Fibrewise equivariant compactifications under étale groupoid actions

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Abstract. The amenability of the natural action of a discrete group on its Alexandroff (respectively on its Stone-Čech) compactification is equivalent to the amenability (respectively the exactness) of the group. Whereas amenability of groupoids has been widely studied, very few results are known concerning exactness of groupoids. Our purpose here is mainly to provide the topological background needed to investigate this notion. To that end, we study first the fibrewise compactifications of fibre spaces and in particular the Alexandroff and the Stone-Čech fibrewise compactifications. Starting from $\mathcal{G}$-spaces, where $\mathcal{G}$ is an étale groupoid, we show that these two compactifications are $\mathcal{G}$-spaces in a natural way. Applications to groupoids will be developed in a separate paper.

Introduction

A *compactification* of a locally compact (Hausdorff) space Y is a pair (Z, φ) where Z is a compact space and φ is a homeomorphism from Y onto a dense open subspace of Z. If (Z, φ) and (Z_1, φ_1) are two compactifications of Y, we say that (Z, φ) is smaller than (Z_1, φ_1) if there exists a continuous map $\psi : Z_1 \to Z$ such that $\psi \circ \varphi_1 = \varphi$. Recall that Y has a greatest compactification, namely its Stone-Čech compactification βY, and a smallest one, namely its Alexandroff compactification Y^+. The Stone-Čech compactification is the Gelfand spectrum of the abelian C^*-algebra $\mathcal{C}_b(Y)$ of continuous bounded functions from Y to $\mathbb{C}$ and the Alexandroff compactification is the spectrum of its subalgebra formed by the functions that have a limit at infinity.

Let now G be a discrete group acting by homeomorphisms on Y, in which case Y is called a *G-space* and the action is said to be continuous. Then the interesting compactifications are those to which the initial G-action extends to a continuous one. Among them we still find βY and Y^+. Although we don't want to enter here into the details, let us mention that some important intrinsic properties of the group G are characterized by properties of continuous actions it may have. In particular, G is amenable if and only if it acts amenably on G^+ and it is exact if and

2010 *Mathematics Subject Classification.* Primary 55R70; Secondary 54D35, 55R91.
Key words and phrases. Fibrewise compactifications, étale groupoid actions.

only if it acts amenably on a compact space, or equivalently on βG. We refer the interested reader to Ozawa's survey [12] for definitions, results and bibliography.

A natural generalization of the notion of discrete group is that of locally compact étale groupoid $\mathcal{G}$ (see Subsection 2.1). Dynamics furnish many examples of such groupoids. Our aim in this paper is, in particular, to provide the topological background necessary to study afterwards such notions as exactness for this class of groupoids. We are first led to consider the following situation. Let $p : Y \to X$ be a continuous surjective map between locally compact (Hausdorff) spaces, in which case we say that (Y, p) is a *fibre space over* X. We assume that X is the space of units of $\mathcal{G}$ and that $\mathcal{G}$ acts continuously on Y (*i.e.*, (Y, p) is a $\mathcal{G}$*-space*, see Definition 2.1). We need to study the triples (Z, φ, q) such that

- (a) Z is a locally compact space and φ is a homeomorphism from Y onto a dense open subspace of Z;
- (b) the $\mathcal{G}$-action on Y extends to a continuous $\mathcal{G}$-action on Z, turning (Z, q) into a $\mathcal{G}$-space;
- (c) q is a *proper* continuous map from Z onto X with $q \circ \varphi = p$.

Recall that q is said to be *proper* if for every compact subset K of X, the inverse image $q^{-1}(K)$ is proper[1]. In particular, Z is compact if and only if X is compact. However, although Z is not always compact, we call (Z, φ, q) a $\mathcal{G}$*-equivariant fibrewise compactification* of (Y, p). The fibrewise compactifications of $\mathcal{G}$ are obtained by taking $(Y, p) = (\mathcal{G}, r)$ where r is the range map and $\mathcal{G}$ acts on itself by multiplication. The group case corresponds to X reduced to a single point.

A a first step, in Section 1 we consider the case where $\mathcal{G}$ is trivial (*i.e.*, reduced to its space X of units. This means that we are looking for triples (Z, φ, q) satisfying Conditions (a) and (c) above. In this purely topological context, without dynamics, this problem seems to have been first considered by Whyburn [17, 18]. We refer to the book [10] for more on this subject. In [17, 18] the author has studied a fibrewise compactification of $p : Y \to X$ that he called the unified space for p. It turns out that it is what we call below the Alexandroff fibrewise compactification. In [10, Definition 3.1], a fibre space $q : Z \to X$ (where Z and X are general topological spaces) such that q is proper is said to be fibrewise compact. We have kept this terminology. However we consider here a more restrictive situation since we want our fibrewise compactifications to be locally compact and to contain Y as an open subspace. Moreover, the compactifications by adding appropriate ultrafilters that are constructed in [10] are not convenient for our purpose. Our compactifications are obtained as Gelfand spectra of well-chosen abelian C^*-algebras that we characterize. In particular, there is a smallest one and a greatest one. They give

[1] or equivalently, if q is closed and each fibre $q^{-1}(x)$ is compact, since X is locally compact and Y is Hausdorff (see [4, Chap. I, §10]).

rise respectively to the Alexandroff fibrewise compactification (Y_p^+, p^+) and to the Stone-Čech fibrewise compactification $(\beta_p Y, p_\beta)$ of (Y, p).

A fibrewise compactification (Z, φ, q) provides a field $(Z^x = q^{-1}(x))_{x\in X}$ of compact spaces over X. Each fibre Z^x contains $Y^x = p^{-1}(x)$ as as open subset, but in general Y^x is not dense into Z^x. In Section 1.2 we describe in details the fibres of the Alexandroff fibrewise compactification Y_p^+. Concerning the general case, we give some more insight about the fibres in Subsections 1.3 and 1.4.

The $\mathcal{G}$-equivariant case is studied in Section 2. If (Z, φ, q) is a fibrewise compactification of a $\mathcal{G}$-space (Y, p), where $\mathcal{G}$ is an étale groupoid, we give in Theorem 2.5 a condition sufficient for the $\mathcal{G}$-action on (Y, p) to extend as a $\mathcal{G}$-action on Z. In particular (Y_p^+, p^+) and $(\beta_p Y, p_\beta)$ have a natural structure of $\mathcal{G}$-space. The applications of these results are postponed to a subsequent paper [1], since they require a more specialized knowledge about locally compact groupoids and their C^*-algebras.

Without further mention, all topological spaces considered in this paper are assumed to be Hausdorff.

1. Fibrewise compactifications

In this section we are given a *fibre space* (Y, p) over a locally compact space X, that is the data of a continuous surjective map p from a locally compact space Y onto X. For $x \in X$ we denote by Y^x the *fibre* $p^{-1}(x)$. We say that (Y, p) is *fibrewise compact* if p is proper. A *morphism* from the fibre space (Y_1, p_1) over X into the fibre space (Y_2, p_2) over the same space is a continuous map $\varphi : Y_1 \to Y_2$ such that $p_2 \circ \varphi = p_1$.

Definition 1.1. A *fibrewise compactification* of (Y, p) is a triple (Z, φ, q) where Z is a locally compact space, $q : Z \to X$ a continuous *proper* map and $\varphi : Y \to Z$ a homeomorphism onto an open dense subset of Z such that $p = q \circ \varphi$.

If (Z, φ, q) and (Z_1, φ_1, q_1) are two fibrewise compactifications of (Y, p), we say that (Z, φ, q) is smaller than (Z_1, φ_1, q_1) if there exists a continuous map $\psi : Z_1 \to Z$ such that $\psi \circ \varphi_1 = \varphi$.

Note that $q \circ \psi \circ \varphi_1 = q \circ \varphi = p = q_1 \circ \varphi_1$ and so $q \circ \psi = q_1$, that is, ψ is a morphism of fibre spaces.

1.1. Construction of fibrewise compactifications.

We denote by $\mathcal{C}_b(Y)$ (resp. $\mathcal{C}_0(Y)$) the C^*-algebra of continuous bounded (resp. vanishing at infinity) functions from Y to $\mathbb{C}$, and $\mathcal{C}_c(Y)$ will be the involutive subalgebra of continuous functions with compact support. We set

$$p^*\mathcal{C}_0(X) = \{f \circ p : f \in \mathcal{C}_0(X)\}.$$

Given a fibrewise compactification (Z, φ, q) of (Y, p), we usually identify Y with the open subset $\varphi(Y)$ of Z. Then $\varphi^* : g \mapsto g|_Y$ is an isomorphism from $\mathcal{C}_0(Z)$ onto a C^*-subalgebra of $\mathcal{C}_b(Y)$ which obviously contains $\mathcal{C}_0(Y)$ and also $p^*\mathcal{C}_0(X)$ since q is proper. We shall freely identify $\mathcal{C}_0(Z)$ to $\varphi^*\mathcal{C}_0(Z)$.

Let us denote by $\mathcal{C}_0(Y, p)$ the closure of

$$\mathcal{C}_c(Y, p) = \{g \in \mathcal{C}_b(Y) : \exists K \text{ compact } \subset X, \operatorname{Supp} g \subset p^{-1}(K)\},$$

where Supp is the support of g. Note that $\mathcal{C}_0(Y, p)$ is the C^*-algebra of continuous bounded functions g on Y such that for every $\varepsilon > 0$ there exists a compact subset K of X satisfying $|g(y)| \leq \varepsilon$ if $y \notin p^{-1}(K)$. One immediately checks that

$$p^*\mathcal{C}_0(X) + \mathcal{C}_0(Y) \subset \varphi^*\mathcal{C}_0(Z) \subset \mathcal{C}_0(Y, p).$$

Proposition 1.2. *The fibrewise compactifications of (Y, p) are in bijective correspondence with the C^*-subalgebras A of $\mathcal{C}_0(Y, p)$ containing $p^*\mathcal{C}_0(X) + \mathcal{C}_0(Y)$. More precisely, to A we associate (Z, φ, q), where Z is the Gelfand spectrum of A, φ is defined by the essential embedding of $\mathcal{C}_0(Y)$ into A, and q is defined by the embedding of $\mathcal{C}_0(X)$ into $\mathcal{C}_0(Z)$ via p^*. The inverse map sends (Z, φ, q) to $\varphi^*\mathcal{C}_0(Z)$.*

Proof. Let A be as in the above statement and denote by Z its Gelfand spectrum. Observe that $\mathcal{C}_0(Y)$ is an ideal of A and therefore Y is canonically identified to an open subspace of Z. Moreover, Y is dense in Z since $\mathcal{C}_0(Y)$ is an essential ideal of A.

Now, let us fix $z \in Z$ and consider the map $f \mapsto (f \circ p)(z)$ defined on $\mathcal{C}_0(X)$. It is a homomorphism. Let us check that it is non-zero, hence a character of $\mathcal{C}_0(X)$. Indeed, let $\mathcal{K}$ be the family of compact subsets of X, ordered by inclusion. For $K \in \mathcal{K}$ we choose $u_K : X \to [0, 1]$ in $\mathcal{C}_c(X)$ such that $u_K(x) = 1$ if $x \in K$. Then $(u_K \circ p)_{K \in \mathcal{K}}$ is an approximate unit of $\mathcal{C}_0(Y, p)$. Therefore, if $(u_K \circ p)(z) = 0$ for all K, we get $g(z) = 0$ for every $g \in A$, a contradiction. It follows that there exists a unique element in X, that we denote by $q(z)$, such that $(f \circ p)(z) = f(q(z))$ for every $f \in \mathcal{C}_0(X)$. In particular, we get $p(z) = q(z)$ when $z \in Y$. Moreover, since $f \circ q$ is continuous for every $f \in \mathcal{C}_0(X)$, we see that q is continuous. Finally, q is a proper map because $f \circ q$ vanishes at infinity for every $f \in \mathcal{C}_0(X)$.

The other assertions of the proposition are immediate. □

Remarks 1.3. Observe that $p^*\mathcal{C}_0(X) + \mathcal{C}_0(Y)$ is a C^*-algebra (see [7, Corollary 1.8.4]). So, there is a smallest fibrewise compactification, which is given by the spectrum of $p^*\mathcal{C}_0(X) + \mathcal{C}_0(Y)$. We denote it by (Y_p^+, p^+) and call it the *fibrewise Alexandroff compactification*. The largest fibrewise compactification is given by the spectrum of $\mathcal{C}_0(Y, p)$. We denote it by $(\beta_p Y, p_\beta)$ and call it the *fibrewise Stone-Čech compactification*.

We shall use several times the fact, remarked in the proof of the previous proposition, that $p^*\mathcal{C}_0(X)$ contains an approximate unit of $\mathcal{C}_0(Y, p)$.

When X is reduced to a point, the fibrewise compactifications are the usual compactifications. When X is compact, we have $\mathcal{C}_c(Y,p) = \mathcal{C}_b(Y)$. Therefore $\beta_p Y$ is the usual Stone-Čech compactification βY. When $Y = X \times V$ is a product with X compact, the Alexandroff fibrewise compactification of Y with respect to the first projection is $X \times V^+$. On the other hand, when X is infinite, compact, and V is not compact, we have $\beta_p(X \times V) = \beta(X \times V)$, which differs from $X \times \beta V$ (see [8]). Finally, let us observe that whenever p is proper we have $\mathcal{C}_0(Y) = \mathcal{C}_0(Y,p)$. So, in this case (Y,p) has only one fibrewise compactification, namely itself.

The next proposition shows that $\beta_p Y$ is the solution of an universal problem.

Proposition 1.4. *Let (Y,p) and (Y_1,p_1) be two fibre spaces over X, where (Y_1,p_1) is fibrewise compact. Let $\varphi_1 : (Y,p) \to (Y_1,p_1)$ be a morphism. There exists a unique continuous map $\Phi_1 : \beta_p Y \to Y_1$ which extends φ_1. Moreover, Φ_1 is a morphism of fibre spaces, that is, $p_\beta = p_1 \circ \Phi_1$.*

Proof. We have $\varphi_1^* \mathcal{C}_0(Y_1) \subset \mathcal{C}_0(Y,p) = \mathcal{C}_0(\beta_p Y)$. For $z \in \beta_p Y$, we consider the homomorphism $f \mapsto (f \circ \varphi_1)(z)$ defined on $\mathcal{C}_0(Y_1)$. Since p_1 is proper, we have $p_1^* \mathcal{C}_0(X) \subset \mathcal{C}_0(Y_1)$ and therefore

$$p^* \mathcal{C}_0(X) = \varphi_1^* \big(p_1^* \mathcal{C}_0(X) \big) \subset \varphi_1^* \mathcal{C}_0(Y_1).$$

As shown in the proof of the previous proposition, $p^* \mathcal{C}_0(X)$ contains an approximate unit for $\mathcal{C}_0(Y,p)$. It follows that $f \mapsto (f \circ \varphi_1)(z)$ is a non-zero homomorphism, thus is of the form $f \mapsto f(\Phi_1(z))$ for a unique $\Phi_1(z) \in Y_1$. Of course, Φ_1 extends φ_1 and is continuous. Finally, observe that for $y \in Y$ and $f \in \mathcal{C}_0(X)$, we have

$$f \circ p_\beta(y) = f \circ p(y) = f \circ p_1 \circ \varphi_1(y) = f \circ p_1 \circ \Phi_1(y)$$

and so $p_\beta(y) = p_1 \circ \Phi_1(y)$. Since Y is dense in $\beta_p Y$ we get $p_\beta = p_1 \circ \Phi_1$. □

Remark 1.5. Let (Z,φ,q) be a fiberwise compactification of (Y,p). Then Y^x is an open subset of the compact space Z^x, but in general Y^x is not dense in Z^x, as shown by the following elementary example, and therefore Z^x is not a compactification of Y^x in the usual sense. We take $X = [0,1]$, $Y = (X \times \{0\}) \cup (]0,1] \times \{1\}) \subset \mathbb{R}^2$ with the topology induced by $\mathbb{R}^2$, and p is the projection on X. Then $Y_p^+ = X \times \{0,1\}$ and p^+ is still the first projection. In particular, the fibre of Y_p^+ above 0 is $Y^0 \cup \{(0,1)\}$ with $Y^0 = \{(0,0)\}$. Note also that $\beta_p Y = \beta Y$ since X is compact. Then $p_\beta^{-1}(0)$ is the disjoint union of the two compact spaces Y^0 and $\beta(]0,1]) \setminus]0,1]$, whereas $p_\beta^{-1}(x) = Y^x$ for $x \in]0,1]$.

Coming back to the general situation, the following elementary observation will be useful.

Lemma 1.6. *Let (Z,φ,q) be a fibrewise compactification of (Y,p). Let V be an open subset of X. Then $p^{-1}(V)$ is dense in $q^{-1}(V)$.*

Proof. This follows immediately from the density of Y in Z. □

Remark 1.7. Let (Y, p) be a fibre space over X. We observe that *there is a greatest open subset U of X such that the restriction of p to $p^{-1}(U)$ is proper.* Indeed, let $(U_i)_{i\in I}$ be a family of open subsets with this property. Then $\cup_{i\in I} U_i$ still has this property. To show this fact, let K be a compact subset of $\cup_{i\in I} U_i$. Every $x \in K$ has a compact neighborhood which is contained in some U_i. Then we can cover K by a finitely many compact sets $C_1, \dots, C_n$ with C_k contained in some U_{i_k}, for $k = 1, \dots, n$. It follows that $p^{-1}(K) = \cup_{k=1}^n p^{-1}(K \cap C_k)$ is compact.

Note that U is described as the set of elements of X having a compact neighborhood K such that $p^{-1}(K)$ is compact.

Proposition 1.8. *Let (Y, p) be a fibre space over X and U as above. Let (Z, φ, q) be a fibrewise compactification of (Y, p). Then $q^{-1}(U) = p^{-1}(U)$. So, Z and Y have the same fibres over $x \in U$.*

Proof. Let V be an open set contained in a compact subset K of U. Then we have $p^{-1}(V) \subset p^{-1}(K)$. Observe that $p^{-1}(K) \subset p^{-1}(U)$ is a compact subset of $q^{-1}(U)$ and that $p^{-1}(V)$ is dense into $q^{-1}(V)$. It follows that $q^{-1}(V) = p^{-1}(V)$ and so $q^{-1}(U) = p^{-1}(U)$ since U is the union of such open sets V. □

Remark 1.9. Even if $x \in X$ is so that Y^x is compact, it may happen that Z^x is strictly bigger, and even be very huge as seen in Remark 1.5.

1.2. **The Alexandroff fibrewise compactification.** Its fibres have a simple description, in contrast with the Stone-Čech situation.

Proposition 1.10. *Let (Y, p) be a fibre space over X and let U be the greatest open subset of X such that the restriction of p to $p^{-1}(U)$ is proper.*

(i) *The fibre $(Y_p^+)^x$ of the Alexandroff fibrewise compactification of (Y, p) is of the following form:*
  - *$(Y_p^+)^x = Y^x$ if $x \in U$;*
  - *$(Y_p^+)^x = (Y^x)^+ = Y^x \cup \{\omega_x\}$, the Alexandroff compactification of Y^x, if Y^x is not compact;*
  - *$(Y_p^+)^x$ is the disjoint union of Y^x and a singleton $\{\omega_x\}$ when $x \notin U$ with Y^x compact.*

(ii) *Y_p^+ is the disjoint union of Y and $F = \{\omega_x : x \in X \setminus U\}$. Moreover, Y is a dense open subset of Y_p^+ and F, with the induced topology, is canonically homeomorphic to $X \setminus U$.*

Proof. Recall that Y_p^+ is the spectrum of the the abelian C^*-algebra $A = I + B$ where I is the ideal $\mathcal{C}_0(Y)$ and $B = p^*\mathcal{C}_0(X)$. It follows that the spectrum Y of I is identified with the open subset of characters of A that are non-zero on I, and

$F = Y_p^+ \setminus Y$ is the set of characters whose kernel contains I, that is, the set of characters of A/I. Since A/I is isomorphic to $B/(B \cap I)$, we have first to determine $B \cap I$.

Let $g = f \circ p$ with $f \in \mathcal{C}_0(X)$ and assume that $g \in\in \mathcal{C}_0(Y)$. We claim that $f(x) = 0$ when $x \notin U$. Indeed, assume on the contrary that $f(x) \neq 0$. Set $\varepsilon = |f(x)|/2$. Observe that $C = \{y \in Y : |f \circ p(y)| \geq \varepsilon\}$ is compact and that $p(C)$ is a compact neighborhood of x. Denote by V its interior. Since $p^{-1}(p(C)) = C$ is compact, we see that the restriction of p to $p^{-1}(V)$ is proper, and so $V \subset U$, a contradiction. It follows that $B \cap I$ is contained into $p^*\mathcal{C}_0(U)$.

On the other hand, since the restriction of p to $p^{-1}(U)$ is proper we have $p^*\mathcal{C}_0(U) \subset \mathcal{C}_0(Y)$. It follows that $B \cap I = p^*\mathcal{C}_0(U)$.

Then $B/(B \cap I) = p^*\mathcal{C}_0(X)/p^*\mathcal{C}_0(U)$ is isomorphic to $p^*\mathcal{C}_0(X \setminus U)$ and thus to $\mathcal{C}_0(X \setminus U)$. The spectrum F of $B/(B \cap I)$ is thus homeomorphic to $X \setminus U$. The inverse homeomorphism sends $x \in X \setminus U$ onto the character ω_x defined by

$$\omega_x(h) = f(x)$$

for any decomposition $h = g + f \circ p$ as a sum of an element $g \in I$ and an element $f \circ p \in B$. This is not ambiguous since $B \cap I = p^*\mathcal{C}_0(U)$.

The rest of the proposition is now immediate. □

Remark 1.11. The topology of Y_p^+ is described in the following way: a subset Ω of Y_p^+ is open if and only if

- $\Omega \cap Y$ is open in Y and $\Omega \cap F$ is open in F (which is canonically homeomorphic to $X \setminus U$);
- for every compact subset K of X, then $(p^+)^{-1}(K) \cap (Y \setminus \Omega)$ is compact in Y.

The proof follows the same lines as the proof given in [17] establishing that the unified space Z of p is fibrewise compact. In fact, as a set, $Z = Y \sqcup X$ and Y_p^+ is nothing else than the closure of Y in the topological space Z. We shall not need these observations in the sequel. The details are left as an exercise.

1.3. **The case of étale fibre spaces.** As already said, the example most important for us is that of an étale groupoid and $p = r$, the range map, which is a local homeomorphism (see Subsection 2.1). In the general case of an étale fibre space, we give in Proposition 1.13 a concrete description the above open subset U of X.

Definition 1.12. We say that a fibre space (Y, p) over X is *étale*[2] if every $y \in Y$ has an open neighborhood S such that $p(S)$ is open and the restriction of p to S is a homeomorphism onto $p(S)$. We denote by p_S^{-1} its inverse map, which is defined on $p(S)$. Such a set S is called an *open section*.

[2]In [10, Definition 1.19] such fibre spaces are called fibrewise discrete.

We collect several consequences of the definition. First, p is an open map and the fibres Y^x with their induced topology are discrete. Moreover, the topology of Y has a basis of open sections. Finally, for every $g \in \mathcal{C}_c(Y)$, let us set $h(x) = \sum_{y \in Y^x} g(y)$. Then $h \in \mathcal{C}_c(X)$. Indeed, using a partition of unity, it suffices to consider the case where g has its compact support in some open section S. Then we have $h(x) = f \circ p_S^{-1}(x)$ if $x \in p(S)$ and $h(x) = 0$ otherwise. The conclusion follows immediately.

Proposition 1.13. *Let (Y, p) be an étale fibre space over X. Let W be the open subset of X formed by the elements x such that the cardinal of the fibres of Y is finite and constant in a neighborhood of x. Then, W is the greatest open subset U of X such that the restriction of p to $p^{-1}(U)$ is proper.*

Proof. Assume that $x \in W$ and write $Y^x = \{y_1, \dots, y_n\}$. We choose disjoint open sections $S_1, \dots, S_n$ with $y_i \in S_i$ for $i = 1, \dots, n$. We may choose $S_1, \dots, S_n$ small enough so that they have the same image V under p and that $p^{-1}(V) = \cup_{i=1}^n S_i$ since the fibres of Y have the same cardinal n in a neighborhood of x. Let K be a compact neighborhood of x contained in V. Then $p^{-1}(K) = \cup_{i=1}^n p^{-1}(K) \cap S_i$ is a finite union of compact sets and therefore is compact. Therefore $x \in U$ (see Remark 1.7). Conversely, assume that $x \in U$. It has a compact neighborhood K such that $p^{-1}(K)$ is compact. Let $f \in \mathcal{C}_c(X)$ be with support in K and equal to 1 in a neighborhood of x. Then $f \circ p$ belongs to $\mathcal{C}_c(Y)$. For $x_1 \in X$, let us set $h(x_1) = \sum_{y \in Y^{x_1}} f \circ p(y)$. Then $h : X \to \mathbb{R}^+$ is continuous and is equal to the cardinal of Y^{x_1} in a neighborhood of x. Thus, we have $x \in W$. □

1.4. **Fibres in the general case.** Let (Z, q) be a fibre space over X. Then $\mathcal{C}_0(Z)$ is a $C_0(X)$-algebra. Let us recall the definition and some facts we shall need about such algebras. For more details we refer to [3], [16].

Definition 1.14. A $\mathcal{C}_0(X)$-*algebra* is a C^*-algebra A equipped with a non-degenerate $*$-homomorphism from $\mathcal{C}_0(X)$ into the center of the multiplier algebra of A.

We denote by $f \cdot a$ the product of $f \in \mathcal{C}_0(X)$ with $a \in A$. Recall that the action of $\mathcal{C}_0(X)$ on A is non-degenerate if there exists an approximate unit (u_λ) of $\mathcal{C}_0(X)$ such that $\lim_\lambda u_\lambda a = a$ for every $a \in A$. Here A will be $\mathcal{C}_0(Z)$ and therefore its multiplier algebra is $\mathcal{C}_b(Z)$.

Given $x \in X$, let us denote by $\mathcal{C}_x(X)$ the ideal in $\mathcal{C}_0(X)$ formed by its elements f such that $f(x) = 0$. Let I_x be the closed linear span of $\{fa : f \in \mathcal{C}_x(X), a \in A\}$. It is a closed ideal A and in fact, we have $I_x = \mathcal{C}_x(X)A = \{fa : f \in \mathcal{C}_x(X), a \in A\}$ (see [3, Corollaire 1.9]). We denote by $A_x = A/I_x$ the quotient C^*-algebra and by a_x the image of $a \in A$ in the quotient. Then A appears as an upper semi-continuous field of C^*-algebras, in the sense that for $a \in A$ the function $x \mapsto \|a_x\|$ is upper semi-continuous [16, Proposition 1.2].

In the case of $A = \mathcal{C}_0(Z)$ where (Z, q) is a fibre space over X, we define a structure of $\mathcal{C}_0(X)$-algebra on $\mathcal{C}_0(Z)$ by setting $f \cdot g = (f \circ q)g$.

Proposition 1.15. *Let (Z, q) as above. For $x \in X$, the fibre Z^x is the Gelfand spectrum of $A_x = \mathcal{C}_0(Z)/\mathcal{C}_x(X)\mathcal{C}_0(Z)$.*

Proof. We set $I_x = \mathcal{C}_x(X)\mathcal{C}_0(Z)$. We denote by θ_x the map from $\mathcal{C}_0(Z)$ onto $\mathcal{C}_0(Z^x)$ which sends $g \in \mathcal{C}_0(Z)$ onto its restriction to Z^x. Let us show that $I_x = \ker \theta_x$. The inclusion $I_x \subset \ker \theta_x$ is immediate since every element of I_x is of the form $(f \circ q)g$ with $f(x) = 0$. Take now $g \in \ker \theta_x$. Given $\varepsilon > 0$, let $K = \{z \in Z : |g(z)| \geq \varepsilon\}$. Let us choose $f \in \mathcal{C}_x(X)$ such that $f(x') = 1$ if $x' \in q(K)$ and $f(X) \subset [0, 1]$. Then $(f \circ q)g \in I_x$ and $\|(f \circ q)g - g\| \leq \varepsilon$. So, we have $g \in I_x$. □

Remark 1.16. Let us consider the more specific case where (Z, φ, q) is a fibrewise compactification of (Y, p). We identify $\mathcal{C}_0(Z)$ with the C^*-algebra $\varphi^*\mathcal{C}_0(Z)$ obtained by restriction to Y. We may consider the map $\tilde{\theta}_x : \varphi^*\mathcal{C}_0(Z) \to \mathcal{C}_b(Y^x)$ sending g to its restriction to the closed subset Y^x of Y. Let denote by $\tilde{I}_x$ the kernel of $\tilde{\theta}_x$. Obviously, we have $I_x \subset \tilde{I}_x$.

Let us consider the case where Y^x is not compact and $Z = Y_p^+$. We see first that $\tilde{\theta}_x\big(p^*\mathcal{C}_0(X) + \mathcal{C}_0(Y)\big)$ is the C^*-algebra of continuous functions on Y^x which have a limit at infinity. Second, we have $\tilde{I}_x = I_x$. Indeed, assume that the restriction of $f \circ p + g$ to Y^x is equal to zero, where $f \in \mathcal{C}_0(X)$ and $g \in \mathcal{C}_0(Y)$. Then, since Y^x is not compact, we immediately see that $f \in \mathcal{C}_x(X)$ and thus $f \circ p \in I_x$. Moreover, we get that $g \in \tilde{I}_x$. It is an easy exercise to see that $g \in I_x$, since $g \in \mathcal{C}_0(Y)$ (see the proof of the previous proposition). It follows that $(Y_p^+)^x = (Y^x)^+$, a fact already observed in Proposition 1.10.

Still with Y^x non compact, let us consider the case where $Z = \beta_p Y$. Then, the range of $\tilde{\theta}_x$ is $\mathcal{C}_b(Y)$. It follows that $\beta(Y^x)$ is a closed subset of $(\beta_p Y)^x$. However, we shall see in Section 2.3 that I_x happens to be strictly smaller than $\tilde{I}_x$, and therefore $\beta(Y^x)$ can be strictly included into $(\beta_p Y)^x$.

2. Fibrewise equivariant compactifications of $\mathcal{G}$-spaces

Let G be a discrete group. A *G-space* is a locally compact space Y on which G acts to the left by homeomorphisms. Then G acts on $\mathcal{C}_b(Y)$ by $sf(y) = f(s^{-1}y)$.

A *G-equivariant compactification* of Y is a pair (Z, φ) where Z is a compact G-space and φ is a G-equivariant homeomorphism from Y onto an open dense subspace of Z. The map sending A to its spectrum is a bijection from the set of unital G-invariant sub-C^*-algebras A of $\mathcal{C}_b(Y)$ which contain $\mathcal{C}_0(Y)$ onto the set of G-equivariant compactifications of Y. Note that Y^+ and βY are such equivariant compactifications.

In this section, we extend these observations to the case of étale groupoids.

2.1. **Preliminaries on étale groupoids.** Let us recall some basic notions and notation. For more details we refer to [14], [13]. A *groupoid* is a small category in which every morphism is invertible. More precisely, it consists of a set $\mathcal{G}$ of morphisms and a subset $\mathcal{G}^{(0)}$ of objects (also called *units*) together with *source* and *range* maps $s, r : \mathcal{G} \to \mathcal{G}^{(0)}$, a *composition law* (or *product*) $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)} \mapsto \gamma_1\gamma_2 \in \mathcal{G}$, where

$$\mathcal{G}^{(2)} = \{(\gamma_1, \gamma_2) \in \mathcal{G} \times \mathcal{G} : s(\gamma_1) = r(\gamma_2)\},$$

and an *inverse* map $\gamma \mapsto \gamma^{-1}$ such that

- $s(\gamma_1\gamma_2) = s(\gamma_2)$ and $r(\gamma_1\gamma_2) = r(\gamma_1)$ for $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)}$;
- $s(x) = x = r(x)$ for $x \in \mathcal{G}^{(0)}$;
- $\gamma s(\gamma) = \gamma = r(\gamma)\gamma$ for $\gamma \in \mathcal{G}$;
- $(\gamma_1\gamma_2)\gamma_3 = \gamma_1(\gamma_2\gamma_3)$ whenever $s(\gamma_1) = r(\gamma_2)$ and $s(\gamma_2) = r(\gamma_3)$;
- $s(\gamma) = r(\gamma^{-1})$ and $\gamma\gamma^{-1} = r(\gamma)$ (and so $r(\gamma) = s(\gamma^{-1})$ and $\gamma^{-1}\gamma = s(\gamma)$).

A *locally compact groupoid* is a groupoid $\mathcal{G}$ equipped with a locally compact topology such that the structure maps are continuous, where $\mathcal{G}^{(2)}$ has the topology induced by $\mathcal{G} \times \mathcal{G}$ and $\mathcal{G}^{(0)}$ has the topology induced by $\mathcal{G}$.

An *étale groupoid* is a locally compact groupoid $\mathcal{G}$ such that the range and source maps are local homeomorphisms from $\mathcal{G}$ into $\mathcal{G}^{(0)}$. Observe that the fibres $\mathcal{G}^x = r^{-1}(x)$ are discrete and that $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$ (see [14, Proposition 2.8]). Etale groupoids are sometimes called r-discrete.

Examples of étale groupoids are plentiful. Let us mention groupoids associated with discrete group actions, local homeomorphisms, pseudo-groups of partial homeomorphisms, topological Markov shifts, graphs,... (see [14], [6], [11], [2], [15] for a non exhaustive list). For a brief account on the notion of étale groupoid see also [5, Section 5.6]. Note that a locally compact space X is a particular case of étale groupoid. In this case, $\mathcal{G} = \mathcal{G}^0 = X$, the source and range maps are the identity one, and the product is $(x, x) \mapsto x$. This groupoid is said to be *trivial*.

Given an étale groupoid $\mathcal{G}$, let us observe that $(\mathcal{G}, r)$ is an étale fibre space over the space $\mathcal{G}^{(0)}$ of units. A *bisection* is a subset S of $\mathcal{G}$ such that the restrictions of r and s to S are injective. Given an open bisection S, we shall denote by r_S^{-1} the inverse map, defined on the open subset $r(S)$ of $\mathcal{G}^{(0)}$, of the restriction of r to S. Note that r_S^{-1} is continuous.

An étale groupoid $\mathcal{G}$ has a cover by open bisections. These open bisections form an inverse semigroup with composition law

$$ST = \Big\{\gamma_1\gamma_2 : (\gamma_1, \gamma_2) \in (S \times T) \cap \mathcal{G}^{(2)}\Big\},$$

the inverse S^{-1} of S being the image of S under the inverse map of $\mathcal{G}$ (see [13, Proposition 2.2.4]). A compact subset K of $\mathcal{G}$ is covered by a finite number of open bisections. Thus, using partitions of unity, we see that every element of $\mathcal{C}_c(\mathcal{G})$ is a finite sum of continuous functions whose compact support is contained in some open bisection.

In the sequel, *we fix an étale groupoid and we denote by X its set of units.* If $p_i : Y_i \to X$, $i = 1, 2$, are two maps, we denote by $Y_1 \,{}_{p_1}\!*_{p_2} Y_2$ (or $Y_1 * Y_2$ when there is no ambiguity) the fibred product $\{(y_1, y_2) \in Y_1 \times Y_2 : p_1(y_1) = p_2(y_2)\}$. Whenever Y_1 and Y_2 are topological spaces, we equip $Y_1 \,{}_{p_1}\!*_{p_2} Y_2$ with the topology induced by the product topology.

Definition 2.1. A left $\mathcal{G}$-*space* is a fibre space (Y, p) over $X = \mathcal{G}^{(0)}$, equipped with a continuous map $(\gamma, y) \mapsto \gamma y$ from $\mathcal{G} \,{}_s\!*_p Y$ into Y, satisfying the following conditions:

(i) $p(\gamma y) = r(\gamma)$ for $(\gamma, y) \in \mathcal{G} \,{}_s\!*_p Y$, and $p(y)y = y$ for $y \in Y$;
(ii) if $(\gamma_1, y) \in \mathcal{G} \,{}_s\!*_p Y$ and $(\gamma_2, \gamma_1) \in \mathcal{G}^{(2)}$, then $(\gamma_2\gamma_1)y = \gamma_2(\gamma_1 y)$.

Note that $y \mapsto \gamma y$ is a homeomorphism from $Y^{s(\gamma)}$ onto $Y^{r(\gamma)}$. Right $\mathcal{G}$-spaces are defined similarly. Without further precisions, a $\mathcal{G}$-space will be a left $\mathcal{G}$-space.

A *morphism* $\varphi : (Y, p) \to (Y_1, p_1)$ between two $\mathcal{G}$-spaces is said to be $\mathcal{G}$-*equivariant* if $\varphi(\gamma y) = \gamma\varphi(y)$ for every $(\gamma, y) \in \mathcal{G} \,{}_s\!*_p Y$.

Let us observe that $(\mathcal{G}, r)$ is a $\mathcal{G}$-space in an obvious way, as well as X. It this latter case, the action of $\gamma \in s^{-1}(x)$ onto $x \in X$ will be denoted by $\gamma \cdot x$, in order to distinguish it from $\gamma x = \gamma$. By definition, we have $\gamma \cdot x = r(\gamma)$. We also note that if (Y, p) is a $\mathcal{G}$-space, then p is $\mathcal{G}$-equivariant: $p(\gamma y) = \gamma \cdot p(y)$.

An open bisection S defines a homeomorphism from $p^{-1}(s(S))$ onto $p^{-1}(r(S))$, by sending y onto γy, where γ is the unique element of S such that $s(\gamma) = p(y)$. We set $Sy = \gamma y$. This applies to $(Y, p) = (\mathcal{G}, r)$ and to $(Y, p) = (X, \mathrm{Id})$. In the latter case, we write $\gamma \cdot x = S \cdot x$. Note that $S \cdot p(y) = p(Sy)$ and thus, for every subset W of $p^{-1}(s(S))$, we have

$$S \cdot p(W) = p(SW).$$

Proposition 2.2. *Let $\mathcal{G}$ be an étale groupoid and (Y, p) a $\mathcal{G}$-space. Let U be the greatest open subset of $X = \mathcal{G}^{(0)}$ such that the restriction of p to $p^{-1}(U)$ is proper. Then U is invariant under the $\mathcal{G}$-action, that is, $r(\gamma) \in U$ if and only if $s(\gamma) \in U$.*

Proof. Let γ be such that $s(\gamma) \in U$ and let S be a compact bisection which is a neighborhood of γ. By Remark 1.7 there exists a compact neighborhood K of $s(\gamma)$ contained into $s(S)$ such that $p^{-1}(K)$ is compact. Then $S \cdot K$ is a compact neighborhood of $r(\gamma)$ and $p^{-1}(S \cdot K) = Sp^{-1}(K)$ is compact. Therefore we have $r(\gamma) \in U$, again using Remark 1.7. $\square$

2.2. **Construction of $\mathcal{G}$-equivariant fibrewise compactifications.** Let (Y,p) be a $\mathcal{G}$-space. We want to study the equivariant fibrewise compactifications of (Y,p) as defined below. The groupoid $\mathcal{G}$ will still be étale and X will denote the set of units of $\mathcal{G}$.

Definition 2.3. A *$\mathcal{G}$-equivariant fibrewise compactification* of the $\mathcal{G}$-space (Y,p) is a fibrewise compactification (Z,φ,q) of (Y,p) such that (Z,q) is a $\mathcal{G}$-space satisfying $\varphi(\gamma y)=\gamma\varphi(y)$ for every $(\gamma,y)\in\mathcal{G}\,{}_s{*}_p\,Y$.

Via φ we shall usually identify Y to a $\mathcal{G}$-subspace of Z. The fibrewise compactifications were characterized in terms of specific C^*-subalgebras of $\mathcal{C}_b(Y)$. The $\mathcal{G}$-equivariant ones are characterized by those C^*-subalgebras that are $\mathcal{G}$-invariant. This notion of invariance is defined by using convolution products.

For $g\in\mathcal{C}_c(\mathcal{G})$ and $f\in\mathcal{C}_b(Y)$ we define the convolution product $g*f$ by

$$(g*f)(y)=\sum_{\gamma\in r^{-1}(p(y))} g(\gamma)f(\gamma^{-1}y). \tag{1}$$

Thanks to the fact that g is a finite sum of continuous functions supported in an open bisection, to study the properties of the convolution product it suffices to consider the case where g has a compact support K contained in an open bisection S. Then

$$\begin{aligned}(g*f)(y)&=0 \quad\text{if}\quad y\notin p^{-1}(r(K))\\ &=g(\gamma)f(\gamma^{-1}y)\quad\text{if}\quad y\in p^{-1}(r(K)),\end{aligned}$$

where $\gamma=r_S^{-1}(p(y))$ and $\gamma^{-1}y=S^{-1}y$. It follows that for every $g\in\mathcal{C}_c(\mathcal{G})$ we have $g*f\in\mathcal{C}_c(Y,p)\subset\mathcal{C}_b(Y)$ with $|(g*f)(y)|\leq\big(\sum_{\{\gamma:r(\gamma)=p(y)\}}|g(\gamma)|\big)\|f\|_\infty$.

Let (Z,φ,q) be a fibrewise compactification of (Y,p). Recall that the restriction map $F\mapsto F\circ\varphi$ identifies $\mathcal{C}_0(Z)$ to a subalgebra of $\mathcal{C}_b(Y)$). Therefore, for $g\in\mathcal{C}_c(\mathcal{G})$, the convolution product $g*F$ makes sense as an element of $\mathcal{C}_b(Y)$, when defined by $g*F=g*(F\circ\varphi)$. We say that $\mathcal{C}_0(Z)$ is *stable under convolution by the elements of* $\mathcal{C}_c(\mathcal{G})$ if $g*F\in\mathcal{C}_0(Z)$ for every $F\in\mathcal{C}_0(Z)$. In this case $g*F$ denotes both the function defined on Y and its unique extension to Z. Under this stability assumption, we want to show that there is a unique continuous $\mathcal{G}$-action on Z extending the given $\mathcal{G}$-action on Y. For $(\gamma_0,z_0)\in\mathcal{G}\,{}_s{*}_q\,Z$, we first explain how γ_0z_0 is defined.

Lemma 2.4. *We assume that $\mathcal{C}_0(Z)$ is stable under convolution by the elements of $\mathcal{C}_c(\mathcal{G})$. Let $(\gamma_0,z_0)\in\mathcal{G}\,{}_s{*}_q\,Z$. Let S be an open bisection such that $\gamma_0\in S$ and let a be a continuous function on $X=\mathcal{G}^{(0)}$ with compact support in $s(S)$, such that $a(x)=1$ for x in a neighborhood of $s(\gamma_0)$. We set $h=(a\circ s)\,\chi_S$ and $g(\gamma)=h(\gamma^{-1})$ (where χ_S is the characteristic function of S).*

(i) *Let $F\in\mathcal{C}_0(Z)$. If $z_0\in Y$, we have $(g*F)(z_0)=F(\gamma_0z_0)$.*

(ii) *For $F \in \mathcal{C}_0(Z)$, the complex number $(g * F)(z_0)$ does not depend on the choice of a and S satisfying the above properties, but only on (γ_0, z_0).*
(iii) *$F \in \mathcal{C}_0(Z) \mapsto (g * F)(z_0)$ is a character of $\mathcal{C}_0(Z)$ that we denote by $\gamma_0 z_0$.*

Proof. (i) Observe that $g \in \mathcal{C}_c(\mathcal{G})$ and so $g * F \in \mathcal{C}_0(Z)$. For $y \in Y \subset Z$, we have

$$\begin{aligned}(g * F)(y) &= a \circ p(y) F(Sy) \quad \text{if} \quad p(y) \in \operatorname{Supp} a \subset s(S) \\ &= 0 \quad \text{otherwise.}\end{aligned} \tag{2}$$

Obviously, if $z_0 \in Y$ we have $(g * F)(z_0) = F(\gamma_0 z_0)$.

(ii) Consider two open bisections S_1, S_2 with $\gamma_0 \in S_1 \cap S_2$, and let a_1, a_2 be two continuous functions with compact support in S_1 and S_2 respectively, with constant value 1 on a neighborhood of $s(\gamma_0)$. Then by (2), there is an open neighborhood V of $s(\gamma_0)$ in X such that $g_1 * F$ and $g_2 * F$ coincide on $p^{-1}(V)$. By Lemma 1.6, we see that $g_1 * F = g_2 * F$ on $q^{-1}(V)$ and in particular we have $(g_1 * F)(z_0) = (g_2 * F)(z_0)$.

(iii) To show that $F \mapsto (g * F)(z_0)$ is a character, we first observe that given F and F' in $\mathcal{C}_0(Z)$, we have, for y such that $p(y)$ belongs to a neighborhood V of $s(\gamma_0)$ on which a takes value 1,

$$a \circ p(y)(FF')(Sy) = a \circ p(y) F(Sy)\, a \circ p(y) F'(Sy),$$

that is

$$(g * (FF'))(y) = (g * F)(y)\,(g * F'(y).$$

Once again we conclude thanks Lemma 1.6 that this equality still holds when y is replaced by z_0.

It remains to check that there exists $F \in \mathcal{C}_0(Z)$ such that $(g * F)(z_0) \neq 0$. We take S to be relatively compact. Let V be a compact neighborhood of $s(\gamma_0)$ in X, contained in $s(S)$ and on which $a(x) = 1$. We choose $f \in \mathcal{C}_c(X)$ such that $f(x) = 1$ if $x \in r(SV) = S \cdot V$ and we set $F = f \circ q$. Since q is proper, we note that $F \in \mathcal{C}_c(Z)$. For $y \in p^{-1}(V)$, we have, since p is equivariant,

$$(g * F)(y) = F(Sy) = f \circ p(Sy) = f(S \cdot y) = 1.$$

Once more, we deduce from Lemma 1.6 that $(g * F)(z_0) = 1$. □

Note that, by definition,

$$F(\gamma_0 z_0) = (g * F)(z_0). \tag{3}$$

Theorem 2.5. *Let (Y, p) be a $\mathcal{G}$-space, where $\mathcal{G}$ is an étale groupoid. Let (Z, φ, q) be a fibrewise compactification of (Y, p) such that $\mathcal{C}_0(Z)$ is stable under convolution by the elements of $\mathcal{C}_c(\mathcal{G})$. The map $(\gamma_0, z_0) \mapsto \gamma_0 z_0$ is a continuous $\mathcal{G}$-action on Z, and it is the only continuous $\mathcal{G}$-action extending the action of $\mathcal{G}$ on Y.*

Proof. Since $\mathcal{G}\,{}_s{*}_p Y$ is dense into $\mathcal{G}\,{}_s{*}_q Z$, there is at most one continuous extension of the $\mathcal{G}$-action.

For $(\gamma_0, z_0) \in \mathcal{G}\,{}_s{*}_q\, Z$, let us check that $q(\gamma_0 z_0) = r(\gamma_0)$. Let V be a compact neighborhood of $r(\gamma_0)$ in $X = \mathcal{G}^{(0)}$ and let f be a continuous function supported in V and taking value 1 in a neighborhood of $r(\gamma_0)$. Set $F = f \circ q \in \mathcal{C}_0(Z)$ and take g as in the previous lemma. The choices of a and S are such that $a = 1$ on a neighborhood W of $s(\gamma_0)$ and $f = 1$ on $r(SW)$. We have

$$(g * F)(y) = a \circ p(y) F(Sy) = f \circ p(Sy) = 1$$

for every $y \in Y$ such that $p(y) \in W$. It follows that $(g * F)(z) = 1$ for $z \in q^{-1}(W)$. In particular $F(\gamma_0 z_0) = (g * F)(z_0) = 1$, and therefore $q(\gamma_0 z_0) \in V$. We conclude that $q(\gamma_0 z_0) = r(\gamma_0)$ since this holds for every V in a basis of neighborhoods of $r(\gamma_0)$.

Given $z_0 \in Z$, let us check that $F(q(z_0)z_0) = F(z_0)$ for every $F \in \mathcal{C}_0(Z)$. For the definition of $F(q(z_0)z_0)$ we take $S = X$. Then in (2) we get, for $y \in Y$,

$$(g * F)(y) = a \circ p(y) F(y),$$

and therefore $(g * F)(z) = a \circ q(z) F(z)$ for $z \in Z$. In particular, we have $F(q(z_0)z_0) = (g * F)(z_0) = F(z_0)$. It follows that $q(z_0)z_0 = z_0$. Thus, Condition (i) in the definition 2.1 is fulfilled.

Let us show the continuity of $(\gamma, z) \mapsto \gamma z$. Let $(\gamma_0, z_0) \in \mathcal{G}\,{}_s{*}_q\, Z$ and let W be a neighborhood of $\gamma_0 z_0$. Let us consider $F \in \mathcal{C}_0(Z)$ such that $F(\gamma_0 z_0) = 1$ and $F(z) = 0$ if $z \notin W$. We take S, a and g as in Lemma 2.4. Let U be an open neighborhood of γ_0, contained into S such that $a \circ s(\gamma) = 1$ if $\gamma \in U$. For $\gamma \in U$ and every $z \in Z$ with $q(z) = s(\gamma)$ we have $F(\gamma z) = (g * F)(z)$. Since $g * F$ is continuous and $(g * F)(z_0) = 1$, there is a neighborhood V of z_0 in Z on which $g * F > 0$. It follows that for $(\gamma, z) \in U_s{*}_q V$ we have $\gamma z \in W$.

Finally, let us show the associativity: $(\gamma_1\gamma_2)z = \gamma_1(\gamma_2 z)$ when this expression makes sense. We have to check that $F((\gamma_1\gamma_2)z) = F(\gamma_1(\gamma_2 z))$ for every $F \in \mathcal{C}_0(Z)$. For $i = 1, 2$, we choose an open bisection S_i containing γ_i, a continuous function a_i with compact support in $s(S_i)$ and taking value 1 in a neighborhood of $s(\gamma_i)$. We introduce g_i (with respect now to S_i, γ_i instead of S, γ_0) as in Lemma 2.4. A straightforward computation shows that

$$F((\gamma_1(\gamma_2 z)) = g_2 * (g_1 * F)(z).$$

In the groupoid algebra $\mathcal{C}_c(\mathcal{G})$ the convolution product of g_2 by g_1 is defined as

$$\begin{aligned}(g_2 * g_1)(\gamma) &= \sum_{\{\gamma' : r(\gamma') = r(\gamma)\}} g_2(\gamma') g_1(\gamma'^{-1}\gamma) \\ &= a_2 \circ r(\gamma) \sum_{\{\gamma' \in S_2 : s(\gamma') = r(\gamma\}} a_1 \circ s(\gamma^{-1}\gamma'^{-1}) \chi_{S_1}(\gamma^{-1}\gamma'^{-1}).\end{aligned}$$

It follows that $(g_2 * g_1)(\gamma) = 0$ except possibly whenever $\gamma' \in S_2$ together with $\gamma^{-1}\gamma'^{-1} \in S_1$ and then we get

$$(g_2 * g_1)(\gamma) = a_2 \circ r(\gamma) a_1 \circ r(\gamma').$$

Let θ_2 denote the homeomorphism $x \mapsto S_2 \cdot x$ from $s(S_2)$ onto $r(S_2)$. Then we have

$$a_1 \circ r(\gamma') = a_1 \circ \theta_2 \circ s(\gamma') = a_1 \circ \theta_2 \circ r(\gamma),$$

and therefore

$$(g_2 * g_1)(\gamma) = a_2 \circ r(\gamma)(a_1 \circ \theta_2) \circ r(\gamma) \chi_{S_1 S_2}(\gamma^{-1}).$$

We observe that $a_1 \circ \theta_2$ is equal to 1 in a neighborhood of $s(\gamma_2)$. From the beginning we may have chosen S_2 so that $r(S_2) \subset s(S_1)$ and a_1 with compact support in $r(S_2)$. Then $S_1 S_2$ is a bisection containing $\gamma_1\gamma_2$ with $s(S_1 S_2) = s(S_2)$. Moreover, $a_2(a_1 \circ \theta_2)$ is continuous with compact support in $s(S_2)$ and is equal to 1 in a neighborhood of $s(\gamma_2) = s(\gamma_1\gamma_2)$.

In conclusion, we get

$$F((\gamma_1(\gamma_2 z)) = g_2 * (g_1 * F)(z) = \big((g_2 * g_1) * F\big)(z) = F((\gamma_1\gamma_2)z),$$

where the last equality follows from (3). Since this holds for every $F \in \mathcal{C}_0(Z)$, we obtain $\gamma_1(\gamma_2 z) = (\gamma_1\gamma_2)z$. □

Corollary 2.6. *Let (Y,p) be a $\mathcal{G}$-space, where $\mathcal{G}$ is an étale groupoid. The $\mathcal{G}$-equivariant fibrewise compactifications of (Y,p) are in bijective correspondence with the C^*-subalgebras of $\mathcal{C}_0(Y,p)$ which contain $p^*\mathcal{C}_0(X) + \mathcal{C}_0(Y)$ and are stable under convolution by the elements of $\mathcal{C}_c(\mathcal{G})$. This correspondence is the restriction to the set formed by these C^*-subalgebras of the bijection described in Proposition 1.2.*

Proof. The only fact that remains to show is that if (Z,φ,q) is a $\mathcal{G}$-equivariant fibrewise compactification, then $\varphi^*\mathcal{C}_0(Z)$ is stable under convolution by the elements of $\mathcal{C}_c(\mathcal{G})$. Let $F \in \mathcal{C}_0(Z)$ and $g \in \mathcal{C}_c(\mathcal{G})$. We define $g * F$ by the expression

$$(g * F)(z) = \sum_{\gamma \in r^{-1}(q(z))} g(\gamma) F(\gamma^{-1} z).$$

This function is continuous with support in $q^{-1}(r(K))$ where K is the compact support of g. Since q is proper, the support of $g * F$ is compact.

To conclude, we observe that $(g * F) \circ \varphi = g * (F \circ \varphi)$. □

Proposition 2.7. *Let (Y,p) be a $\mathcal{G}$-space, where $\mathcal{G}$ is an étale groupoid. The structure of $\mathcal{G}$-space of (Y,p) extends in a unique way to the Alexandroff and the Stone-Čech fibrewise compactifications. This makes them $\mathcal{G}$-equivariant fibrewise compactifications.*

Proof. We have to show that the C^*-algebra $\mathcal{C}_0(Y,p)$ and $p^*\mathcal{C}_0(X)+\mathcal{C}_0(Y)$ and stable under convolution by the elements of $\mathcal{C}_c(\mathcal{G})$. The verifications are immediate. □

Proposition 2.8. *Let $\mathcal{G}$ be an étale groupoid and (Y,p), (Y_1,p_1) be two $\mathcal{G}$-spaces. We assume that (Y_1,p_1) is fibrewise compact. Let $\varphi_1:(Y,p)\to(Y_1,p_1)$ be a $\mathcal{G}$-equivariant morphism. The unique continuous map $\Phi_1:\beta_pY\to Y_1$ which extends φ_1 is $\mathcal{G}$-equivariant.*

Proof. The two continuous maps $(\gamma,z)\mapsto\Phi_1(\gamma z)$ and $(\gamma,z)\mapsto\gamma\Phi_1(z)$, which are defined on $\mathcal{G}\,{}_s{*}_{p_\beta}\,\beta_pY$ coincide on $\mathcal{G}\,{}_s{*}_p\,Y$ which is dense into $\mathcal{G}\,{}_s{*}_{p_\beta}\,\beta_pY$. □

Note in particular that $(\mathcal{G},r)$ has two important $\mathcal{G}$-equivariant fibrewise compactifications, its Alexandroff fibrewise compactification $(\mathcal{G}_r^+,r^+)$ and its Stone-Čech fibrewise compactification $(\beta_r\mathcal{G},r_\beta)$.

With techniques similar to those used above, it is possible to construct smaller fibrewise compact $\mathcal{G}$-spaces from a given one (Z,q), when $\mathcal{G}$ is an étale groupoid. This fact of independent interest will be needed in [1].

Proposition 2.9. *Let (Z,q) be a $\mathcal{G}$-space where $q:Z\to X=\mathcal{G}^{(0)}$ is proper. Let A be a C^*-subalgebra of $\mathcal{C}_0(Z)$ which contains $q^*\mathcal{C}_0(X)$ and is stable under convolution by the elements of $\mathcal{C}_c(\mathcal{G})$. Denote by Y the Gelfand spectrum of A and by $p:Y\to X$ (resp. $q_Y:Z\to Y$) the continuous surjective map corresponding to the embedding $\mathcal{C}_0(X)\subset\mathcal{C}_0(Y)$ (resp. $\mathcal{C}_0(Y)\subset\mathcal{C}_0(Z)$. Then (Y,p) has a unique structure of $\mathcal{G}$-space which make q_Y equivariant. Moreover, $p\circ q_Y=q$ and therefore p and q_Y are proper.*

Proof. Since $q^*\mathcal{C}_0(X)$ contains an approximate unit for $\mathcal{C}_0(Z)$, the map q_Y is well defined by the equality $\langle q_Y(z),f\rangle=f(z)$ for every $f\in A$ and $z\in Z$. Similarly, p is defined be the equality $\langle y,f\circ q\rangle=f(p(y))$ for $f\in\mathcal{C}_0(X)$ and $y\in Y$. Taking $y=q_Y(z)$ we get $f\circ q(z)=f((p\circ q_Y)(z))$ for $f\in\mathcal{C}_0(X)$ and $z\in Z$. It follows that $p\circ q_Y=q$.

The uniqueness assertion for the structure of $\mathcal{G}$-space of Y is obvious. Given $(\gamma_0,q_Y(z_0))\in\mathcal{G}_s{*}_p\,Y$, let us explain how $\gamma_0q_Y(z_0)$ is defined. As in the proof of Lemma 2.4, we consider an open bisection S such that $\gamma_0\in S$ and a continuous function a on X with compact support in $s(S)$, such that $a(x)=1$ for x in a neighborhood of $s(\gamma_0)$. We set $h=(a\circ s)\,\chi_S$ and $g(\gamma)=h(\gamma^{-1})$. Let $f\in\mathcal{C}_0(Y)$. Then we have $(g*(f\circ q_Y))(z_0)=f\circ q_Y(\gamma_0z_0)$. It follows that $f\mapsto(g*(f\circ q_Y))(z_0)$ is a character of A and since $g*(f\circ q_Y)$ belongs to $q_Y^*\mathcal{C}_0(Y)$, this character only depends on $q_Y(z_0)$ and (as before) γ_0. We denote it by $\gamma_0q_Y(z_0)$. So, we have $f\circ q_Y(\gamma_0z_0)=f(\gamma_0q_Y(z_0))$ for every $f\in\mathcal{C}_0(Y)$ and therefore $q_Y(\gamma_0z_0)=\gamma_0q_Y(z_0)$. The fact that we define in this way a continuous action is proved with arguments similar to those used in the the proof of Theorem 2.5. □

2.3. **An example.** By an *étale bundle of groups* we mean an étale groupoid $\mathcal{G}$ such that for every $\gamma \in \mathcal{G}$ we have $r(\gamma) = s(\gamma)$. In particular, $r^{-1}(x) = \mathcal{G}(x)$ is a discrete group for every $x \in X = \mathcal{G}^{(0)}$. Thus, $\mathcal{G}$ is the data of a field $x \in X \mapsto \mathcal{G}(x)$ of discrete groups with a suitable topology on the union of these groups.

The following class of étale bundle of groups was introduced in [9] in order to provide examples of groupoids for which the Baum-Connes conjecture fails. We consider an infinite discrete group Γ_∞ and a sequence $(H_n)_{n\in\mathbb{N}}$ of normal subgroups of Γ_∞ of finite index. We set $\Gamma_n = \Gamma_\infty/H_n$ and we denote by $\pi_n : \Gamma \to \Gamma_n$ the quotient map. The identity map of Γ_∞ is denoted by π_∞. Let $\mathcal{G}$ be the quotient of $\mathbb{N}^+ \times \Gamma_\infty$ by the following equivalence relation: $(m, s) \sim (n, t)$ if and only if $n = m$ and $\pi_n(s) = \pi_n(t)$. Then $\mathcal{G}$ is the bundle of groups $n \mapsto \Gamma_n$ over $\mathbb{N}^+$. The range (and also source) map is $r : (n, \pi_n(s)) \mapsto n \in \mathbb{N}^+ = \mathcal{G}^{(0)}$. We endow $\mathcal{G}$ with the quotient topology. Then $r^{-1}(\mathbb{N})$ is a discrete open subset of $\mathcal{G}$. A basis of neighborhoods of (∞, s) is formed by the subsets $\{(n, \pi_n(s)) : n \ge n_0\}$, where n_0 runs over $\mathbb{N}$. We observe that (∞, s) and (∞, t) have disjoint neighborhoods if and only if there exists an integer n_0 such that $\pi_n(s) \neq \pi_n(t)$ for $n \ge n_0$. Therefore, $\mathcal{G}$ is Hausdorff (and obviously an étale groupoid) if and only if for every $s \neq 1$ there exists n_0 such that $s \notin H_n$ for $n \ge n_0$ (in which case Γ_∞ is residually finite). Such examples are provided by taking $\Gamma_\infty = \mathrm{SL}_k(\mathbb{Z})$ and $\Gamma_n = \mathrm{SL}_k(\mathbb{Z}/n\mathbb{Z})$, for $k \ge 2$.

Let $\mathcal{G}$ be such an étale bundle of groups and $\mathcal{G}_r^+$ its Alexandroff fibrewise compactification. We have $(\mathcal{G}_r^+)^n = \Gamma_n$ for $n \in \mathbb{N}$ and $(\mathcal{G}_r^+)^\infty = (\mathcal{G}^\infty)^+ = \Gamma_\infty^+$.

As for $\beta_r\mathcal{G}$, it is equal to $\beta\mathcal{G}$ since the basis $\mathbb{N}^+$ of the bundle is compact. We still have $(\beta_r\mathcal{G})^n = \Gamma_n$ for $n \in \mathbb{N}$. On the other hand $(\beta_r\mathcal{G})^\infty$ is strictly bigger than $\beta(\mathcal{G}^\infty) = \beta\Gamma_\infty$. Indeed, let us write $\Gamma = \{s_i : i \in \mathbb{N}\}$. We choose inductively a subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that $\pi_{n_k}(s_k) \neq \pi_{n_k}(s_i)$ when $i < k$. Let $g \in \mathcal{C}_b(\mathcal{G})$ be defined as follows: we set $g((n_k, \pi_{n_k}(s_k)) = 1$ for every k and $g(y) = 0$ if $y \notin \{(n_k, \pi_{n_k}(s_k) : k \in \mathbb{N}\}$. The continuity of g is due to the choice of $(n_k)_{k\in\mathbb{N}}$. Let ω be a free ultrafilter on $\mathbb{N}$. The map $F \mapsto \lim_\omega F((n_k, \pi_{n_k}(s_k))$ is a character of $\mathcal{C}_b(\mathcal{G})$. This character χ belongs to $(\beta_r\mathcal{G})^\infty \setminus \beta\Gamma_\infty$ since we have $\chi(g) = 1$, whereas $\chi'(g) = 0$ for every $\chi' \in \beta\Gamma_\infty$.

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