APPROXIMATION PROPERTIES FOR COSET SPACES AND THEIR OPERATOR ALGEBRAS

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Abstract. We give a review of the notions of amenability, Haagerup property and weak amenability for coset spaces Γ/Λ when Λ is an almost normal subgroup of a discrete group Γ and we discuss the behaviour of the commutant of the corresponding quasi-regular representation.

Introduction

It is a classical fact that many properties of a discrete group are easily translated into equivalent properties of its reduced C^*-algebra or of its von Neumann algebra. This applies in particular to the group Γ/Λ whenever Λ is a normal subgroup of a discrete group Γ. We want to discuss here the case of coset spaces Γ/Λ where Λ is not necessarily normal.

Amenability is the first notion that was studied in this context and is due to Eymard [16]. It extends the notion of amenability of the group Γ/Λ in case Λ is normal but, however, this notion does not entirely behave as expected. For instance, if Γ_1 is a subgroup of Γ containing Λ, the amenability of Γ/Λ does not always imply the amenability of Γ_1/Λ [27, 29]. The amenability of Γ/Λ has nothing to do with the injectivity of the von Neumann algebra $\lambda_{\Gamma/\Lambda}(\Gamma)''$, where $\lambda_{\Gamma/\Lambda}$ is the quasi-regular representation of Γ on $\ell^2(\Gamma/\Lambda)$ (see Examples 3.2 and 3.3 below).

When Λ is a normal subgroup, $\lambda_{\Gamma/\Lambda}(\Gamma)''$ is the left von Neumann algebra $\mathbf{L}(\Gamma/\Lambda)$ of the group Γ/Λ and $\lambda_{\Gamma/\Lambda}(\Gamma)'$ is its right von Neumann algebra $\mathbf{R}(\Gamma/\Lambda)$, and these two algebras are isomorphic. This is no longer true in general. As observed by many authors, beginning with Mackey [26], in the general situation, the algebra $\lambda_{\Gamma/\Lambda}(\Gamma)''$, or rather its commutant $\lambda_{\Gamma/\Lambda}(\Gamma)'$, is more directly linked with the coset space $\mathcal{C}_\Gamma(\Lambda)/\Lambda$ than to Γ/Λ, where $\mathcal{C}_\Gamma(\Lambda)$ is the commensurator of Λ in Γ (see [13, 3, 2]). We show that $\lambda_{\Gamma/\Lambda}(\Gamma)'$ is injective when $\mathcal{C}_\Gamma(\Lambda)/\Lambda$ is amenable (Proposition 3.5), although the converse is not true (Example 3.3). The amenability of $\mathcal{C}_\Gamma(\Lambda)/\Lambda$ is not implied by that of Γ/Λ (Example 3.2). It so appears that the interesting situation is when Λ is almost normal in Γ, that is, $\Gamma = \mathcal{C}_\Gamma(\Lambda)$. One also says that (Γ, Λ) is a Hecke pair. Amenability passes to sub-coset spaces Γ_1/Λ in this case. This is studied in Section 3.

The analogue for coset spaces of the Haagerup approximation property for groups was introduced in [4] and also discussed in [30, Remark 3.5]. It makes sense only when Λ is

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an almost normal subgroup of Γ. It is studied in Section 4 where we show in particular that it implies that $\lambda_{\Gamma/\Lambda}(\Gamma)'$ has the Haagerup approximation property with respect to its canonical state ω. It is important to have in mind that, in contrast with the case where Λ is a normal subgroup, in general ω is not a trace, and its modular automorphism group can even exhibit very interesting phase transitions with spontaneous symmetry breaking as shown in [5].

In Section 5 we introduce and study the notion of weak amenability for Hecke pairs. This answers a question raised by Quanhua Xu during the 24th International Conference of Operator Theory in Timisoara (July 2012).

Property (T) for coset spaces is briefly discussed in [30, Remarks 5.6]. Its definition does not require that Λ is an almost normal subgroup. Although it is not an approximation property, we discuss this notion in Section 6 because it gives an obstruction for the existence of almost normal subgroups that yield coset spaces with the Haagerup property.

Finally, the last section is devoted to examples and open problems.

The main tool for the study of notions related to an almost normal subgroup Λ of Γ is the Schlichting completion (G', H') of (Γ, Λ). It originates in works of G. Schlichting at the end of the seventies, and was developed by Tzanev [34] (see also [25]). Its advantage is to reduce the study of the pair (Γ, Λ) to that of (G', H'), whose main feature is that H' is a compact open subgroup of G'. This construction is recalled in Section 2. We observe in this text that the coset space Γ/Λ has one of the four above mentioned properties if and only if its associated totally disconnected locally compact group G' has the corresponding property.

The first section is not new either, but is developed for the reader's convenience and to fix notations. It shows that the commutant $\lambda_{\Gamma/\Lambda}(\Gamma)'$ of the quasi-regular representation is generated, as a von Neumann algebra on $\ell^2(\Gamma/\Lambda)$, by the Hecke algebra of the Hecke pair $(\mathcal{C}_\Gamma(\Lambda), \Lambda)$, when Λ is any subgroup of Γ. This dates back to the paper [26] of Mackey. When Λ is almost normal, $\lambda_{\Gamma/\Lambda}(\Gamma)'$ is called the (right) Hecke von Neumann algebra of (Γ, Λ), while the norm closure of the Hecke algebra is called its (right) Hecke reduced C^*-algebra.

Although we have added several new contributions, this paper is mainly expository in nature. The author's aim is to draw the reader's attention to these Hecke operator algebras associated with various kinds of coset spaces (not only the amenable ones), since they provide lot of interesting examples that deserve to be studied. In particular, by [33, Théorème 1.25] they provide examples of every type of infinite dimensional von Neumann factor, including type III_λ, $\lambda \in [0, 1]$.

Notations and conventions. In this text, Γ will be a discrete group and Λ a subgroup of Γ. On the other hand, capital Roman letters such as G, H, L will denote general Hausdorff locally compact groups.

Let H be a closed subgroup of G. Then G/H is the locally compact space of left cosets and $\langle G/H \rangle$ will denote a set of representatives of G/H. Similarly we introduce the notation $\langle H \setminus G\rangle$ and $\langle H \setminus G/H\rangle$ for sets of representatives of the spaces $H \setminus G$ and $H \setminus G/H$ of right and double cosets respectively. Elements of G/H will be denoted gH or $\dot{g}$. The H-orbit of gH in G/H as well as the corresponding subset of G are written HgH.

Given a right H-invariant function f on G, we shall denote by $\tilde{f}$ the function on G/H obtained by passing to the quotient. On the other hand, we shall sometimes identify a function ξ on G/H with the corresponding right H-invariant function on G. This means in particular that we may write $\xi(g)$ instead of $\xi(\dot{g})$ for simplicity.

The characteristic function of a subset E of G is as usual denoted by $\mathbf{1}_E$. For $g \in G$, the function $\mathbf{1}_{gH}$, when viewed as a function on G/H, will sometimes be written $\delta_{\dot{g}}$.

1. The commutant of a quasi-regular representation

Recall that the *quasi-regular representation* $\lambda_{\Gamma/\Lambda}$ is defined by

$$\lambda_{\Gamma/\Lambda}(g)\xi(\dot{k}) = \xi(g^{-1}\dot{k})$$

for $g, k \in \Gamma$ and $\xi \in \ell^2(\Gamma/\Lambda)$.

Let $T \in \lambda_{\Gamma/\Lambda}(\Gamma)'$. This operator is completely determined by the function

$$\widehat{T} : g \mapsto T(\delta_{\dot{e}})(\dot{g}),$$

where e is the unit of Γ. Indeed, since T commutes with the quasi-regular representation we get, for $g, k \in \Gamma$,

$$T\delta_{\dot{k}} = \sum_{g\in\langle\Gamma/\Lambda\rangle} \widehat{T}(k^{-1}g)\delta_{\dot{g}}. \tag{1.1}$$

We immediately see that $\widehat{T}$ is Λ-bi-invariant: the right invariance is obvious and the left invariance follows from the fact that the operator T commutes with the quasi-regular representation.

We have $\sum_{g\in\langle\Gamma/\Lambda\rangle} \left|\widehat{T}(g)\right|^2 < +\infty$, and so $\widehat{T}(g) = 0$ whenever the Λ-orbit of $g\Lambda$ in Γ/Λ is infinite. Since

$$\widehat{T^*}(g) = (T^*\delta_{\dot{e}})(\dot{g}) = \overline{\widehat{T}(g^{-1})},$$

we also see that $\widehat{T}(g) = 0$ whenever the Λ-orbit of $g^{-1}\Lambda$ in Γ/Λ is infinite. This implies that the support of $\widehat{T}$ is contained in the commensurator of Λ in Γ, whose definition we recall now.

1.1. Commensurators and Hecke pairs.

Definition 1.1. Let H be a subgroup of a group G. The *commensurator*[1] *of H in G* is the set of $g \in G$ such that the subgroup $H \cap gHg^{-1}$ has a finite index in H and gHg^{-1}. We denote it by $\mathcal{C}_G(H)$.

[1]The name of quasi-normalizer is also used.

Since the map $h \mapsto hgH$ induces a bijection from $H/(H \cap gHg^{-1})$ onto the orbit HgH of gH in G/H, we see $H \cap gHg^{-1}$ has a finite index in H if and only if this orbit is finite, that is, HgH is a finite union of left cosets. It follows that $\mathcal{C}_G(H)$ is a subgroup of G which obviously contains the *normalizer* $\mathcal{N}_G(H)$ of H in G.

Given $g \in \mathcal{C}_H(G)$, we denote by $L(g)$ the cardinal of the H-orbit of gH and we set $R(g) = L(g^{-1})$. The function $g \mapsto L(g)/R(g)$ is a group homomorphism from $\mathcal{C}_G(H)$ into $\mathbb{Q}_+^*$ (see [34, Proposition 2.1]).

We say that H is *almost normal in* G, or that (G, H) is a *Hecke pair*, if $\mathcal{C}_G(H) = G$.

Examples 1.2. Examples of Hecke pairs (G, H) are plentiful. Let us begin by two trivial ones:

(a) H is a normal subgroup of G. Here $L(g) = R(g) = 1$ for every $g \in G$;
(b) H is either a finite subgroup or a subgroup of finite index of G.

As we shall see in Section 2, the next example is the prototype of Hecke pairs:

(c) H is a compact open subgroup of a locally compact group G (*e.g.* $G = SL_n(\mathbb{Q}_p)$ and $H = SL_n(\mathbb{Z}_p)$, where $\mathbb{Z}_p$ is the ring of p-adic integers for a given prime number p).

Classical examples of Hecke pairs (Γ, Λ) made of countable groups are the following:

(d) $(\Gamma = SL_n(\mathbb{Z}[1/p]), \Lambda = SL_n(\mathbb{Z}))$;
(e) $(\Gamma = SL_n(\mathbb{Q}), \Lambda = SL_n(\mathbb{Z}))$;
(f) $(\Gamma = \mathbb{Q} \rtimes \mathbb{Q}_+^*, \Lambda = \mathbb{Z} \rtimes \{1\})$;
(g) $(\Gamma = BS(m,n) = \langle t, x : t^{-1}x^m t = x^n \rangle, \Lambda = \langle x \rangle)$.

Here, $BS(m;n)$ is the Baumslag-Solitar group with parameters m, n. Example (f) is the famous example of Bost and Connes from which they constructed a dynamical system with spontaneous symmetry breaking.

Remark 1.3. An easy way to construct examples is as follows. Assume that Γ acts by isometries on a locally finite metric space[2] (*e.g.*, by automorphisms of a locally finite connected graph without loops and multiple edges, when considering the minimal path length metric). Then, obviously the stabilizer Λ of a vertex is almost normal in Γ. In fact this is the most general way to construct almost normal subgroups (see [1, Theorem 2.15]). For instance, the case of the Baumslag-Solitar group, which is an HNN-extension, can be established by considering its action on the corresponding Bass-Serre tree (see Examples 2.5 for details).

1.2. **Hecke algebras and the commutant of the quasi-regular representation.** Let Λ be any subgroup of a group Γ. We denote by $\mathbb{C}[\Gamma; \Lambda]$ the linear span of the set $\{\mathbf{1}_{\Lambda g \Lambda} : g \in \mathcal{C}_\Gamma(\Lambda)\}$. It is the space of Λ-bi-invariant functions on Γ which are finitely supported on $\mathcal{C}_\Gamma(\Lambda)/\Lambda$ (and equivalently on $\Lambda \setminus \mathcal{C}_\Gamma(\Lambda)/\Lambda$) after passing to the quotient[3]. It is easy to guess from the observations preceding Definition 1.1 that this space plays an

[2] A metric space is *locally finite* if its balls have a finite number of elements.

[3] Note that $\mathbb{C}[\Gamma; \Lambda] = \mathbb{C}[\mathcal{C}_\Gamma(\Lambda); \Lambda]$. When Λ is almost normal in Γ, the notation $\mathcal{H}(\Gamma, \Lambda)$ is often used instead of $\mathbb{C}[\Gamma; \Lambda]$.

important role in the study of $\lambda_{\Gamma/\Lambda}(\Gamma)'$. We first recall that it has a natural structure of involutive algebra, called the *Hecke algebra* of the Hecke pair $(\mathcal{C}_\Gamma(\Lambda),\Lambda)$, when equipped with the convolution product

$$(f_1 * f_2)(g) = \sum_{g_1\in\langle\Gamma/\Lambda\rangle} f_1(g_1)f_2(g_1^{-1}g) = \sum_{g_2\in\langle\Lambda\backslash\Gamma\rangle} f_1(gg_2^{-1})f_2(g_2),$$

and the involution

$$f^*(g) = \overline{f(g^{-1})}$$

(see [22, Section 2.2] for detailed proofs).

Whenever Λ is a normal subgroup of $\mathcal{C}_\Gamma(\Lambda)$, the corresponding Hecke algebra $\mathbb{C}[\Gamma;\Lambda]$ is the goup algebra $\mathbb{C}[\mathcal{C}_\Gamma(\Lambda)/\Lambda]$ of the group $\mathcal{C}_\Gamma(\Lambda)/\Lambda$ and we have

$$\mathbf{1}_{\Lambda g\Lambda} * \mathbf{1}_{\Lambda h\Lambda} = \mathbf{1}_{\Lambda gh\Lambda}$$

for $g,h\in\mathcal{C}_\Gamma(\Lambda)$. In general, we get the more complicated formula

$$\mathbf{1}_{\Lambda g\Lambda} * \mathbf{1}_{\Lambda h\Lambda} = \sum_{g_i\Lambda\subset\Lambda g\Lambda,\, h_j\subset\Lambda h\Lambda} \frac{1}{L(g_ih_j)}\mathbf{1}_{\Lambda g_ih_j\Lambda},$$

where $g_i\Lambda\subset\Lambda g\Lambda$ means that g_i runs over some $L(g)$ representatives of left Λ-cosets included into $\Lambda g\Lambda$ (see [22, page 15] or [34]).

We define a representation $\mathbf{R}$ of the opposite algebra of $\mathbb{C}[\Gamma;\Lambda]$ on $\ell^2(\Gamma/\Lambda)$ by the formula

$$\mathbf{R}(f)\xi(\dot g) = \sum_{k\in\langle\Gamma/\Lambda\rangle} \xi(\dot k)f(k^{-1}g)$$

for $f\in\mathbb{C}[\Gamma;\Lambda]$ and $\xi\in\ell^2(\Gamma/\Lambda)$.

To check that $\mathbf{R}(f)$ is a bounded operator, it suffices to consider $f=\mathbf{1}_{\Lambda g\Lambda}$ with $g\in\mathcal{C}_\Gamma(\Lambda)$. We observe that the operator $\mathbf{R}(f)$ is represented, in the canonical basis of $\ell^2(\Gamma/\Lambda)$, by the matrix $[T_{\dot h,\dot k}]$ with $T_{\dot h,\dot k}=\mathbf{1}_{\Lambda g\Lambda}(k^{-1}h)$.

We have $\sup_h\sum_{k\in\langle\Gamma/\Lambda\rangle}\left|T_{\dot h,\dot k}\right|\le R(g)$ and $\sup_k\sum_{h\in\langle\Gamma/\Lambda\rangle}\left|T_{\dot h,\dot k}\right|\le L(g)$. It follows that the matrix $[T_{\dot h,\dot k}]$ defines a bounded operator from $\ell^\infty(\Gamma/\Lambda)$ into itself with $\left\|[T_{\dot h,\dot k}]\right\|_{\ell^\infty\to\ell^\infty}\le R(g)$, and that similarly $\left\|[T_{\dot h,\dot k}]\right\|_{\ell^1\to\ell^1}\le L(g)$. By the Riesz-Thorin interpolation theorem we get $\|\mathbf{R}(f)\|\le L(g)^{1/2}R(g)^{1/2}$.

Straightforward computations show that $\mathbf{R}(f_1*f_2)=\mathbf{R}(f_2)\mathbf{R}(f_1)$ and $\mathbf{R}(f^*)=\mathbf{R}(f)^*$.

We are now ready to show the following result ([3, Theorem 2.2]).

Theorem 1.4. *The commutant $\lambda_{\Gamma/\Lambda}(\Gamma)'$ of the quasi-regular representation is the weak closure N of* $\mathbf{R}(\mathbb{C}[\Gamma;\Lambda])$.

Proof. Obviously, $\mathbf{R}(\mathbb{C}[\Gamma;\Lambda])$ commutes with $\lambda_{\Gamma/\Lambda}(\Gamma)$. It remains to prove the inclusion $\lambda_{\Gamma/\Lambda}(\Gamma)'\subset N$. Let $T\in\lambda_{\Gamma/\Lambda}(\Gamma)'$. Recall that for $y\in\Gamma$ we have

$$\sum_{g\in\langle\Gamma/\Lambda\rangle}\left|\widehat{T}(g)\right|^2 = \sum_{g\in\langle\Gamma/\Lambda\rangle}\left|\widehat{T}(y^{-1}g)\right|^2 = \|T\delta_{\dot y}\|_2^2 < +\infty,$$

and after replacing T by T^*,

$$\sum_{g\in\langle\Gamma/\Lambda\rangle} \left|\widehat{T}(g^{-1}y)\right|^2 < +\infty.$$

For $\xi \in \ell^2(\Gamma/\Lambda)$, we have

$$T\xi = \sum_{k\in\langle\Gamma/\Lambda\rangle} \xi(\dot{k}) \sum_{g\in\langle\Gamma/\Lambda\rangle} \widehat{T}(k^{-1}g)\delta_{\dot{g}},$$

so

$$T\xi(\dot{g}) = \sum_{k\in\langle\Gamma/\Lambda\rangle} \xi(\dot{k})\widehat{T}(k^{-1}g)$$

where the series is absolutely convergent.

Write $\mathcal{C}_\Gamma(\Lambda)$ as the union of an increasing net K_i where each K_i is a finite union of double Λ-cosets of elements of $\mathcal{C}_\Gamma(\Lambda)$. Set $T_i = \mathbf{1}_{K_i}T \in \mathbb{C}[\Gamma;\Lambda]$. If follows from the above observations that $\lim_i \left\|(T - \mathbf{R}(T_i))\delta_{\dot{y}}\right\|_2 = 0$ for every $y \in \Gamma$ and that $\lim_i T\xi(\dot{g}) - \mathbf{R}(T_i)\xi(\dot{g}) = 0$ for every $\xi \in \ell^2(\Gamma/\Lambda)$ and $g \in \Gamma$. Let $S \in N'$. Then

$$\lim_i S\mathbf{R}(T_i)\delta_{\dot{y}} = ST\delta_{\dot{y}} \quad \text{in} \quad \|\cdot\|_2\text{-norm},$$

and

$$\lim_i S\mathbf{R}(T_i)\delta_{\dot{y}} = \lim_i \mathbf{R}(T_i)S\delta_{\dot{y}} = TS\delta_{\dot{y}} \quad \text{pointwise.}$$

Therefore, $T \in N'' = N$. □

So, $\mathbf{R}(\mathbb{C}[\Gamma;\Lambda])$ is weaky dense in the commutant of $\lambda_{\Gamma/\Lambda}(\Gamma)$. The algebra N is called the (right) *von Neumann algebra of the pair* (Γ,Λ) and is also written $\mathbf{R}(\Gamma,\Lambda)$. The norm closure $\mathbf{C}^*_\rho(\Gamma,\Lambda)$ of $\mathbf{R}(\mathbb{C}[\Gamma;\Lambda])$ is called the (right) *reduced C^*-algebra* of this pair. When (Γ,Λ) is a Hecke pair of groups, we shall speak of *Hecke von Neumann algebra* and *Hecke reduced C^*-algebra* respectively. When Λ is a normal subgroup of Γ, they are respectively the right von Neumann algebra $\mathbf{R}(\Gamma/\Lambda)$ and the right C^*-algebra $\mathbf{C}^*_\rho(\Gamma/\Lambda)$ of the group Γ/Λ.

Corollary 1.5 ([26])**.** *The quasi-regular representation $\lambda_{\Gamma/\Lambda}$ is irreducible if and only if $\mathcal{C}_\Gamma(\Lambda) = \Lambda$.*

1.3. **The canonical state on $\mathbf{R}(\mathbb{C}[\Gamma;\Lambda])$.** We shall denote by ω the vector state on $N = \mathbf{R}(\mathbb{C}[\Gamma;\Lambda])$ induced by $\delta_{\dot{e}}$. Note that this state is faithful since $\delta_{\dot{e}}$ is separating for N. However, it is not cyclic when Λ is not a normal subgroup in Γ. We denote by $\mathcal{N}$ the closure of $N\delta_{\dot{e}}$ in $\ell^2(\Gamma/\Lambda)$ and by p the orthogonal projection on $\mathcal{N}$. It is an element of $\lambda_{\Gamma/\Lambda}(\Gamma)''$ whose central support is one. In particular, the map $T \mapsto Tp$ is an isomorphism from N onto the induced von Neumann algebra Np.

It is readily checked that $\mathcal{N}$ is the Hilbert subspace of $\ell^2(\mathcal{C}_\Gamma(\Lambda)/\Lambda)$ consisting in its Λ-invariant vectors, and that $\left\{L(g)^{-1/2}\delta_{\ddot{g}} : g \in \langle\Lambda \setminus \mathcal{C}_\Gamma(\Lambda)/\Lambda\rangle\right\}$ is an orthonormal basis of $\mathcal{N}$, where $\delta_{\ddot{g}}$ denote $\mathbf{1}_{\Lambda g\Lambda}$ when viewed as a function on Γ/Λ.

We also observe that $T \mapsto T\delta_{\dot e}$ induces an isometry from $L^2(N,\omega)$ onto $\mathcal{N}$: for $f_1, f_2 \in \mathbb{C}[\Gamma;\Lambda]$ we have
$$\langle \mathbf{R}(f_1), \mathbf{R}(f_2)\rangle_{L^2(N,\omega)} = \langle f_1, f_2\rangle_{\ell^2(\Gamma/\Lambda)}.$$
Thus, N is in standard form on $\mathcal{N}$. We note that, in contrast with the case where Λ is a normal subgroup of Γ, in general ω is not a trace on N. In [2], Binder proved that ω is a trace if and only if $L(g) = L(g^{-1})$ for every $g \in \mathcal{C}_\Gamma(\Lambda)$. There (and this was further developed in [5]) the modular automorphism group relative to the state ω on N or $\mathbf{C}^*_\rho(\Gamma,\Lambda)$ was identified as follows. Set $\Delta(g) = L(g)/R(g)$ and view this Λ-bi-invariant function Δ as the (unbounded) multiplication operator on $\mathcal{N}$, defined by
$$\Delta(\delta_{\dot g}) = \Delta(g)\delta_{\dot g}, \quad \forall g \in \mathcal{C}_\Gamma(\Lambda).$$
Then, for $T \in N = \mathbf{R}(\Gamma,\Lambda)$ we have $\sigma_t(T) = \Delta^{-it}T\Delta^{it}$ (see also [34]). Example 1.2 (f) has bee studied in detail in [5]. In particular, N is the type III_1 hyperfinite factor in this case.

Remark 1.6. As made obvious above, the commutant of $\lambda_{\Gamma/\Lambda}(\Gamma)$ only depends on the Hecke pair $(\mathcal{C}_\Gamma(\Lambda),\Lambda)$. So it will suffice for our purpose to assume in the rest of the paper that Λ is almost normal in Γ.

2. The Schlichting completion

This completion, which dates back to Schlichting's papers is described in [34]. It is a useful tool in order to understand Hecke pairs. For the reader's convenience, we recall its construction as well as some of its features that we shall need in the sequel.

Let G be a group and H a subgroup. We say that the pair (G,H) is *reduced* if the left action of G on G/H is effective. Replacing G by $G_1 = G/L$ and H by $H_1 = H/L$ where $L = \cap_{g\in G} gHg^{-1}$, we get an effective action of the group G_1 on the coset space G_1/H_1. This latter space is in canonical bijective correspondance with G/H. Clearly, H is almost normal in G if and only if H_1 is almost normal in G_1. There is no loss of generality to assume effectiveness.

Proposition 2.1. *Let Λ be an almost normal subgroup of Γ. There exists a unique (up to isomorphism) triple (G',H',θ) such that the pair (G',H') is reduced and*

(a) *G' is a locally compact group, H' is a compact open subgroup of G';*
(b) *θ is a homomorphism from Γ into G' with $\overline{\theta(\Gamma)} = G'$ and $\theta^{-1}(H') = \Lambda$.*

Proof. We may assume that the pair (Γ,Λ) is reduced, without loss of generality. Then Γ is viewed as a subgroup of the group $\mathrm{Bij}(\Gamma/\Lambda)$ of bijections of Γ/Λ. Let us denote by G' the closure of Γ in the space of all maps from Γ/Λ into itself, equipped with the topology of pointwise convergence.

We first check that G' is contained into $\mathrm{Bij}(\Gamma/\Lambda)$. Obviously, the elements of G' are injective maps, since Γ/Λ is discrete. We now take $f \in G'$ and show that f is surjective. Let (g_i) be a net in Γ such that $f = \lim_i g_i$ in the topology of pointwise convergence. Let i_0 be such that $f(\dot e) = g_i\dot e$ for $i \geq i_0$. We set $g = g_{i_0}$. So $h_i = g^{-1}g_i \in \Lambda$ whenever $i \geq i_0$. Observe that $\lim_i h_i = g^{-1}f$. In particular, $f_1 = g^{-1}f$ is an injective map and for $y \in \Gamma/\Lambda$,

$h \in \Lambda$, we have $f_1(hy) = (g^{-1}f)(hy) = \lim_i h_i hy \subset \Lambda y$. Hence, $f_1(\Lambda y) \subset \Lambda y$. Since Λy is a finite set, it follows that y is in the range of f_1. Thus, we see that $f_1 \in \mathrm{Bij}(\Gamma/\Lambda)$, and $f \in \mathrm{Bij}(\Gamma/\Lambda)$ too.

Now, we want to show that G' is a locally compact group. For $x \in \Gamma/\Lambda$, we set $G'_x = \{g' \in G' : g'(x) = x\}$. It is easily checked that $G'_{\dot{e}}$ is the closure of Λ. Let us prove that Λ is relatively compact. This follows from the fact that Λ is a subset of $\Pi_{y\in\Gamma/\Lambda}\Lambda y$, which is compact, by the Tychonov theorem. Since finite intersections of such stabilizers G'_x form a basis of neighbourhoods of the identity in G', we see that G' is a locally compact group. Note that the G'_x are both compact and open, and so G' is totally disconnected.

We set $H' = G'_{\dot{e}}$ and denote by θ the inclusion of Γ into G'. We immediately see that (G', H', θ) fulfils the required conditions. Note that $H' = \overline{\Lambda}$.

Let (G'_1, H'_1, θ_1) be another triple with the same properties. Observe that $G'_1 = \theta_1(\Gamma)H'_1$ and $G' = \theta(\Gamma)\overline{\Lambda}$. Moreover, the map $\alpha : \Gamma/\Lambda \to G'_1/H'_1$ sending $\dot{g}$ onto $\theta_1(g)H'_1 \in G'_1/H'_1$ is a bijection, which induces an effective action ι of G'_1 on Γ/Λ. We have $\iota(\theta_1(g)) = \theta(g)$ for $g \in \Gamma$ and so $\iota(G'_1) \subset G'$. Moreover, $\iota(H'_1)$ is contained into the stabilizer of $\dot{e}$, that is $\overline{\Lambda}$. On the other hand, we have $\Lambda \subset \iota(H'_1)$, and therefore $\overline{\Lambda} = \iota(H'_1)$. It follows that ι is an isomorphism from G'_1 onto G'. Moreover it is an homeomorphism because it sends continuously the compact open set H'_1 onto the open set $\overline{\Lambda}$. □

The pair (G', H') is called the *Schlichting completion of* (Γ, Λ). We keep this notation in the rest of the paper. We shall use the following obvious facts.

Lemma 2.2. *The Schlichting completion (G', H') has the following properties:*

(i) $\overline{\theta(\Lambda)} = H'$;
(ii) *$G' = \theta(\Gamma)H'$ and for every $g' \in G'$ there exists a unique left Λ-coset $g\Lambda$ such that $g' = \theta(g)h'$ with $h' \in H'$.*
(iii) *θ induces a bijection $\widetilde{\theta} : g\Gamma \to \theta(g)H'$ from Γ/Λ onto G'/H' such that $\widetilde{\theta}(gg_1\Lambda) = \theta(g)\widetilde{\theta}(g_1\Lambda)$ for $g, g_1 \in \Gamma$, that is, a Γ-equivariant bijection. This induces a unitary operator $U : \ell^2(\Gamma/\Lambda) \to \ell^2(G'/H')$ such that $U\lambda_{\Gamma/\Lambda}(g) = \lambda_{G'/H'}(\theta(g))U$.*
(iv) *$\psi' \mapsto \psi = \psi' \circ \theta$ is a bijection from the space of (continuous) right H'-invariant functions on G' onto the space of right Λ-invariant functions on Γ. The inverse bijection is $\psi \mapsto \psi'$ where $\psi'(g') = \psi(g)$ for $g' = \theta(g)h'$.*

Similarly, for double cosets we shall use the following lemma.

Lemma 2.3. *The map $\Lambda g\Lambda \mapsto H'\theta(g)H'$ is a bijection from the set $\Lambda \setminus \Gamma/\Lambda$ onto $H' \setminus G'/H'$. It follows that $\psi' \mapsto \psi = \psi' \circ \theta$ is a bijection from the space of (continuous) H'-bi-invariant functions on G' onto the space of Λ-bi-invariant functions on Γ. Moreover, ψ' is positive definite on G' if and only if ψ is positive definite on Γ.*

Proof. Obviously the map $\Lambda g\Lambda \mapsto H'\theta(g)H'$ is surjective. Let us show that it is injective. Assume that $H'\theta(g)H' = H'\theta(k)H'$. Then, we have $\theta(g)H'\theta(k)^{-1} \cap H' \neq \emptyset$. Since $\theta(g)H'\theta(k)^{-1}$ is open, it contains some $\theta(h)$ with $h \in \Lambda$ and so $\theta(g)H' = \theta(hk)H'$. By condition (b) of Proposition 2.1, we see that $k^{-1}h^{-1}g \in \Lambda$ and therefore $\Lambda g\Lambda = \Lambda k\Lambda$.

Assume now that ψ is positive definite. We take $\lambda_1, \dots, \lambda_n$ in $\mathbb{C}$ and $g'_1, \dots, g'_n \in G'$ and we write $g'_i = \theta(g_i)h'_i$. Then we have

$$\sum_{1\le i,j\le n} \lambda_i \overline{\lambda_j} \psi'((g'_i)^{-1} g'_j) = \sum_{1\le i,j\le n} \lambda_i \overline{\lambda_j} \psi(g_i^{-1} g_j) \ge 0.$$

The converse is obvious. □

Remark 2.4. One may defines the Hecke algebra of the pair (G', H') exactly as for (Γ, Λ). The bijection of the previous lemma induces an isomorphism of involutive algebras from $\mathbb{C}[G'; H']$ onto $\mathbb{C}[\Gamma; \Lambda]$.

Examples 2.5. If Λ is a normal subgroup of Γ, its Schlichting completion is $(\Gamma/\Lambda, \{\dot{e}\})$.

The Schlichting completion of $(SL(n, \mathbb{Z}[1/p]), SL_n(\mathbb{Z}))$ is $(SL_n(\mathbb{Q}_p), SL_n(\mathbb{Z}_p))$.

The Schlichting completion for Examples 1.2 (e), (f) involves more sophisticated objects from number theory, namely the ring of finite adèles and its subring of integers (see [5, 33]).

Example 1.2 (g) is a particular case of HNN-extension. Consider a group Λ and an isomorphism $\sigma : H \to K$ between two subgroups of Λ. Denote by Γ the HNN-extension relative to (Λ, H, σ), that is, Γ has the following presentation

$$\Gamma = \langle \Lambda, t | t^{-1} h t = \sigma(h), \forall h \in H \rangle.$$

(The group $BS(m, n)$ corresponds to $\Lambda = \mathbb{Z}$, $H = m\mathbb{Z}$, $K = n\mathbb{Z}$ and $\sigma(mz) = nz$ for every $z \in \mathbb{Z}$.)

The group Γ acts transitively, with the obvious action, on its Bass-Serre tree $\mathcal{T}$, which is defined as follows (see [31]): the set $\mathcal{T}^0$ of vertices is Γ/Λ and the set $\mathcal{T}^1$ of edges is $(\Gamma/H) \sqcup (\Gamma/K)$, the sources of gH and gK are $g\Lambda$, their targets are $gt\Lambda$ and $gt^{-1}\Lambda$ respectively. In particular each vertex has degree $[\Lambda : H] + [\Lambda : K]$. So, $\mathcal{T}$ is locally finite if and only if the subgroups H and K have a finite index in Λ. In this case, as observed in Remark 1.3, (Γ, Λ) is a Hecke pair, since Λ is a vertex stabilizer. Then, the Schlichting completion of (Γ, Λ) is described as a particular case of the following result.

Proposition 2.6. *Let Γ be a group acting by isometries on a locally finite metric space X and Λ_x be the stabilizer of an element $x \in X$. We endow the group* $\mathrm{Iso}(X)$ *of isometries of X with the topology of pointwise convergence and denote by $\theta : \Gamma \to \mathrm{Iso}(X)$ the group homomorphism corresponding to the action. The Schlichting completion of (Γ, Λ_x) is the pair $(G/L, H/L)$ where G and H are the closures of $\theta(\Gamma)$ and $\theta(\Lambda_x)$ respectively in* $\mathrm{Iso}(X)$ *and where $L = \cap_{g\in G} gHg^{-1}$.*

Proof. Since X is locally finite, it is known that $\mathrm{Iso}(X)$ is a locally compact group which acts properly on X (see [1] for instance). Therefore the group G defined above acts properly on X, and H, which is easily seen to be the stabilizer of x, is compact. It is open since X is discrete. The pair $(G' = G/L, H' = H/L)$ is reduced and the conditions of Proposition 2.1 are obviously satisfied. □

Remark 2.7. We observe that the group G in the above proposition does not depend on the choice of $x \in X$ and therefore the group G' of the Schlichting completion of (Γ, Λ_x) only depends on x through the quotient of G by a compact normal subgroup.

3. Co-amenability

Co-amenability is the first relative property that was studied for a pair (G, H) where H is a closed subgroup of the locally compact group G. It is due to Eymard [16] and can be defined by several properties he proved to be equivalent. We just recall the following ones.

Theorem 3.1 ([16]). *The following properties are equivalent:*

(i) *there exists a G-invariant mean on $L^\infty(G/H)$, that is, a G-invariant state on $L^\infty(G/H)$;*
(ii) *the trivial representation of G is weakly contained into $\lambda_{G/H}$, that is, for every $\varepsilon > 0$ and every compact subset K of G, there exists a unit function ξ in $L^2(G/H)$ such that $\sup_{s\in K} \left\|\lambda_{G/H}(s)\xi - \xi\right\|_2 \leq \varepsilon$;*
(iii) *the pair (G, H) has the conditional fixed point property, that is, for any compact convex subset Q of a locally convex topological vector space, if G acts continuously and affinely on Q in such a way that there is a H-fixed point, then there also exists a G-fixed point.*

When these properties are satisfied, we say that H is *co-amenable in G*, or that the *coset space G/H is amenable*. Other terminologies found in the literature are that the *pair (G, H) is amenable* or that H is *co-Følner in G*. In case H is reduced to the identity, we say that G is amenable.

When H is a normal subgroup of G, the amenability of the coset space G/H is the same as the amenability of the group G/H. In the discrete case and when Λ is a normal subgroup of Γ, it is well-known that the group Γ/Λ is amenable if and only if the reduced group C^*-algebra of Γ/Λ is nuclear, or equivalently, if and only if its von Neuman algebra is injective. Without the assumption of normality, these algebras are respectively replaced by $\mathbf{C}^*_\rho(\Gamma, \Lambda)$ and $\mathbf{R}(\Gamma, \Lambda)$. We now give examples showing that the amenability of Γ/Λ cannot in general be read on these algebras.

Example 3.2. In [27] (see also [29]), we find the following nice example of co-amenable subgroup. Consider any non-trivial discrete group Q. Set $\Lambda = \oplus_{n\geq 0} Q$, $\Gamma_1 = \oplus_{n\in\mathbb{Z}} Q$, $\Gamma = \Gamma_1 \rtimes \mathbb{Z} = Q \wr \mathbb{Z}$, the wreath product of Q by $\mathbb{Z}$. Monod and Popa proved that Λ is co-amenable in Γ, whatever Q is. As a consequence, one sees that co-amenability is not hereditary in the following sense: Γ/Λ is amenable but Γ_1/Λ is amenable only if Q is amenable. This fact contrasts with the hereditary property that holds when Λ is a normal subgroup of Γ. Here the commensurator of Λ in Γ is Γ_1 and Γ_1/Λ is the group $\oplus_{n<0} Q$. It follows that $\lambda_{\Gamma/\Lambda}(\Gamma)'$, which is isomorphic to the right von Neumann algebra of the group Γ_1/Λ, is isomorphic to the von Neumann tensor product $\overline{\otimes}_{n<0}\mathbf{R}(Q)$, where $\mathbf{R}(Q)$ is the right von Neumann algebra of the group Q. It is injective if and only if this group Q is amenable.

Example 3.3. Consider the Hecke pair $(\Gamma = SL_2(\mathbb{Q}), \Lambda = SL_2(\mathbb{Z}))$. Its Hecke algebra is abelian (see [5]), hence the von Neumann algebra $\lambda_{\Gamma/\Lambda}(\Gamma)''$ is of type I. However, Λ is not co-amenable in Γ. Otherwise its Schlichting completion G' would be an amenable locally compact group (see below). But $G' = SL_2(\mathcal{A})$, where $\mathcal{A}$ is the ring of finite adèles, is not amenable.

Another example of the same kind is the Hecke pair $(\Gamma = SL_n(\mathbb{Z}[1/p]), \Lambda = SL_n(\mathbb{Z}))$. Its Schlichting completion is $(G' = SL_n(\mathbb{Q}_p), H' = SL_n(\mathbb{Z}_p))$, which is a Gelfand pair. It follows that its Hecke pair is abelian (indeed an algebra of polynomials by the Satake isomorphism [22, page 19]), although G' is not amenable.

Proposition 3.4. *Let (Γ, Λ) be a Hecke pair and (G', H') its Schlichting completion. The following conditions are equivalent:*

(i) *Λ is co-amenable in Γ;*
(ii) *H' is co-amenable in G';*
(iii) *G' is amenable.*

Proof. This result is contained in [34, Proposition 5.1]. Let us first recall the proof of (i) $\Rightarrow$ (ii). One uses the characterization of co-amenability in terms of conditional fixed point property. Let $G' \curvearrowright C$ be a continuous affine action on a compact convex set which has a H'-fixed point. Since Λ is co-amenable in Γ we deduce the existence of a $\theta(\Gamma)$-fixed point in C, and since $\theta(\Gamma)$ is dense in G', this point is also G'-fixed. To prove the converse, we observe that if the trivial representation of G' is weakly contained in the quasi-regular representation $\lambda_{G'/H'}$, then it follows immediately from Lemma 2.2 (iii) that the trivial representation of Γ is weakly contained in $\lambda_{\Gamma/\Lambda}$.

The equivalence between (ii) and (iii) is obvious, using the fixed point characterizations and the fact that H' is compact. □

Proposition 3.5. *Let Λ be a subgroup of Γ which is co-amenable in its commensurator. Then $\mathbf{C}^*_\rho(\Gamma, \Lambda)$ is nuclear and $\mathbf{R}(\Gamma, \Lambda)$ is injective, and thus $\lambda_{\Gamma/\Lambda}(\Gamma)''$ is injective.*

As already observed, it suffices to consider the case where Λ is almost normal in Γ. One way to prove this result is to observe that $\mathbf{C}^*_\rho(\Gamma, \Lambda)$ and $\mathbf{R}(\Gamma, \Lambda)$ are corners of $\mathbf{C}^*_\rho(G')$ and $\mathbf{R}(G')$ respectively, where $\mathbf{C}^*_\rho(G')$ is the (right) reduced C^*-algebra of G' and $\mathbf{R}(G')$ its weak closure (see [34]). We give here another proof. For further purposes, we state first an easy fact, more general that what is immediately needed.

Lemma 3.6. *Let Λ be an almost normal subgroup of Γ and let ψ be a Λ-bi-invariant function from Γ to $\mathbb{C}$. We assume that there exist two bounded functions ξ, η from Γ/Λ into a Hilbert space $\mathcal{K}$ such that for $g, h \in \Gamma$ we have $\psi(h^{-1}g) = \left\langle \xi(\dot{g}), \eta(\dot{h}) \right\rangle$. Then there exists a unique normal map $\Psi : \mathbf{R}(\Gamma, \Lambda) \to \mathbf{R}(\Gamma, \Lambda)$ such that $\Psi(\mathbf{R}(f)) = \mathbf{R}(\psi f)$ for every $f \in \mathbb{C}[\Gamma; \Lambda]$. Moreover Ψ is completely bounded[4] with $\|\Psi\|_{\mathrm{cb}} \leq \|\xi\|_\infty \|\eta\|_\infty$ and whenever $\xi = \eta$, then Ψ is completey positive with $\|\Psi\|_{\mathrm{cb}} = \|\xi\|^2_\infty$.*

Proof. We define two bounded operators $S, T : \ell^2(\Gamma/\Lambda) \to \ell^2(\Gamma/\Lambda, \mathcal{K})$ by

$$Sl(\dot{g}) = \xi(\dot{g})l(\dot{g}), \quad Tl(\dot{g}) = \eta(\dot{g})l(\dot{g}).$$

[4]We denote by $\|\cdot\|_{\mathrm{cb}}$ its completely bounded norm.

Then, straightforward computations show that, for $f \in \mathbb{C}[\Gamma;\Lambda]$, we have

$$\begin{aligned} S^*\big(\mathbf{R}(f)\otimes 1\big)Tl(\dot g) &= \sum_{h\in\langle\Gamma/\Lambda\rangle} \Big\langle \xi(\dot g), \eta(\dot h)\Big\rangle l(\dot h) f(h^{-1}g)) \\ &= \sum_{h\in\langle\Gamma/\Lambda\rangle} l(\dot h)\psi(h^{-1}g)f(h^{-1}g)) \\ &= \mathbf{R}(\psi f) l(\dot g). \end{aligned}$$

The other assertions follow immediately. □

Note that the restriction of Ψ to $\mathbf{C}^*_\rho(\Gamma,\Lambda)$ is a completely bounded map from this C^*-algebra into itself.

We observe that $\omega\circ\Psi = \psi(e)\,\omega$ and

$$\omega\Big(\big(\Psi(\mathbf{R}(f)) - \mathbf{R}(f)\big)^*\big(\Psi(\mathbf{R}(f)) - \mathbf{R}(f)\big)\Big) = \|\psi f - f\|^2_{\ell^2(\Gamma/\Lambda)}. \tag{3.2}$$

We set $\|\psi\|_{\rm cb} = \inf \|\xi\|_\infty \|\eta\|_\infty$ where (ξ,η) runs over all pairs satisfying the properties of the lemma. We have $\|\Psi\|_{\rm cb} \le \|\psi\|_{\rm cb}$. When Λ is a normal subgroup, it is well-known that these two quantities are the same (see [23] for instance). We do not know whether this is true in general.

Our second observation is a characterization of co-amenability in terms of positive definite functions when the subgroup is almost normal. No such characterization exists for general subgroups. We begin by giving a sufficient condition for a positive definite function to be a coefficient of a quasi-regular representation. This extends a result of Godement [18] stating that a square integrable continuous positive definite function on a locally compact group is a coefficient of its regular representation.

Theorem 3.7. *Let H be a closed subgroup of a locally compact group G such that there exists a G-invariant measure on G/H. Let φ be a continuous positive definite function on G. We assume that φ is H-bi-invariant and that $\tilde\varphi$ obtained by passing to the quotient is in $L^2(G/H)$. Then φ is a coefficient of the quasi-regular representation.*

Proof. We have fixed a left-invariant measure μ on G/H, denoted also $\mathrm{d}\,\dot t$. We use arguments similar to those of Godement's proof, which concerns the case where H is the trivial subgroup $\{e\}$. Given a continuous function with compact support $\xi \in C_c(G/H)$, we define the function $\rho(\varphi)\xi$ by

$$\rho(\varphi)\xi(\dot s) = \int_{G/H} \varphi(s^{-1}t)\xi(\dot t)\,\mathrm{d}\dot t.$$

This function belongs to $L^2(G/H)$. Indeed, if K is a compact subset of G/H containing the support of ξ, by using the Cauchy-Schwarz inequality, the invariance of μ and the fact that $\varphi(s) = \overline{\varphi(s^{-1})}$, we have

$$\begin{aligned}\int_{G/H} |\rho(\varphi)\xi(\dot{s})|^2 \,\mathrm{d}\dot{s} &\leq \int_{G/H} \Big(\int_{G/H} \big|\varphi(s^{-1}t)\big|\big|\xi(\dot{t})\big| \,\mathrm{d}\dot{t} \Big)^2 \,\mathrm{d}\dot{s} \\ &\leq \Big(\int_{G/H\times G/H} \mathbf{1}_K(\dot{t})\big|\varphi(s^{-1}t)\big|^2 \,\mathrm{d}\dot{t}\,\mathrm{d}\dot{s}\Big) \|\xi\|_2^2 = \mu(K)\|\varphi\|_2^2\|\xi\|_2^2.\end{aligned}$$

Therefore, $\rho(\varphi)$ is an operator which has the space $C_c(G/H)$ of compactly supported continuous functions on G/H as domain. It is unbounded in general and non-negative, since for every $\xi \in C_c(G/H)$ we have, due to the fact that φ is positive definite,

$$\langle \xi, \rho(\varphi)\xi\rangle = \int_{G/H\times G/H} \overline{\xi(\dot{s})}\varphi(s^{-1}t)\xi(\dot{t}) \,\mathrm{d}\dot{t}\,\mathrm{d}\dot{s} \geq 0.$$

We still denote by $\rho(\varphi)$ its Friedrichs extension. It is a non-negative self-adjoint operator. If $\xi \in C_c(G/H)$ and $s \in G$, a routine computation shows that

$$\lambda_{G/H}(s)\rho(\varphi)\xi = \rho(\varphi)\lambda_{G/H}(s)\xi$$

and therefore $\lambda_{G/H}(s)$ commutes with the Friedrichs extension.

Let (V_i) be a decreasing net of compact neighbourhoods of $\dot{e}$ in G/H, such that $\cap_i V_i = \{\dot{e}\}$. We set $\xi_i = \rho(\varphi)^{1/2} f_i$ where f_i is a non-negative function with $\int_{G/H} f_i(\dot{t}) \,\mathrm{d}\,\dot{t} = 1$ and such that $f_i(\dot{t}) = 0$ outside V_i. We have

$$\langle \xi_i, \xi_j\rangle = \langle f_i, \rho(\varphi)f_j\rangle = \int_{G/H\times G/H} f_i(\dot{t})\varphi(t^{-1}s)f_j(\dot{s}) \,\mathrm{d}\dot{t}\,\mathrm{d}\dot{s},$$

and so

$$|\langle \xi_i, \xi_j\rangle - \varphi(e)| \leq \int_{G/H\times G/H} f_i(\dot{t})\big|\varphi(t^{-1}s) - \varphi(e)\big|f_j(\dot{s}) \,\mathrm{d}\dot{t}\,\mathrm{d}\dot{s}.$$

We remark that φ, being continuous and positive definite is uniformly continuous. It follows that $\lim_{i,j} \langle \xi_i, , \xi_j\rangle = \varphi(e)$, from which we deduce that (ξ_i) is a Cauchy net. We denote by ξ its limit.

We shall now show that φ is the coefficient of $\lambda_{G/H}$ associated with ξ. We have

$$\begin{aligned}\big\langle \xi, \lambda_{G/H}(s)\xi\big\rangle &= \lim_i \big\langle \xi_i, \lambda_{G/H}(s)\xi_i\big\rangle \\ &= \lim_i \Big\langle \rho(\varphi)^{1/2} f_i, \lambda_{G/H}(s)\rho(\varphi)^{1/2} f_i\Big\rangle \\ &= \lim_i \big\langle f_i, \lambda_{G/H}(s)\rho(\varphi)f_i\big\rangle \\ &= \lim_i \int_{G/H\times G/H} f_i(\dot{t})\varphi(t^{-1}su)f_i(\dot{u}) \,\mathrm{d}\dot{t}\,\mathrm{d}\dot{u},\end{aligned}$$

since $\lambda_{G/H}(s)$ commutes with $\rho(\varphi)$.

We have

$$\left|\int_{G/H\times G/H} f_i(\dot t)\varphi(t^{-1}su)f_i(\dot u)\,\mathrm{d}\dot t\,\mathrm{d}\dot u-\varphi(s)\right|$$
$$\leq \int_{G/H\times G/H} f_i(\dot t)f_i(\dot u)\big|\varphi(t^{-1}su)-\varphi(s)\big|\,\mathrm{d}\dot t\,\mathrm{d}\dot u.$$

Using again the uniform continuity of φ, we see that

$$\langle \xi,\lambda_{G/H}(s)\xi\rangle=\lim_i\langle \xi_i,\lambda_{G/H}(s)\xi_i\rangle=\varphi(s).$$

□

Note that is $\varphi(s)=\langle \xi,\lambda_{G/H}(s)\xi\rangle$ for all s, then φ is H-bi-invariant if and only if the vector ξ is H-invariant. If moreover $\tilde\varphi$ is compactly supported, then there exists a compact subset K of G such that the support of φ is contained in KH.

We are now able to give a characterization of co-amenability in terms of positive definite functions when Λ is almost normal in Γ.

Theorem 3.8. *Let Λ be an almost normal subgroup of Γ. The following conditions are equivalent:*

(i) *Λ is co-amenable in Γ ;*
(ii) *for every $\varepsilon>0$ and every finite subset K of Γ, there exists a Λ-invariant unit function ξ with finite support on Γ/Λ such that $\sup_{g\in K}\big\|\lambda_{\Gamma/\Lambda}(g)\xi-\xi\big\|_2\leq\varepsilon$;*
(iii) *for every $\varepsilon>0$ and every finite subset K of Γ, there exists a Λ-bi-invariant positive definite function on Γ whose support is a finite union of double cosets, such that $\sup_{g\in K}|\varphi(g)-1|\leq\varepsilon$.*

Proof. (ii) ⇒ (iii). Assume that (ii) holds. Given $\varepsilon>0$ and K, let ξ be as in (ii). We set $\varphi(g)=\langle\xi,\lambda_{\Gamma/\Lambda}(g)\xi\rangle$. Then φ is Λ-bi-invariant and we have

$$\sup_{g\in K}|\varphi(g)-1|\leq\sup_{g\in K}\big\|\lambda_{\Gamma/\Lambda}(g)\xi-\xi\big\|_2\leq\varepsilon.$$

(iii) ⇒ (i) is also very easy, thanks to Theorem 3.7 and the classical inequality

$$\big\|\lambda_{\Gamma/\Lambda}(g)\xi-\xi\big\|_2^2\leq 2\big|\langle\xi,\lambda_{\Gamma/\Lambda}(g)\xi-\xi\rangle\big|.$$

It remains to prove (i) ⇒ (ii). We shall use the Schlichting completion and the following easy lemma.

Lemma 3.9. *Let H be a compact and co-amenable subgroup of a locally compact group G. For every $\varepsilon>0$ and every compact subset K of G, there exists a H-invariant unit function ξ in $L^2(G/H)$ such that $\sup_{g\in K}\big\|\lambda_{G/H}(g)\xi-\xi\big\|_2\leq\varepsilon$. Moreover we may choose ξ to be compactly supported.*

Proof. We may assume that K contains H. Take $\varepsilon'<1/2$ such that $5\varepsilon'\leq\varepsilon$. There exists a unit vector $\eta\in L^2(G/H)$ such that

$$\sup_{g\in K}\big\|\lambda_{G/H}(g)\eta-\eta\big\|_2\leq\varepsilon'.$$

We set $\eta' = \int_H \lambda_{G/H}(h)\eta\,\mathrm{d}h$, the integration being with respect to the Haar probability measure on H. We have $\|\eta' - \eta\|_2 \leq \varepsilon'$ and $\|\xi - \eta\|_2 \leq 2\varepsilon'$, where $\xi = \eta'/\|\eta'\|_2$. Then ξ is H-invariant and $\sup_{g\in K} \left\|\lambda_{G/H}(g)\xi - \xi\right\|_2 \leq 5\varepsilon' \leq \varepsilon$. Moreover, if we have started with a compactly supported function η, so is ξ. $\square$

Proof of (i) $\Rightarrow$ (ii) *in Theorem 3.8.* By Proposition 3.4, H' is co-amenable in G'. Then by Lemma 3.9, there is a net (ξ_i) of H'-invariant unit vectors in $\ell^2(G'/H')$, finitely supported, such that $\lim_i \left\|\lambda_{G'/H'}(s)\xi_i - \xi_i\right\|_2 = 0$ uniformly on compact subsets of G'. Let $U : \ell^2(\Gamma/\Lambda) \to \ell^2(G'/H')$ be as in Lemma 2.2 (iii). Then the net $(U^{-1}\xi_i)$ is made of Λ-invariant finitely supported unit vectors such that $\lim_i \left\|\lambda_{\Gamma/\Lambda}(s)U^{-1}\xi_i - U^{-1}\xi_i\right\|_2 = 0$ pointwise. $\square$

Theorem 3.8 has the following immediate consequence.

Corollary 3.10. *Let Λ be an almost normal co-amenable subgroup of Γ Then Λ is co-amenable in any subgroup Γ_1 of Γ containing Λ.*

This is obvious with the characterization (iii) of Theorem 3.8. This result has to be compared with Example 3.2. Proposition 3.5 is also an easy consequence of this characterization.

Proof of Proposition 3.5. Let (φ_i) be a net of Λ-bi-invariant positive definite functions on Γ, supported on finite unions of double cosets, that converges to 1 pointwise. We may assume that $\varphi_i(e) = 1$. Each φ_i can be written as $\varphi_i(g) = \left\langle \xi_i, \lambda_{\Gamma/\Lambda}(g)\xi_i \right\rangle$ where ξ_i is a unit Λ-invariant vector of $\ell^2(\Gamma/\Lambda)$. By Lemma 3.6 there exists a normal unital completely positive map Φ_i from $\mathbf{R}(\Gamma,\Lambda)$ into itself such that $\Phi_i(\mathbf{R}(f)) = \mathbf{R}(\varphi_i f)$ for every $f \in \mathbb{C}[\Gamma,\Lambda]$. Obviously, Φ_i has a finite rank. It remains to show that for $T \in \mathbf{R}(\Gamma,\Lambda)$ we have $\lim_i \Phi_i(T) = T$ in the weak topology. By approximation it suffices to take $T = \mathbf{R}(f)$. The conclusion follows from the equality (see (3.2)),

$$\|\Phi_i(\mathbf{R}(f)) - \mathbf{R}(f)\|_{L^2}^2 = \sum_{g\in\langle\Gamma/\Lambda\rangle} |\varphi_i(g) - 1|^2 |f(g)|^2.$$

$\square$

4. Co-Haagerup property

The now called Haagerup property was first detected by this author for free groups [21]. Since then, this property proved to be very useful in various branches of mathematics (see [8]). Close connections with approximation properties of finite von Neumann algebras were discovered in [10, 9]). We begin by recalling some definitions.

Definition 4.1. We say that a locally compact group G has the *Haagerup* (approximation) *property* if there exists a net (φ_i) of normalized[5] continuous positive definite functions on G, vanishing at infinity, which converges to 1 uniformly on compact subsets of G.

[5] *i.e.* such that $\varphi_i(e) = 1$.

Let M be a von Neumann algebra, equipped with a normal faithful state ω. A completely positive map $\Phi : M \to M$ such that $\omega \circ \Phi = \omega$ gives rise to a bounded operator $\widehat{\Phi} : L^2(M,\omega) \to L^2(M,\omega)$ in the following way. Given $x \in M$, we denote by $\widehat{x}$ this element when we want to stress the fact that x is viewed as an element of $L^2(M,\omega)$. We have

$$\omega(\Phi(x)^*\Phi(x)) \leq \|\Phi\| \, \omega(\Phi(x^*x)) \leq \|\Phi\| \, \omega(x^*x).$$

Then $\widehat{\Phi}$ is well defined by $\widehat{\Phi}(\widehat{x}) = \widehat{\Phi(x)}$.

Definition 4.2. We say that M has the *Haagerup* (approximation) *property with respect to* ω if there exists a net (Φ_i) of unital completely positive maps from M to M such that

(a) $\omega \circ \Phi_i = \omega$ for every i ;
(b) $\widehat{\Phi_i}$ is a compact operator for every i ;
(c) $\lim_i \|\Phi_i(x) - x\|_{L^2(M,\omega)} = 0$ for every $x \in M$.

This property has been previously defined under the assumption that ω is a trace. In this case, Jolissaint proved that it does not depend on the choice of the faithful tracial state [24]. Moreover, when M is the tracial von Neumann algebra $\mathbf{L}(\Gamma)$ of a group Γ, then Γ has the Haagerup property if and only if $\mathbf{L}(\Gamma)$ has the Haagerup property [9].

Given an almost normal subgroup Λ of Γ, the notion of Haagerup property relative to Λ appears in [30, Remarks 3.5] (see also [12]). No definition is known without the assumption of almost normality. Observe that this latter property is implied by Conditions (a) and (b) of the next definition.

Definition 4.3. Let Λ be an almost normal subgroup of a group Γ. We say that Λ is *co-Haagerup in* Γ, or that the *coset space* Γ/Λ, or the *Hecke pair* (Γ,Λ)*, has the Haagerup property*, if there exists a net (φ_i) of normalized positive definite functions on Γ such that

(a) for all i the function φ_i is Λ-bi-invariant and, passing to the quotient, $\tilde{\varphi}_i$ is in the space $c_0(\Gamma/\Lambda)$ of functions on Γ/Λ that vanish at infinity;
(b) $\lim_i \varphi_i = 1$ pointwise.

Proposition 4.4. *Let Λ be an almost normal subgroup of Γ which is co-Haagerup. Then $\mathbf{R}(\Gamma,\Lambda)$ has the Haagerup property with respect to its canonical state ω.*

Proof. Let (φ_i) be a net of positive definite functions as in Definition 4.3. By the classical GNS construction, for each i there is a representation $(\pi_i, \mathcal{K}_i)$ of Γ and a unit vector $\xi_i \in \mathcal{K}_i$ such that $\varphi_i(g) = \langle \xi_i, \pi_i(g)\xi_i\rangle$ for $g \in \Gamma$. Morever, ξ_i is Λ-invariant since φ_i is Λ-bi-invariant. By Lemma 3.6, there is a unique normal completely positive map Φ_i from $N = \mathbf{R}(\Gamma,\Lambda)$ into itself such that $\Phi_i(\mathbf{R}(f)) = \mathbf{R}(\varphi_i f)$ for $f \in \mathbb{C}[\Gamma;\Lambda]$. Moreover Φ_i preserves the state ω, and as in the proof of Proposition 3.5 we see that $\lim_i \|\Phi_i(T) - T\|_{L^2} = 0$ for every $T \in \mathbf{R}(\Gamma,\Lambda)$.

Recall that $\left\{L(g)^{-1/2}\delta_{\ddot{g}} : g \in \langle \Lambda \setminus \Gamma/\Lambda\rangle\right\}$ is an orthonormal basis of $L^2(N,\omega)$. We have $\widehat{\Phi_i}(\delta_{\ddot{g}}) = \varphi_i(g)\delta_{\ddot{g}}$. It follows that $\widehat{\Phi_i}$ is a compact operator since φ_i, when viewed as a function on $\Lambda \setminus \Gamma/\Lambda$, belongs to $c_0(\Lambda \setminus \Gamma/\Lambda)$. $\square$

Proposition 4.5. *Let (Γ,Λ) be a Hecke pair and (G',H') its Schlichting completion. Then Γ/Λ has the Haagerup property if and only if the group G' has the Haagerup property.*

Proof. Assume that Γ/Λ has the Haagerup property. Let (φ_i) be a net of positive definite functions on Γ as in Definition 4.3. We define φ_i' on G' by the expression $\varphi_i'(g') = \varphi_i(g)$, where $g\Lambda$ is the unique element of Γ/Λ such that $g'H' = \theta(g)H'$. It is a H'-bi-invariant positive definite function on G', by Lemma 2.3. Since H' is compact, we see that φ_i' vanishes at infinity. Obviously, the net (φ_i') converges to 1 uniformly on compact subsets of G', and therefore this group has the Haagerup property.

Conversely, assume that G' has the Haagerup property. Then there exists a unitary C_0-representation[6] π on a Hilbert space $\mathcal{H}$, which contains weakly the trivial representation (see [8, Theorem 2.1.2]). Let $0 < \varepsilon \leq 1/2$ and a compact subset K of G' containing H' be given. There exists a unit vector $\xi \in \mathcal{H}$ such that $\sup_{g \in K} \|\pi(g)\xi - \xi\| \leq \varepsilon$. We set $\xi' = \int_{H'} \pi(h)\xi \,\mathrm{d}h$ (where we integrate with respect to the Haar probability measure on H') and $\xi'' = \xi'/\|\xi'\|$. Then we have $\|\xi' - \xi\| \leq \varepsilon$ and $\|\xi'' - \xi\| \leq 2\varepsilon$. It follows that the coefficient φ' of π defined on G' by $\varphi'(g) = \langle \xi'', \pi(g)\xi'' \rangle$ is H'-bi-invariant, vanishes at infinity and satisfies, for $g \in K$,

$$|\varphi'(g) - 1| \leq \|\pi(g)\xi'' - \xi''\| \leq 5\varepsilon.$$

By passing to the quotient, we get $\widetilde{\varphi'} \in c_0(G'/H')$.

We define a Λ-bi-invariant positive definite function φ on Γ by

$$\varphi(g) = \varphi'(\theta(g)).$$

Since $\widetilde{\varphi}$ is obtained from $\widetilde{\varphi'}$ via the natural bijection from Γ/Λ onto G'/H', we see that $\widetilde{\varphi} \in c_0(\Gamma/\Lambda)$.

Now, starting with a net (ξ_i) of almost invariant unit vectors in $\mathcal{H}$ (*i.e.* such that $\lim_i \|\pi(g)\xi_i - \xi_i\| = 0$ uniformly on compact subsets of G'), this construction provides us with a net (φ_i) of Λ-bi-invariant positive definite functions vanishing at infinity on Γ, which converges to one pointwise. $\square$

5. Weak co-amenability

The notion of weak amenability stems also from the seminal paper [21] although the terminology was introduced later [14]. Let G be a locally compact group and $\mathbf{A}(G)$ its Fourier algebra. Recall that $\mathbf{A}(G)$ is the predual of the von Neumann algebra $\mathbf{L}(G)$ of G. A multiplier ψ of $\mathbf{A}(G)$ is said to be a *completely bounded multiplier* if its transposed operator is completely bounded. We denote by $\|\psi\|_{\mathrm{cb}}$ the completely bounded norm of this operator. A completely bounded multiplier ψ is characterized by the following property: there exist two continuous bounded functions ξ, η from G into some Hilbert space $\mathcal{K}$ such that $\psi(k^{-1}g) = \langle \xi(g), \eta(k) \rangle$ for $g, k \in G$. Moreover we have $\|\psi\|_{\mathrm{cb}} = \inf \|\xi\|_\infty \|\eta\|_\infty$ where ξ, η runs over all possible such decompositions (see [6], and [23] for a simple proof).

Definition 5.1. A locally compact group G is said to be *weakly amenable* if there exists a net (ψ_i) in $\mathbf{A}(G)$ which converges to 1 uniformly on compact subsets of G and such that there exists a constant C with $\|\psi_i\|_{\mathrm{cb}} \leq C$ for all i. The *Cowling-Haagerup constant* $\Lambda_{\mathrm{cb}}(G)$ of G is the infimum of such C for all possible such nets.

[6]Recall that a representation is C_0 if all its coefficients vanish at infinity.

In fact, by [14, Proposition 1.1], we may assume that the ψ_i's have a compact support.

The previous definition has to be compared with Leptin's characterization of amenability, which requires the stronger condition that $\sup_i \|\psi_i\|_{A(G)} < +\infty$. Recall that $\|\psi_i\|_{\mathrm{cb}} \leq \|\psi_i\|_{A(G)}$.

Definition 5.2. We say that a C^*-algebra (resp. a von Neumann algebra) B has the *completely bounded approximation property* (CBAP) (resp. the *weak* completely bounded approximation property* (W*CBAP)) if there exists a net of finite rank maps $\Psi_i : B \to B$ (resp. normal finite rank maps) that converges to the identity map in the point-norm (resp. point-weak*) topology and such that $\sup_i \|\Psi_i\|_{\mathrm{cb}} \leq C$.

The *Haagerup constant* $\Lambda_{\mathrm{cb}}(B)$ is the infimum such C for all possible such nets.

Let Γ be a discrete group. Then Γ is weakly amenable if and only if its reduced C^*-algebra has the CBAP, and also if and only if its von Neumann algebra has the W*CBAP. Moreover, the corresponding constants are the same. For a proof of this result due to Haagerup, see [7, Theorem 12.3.10].

Definition 5.3. Let Λ be an almost normal subgroup of a group Γ. We say that Λ is *weakly co-amenable in* Γ, or that the *coset space* Γ/Λ, or the *Hecke pair* (Γ, Λ)*, is weakly amenable*, if there exists a net (ψ_i) of complex-valued functions on Γ such that

(a) for all i the function ψ_i is Λ-bi-invariant and, by passing to the quotient, $\tilde{\psi_i}$ has a finite support on Γ/Λ ;
(b) for all i there exist two bounded functions ξ_i, η_i from Γ/Λ into some Hilbert space $\mathcal{K}_i$ such that $\psi_i(k^{-1}g) = \left\langle \xi_i(\dot{g}), \eta_i(\dot{k}) \right\rangle$ for all $g, k \in \Gamma$;
(c) $\lim_i \psi_i = 1$ pointwise;
(d) $\sup_i \|\psi_i\|_{\mathrm{cb}} \leq C$.

We denote by $\Lambda_{\mathrm{cb}}(\Gamma, \Lambda)$ the infimum such C for all possible such nets[7] .

Proposition 5.4. *Let Λ be a weakly co-amenable almost normal subgroup of Γ. Then $\mathbf{R}(\Gamma, \Lambda)$ has the W*CBAP and $\mathbf{C}^*_\rho(\Gamma, \Lambda)$ has the CBAP. Moreover we have $\Lambda_{\mathrm{cb}}(\mathbf{R}(\Gamma, \Lambda)) \leq \Lambda_{\mathrm{cb}}(\Gamma, \Lambda)$ and $\Lambda_{\mathrm{cb}}(\mathbf{C}^*_\rho(\Gamma, \Lambda)) \leq \Lambda_{\mathrm{cb}}(\Gamma, \Lambda)$.*

Proof. Let (ψ_i) be a net satisfying the conditions of Definition 5.3 for the pair (Γ, Λ). Let Ψ_i be the completely bounded function from $\mathbf{R}(\Gamma, \Lambda)$ into itself constructed from ψ_i in Lemma 3.6. This map has a finite rank and $\|\Psi_i\|_{\mathrm{cb}} \leq \|\psi_i\|_{\mathrm{cb}}$. Moreover, again as in the proof of Proposition 3.5, we see that $\lim_i \|\Psi_i(T) - T\|_{L^2} = 0$ for every $T \in \mathbf{R}(\Gamma, \Lambda)$. $\square$

Proposition 5.5. *Let (Γ, Λ) be a Hecke pair and (G', H') its Schlichting completion. Then Γ/Λ is weakly amenable if and only if the group G' is weakly amenable. In this case, we have $\Lambda_{\mathrm{cb}}(G') = \Lambda_{\mathrm{cb}}(\Gamma, \Lambda)$.*

Proof. Let ψ, ξ, η as in Lemma 3.6 and let ψ' be the corresponding H'-bi-invariant function on G'. Recall that $\psi'(g') = \psi(g)$ where $g' = \theta(g)h'$, $h' \in H'$. We set

$$\xi'(\theta(g)H') = \xi(g\Lambda), \quad \eta'(\theta(g)H') = \eta(g\Lambda).$$

[7] Recall that $\|\psi_i\|_{\mathrm{cb}} = \inf \|\xi_i\|_\infty \|\eta_i\|_\infty$ where (ξ_i, η_i) runs over all pairs as in (b).

For $g' = \theta(g)h'$ and $g'_1 = \theta(g_1)h'_1$, we have

$$\begin{aligned}\psi'((g'_1)^{-1}g') &= \psi(g_1^{-1}g) = \langle \xi(g\Lambda), \eta(g_1\Lambda)\rangle \\ &= \langle \xi'(g'H'), \eta'(g'_1H')\rangle = \langle \xi''(g'), \eta''(g'_1)\rangle,\end{aligned}$$

where $\xi''(g') = \xi'(g'H')$ and similarly for η''.

These functions ξ'' and η'' being continuous and bounded on G' by $\|\xi\|_\infty$ and $\|\eta\|_\infty$ respectively, we conclude that ψ' is a completely bounded multiplier of G' with $\|\psi'\|_{\mathrm{cb}} \leq \|\psi\|_{\mathrm{cb}}$.

Since, after passing to quotient, $\widetilde{\psi}'(\theta(g)H') = \widetilde{\psi}(g\Lambda))$ for $g \in \Gamma$, we easily conclude that ψ' is compactly supported whenever $\widetilde{\psi}$ has a finite support.

Finally, if (ψ_i) is a net as in Definition 5.3, we see that the corresponding net (ψ'_i) converges to 1 uniformly on compact subsets of G', and it follows from the above observations that G' is weakly amenable. Moreover, we get $\Lambda_{\mathrm{cb}}(G') \leq \Lambda_{\mathrm{cb}}(\Gamma, \Lambda)$.

To prove the converse, let us first start with a completely bounded multiplier ψ' of G' such that for some compact subset K of G' and some $\varepsilon > 0$ we have $|\psi'(h'kh'') - 1| \leq \varepsilon$ for $k \in K$, $h', h'' \in H'$. We denote by ξ', η' two continuous bounded functions from G' into some Hilbert space $\mathcal{H}$ such that $\psi'(k^{-1}g) = \langle \xi'(g), \eta'(k)\rangle$ for $g, k \in G'$.

We set $\psi''(g) = \int_{H'\times H'} \psi'(h'gh'')\,\mathrm{d}h'\,\mathrm{d}h''$ where the integrations are with respect to the Haar probability measure on H'. Of course, ψ'' is H'-bi-invariant, and we have $\sup_{k\in K} |\psi'(k) - \psi''(k)| \leq \varepsilon$. If ξ'' and η'' are defined on G'/H' by

$$\xi''(gH') = \int_{H'} \xi'(gh')\,\mathrm{d}h', \quad \eta''(gH') = \int_{H'} \eta'(gh')\,\mathrm{d}h',$$

we have $\psi''(k^{-1}g) = \langle \xi''(gH'), \eta''(kH')\rangle$ for $g, k \in G'$.

Now, if we define $\psi : \Gamma \to \mathbb{C}$ and $\xi, \eta : \Gamma/\Lambda \to \mathcal{H}$ by

$$\psi(g) = \psi''(\theta(g)), \quad \xi(g\Lambda) = \xi''(\theta(g)H'), \quad \eta(g\Lambda) = \eta''(\theta(g)H'),$$

we see that ψ is a Λ-bi-invariant completely bounded multiplier on Γ such that $\psi(k^{-1}g) = \langle \xi(g\Lambda), \eta(k\Lambda)\rangle$ for $g, k \in \Gamma$. Moreover, if ψ' is compactly supported, so is ψ'' and the support of $\widetilde{\psi}$ is finite.

Finally, starting from a net (ψ'_i) of compactly supported completely bounded multipliers of G' with $\sup_i \|\psi'_i\|_{\mathrm{cb}} = c < +\infty$, converging to 1 uniformly on compact subsets of G', we get from the above considerations a net (ψ_i) which satisfies the conditions of Definition 5.3 with $\sup_i \|\psi_i\|_{\mathrm{cb}} \leq c$. Hence we have $\Lambda_{\mathrm{cb}}(\Gamma, \Lambda) \leq \Lambda_{\mathrm{cb}}(G')$. □

6. Co-rigidity

We first recall the now classical notion of property (T) for a locally compact group G. Let $(\pi, \mathcal{H})$ be a unitary representation of a locally compact group G. Given $\varepsilon > 0$ and a compact subset K of G, recall that an (ε, K)-invariant vector is a unit vector $\xi \in \mathcal{H}$ such that $\sup_{g\in K} \|\pi(g)\xi - \xi\| \leq \varepsilon$. One says that π *almost contains invariant vectors* (or that the trivial representation of G is weakly contained in π) if it contains (ε, K)-invariant vectors for every $\varepsilon > 0$ and every compact subset K of G.

Definition 6.1. We say that G has the *Kazdhan property* (T), or is *rigid*, if every unitary representation of G that almost contains invariant vectors actually contains a non-zero invariant vector.

Proposition 6.2. *The following conditions are equivalent:*

(i) *G is rigid;*
(ii) *for every $0 < \delta < 2$ there exist $\varepsilon > 0$ and a compact subset K of G such that for every unitary representation $(\pi, \mathcal{H})$ of G and every unit (ε, K)-invariant vector $\xi \in \mathcal{H}$, there is a G-invariant unit vector η with $\|\xi - \eta\| \leq \delta$;*
(iii) *every net (φ_i) of normalized continuous positive definite functions on G that converges to 1 uniformly on compact subsets of G also converges to 1 uniformly on G.*

For proofs, we refer for instance to [15, Proposition 1.16 and Théorème 5.11].

Definition 6.3. Let H be a closed subgroup of a locally compact group G. We say that a unitary representation of G *almost contains invariant vectors which are H-invariant* if for every $\varepsilon > 0$ and every compact subset K of G, there is a H-invariant vector which is (ε, K)-invariant.

We say that G has *property* (T) *relative to H* (or that H is *co-rigid in G*, or that *the coset space G/H has the property* (T)) if every unitary representation of G which almost contains invariant vectors which are H-invariant actually contains a non-zero G-invariant vector.

Proposition 6.4. *The following conditions are equivalent:*

(i) *H is co-rigid in G;*
(ii) *for every $0 < \delta < 2$ there exist $\varepsilon > 0$ and a compact subset K of G such that for every unitary representation $(\pi, \mathcal{H})$ of G and every unit (ε, K)-invariant and H-invariant vector $\xi \in \mathcal{H}$, there is a G-invariant unit vector η with $\|\xi - \eta\| \leq \delta$.*
(iii) *every net (φ_i) of normalized H-bi-invariant continuous positive definite functions on G that converges to 1 uniformly on compact subsets of G also converges to 1 uniformly on G.*

The proof is similar to that of Proposition 6.2.

Proposition 6.5. *Let Λ be an almost normal subgroup of Γ. Then the coset space Γ/Λ has the property (T) if and only if the group G' has the property (T).*

Proof. Assume that G' has Property (T). Let (φ_i) be a net of Λ-bi-invariant positive definite functions on Γ that converges pointwise to 1. The corresponding net (φ_i') on G' (see Lemma 2.3) converges to 1 uniformly on compact subsets of G' and so uniformly on G', from which is follows that (φ_i) converges to 1 uniformly on Γ.

Conversely, assume that the coset space Γ/Λ has the property (T). Let $(\pi, \mathcal{H})$ be a unitary representation of G' that almost contains invariant vectors. Using the fact that H' is compact, it is easy to see that it almost contains invariant vectors which are H'-invariant. Then the representation $\pi \circ \theta$ of Γ almost contains invariant vectors which are Λ-invariant. It follows that π has a non-zero $\theta(\Gamma)$-invariant vector, which is also G'-invariant by the density of $\theta(\Gamma)$ in G'. □

Remark 6.6. Reference to this notion of co-rigidity is made in [30, Remarks 5.6]. It should not be confused with the more familiar notion of *rigid inclusion* of a closed subgroup H in a locally compact group G, due to Kazhdan and Margulis, which reads as follows: $H \subset G$ is rigid if every unitary representation of G which almost contains invariant vectors actually contains a non-zero H-invariant vector.

7. Examples and problems

7.1. **Amenable coset spaces.** Note that a coset space Γ/Λ is automatically amenable when Γ is amenable. On the other hand, if Γ has the property (T), then Γ/Λ is amenable if and only if Λ has a finite index in Γ.

It was left as an open problem by Greenleaf whether the existence of a reduced pair[8] (Γ, Λ) such that Γ/Λ is amenable implies the amenability of Γ [19, Problem, page 18]. The first examples were found only some 20 years later by van Douwen: he proved that every finitely generated non-abelian free group contains such a co-amenable subgroup. Later, other examples were provided in [27, 29] (see 3.2). Thereafter, this problem has been studied by many researchers [17, 20, 28]. Most of their examples are built on free or amalgamed free products of groups.

As for reduced Hecke pairs (Γ, Λ) such that Γ/Λ is amenable, there is the famous example $(\Gamma = \mathbb{Q} \rtimes \mathbb{Q}_+^*, \Lambda = \mathbb{Z} \rtimes \{1\})$ studied by Bost and Connes. But it is amenable for the obvious reason that Γ is amenable.

Problem 1. Greenleaf's problem is still open in this setting : find examples of reduced Hecke pairs (Γ, Λ) such that Γ/Λ is amenable without Γ being so.

7.2. **Coset spaces with the Haagerup property.** We first observe that the Haagerup property of Γ/Λ has nothing to do with the fact that Γ possesses or not this property. For instance any finitely generated group is a quotient of a free group, and free groups have the Haagerup property. On the other hand, $SL_2(\mathbb{Z})$, which has the Haagerup property is a quotient of $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ which has not this property.

An easy way of constructing examples of Hecke pairs (Γ, Λ) such that Γ/Λ has the Haagerup property is as follows.

Proposition 7.1. *Assume that Γ acts on a locally finite tree $\mathcal{T}$ and let Λ be the stabilizer of some vertex v_0. Then Λ is co-Haagerup in Γ.*

Proof. Denote by d the length metric on $\mathcal{T}$. The function $\psi : g \mapsto d(v_0, gv_0)$ is conditionally negative definite and Λ-bi-invariant. Moreover it is a proper function after passing to the quotient on Γ/Λ. Then the sequence (φ_n), where $\varphi_n = \exp(-\psi/n)$ satisfies the properties of Definition 4.3. □

[8] In order to avoid trivial examples we are only interested in reduced pairs. Indeed, otherwise it is easy to construct artificially many examples by starting with a surjective homomorphism α from a non-amenable group Γ onto an amenable group Γ', then take any subgroup Λ' of Γ' and consider $(\Gamma, \Lambda = \alpha^{-1}(\Lambda'))$.

For instance, let Γ be the HNN-extension relative to (Λ, H, σ) where $\sigma : H \to K$ is an isomorphism between two subgroups of finite index in Λ (see Examples 2.5). Then Γ/Λ has the Haagerup property.

Another example is given by $(SL_2(\mathbb{Z}[1/p]), SL_2(\mathbb{Z}))$. One may use the action of $SL_2(\mathbb{Z}[1/p])$ on the $(p+1)$-regular tree. Another proof is to observe that the group G' of the Schlichting completion of this pair is $SL_2(\mathbb{Q}_p)$ which has the Haagerup property and then use Proposition 4.5.

7.3. **Weakly amenable coset spaces.** The same preliminary observation applies: the weak amenability of Γ/Λ has nothing to do with that of Γ.

Proposition 7.2. *Assume that Γ acts on a locally finite tree $\mathcal{T}$ and let Λ be the stabilizer of some vertex v_0. Then Λ is weakly co-amenable in Γ and $\Lambda_{\mathrm{cb}}(\Gamma, \Lambda) = 1$.*

Proof. Using Proposition 5.5, it suffices to show that the group G' of the Schlichting completion of (Γ, Λ) is weakly amenable with constant $\Lambda_{\mathrm{cb}}(\Gamma) = 1$. By Proposition 2.6, we have $G' = G/L$ where G acts properly on the tree $\mathcal{T}$ and L is a normal compact subgroup of G. We know that G and G/L are simultaneously weakly amenable with the same constant Λ_{cb} (see [14, Proposition 1.3]). To conclude, we use Theorem 6 of [32] which states that a locally compact group acting on tree is weakly amenable with constant 1 as soon as the stabilizer of one vertex is compact. □

Thus the pair $(SL_2(\mathbb{Z}[1/p]), SL_2(\mathbb{Z}))$ and the HNN-extensions of the previous subsection provide weakly amenable coset spaces with constant 1.

Problem 2. Let Λ be a weakly co-amenable subgroup of Γ. We have shown in Proposition 5.4 that $\Lambda_{\mathrm{cb}}(\mathbf{R}(\Gamma, \Lambda)) \leq \Lambda_{\mathrm{cb}}(\Gamma, \Lambda)$. Are there examples where the inequality is strict ? Same question for $\Lambda_{\mathrm{cb}}(\mathbf{C}^*_\rho(\Gamma, \Lambda))$. Have we $\Lambda_{\mathrm{cb}}(\mathbf{C}^*_\rho(\Gamma, \Lambda)) = \Lambda_{\mathrm{cb}}(\mathbf{R}(\Gamma, \Lambda))$?

7.4. **Co-rigidity.** Obviously, if Γ has the property (T), then every subgroup Λ is co-rigid in Γ. In contrast with the amenable case, it is easy to construct co-rigid reduced examples where Γ has not the property (T) and may even be amenable. The following example is due to S. Popa [30, Remark 5.6.2']. It is based on a simple observation: assume that there exist a finite subset F of Γ and an integer $n \geq 1$ such that every $g \in \Gamma$ belongs to some $g_1 \Lambda g_2 \Lambda \cdots g_n \Lambda$ with $g_i \in F$ for $i = 1, \dots, n$. Then Λ is co-rigid in Γ. Indeed, let $(\pi, \mathcal{H})$ be a unitary representation of Γ such that there exists a unit Λ-invariant vector $\xi \in \mathcal{H}$ with $\sup_{g \in F} \|\pi(g)\xi - \xi\| \leq 1/2n$. Then we have $\|\pi(g)\xi - \xi\| \leq 1/2$ for every $g \in \Gamma$, and the element of smallest norm in the closed convex envelope of $\pi(\Gamma)\xi$ is non-zero and Γ-invariant. An example of this kind is $\Gamma = \mathbb{Q} \rtimes \mathbb{Q}^*_+$, $\Lambda = \mathbb{Q}^*_+$ and $F = (1, 1)$, since $\Lambda F \Lambda = \Gamma$.

In the setting of Hecke pairs, the situation is quite different. Assume that for such a pair (Γ, Λ), the coset space Γ/Λ has the property (T). Whenever Γ is amenable or even only has the Haagerup property, it follows from Theorem 3.8 (or Definition 4.3) and Proposition 6.4 that Λ has a finite index in Γ.

Problem 3. Find examples of reduced Hecke pairs (Γ, Λ) such that Γ/Λ has the property (T) without Γ having this property.

When Λ is a normal subgroup of Γ, then Γ/Λ has the property (T) if and only if its von Neumann algebra has this property [11].

Problem 4. When Λ is almost normal and co-rigid in Γ, is there an analogue of property (T) for $\mathbf{R}(\Gamma, \Lambda)$?

7.5. **Final remark.** Let Γ be acting by isometries on a locally finite metric space X and denote by Λ_x the stabilizer of $x \in X$. We have observed in Remark 2.7 that the group G' of the Schlichting completion of (Γ, Λ_x) is the quotient of a group G, independent of x by a normal compact subgroup which may depend on x. The four properties considered in this paper (amenability, Haagerup property, weak amenability, rigidity) are stable by passing to such quotients. Since the corresponding properties of Γ/Λ_x are read on G' (Propositions 3.4, 4.5, 5.5, 6.5) we conclude that they do not depend on the choice of x. More precisely, with the notation of Proposition 2.6, we see that Γ/Λ_x has one of the above mentioned properties if and only if the closure G of $\theta(\Gamma)$ in $\mathrm{Iso}(X)$ has the same property.

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