{Graft}_n(T, \mathcal{M}') \quad \text{for} \quad \mathrm{Graft}_n\left(T, (\boldsymbol{\mathcal{M}}_A)_{A\in\mathcal{P}_n^+}\right). \tag{7.2}$$

We shall stress here that the definition of $\mathrm{Graft}_n(T, \mathcal{M}')$ is abusive because the measure $\mathcal{M}'$ is not clearly defined as T is an equivalence class of trees and that furthermore there is no clear measurability property in T, which is mandatory as we want to consider T a random variable in the n-leaves generalized decomposition from Theorem 5.10. So in conclusion, the notation:

$$\mathrm{Graft}_n(T, \mathcal{M}') \tag{7.3}$$

where, conditionally on T, the random measure $\mathcal{M}'$ a Poisson point measure on $T \times \mathbb{T}_{\mathrm{loc-K}}$ with intensity $2\beta\, d\mathscr{L}^T(\mathrm{d}x) \mathbb{N}^\theta[\mathrm{d}\mathcal{T}]$ is an abusive shortcut for:

$$\mathrm{Graft}_n\big(T, \mathcal{T}^*\big) \tag{7.4}$$

with $\mathcal{T}^* = (\mathcal{T}'_A)_{A\in\mathcal{P}_n^+}$ independent Kesten trees with their distinguished spines.

Thanks to the measurability property of Graft_n in its two arguments given in Lemma 6.29, the discrete tree T in (7.4) can be a $\mathbb{T}^{(n)}_{\mathrm{dis}}$-valued random variable. In the setting of the present paper the random variables T and $\mathcal{T}^*$ will be independent.

7.3 Planar trees (Section 5.3)

Recall $\mathbb{T}^{(n)}_{\mathrm{dis}} \subset \mathbb{T}^{(n)}_{\mathrm{loc-K}}$ is the closed subset of (equivalence classes of) discrete trees, that is, compact trees with all the leaves being distinguished, see (6.11). Let $(\mathbf{t}, \mathbf{v}) \in \mathbb{T}^{(n)}_{\mathrm{dis}}$ with $\mathbf{v} = (v_0 = \varrho, \ldots, v_n)$. (Notice that the tree $\mathbf{t}$ has at most n leaves.) For $k \in \{1, \ldots, n-1\}$, let p_{k+1} denote the projection of v_{k+1} on $\mathrm{Span}(\mathbf{t}, (v_0, \ldots, v_k))$, that is the only point on $[\![\varrho, v_{k+1}]\!]$ such that $[\![\varrho, p_{k+1}]\!] = [\![\varrho, v_{k+1}]\!] \cap \mathrm{Span}(\mathbf{t}, (v_0, \ldots, v_k))$. The discrete tree $(\mathbf{t}, \mathbf{v})$ is planar if $p_{k+1} \in [\![\varrho, v_k]\!]$ for all $k \in \{1, \ldots, n-1\}$. It is easy to check this condition is equivalent to the condition used in Section 5.3: for all $x \in \mathbf{t}$, there exists $0 \leq i_{\mathrm{g}} \leq i_{\mathrm{d}} \leq n$ such that $v_i \in \mathbf{t}_x$ if and only if $i_{\mathrm{g}} \leq i \leq i_{\mathrm{d}}$.

Let $\mathbb{T}^{(n)}_{\mathrm{plan}} \subset \mathbb{T}^{(n)}_{\mathrm{dis}}$ be the set of (equivalence classes of) n-pointed planar trees. It is elementary to check that for a discrete tree $(\mathbf{t}, \mathbf{v}) \in \mathbb{T}^{(n)}_{\mathrm{dis}}$ there exists a permutation (which is not unique) π such that the discrete tree $(\mathbf{t}, \mathbf{v}^\pi)$ is planar. Arguing as in the proof of Lemma 6.2, on get that the map $(\mathbf{t}, \mathbf{v}) \mapsto (\mathbf{t}, \mathbf{v}_k)$ with $\mathbf{v}_k = (v_0, \ldots, v_n, p_k)$ is 5/2-Lipschitz from $\mathbb{T}^{(n)}_{\mathrm{dis}}$ to $\mathbb{T}^{(n)}_{\mathrm{dis}}$. Then, since the application $(\mathbf{t}, \mathbf{v}_k) \mapsto d(\varrho, p_k) + d(p_k, v_k) - d(\varrho, v_k)$ is clearly continuous and the latter quantity is zero if and only if $p_k \in [\![\varrho, v_k]\!]$, we deduce that $\mathbb{T}^{(n)}_{\mathrm{plan}}$ is a closed subset of $\mathbb{T}^{(n)}_{\mathrm{dis}}$ and thus a closed subset of $\mathbb{T}^{(n)}_{\mathrm{K}}$.

7.4 Oriented grafting on discrete trees (Section 5.3)

When considering planar trees in Section 7.3, we shall also be interested in a grafting on the left or on the right of $i \in \{1, \ldots, n\}$, which is the same as the grafting (6.14), but for the order of the coordinates of the vector $\mathbf{v} \circledast \mathbf{v}'$. Recall that for $h \geq 0$ and $(T, \mathbf{v}) \in \mathbb{T}^{(n)}_{\mathrm{dis}}$, the vertex $x_{i,h}$ the unique vertex of T that satisfies $x_{i,h} \in [\![\varrho, v_i]\!]$ and $H(x_{i,h}) = H(v_i) \wedge h$, see Section 6.3. For $\epsilon \in \{\mathrm{g}, \mathrm{d}\}$, we define the grafting map $\circledast^{\epsilon}_{i,h}$ by (5.8) with $x = x_{i,h}$ and (5.9), (5.10) and (5.11), using the convention stated thereafter when $i_{\mathrm{g}} = 0$ (that is, $x_{i,h} = \varrho$) and $i_{\mathrm{d}} = n$. Let us recall that $i_{\mathrm{g}} = \min\{j \in \{0, \ldots, n\} : v_j \in T_{x_{i,h}}\}$ (resp. $i_{\mathrm{d}} = \max\{j \in \{0, \ldots, n\} : v_j \in T_{x_{i,h}}\}$) is the leftmost (resp. rightmost) distinguished vertex being a descendant of $x_{i,h}$.

Lemma 7.3 (Measurability of the left/right grafting maps)**.** *Let $n, k \in \mathbb{N}$, $i \in \{0, \ldots, n\}$ and $\epsilon \in \{\mathrm{g}, \mathrm{d}\}$. The map $\big(h, (T, \mathbf{v}), (T', \mathbf{v}')\big) \mapsto T \circledast^{\epsilon}_{i,h} T'$ is measurable from $\mathbb{R}_+ \times \mathbb{T}^{(n)}_{\mathrm{loc-K}} \times \mathbb{T}^{(k)}_{\mathrm{loc-K}}$ to $\mathbb{T}^{(n+k)}_{\mathrm{loc-K}}$.*

Proof. We recall that the map $\big(h, (T, \mathbf{v})\big) \mapsto \big(T, (\mathbf{v}, x_{i,h})\big)$ is continuous from $\mathbb{R}_+ \times \mathbb{T}^{(n)}_{\mathrm{loc-K}}$ to $\mathbb{T}^{(n+1)}_{\mathrm{loc-K}}$, see Section 6.3, and that the grafting map $\circledast_{i,h}$ is continuous, see Lemma 6.15 therein. Thanks to the continuity of the permutation of the distinguished vertices (so that i_{g} and i_{d} play a similar role by considering the permutation π on $\{0,, \ldots, n\}$ such that $\pi(0) = 0$ and $\pi(j) = n + 1 - j$ otherwise) and of the removing of distinguished vertices (so that $x_{i,h}$ can be removed from the distinguished vertices of $\big(T, (\mathbf{v}, x_{i,h})\big)$), see Lemmas 6.9 and 6.6, we only need to prove that the map $(T, \mathbf{v}) \mapsto i_{\mathrm{g}}$, with $i = n$ and $h = 0$ or equivalently $x_{i,h} = v_n$, is a measurable function from $\mathbb{T}^{(n)}_{\mathrm{dis}}$ to $\{0, \ldots, n\}$ for $n \in \mathbb{N}^*$. This latter result is obvious as $\{i_{\mathrm{g}} > k\} = \bigcap_{j=0}^{k} \{v_j \not\in T_{v_n}\}$ and as v_j belongs to T_{v_n} if and only if $d(\varrho, v_i) = d(\varrho, v_n) + d(v_n, v_i)$ and the map $(T, \mathbf{v}) \mapsto (d(v_i, v_j), 0 \leq i \leq j \leq n)$ is trivially continuous. □

7.5 The Growth_n function from (5.22)

Let $n \in \mathbb{N}^*$. We consider the function Growth_n defined in (5.22), which formally is written as first attaching successively a branch $([0, h], (0, h)) \in \mathbb{T}^{(1)}_{\mathrm{dis}}$ simply denoted $[0, h]$ to each distinguished vertex $\mathbf{v}^*$ of $(T, \mathbf{v})$, but the root, (notice that there are then $2n + 1$ distinguished vertices) and then forgetting all the n distinguished vertices $\mathbf{v}^*$ so that there are only $n + 1$ distinguished vertices:

$$\mathrm{Growth}_n\big((T, \mathbf{v}), h\big) = \Pi^{\circ, A_n}_{2n} \circ \mathrm{Growth}'_{n,n}\big((T, \mathbf{v}), h\big),$$

where Π°, A_n}_{2n} is defined in (6.9) with $A_n = (0, n + 1, \ldots, 2n)$ and for $i = 1, \ldots, n$:

$$\mathrm{Growth}'_{n,i}\big((T, \mathbf{v}), h\big) = \mathrm{Growth}'_{n,i-1}\big((T, \mathbf{v}), h\big) \circledast_i [0, h],$$

with the convention $\mathrm{Growth}'_{n,0}\big((T,\mathbf{v}),h\big)=(T,\mathbf{v})$. Using the continuity of the grafting procedure (see Lemma 6.15) and the continuity of Π_{2n}°,A_n} (see Lemma 6.6), we get the following result.

Lemma 7.4 (Continuity of the map Growth_n)**.** *Let $n\in\mathbb{N}^*$. The map Growth_n is continuous from $\mathbb{T}^{(n)}_{\mathrm{loc-K}}\times\mathbb{R}_+$ to $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$.*

7.6 A detail of the proof of Corollary 5.9

Recall $\mathbb{T}^{(1),0,*}_{\mathrm{loc-K}}$ defined in (6.41) is the Borel subset of $\mathbb{T}^{(1)}_{\mathrm{loc-K}}$ of the trees such that the root is not a branching vertex and the distinguished vertex is distinct from the root. The map $g:\ (T,\mathbf{v})\mapsto\big(d(\varrho,v),\boldsymbol{\mathcal{M}}(T,\mathbf{v})\big)$, with $\mathbf{v}=(\varrho,v)$, defined on $\mathbb{T}^{(1),0,*}_{\mathrm{loc-K}}$ is injective and bi-measurable, see Corollary 6.34. We deduce that $(T,\mathbf{v})$ is a measurable function of $\big(d(\varrho,v),\boldsymbol{\mathcal{M}}(T,\mathbf{v})\big)$ on the image of $\mathbb{T}^{(1),0,*}_{\mathrm{loc-K}}$ by g.

Furthermore the set $\mathbb{T}^{(1),0,*}_{\mathrm{loc-K}}$ is of full measure with respect to the distribution of $(\mathcal{T},\mathbf{v})$ under $\mathbb{N}^\theta[\mathrm{d}\mathcal{T}]\,\Lambda_t(dv)$, with $\mathbf{v}=(\varrho,v)$, as $\mathbb{N}^\theta$-a.e. the root of $\mathcal{T}$ is not a branching vertex and $d(\varrho,v)=t>0$. Thus, as $t>0$ is fixed, we get that $(\mathcal{T},\mathbf{v})$ is a measurable function of $\boldsymbol{\mathcal{M}}(\mathcal{T},\mathbf{v})$.

7.7 Construction of the continuum random tree $\mathcal{T}^{\alpha,\theta}$

Let $\beta>0$, $\theta,\alpha\in\mathbb{R}_+$ and let $S^{\alpha,\theta}$ be a Poisson point measure on $[0,\infty)$ with intensity measure $f_{\mathrm{int}}(t)\,\mathrm{d}t$ and f_{int} given by (5.32). We first consider the case $\alpha>0$. Denote by $(\xi_i,i\in\mathbb{N}^*)$ the increasing sequence of jumping times of the inhomogeneous Poisson process $(N_t^{\alpha,\theta}=S^{\alpha,\theta}([0,t]),t\geq 0)$. We consider the $\mathbb{T}^{(n)}_{\mathrm{dis}}$-valued random variable $\mathfrak{T}_{\xi_n}$ of Section 5.4.3 for $n\geq 1$ associated to f_{int}. In particular, recall that, for every $n\geq 1$, $\mathfrak{T}_{\xi_n}$ is a discrete tree with n distinguished leaves, where all of them are at height ξ_n.

For every $n\geq 1$, let $\mathcal{T}^{n,*}=(\mathcal{T}_A,A\in\mathcal{P}_n^+)$ be a family of independent Kesten trees with parameter (β,α), independent of the tree $\mathfrak{T}_{\xi_n}$. We define the random marked tree:

$$\mathcal{T}^{(n)}=\Big(\Pi_n^\circ(\tilde{\mathcal{T}}^{(n)}),\mathrm{Span}^\circ(\tilde{\mathcal{T}}^{(n)})\Big)\quad\text{with}\quad\tilde{\mathcal{T}}^{(n)}=\mathrm{Graft}_n\left(\mathfrak{T}_{\xi_n},\mathcal{T}^{n,*}\right).$$

Thanks to Lemma 6.18 and Lemma 6.29 on the measurability of the grafting function, we deduce that $\mathcal{T}^{(n)}$ is a $\mathbb{T}^{[2]}_{\mathrm{loc-K}}$-valued random variable. The family of the distributions of the $\mathbb{T}^{[2]}_{\mathrm{loc-K}}$-valued random trees $(\mathcal{T}^{(n)},n\geq 1)$ is consistent in the sense that, for every $n\geq 1$ and every $t\leq\xi_n$, $r_t^{[2]}(\mathcal{T}^{(n)})\overset{(d)}{=}r_t^{[2]}(\mathcal{T}^{(n+1)})$. It is in particular a Cauchy sequence in $\mathbb{T}^{[2]}_{\mathrm{loc-K}}$, and we denote by $(\mathcal{T}^{\alpha,\theta},\mathfrak{T}^{\alpha,\theta})$ its limit which is thus a $\mathbb{T}^{[2]}_{\mathrm{loc-K}}$-valued random variable. By construction, $\mathfrak{T}^{\alpha,\theta}$ and $\mathfrak{T}^{\mathrm{ske}}$ have the same distribution. This construction is a formal way to define the tree obtained by grafting on the infinite discrete tree $\mathfrak{T}^{\mathrm{ske}}$ (which serves as a backbone) at x_i a tree $\mathcal{T}_i$ where $((x_i,\mathcal{T}_i),i\in I)$ are the atoms of a Poisson point measure of intensity $2\beta\mathscr{L}(\mathrm{d}x)\mathbb{N}^\theta(\mathrm{d}\mathcal{T})$, where $\mathscr{L}$ is the length measure on $\mathfrak{T}^{\mathrm{ske}}$.

For $\alpha=0$, we simply define $(\mathcal{T}^{\emptyset,\theta},\mathfrak{T}^{0,\theta})$ as the Kesten tree with parameter (β,α).

We then define the $\mathbb{T}^{[2]}_{\mathrm{loc-K}}$-valued random process $\Big((\mathcal{T}_t^{\alpha,\theta},\mathfrak{T}_t^{\alpha,\theta}),t\geq 0\Big)$ by setting:

$$\mathcal{T}_t^{\alpha,\theta}=r_t(\mathcal{T}^{\alpha,\theta})\quad\text{and}\quad\mathfrak{T}_t^{\alpha,\theta}=r_t(\mathfrak{T}_t^{\alpha,\theta}).$$

In particular, thanks to Lemma 6.13, the random variable $(\mathcal{T}_t^{\alpha,\theta},\tilde{N}_t(\mathfrak{T}^{\alpha,\theta}))$ is well defined.

8 Proof of Theorem 5.10

We prove Formula (5.31) by induction. For $n=1$, as $\mathbf{T}_1=[0,t]$ (with root $\varrho=0$ and distinguished vertex $v_1=t$), this is Corollary 5.9.

Let $k \in \mathbb{N}^*$. Recall the maps $\mathbf{L}_k$, from (6.28) in Section 6.6, and Split_k from (6.31) in Section 6.7. For $(T, \mathbf{v}) \in \mathbb{T}^{(k)}_{\mathrm{loc-K}}$ and $A \in \mathcal{P}_k^+$, we write $\mathcal{M}_A[T, \mathbf{v}](\mathrm{d}h, \mathrm{d}\mathbf{t})$ for the measure $\boldsymbol{\mathcal{M}}(\hat{T}_A(T, \mathbf{v}))$ on $E = \mathbb{R}_+ \times \mathbb{T}^*_{\mathrm{loc-K}}$, where $(\hat{T}_A(T, \mathbf{v}), A \in \mathcal{P}_k) = \mathrm{Split}_k(T, \mathbf{v})$ and the measure $\boldsymbol{\mathcal{M}}(T, \mathbf{v})$ is defined at the end of Section 6.9. We also recall the notation $(\ell_A(T, \mathbf{v}), A \in \mathcal{P}_k^+) = \mathbf{L}_k(T, \mathbf{v})$, and notice that $\ell_A(T, \mathbf{v}) = 0$ implies that $\mathcal{M}_A[T, \mathbf{v}] = 0$. Let $n \in \mathbb{N}^*$ and $(\Phi_A, A \in \mathcal{P}_n^+)$ be a family of non-negative measurable functions defined on E. Let f be a bounded non-negative measurable function defined on $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$ (or more simply on $\mathbb{T}^{(n)}_{\mathrm{dis}}$). We shall first prove (5.31) for a non-negative function F defined on $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$ of the form:

$$F(T, \mathbf{v}) = f(\mathrm{Span}(T, \mathbf{v})) \exp\Big\{ - \sum_{A \in \mathcal{P}_n^+} \big\langle \Phi_A, \mathcal{M}_A[T, \mathbf{v}] \big\rangle \Big\}.$$

Let $n \geq 2$ and suppose that (5.31) holds for $n-1$. For $k \in \{1, \ldots, n\}$, we denote by $\mathcal{T}^{[k]}$ the tree $\mathrm{Span}(\mathcal{T}, \mathbf{v}_k) \in \mathbb{T}^{(k)}_{\mathrm{loc-K}}$, where $\mathbf{v}_k = (v_0 = \varrho, \mathbf{v}_k^*)$ and $\mathbf{v}_k^* = (v_1, \ldots, v_k)$; and we simply write $\mathcal{M}_A^{[k]}$ for $\mathcal{M}_A[\mathcal{T}, \mathbf{v}_k]$ and $\ell_A^{[k]}$ for $\ell_A(\mathcal{T}, \mathbf{v}_k)$, so that under $\mathbb{N}^\theta[\mathrm{d}\mathcal{T}]\, \ell_t^{\otimes n}(\mathrm{d}\mathbf{v}^*)$:

$$F(\mathcal{T}, \mathbf{v}_n) = f(\mathcal{T}^{[n]}) \exp\Big\{ - \sum_{A \in \mathcal{P}_n^+} \big\langle \Phi_A, \mathcal{M}_A^{[n]} \big\rangle \Big\}.$$

We also write $v_A^{[k]}$ and $w_A^{[k]}$ for v_A and w_A from (6.26) and (6.27) with $(T, \mathbf{v})$ replaced by $(\mathcal{T}^{[k]}, \mathbf{v}_k)$; and thus we have $\ell_A^{[k]} = d(w_A^{[k]}, v_A^{[k]})$.

Similarly, under $\mathbb{E}^{\theta, t}$, for $k \geq 2$, we write also $\hat{\mathcal{M}}_A^{[k]}$ for the measure $\boldsymbol{\mathcal{M}}(T_A^*)$ restricted to $[0, \ell_A(\mathbf{T}_k)] \times \mathbb{T}^*_{\mathrm{loc-K}}$, $\hat{v}_A^{[k]}$ and $\hat{w}_A^{[k]}$ for v_A and w_A from (6.26) and (6.27) with $(T, \mathbf{v})$ replaced by $(\mathbf{T}_k, \mathbf{v}_k)$, and $\hat{\ell}_A^{[k]} = d(\hat{w}_A^{[k]}, \hat{v}_A^{[k]}) = \ell_A(\mathbf{T}_k)$. For $n \geq 2$, simply writing $\mathbf{T}_n$ for $(\mathbf{T}_n, \mathbf{v}_n)$, we have:

$$F\big(\mathrm{Graft}_n(\mathbf{T}_n, \mathcal{T}^*)\big) = f(\mathbf{T}_n) \exp\Big\{ - \sum_{A \in \mathcal{P}_n^+} \big\langle \Phi_A, \hat{\mathcal{M}}_A^{[n]} \big\rangle \Big\}.$$

Using the definition of the Kesten tree via Poisson point measures and the definition of the function Graft_n, we obtain in particular that:

$$\mathbb{E}^{\theta, t}\Big[F\big(\mathrm{Graft}_n(\mathbf{T}_n, \mathcal{T}^*)\big)\Big] = \mathbb{E}^{\theta, t}\left[F'(\mathbf{T}_n)\right], \tag{8.1}$$

where

$$F'(\mathbf{T}_n) = f(\mathbf{T}_n) \exp\left\{ -2\beta \sum_{A \in \mathcal{P}_n^+} \int_0^{\hat{\ell}_A^{[n]}} \mathrm{d}a\, \mathbb{N}^\theta\left[1 - \mathrm{e}^{-\Phi_A(a, \mathcal{T})}\right] \right\}. \tag{8.2}$$

Recall (6.2). Set $p_n = p_{\mathbf{v}_{n-1}}(v_n)$ for the projection of v_n on $\mathcal{T}^{[n-1]}$. Since $\mathbb{N}^\theta$-a.e. $p_n \neq \varrho$, we deduce that there exists $\mathbb{N}^\theta$-a.e. a unique $B \in \mathcal{P}_{n-1}^+$ such that $p_n \in]\!] w_B^{[n-1]}, v_B^{[n-1]}]\!] \subset \mathcal{T}^{[n-1]}$, and write $h_n = d(p_n, w_B^{[n-1]})$. Recall the function Tree, defined in Section 6.9 just before Lemma 6.32, from $\mathbb{M}(E)$ into $\mathbb{T}^{[2]}_{\mathrm{loc-K}}$ and the projection $\tilde{\Pi}$ from $\mathbb{T}^{[2]}_{\mathrm{loc-K}}$ to $\mathbb{T}_{\mathrm{loc-K}}$, defined just before Lemma 6.24, which forgets about the marked subtree defined in Section 6.5. We simply write $\mathrm{Tree}' = \tilde{\Pi} \circ \mathrm{Tree}$. On the one hand, we have:

$$\begin{aligned} \mathcal{T}^{[n]} &= \mathcal{T}^{[n-1]} \circledast_{\min B, H(p_n)} \big[0, t - H(p_n)\big], \\ \ell_B^{[n-1]} &= \ell_B^{[n]} + \ell_{B \cup \{n\}}^{[n]}, \\ \mathcal{M}_B^{[n-1]} &= \mathcal{M}_{B \cup \{n\}}^{[n]} + \mathcal{M}_B^{[n]}(\cdot + h_n, \cdot) + \delta_{\left(h_n, \mathrm{Tree}'\left(\mathcal{M}_{\{n\}}^{[n]}\right)\right)}; \end{aligned} \tag{8.3}$$

and, to fix notation, we shall write:

$$\mathcal{M}_B^{[n-1]} = \mathcal{M}_B[\mathcal{T}, \mathbf{v}_{n-1}] = \sum_{i \in I_{n-1}^B} \delta_{\mathrm{h}_i^{[n-1],B}, \mathcal{T}_i^{[n-1],B}}.$$

On the other hand, for $A \in \mathcal{P}_{n-1}^+$ and $A \neq B$, we have:

$$B \subset A \implies \mathcal{M}_A^{[n-1]} = \mathcal{M}_{A\cup\{n\}}^{[n]}, \quad \mathcal{M}_A^{[n]} = 0, \quad \ell_A^{[n-1]} = \ell_{A\cup\{n\}}^{[n]} \quad \text{and} \quad \ell_A^{[n]} = 0, \tag{8.4}$$

$$A \cap B \in \{\emptyset, A\} \implies \mathcal{M}_A^{[n-1]} = \mathcal{M}_A^{[n]}, \quad \mathcal{M}_{A\cup\{n\}}^{[n]} = 0, \quad \ell_A^{[n-1]} = \ell_A^{[n]} \quad \text{and} \quad \ell_{A\cup\{n\}}^{[n]} = 0, \tag{8.5}$$

$$A \cap B \notin \{\emptyset, B, A\} \implies \mathcal{M}_A^{[n-1]} = \mathcal{M}_A^{[n]} = \mathcal{M}_{A\cup\{n\}}^{[n]} = 0 \quad \text{and} \quad \ell_A^{[n-1]} = \ell_A^{[n]} = \ell_{A\cup\{n\}}^{[n]} = 0. \tag{8.6}$$

It is also easy to rebuild $(\mathcal{M}_A^{[n]}, A \in \mathcal{P}_n^+)$ from $(\mathcal{M}_A^{[n-1]}, A \in \mathcal{P}_{n-1}^+)$ and v_n.

Set

$$F_n = \mathbb{N}^\theta \left[\int_{\mathcal{T}^n} \Lambda_t^{\otimes n}(\mathrm{d}\mathbf{v}_n^*) \, F\Big(\mathcal{T}, \mathbf{v}_n\Big) \right].$$

Considering that $\mathcal{T}_i^{[n-1],B}$ is a subset of $\mathcal{T}$, we have:

$$\begin{aligned}
F_n &= \mathbb{N}^\theta \left[\int_{\mathcal{T}^{n-1}} \Lambda_t^{\otimes(n-1)}(\mathrm{d}\mathbf{v}_{n-1}^*) \sum_{B \in \mathcal{P}_{n-1}^+} \sum_{i \in I_{n-1}^B} \int_{\mathcal{T}_i^{[n-1],B}} \ell_t(dv_n) \, F(\mathcal{T}, \mathbf{v}_n) \right] \\
&= \mathbb{N}^\theta \Bigg[\int_{\mathcal{T}^{n-1}} \Lambda_t^{\otimes(n-1)}(\mathrm{d}\mathbf{v}_{n-1}^*) \\
&\qquad \sum_{B \in \mathcal{P}_{n-1}^+} \sum_{i \in I_{n-1}^B} \Gamma_B \left(\mathcal{T}^{[n-1]}, H(w_B^{[n-1]}), \mathcal{M}_{B,i}^{[n-1]}, H(w_B^{[n-1]}) + \mathrm{h}_i^{[n-1],B}, \mathcal{T}_i^{[n-1],B} \right) \\
&\qquad \times \exp\Bigg\{ - \sum_{A \in \mathcal{P}_{n-1}^+ \setminus \{B\}} \Big\langle \mathbf{1}_{\{B \subset A\}} \, \Phi_{A\cup\{n\}} + \mathbf{1}_{\{A\cap B = \emptyset \text{ or } A\}} \, \Phi_A, \mathcal{M}_A^{[n-1]} \Big\rangle \Bigg\} \Bigg],
\end{aligned}$$

where the measure $\mathcal{M}_{B,i}^{[n-1]}$ is the measure $\mathcal{M}_B^{[n-1]}$ without its atom at $(\mathrm{h}_i^{[n-1],B}, \mathcal{T}_i^{[n-1],B})$:

$$\mathcal{M}_{B,i}^{[n-1]} = \mathcal{M}_B^{[n-1]} - \delta_{(\mathrm{h}_i^{[n-1],B}, \mathcal{T}_i^{[n-1],B})},$$

and, for $(T, \mathbf{w}) \in \mathbb{T}_{\mathrm{loc-K}}^{(n-1)}$, $(T', \varrho') \in \mathbb{T}_{\mathrm{loc-K}}$, $\nu \in \mathbb{M}(E)$ and $\mathrm{h}' \geq h \geq 0$:

$$\begin{aligned}
\Gamma_B\big((T, \mathbf{w}), h, \nu, \mathrm{h}', T'\big) &= f\big(T \circledast_{\min B, \mathrm{h}'} [0, t - \mathrm{h}']\big) \, \exp\left\{ -\langle \Phi_{B, \mathrm{h}'-h}, \nu \rangle \right\} \\
&\qquad \times \int_{T'} \Lambda_{t-\mathrm{h}'}(dv) \, \exp\Big\{ -\Big\langle \Phi_{\{n\}}, \boldsymbol{\mathcal{M}}\big(T', (\varrho', v)\big) \Big\rangle \Big\},
\end{aligned}$$

with:

$$\Phi_{B,\mathrm{h}''}(s, \mathbf{t}) = \mathbf{1}_{\{s \leq \mathrm{h}''\}} \Phi_{B\cup\{n\}}(s, \mathbf{t}) + \mathbf{1}_{\{s > \mathrm{h}''\}} \Phi_B(s - \mathrm{h}'', \mathbf{t}). \tag{8.7}$$

For $B \in \mathcal{P}_{n-1}^+$, using the notation $\hat{\mathcal{M}}_B^{[n]} = \sum_{i \in \hat{I}_{n-1}^B} \delta_{(\hat{\mathrm{h}}_i^{[n-1],B}, \hat{\mathcal{T}}_i^{[n-1],B})}$, we set for $i \in \hat{I}_{n-1}^B$:

$$\hat{\mathcal{M}}_{B,i}^{[n-1]} = \hat{\mathcal{M}}_B^{[n-1]} - \delta_{\left(\hat{\mathrm{h}}_i^{[n-1],B}, \hat{\mathcal{T}}_i^{[n-1],B}\right)}.$$

We deduce from the induction hypothesis (*i.e.* Equation (5.31) with $n-1$ instead of n) and the definition of Kesten tree, with $F_n = (n-1)!(\tilde{c}_t^\theta)^{2-n}\, \mathrm{e}^{-2\beta\theta t}\, G_n$ that: G_n is equal to

$$\mathbb{E}^\theta\Bigg[\sum_{B\in\mathcal{P}_{n-1}^+}\sum_{i\in\hat{I}_{n-1}^B}\Gamma_B\left(\mathbf{T}_{n-1}, H(\hat{w}_{[n-1],B}), \hat{\mathcal{M}}_{B,i}^{[n-1]}, H(\hat{w}_{[n-1],B}) + \hat{\mathrm{h}}_i^{[n-1],B}, \hat{\mathcal{T}}_i^{[n-1],B}\right) \\ \times\ \exp\bigg\{-\sum_{A\in\mathcal{P}_{n-1}^+\setminus\{B\}}\left\langle \mathbf{1}_{\{B\subset A\}}\,\Phi_{A\cup\{n\}} + \mathbf{1}_{\{A\cap B=\emptyset \text{ or } A\}}\,\Phi_A, \hat{\mathcal{M}}_A^{[n-1]}\right\rangle\bigg\}\Bigg].$$

Since for $A\in\mathcal{P}_{n-1}^+$, the random measure $\boldsymbol{\mathcal{M}}(T_A^*, \hat{\ell}_A^{[n-1]})(\mathrm{d}h', \mathrm{d}\mathcal{T}')$ is conditionally given $\hat{\ell}_A^{[n-1]}$ a Poisson point measure on $[0, \hat{\ell}_A^{[n-1]}]\times\mathbb{T}_{\mathrm{loc-K}}$ with intensity $2\beta\mathrm{d}h'\,\mathbb{N}^\theta[\mathrm{d}\mathcal{T}']$, we deduce from Palm formula that: G_n is equal to

$$\mathbb{E}^\theta\Bigg[\sum_{B\in\mathcal{P}_{n-1}^+} 2\beta\int_0^{\hat{\ell}_B^{[n-1]}}\mathrm{d}r\int\mathbb{N}^\theta[\mathrm{d}\mathcal{T}]\,\Gamma_B\left(\mathbf{T}_{n-1}, H(\hat{w}_B^{[n-1]}), \hat{\mathcal{M}}_B^{[n-1]}, H(\hat{w}_B^{[n-1]}) + r, \mathcal{T}\right) \\ \times\ \exp\bigg\{-\sum_{A\in\mathcal{P}_{n-1}^+\setminus\{B_x\}}\left\langle \mathbf{1}_{\{B_x\subset A\}}\,\Phi_{A\cup\{n\}} + \mathbf{1}_{\{A\cap B_x=\emptyset \text{ or } A\}}\,\Phi_A, \hat{\mathcal{M}}_A^{[n-1]}\right\rangle\bigg\}\Bigg] \\ = \mathbb{E}^\theta\Bigg[2\beta\int_{\mathbf{T}_{n-1,t}}\mathscr{L}(\mathrm{d}x)\int\mathbb{N}^\theta[\mathrm{d}\mathcal{T}]\,\Gamma_{B_x}\left(\mathbf{T}_{n-1}, H(\hat{w}_{B_x}^{[n-1]}), \hat{\mathcal{M}}_{B_x}^{[n-1]}, H(x), \mathcal{T}\right) \\ \times\ \exp\bigg\{-\sum_{A\in\mathcal{P}_{n-1}^+\setminus\{B_x\}}\left\langle \mathbf{1}_{\{B_x\subset A\}}\,\Phi_{A\cup\{n\}} + \mathbf{1}_{\{A\cap B_x=\emptyset \text{ or } A\}}\,\Phi_A, \hat{\mathcal{M}}_A^{[n-1]}\right\rangle\bigg\}\Bigg],$$

where B_x is the only element B of $\mathcal{P}_{n-1}^+$ such that x belongs to the branch B of $\mathbf{T}_{n-1}$: $x\in\rrbracket\hat{w}_B^{[n-1]}, \hat{v}_B^{[n-1]}\rrbracket$, where, as $\mathbf{T}_{n-1}$ is discrete, we recall that $\mathrm{Split}_{n-1}(\mathbf{T}_{n-1}) = \left(\llbracket\hat{w}_A^{[n-1]}, \hat{v}_A^{[n-1]}\rrbracket, A\in\mathcal{P}_{n-1}\right)$ with $\mathcal{P}_{n-1} = \mathcal{P}_{n-1}^+\cup\big\{\{0\}\big\}$. Using (5.31) again for $n=1$ (or Corollary 5.9) gives:

$$\int\mathbb{N}^\theta[\mathrm{d}\mathcal{T}]\,\Gamma_B(\mathbf{T}_{n-1,t}, h, \nu, \mathrm{h}', \mathcal{T}) = f\big(\mathbf{T}_{n-1}\circledast_{\min B,\mathrm{h}'}[0, t-\mathrm{h}']\big)\,\mathrm{e}^{-\langle\Phi_{\mathrm{h}'-h},\nu\rangle} \\ \times\exp\bigg\{-2\beta\theta(t-\mathrm{h}') - 2\beta\int_0^{t-\mathrm{h}'}\mathrm{d}a\,\mathbb{N}^\theta\left[1-\mathrm{e}^{-\Phi_{\{n\}}(a,\mathcal{T})}\right]\bigg\}.$$

With x chosen according to the length measure $\mathscr{L}(\mathrm{d}x)$ on $\mathbf{T}_{n-1}$, the tree $\mathbf{T}_{n-1}\circledast_{\min B_x, H(x)}\big[0, t-H(x)\big]$ is obtained by grafting a branch of length $t-H(x)$ at x on $\mathbf{T}_{n-1}$ and thus will simply be denoted as $\mathbf{T}_{n-1}\circledast_x\big[0, t-H(x)\big]$ (see also Remark 5.2 for similar notation).

Therefore, we obtain:

$$\begin{aligned} G_n = \mathbb{E}^\theta \Bigg[2\beta \int_{\mathbf{T}_{n-1}} &\mathscr{L}(\mathrm{d}x)\, f\Big(\mathbf{T}_{n-1} \circledast_x \big[0, t - H(x)\big]\Big) \exp\Big\{-2\beta\big(t - H(x)\big)\Big\} \\ &\times \exp\Bigg\{-2\beta \sum_{A \in \mathcal{P}_{n-1}^+ \setminus \{B_x\}} \mathbf{1}_{\{B_x \subset A\}} \int_0^{\hat{\ell}_A^{[n-1]}} \mathrm{d}a\, \mathbb{N}^\theta \Big[1 - \mathrm{e}^{-\Phi_{A\cup\{n\}}(a,\mathcal{T})}\Big]\Bigg\} \\ &\times \exp\Bigg\{-2\beta \sum_{A \in \mathcal{P}_{n-1}^+ \setminus \{B_x\}} \mathbf{1}_{\{A \cap B_x = \emptyset \text{ or } A\}} \int_0^{\hat{\ell}_A^{[n-1]}} \mathrm{d}a\, \mathbb{N}^\theta \Big[1 - \mathrm{e}^{-\Phi_{A}(a,\mathcal{T})}\Big]\Bigg\} \\ &\times \exp\Bigg\{-2\beta \int_0^{H(x) - H\left(w_{B_x}^{[n-1]}\right)} \mathrm{d}a\, \mathbb{N}^\theta \Big[1 - \mathrm{e}^{-\Phi_{B_x\cup\{n\}}(a,\mathcal{T})}\Big]\Bigg\} \\ &\times \exp\Bigg\{-2\beta \int_0^{H\left(v_{B_x}^{[n-1]}\right) - H(x)} \mathrm{d}a\, \mathbb{N}^\theta \Big[1 - \mathrm{e}^{-\Phi_{B_x}(a,\mathcal{T})}\Big]\Bigg\} \\ &\times \exp\Bigg\{-2\beta \int_0^{t - H(x)} \mathrm{d}a\, \mathbb{N}^\theta \Big[1 - \mathrm{e}^{-\Phi_{\{n\}}(a,\mathcal{T})}\Big]\Bigg\}\Bigg]. \end{aligned}$$

We deduce from Lemma 5.1 with the density:

$$f_{\mathrm{dens}}(s) = \frac{2\beta\theta\, \mathrm{e}^{2\beta\theta s}}{\mathrm{e}^{2\beta\theta t} - 1}\, \mathbf{1}_{[0,t]}(s) = \tilde{c}_t^\theta\, \beta\, \mathrm{e}^{-2\beta\theta(t-s)}\, \mathbf{1}_{[0,t]}(s)$$

that for a non-negative measurable function F'' defined on $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ (or $\mathbb{T}_{\mathrm{dis}}^{(n)}$):

$$\mathbb{E}^{\theta,t}\left[2\beta \int_{\mathbf{T}_{n-1}} \mathscr{L}(\mathrm{d}x)\, F''\Big(\mathbf{T}_{n-1} \circledast_x [0, t - H(x)]\Big)\, \mathrm{e}^{-2\beta\theta(t - H(x))}\right] = (\tilde{c}_t^\theta)^{-1}\, n\, \mathbb{E}^{\theta,t}\left[F''(\mathbf{T}_n)\right].$$

Using similar equations as (8.3), (8.4), (8.5) and (8.6) stated with $\mathbf{T}_n$ instead of $(\mathcal{T}, \mathbf{v}_n)$ as well as an obvious choice of F'', we obtain that:

$$G_n = (\tilde{c}_t^\theta)^{-1}\, n\, \mathbb{E}^{\theta,t}\left[F'(\mathbf{T}_n)\right],$$

where $F'(\mathbf{T}_n)$ is given by (8.2). Then, we deduce from (8.1) that:

$$G_n = (\tilde{c}_t^\theta)^{-1}\, n\, \mathbb{E}^{\theta,t}\Big[F\big(\mathrm{Graft}_n(\mathbf{T}_n, \mathcal{T}^*)\big)\Big].$$

This gives:

$$\begin{aligned} \mathbb{N}^\theta\left[\int_{\mathcal{T}^n} \Lambda_t^{\otimes n}(\mathrm{d}\mathbf{v}_n^*)\, F(\mathcal{T}, \mathbf{v})\right] = F_n &= (n-1)!(\tilde{c}_t^\theta)^{2-n}\, \mathrm{e}^{-2\beta\theta t}\, G_n \\ &= n!\, \left(\tilde{c}_t^\theta\right)^{1-n} \mathrm{e}^{-2\beta\theta t}\, \mathbb{E}^{\theta,t}\Big[F\big(\mathrm{Graft}_n(\mathbf{T}_n, \mathcal{T}^*)\big)\Big]. \end{aligned}$$

Thus, Equation (5.31) holds for the functionals F we considered.

Recall that $\mathbb{T}_{\mathrm{loc-K}}^{(n),0,*}$ is the Borel subset of $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$ of the trees such that the root is not a branching vertex and the distinguished vertices (but the root) are distinct from the root. The map:

$$(T, \mathbf{v}) \mapsto \Big(\mathrm{Span}(T, \mathbf{v}), \big(\mathcal{M}_A[T, \mathbf{v}], A \in \mathcal{P}_n^+\big)\Big)$$

defined on $\mathbb{T}_{\mathrm{loc-K}}^{(n),0,*}$ is one-to-one onto its image and bi-measurable, see Corollary 6.35. Furthermore the set $\mathbb{T}_{\mathrm{loc-K}}^{(n),0,*}$ is of full measure with respect to the distribution of $(\mathcal{T}, \mathbf{v})$

under $\mathbb{N}^\theta[\mathrm{d}\mathcal{T}]\,\Lambda_t^{\otimes n}(d\mathbf{v}^*)$, with $\mathbf{v} = (\varrho, \mathbf{v}^*)$, as $\mathbb{N}^\theta$-a.e. the root of $\mathcal{T}$ is not a branching vertex. Thus, $(\mathcal{T}, \mathbf{v})$ is a measurable function of $\left(\mathcal{T}^{[n]}, \left(\hat{\mathcal{M}}_A^{[n]}, A \in \mathcal{P}_n^+\right)\right)$. We then conclude by the monotone class theorem that Equation (5.31) holds for any non-negative measurable function F defined on $\mathbb{T}_{\mathrm{loc-K}}^{(n)}$.

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Index of notation

Trees and pointed trees

- T, $\mathbf{t}$, $\mathbf{T}$, $\mathcal{T}$: generic notations for trees (or class of equiv. trees).
- d: generic distance on a tree.
- ϱ: generic notation for the root of trees.
- $H(x) = d(\varrho, x)$: height of the vertex x.
- $H(T)$: height of the tree T.
- T_x: subtree of T above the vertex $x \in T$.
- $[\![x, y]\!]$: the branch joining the vertices x to y.
- T_0: the rooted tree reduced to its root.
- T_1: the rooted infinite branch.
- $\mathscr{L}$ or $\mathscr{L}^T$: length measure on the tree T.
- $\mathbf{v} = (v_0 = \varrho, v_1, \dots, v_n)$: generic notation for distinguished vertices of a tree.
- $(T, \mathbf{v})$ a (or a class of equiv. of) rooted n-pointed tree.
- $(T, S) = (T, S, d, \varrho)$ a (or a class of equiv. of) marked tree with $\varrho \in S \subset T$.

Grafting a tree on a tree

- $(T \circledast_i T', \mathbf{v} \circledast \mathbf{v}')$, also denoted by $T \circledast_i T'$, is the tree obtained by grafting T' on T at the distinguished vertex $v_i \in T$ and identifying the root ϱ' of T' with v_i. The distinguished vertices $\mathbf{v} \circledast \mathbf{v}'$ are the concatenation of the distinguished vertices $\mathbf{v}$ of T and the distinguished vertices $\mathbf{v}'$ (but for the root) of T'.
- $T \circledast_{i,h} T'$, is the tree obtained by grafting T' on T at level h on the branch $[\![\varrho, v_i]\!]$.
- $T \circledast_{i,h}^{\epsilon} T'$, with $\epsilon \in \{\mathrm{g}, \mathrm{d}\}$, same as above but for the distinguished vertices of T' which are inserted on the left (if $\epsilon = \mathrm{g}$) or on the right of v_i (if $\epsilon = \mathrm{d}$).

Spanning and truncation

- $\mathrm{Span}^\circ(T, \mathbf{v})$: the discrete rooted subtree of T spanned by the distinguished vertices $\mathbf{v}$.
- $\mathrm{Span}(T, \mathbf{v})$: the rooted tree $(\mathrm{Span}^\circ(T, \mathbf{v}), \mathbf{v})$ with the distinguished vertices $\mathbf{v}$.
- The map Π_n° removes the distinguished vertices (but the root) from an n-pointed tree: $\Pi_n^\circ(T, \mathbf{v}) = (T, \varrho)$. Thus:

$$\Pi_n^\circ(\mathrm{Span}(T, \mathbf{v})) = \mathrm{Span}^\circ(T, \mathbf{v}).$$

- $r_t(T, \mathbf{v})$: the tree T truncated at level t with the spanned tree $\mathrm{Span}^\circ(T, \mathbf{v})$, and the distinguished vertices $\mathbf{v}$.
- $r_t^{[2]}$, $r_t^{[2],+}$, $r_t^{[2],-}$, $r_*^{[2]}$, $\tilde{r}_t^{[2],+}$: various truncation on marked trees (see Sect. 6.4 and 6.5).

Splitting and grafting

- $\mathbf{L}_n(T, \mathbf{v})$ record the lengths of all the branches of the subtree $\mathrm{Span}(T, \mathbf{v})$ spanned by the n distinguished vertices:

$$\mathbf{L}_n(T, \mathbf{v}) = (\ell_A(T, \mathbf{v}), A \in \mathcal{P}_n^+),$$

with $\mathcal{P}_n^+$ the set of all subsets $A \subset \{1, \dots, n\}$ such that $A \neq \emptyset$.

- $\mathrm{Split}_n(T, \mathbf{v})$ record the subtrees of T associated to all the branches of $\mathrm{Span}(T, bv)$:

$$\mathrm{Split}_n(T, \mathbf{v}) = \left(\hat{T}_A(T, \mathbf{v}), A \in \mathcal{P}_n\right) \quad (8.8)$$

with $\mathcal{P}_n = \mathcal{P}_n^+ \cup \{\{0\}\}$.

- $\mathrm{Graft}_n(T',(T^*_A, A \in \mathcal{P}_n^+))$: replace the branches labeled by A, of the discrete n-pointed tree T' by the trees T^*_A with a marked infinite branch cut at the length $\ell_A(T,\mathbf{v})$. (The discrete tree $(T',\mathbf{v}')$ can be coded/replaced by $\mathbf{L}_n(T',\mathbf{v}')$.)

- Intuitively, we have for $(T,\mathbf{v})$ a n-pointed tree whose root is not a branching vertex (see (6.36)):

$$(T,\mathbf{v}) = \mathrm{Graft}_n\Big(\mathrm{Span}_n(T,\mathbf{v}), \mathrm{Split}_n(T,\mathbf{v})\Big).$$

Set of (equiv. classes of) trees

- $\mathbb{T}_\mathrm{K}$ set of (equiv. classes of) rooted compact trees.

- $\mathbb{T}^{(n)}_\mathrm{K}$ set of (equiv. classes of) rooted n-pointed compact trees; $\mathbb{T}^{(0)}_\mathrm{K} = \mathbb{T}_\mathrm{K}$.

- $d^{(n)}_\mathrm{GH}$ the distance on $\mathbb{T}^{(n)}_\mathrm{K}$; $d^{(0)}_\mathrm{GH} \equiv d_\mathrm{GH}$.

- $\mathbb{T}_{\mathrm{loc-K}}$ set of (equiv. classes of) rooted complete locally compact trees.

- $\mathbb{T}^*_{\mathrm{loc-K}} = \mathbb{T}_{\mathrm{loc-K}} \backslash \{\mathrm{T}_0\}$.

- $\mathbb{T}^0_{\mathrm{loc-K}}$ subset of $\mathbb{T}_{\mathrm{loc-K}}$ of trees whose root is not a branching vertex.

- $\mathbb{T}^{0,*}_{\mathrm{loc-K}} = \mathbb{T}^0_{\mathrm{loc-K}} \cap \mathbb{T}^*_{\mathrm{loc-K}}$.

- $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$ set of (equiv. classes of) rooted n-pointed complete locally compact trees; $\mathbb{T}^{(0)}_{\mathrm{loc-K}} = \mathbb{T}_{\mathrm{loc-K}}$.

- $d^{(n)}_\mathrm{LGH}$ the distance on $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$; $d^{(0)}_\mathrm{LGH} \equiv d_\mathrm{LGH}$.

- $\mathbb{T}^{(n),0}_{\mathrm{loc-K}}$ subset of $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$ of trees whose root is not a branching vertex.

- $\mathbb{T}^{(n),*}_{\mathrm{loc-K}}$ subset of $\mathbb{T}^{(n)}_{\mathrm{loc-K}}$ of trees whose all distinguished vertices (but the root) are distinct from the root.

- $\mathbb{T}^{(n),0,*}_{\mathrm{loc-K}} = \mathbb{T}^{(n),0}_{\mathrm{loc-K}} \cap \mathbb{T}^{(n),*}_{\mathrm{loc-K}}$.

- $\mathbb{T}^{(n)}_\mathrm{dis}$ subset of $\mathbb{T}^{(n)}_\mathrm{K} \subset \mathbb{T}^{(n)}_{\mathrm{loc-K}}$ of discrete trees.

- $\mathbb{T}^{[2]}_{\mathrm{loc-K}}$ set of (equiv. classes of) rooted complete locally compact marked trees.

- $\mathbb{T}^{\mathrm{spine}}_{\mathrm{loc-K}}$ subset of $\mathbb{T}^{[2]}_{\mathrm{loc-K}}$ of marked trees (T,S) such that $S = \mathrm{T}_1$, with T_1 the infinite branch.

Trees with a marked branch and point measures

- $E = \mathbb{R}_+ \times \mathbb{T}^*_{\mathrm{loc-K}}$.

- $\mathbb{M}(E)$ set of point measures on E which are bounded on bounded sets of E.

- $\mathrm{Tree} : \mathbb{M}(E) \to \mathbb{T}^{\mathrm{spine}}_{\mathrm{loc-K}}$ maps the measure $\mathcal{M} = \sum_{i\in I} \delta_{h_i,T_i}$ to the marked tree (T,T_1), with the rooted tree T obtained by grafting the trees T_i on the rooted infinite branch T_1 at level h_i.

- $\boldsymbol{\mathcal{M}} : \mathbb{T}^{\mathrm{spine}}_{\mathrm{loc-K}} \to \mathbb{M}(E)$ maps the marked tree (T,T_1) to the measure $\sum_{i\in I} \delta_{h_i,T_i}$ where $T_i \backslash \{\varrho_i\}$ are the connected component of $T \backslash \mathrm{T}_1$ with root $\varrho_i \in \mathrm{T}_1$ and $h_i = d(\varrho, \varrho_i)$, where ϱ is the common root of T and T_1.

- $\boldsymbol{\mathcal{M}}$ is also defined on $\mathbb{T}^{(1)}_{\mathrm{loc-K}}$.

Reconstruction results

- With Id the identity map:

$$\mathrm{Tree} \circ \boldsymbol{\mathcal{M}} = \mathrm{Id} \quad \text{on } \mathbb{T}^{\mathrm{spine}}_{\mathrm{loc-K}},$$
$$\boldsymbol{\mathcal{M}} \circ \mathrm{Tree} = \mathrm{Id} \quad \text{on } \tilde{\mathbb{M}}(E) = \mathrm{Im}(\boldsymbol{\mathcal{M}}).$$

- $(T,\mathbf{v}) \in \mathbb{T}^{(1),0,*}_{\mathrm{loc-K}}$ can be recovered in a measurable way from $(d(\varrho,v), \boldsymbol{\mathcal{M}}(T,\mathbf{v}))$.

- $(T,\mathbf{v}) \in \mathbb{T}^{(n),0,*}_{\mathrm{loc-K}}$ can be recovered in a measurable way from $(\mathrm{Span}_n(T,\mathbf{v}), (\mathcal{M}_A[T,\mathbf{v}], A \in \mathcal{P}_n^+))$, where $\mathcal{M}_A[T,\mathbf{v}] = \boldsymbol{\mathcal{M}}(\hat{T}_A(T,\mathbf{v}))$, with $\hat{T}_A(T,\mathbf{v}) \in \mathbb{T}^{(1)}_{\mathrm{loc-K}}$ defined by the splitting operation in (8.8).

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