

ON TENSOR PRODUCTS OF GROUP C^* -ALGEBRAS AND RELATED TOPICS

CLAIRE ANANTHARAMAN-DELAROCHE

ABSTRACT. We discuss properties and examples of discrete groups in connection with their operator algebras and related tensor products.

CONTENTS

1. Introduction	2
2. Preliminary background and notations	3
2.1. Maximal and minimal tensor products	3
2.2. Nuclear C^* -algebras	4
2.3. The weak expectation property	6
2.4. Exact C^* -algebras	7
2.5. Groups and operator algebras	8
2.6. Crossed products	10
3. Amenable groups and their C^*-algebras	11
4. Kirchberg factorization property	14
5. Amenable dynamical systems	18
5.1. Definition and basic results	18
5.2. Boundary amenability	19
5.3. Amenable actions on universal compactifications	22
6. The Akemann-Ostrand property (\mathcal{AO}) and related notions	24
6.1. Property (\mathcal{S})	24
6.2. Property (\mathcal{H})	26
6.3. The C^* -algebra of the biregular representation	28
7. Applications of property (\mathcal{AO})	32
7.1. Non-nuclearity in K -theory	32
7.2. Solid von Neumann algebras	33
References	33

The author acknowledges the hospitality and support of the Centre Bernoulli at the EPFL in Lausanne, where she began to write this exposition in January 2007, and of the Fields Institute in Toronto where it was completed in October 2007.

1. Introduction

This exposition is an extended version of notes for a lecture presented at the Centre Bernoulli in January 2007. Our purpose is to review three factorization problems related with the biregular representation of an infinite countable discrete group¹ Γ .

Let λ and ρ be respectively the left and right regular representations of Γ . The biregular representation is the representation $\lambda \cdot \rho : (s, t) \mapsto \lambda(s)\rho(t)$ of $\Gamma \times \Gamma$ in the Hilbert space $\ell^2(\Gamma)$. If $C^*(\Gamma)$, $C_\lambda^*(\Gamma)$, denote respectively the full and reduced C^* -algebras associated with Γ , the biregular representation extends to homomorphisms

$$\begin{aligned} \lambda \cdot \rho &: C^*(\Gamma) \otimes_{\max} C^*(\Gamma) \rightarrow \mathcal{B}(\ell^2(\Gamma)), \\ (\lambda \cdot \rho)_r &: C_\lambda^*(\Gamma) \otimes_{\max} C_\lambda^*(\Gamma) \rightarrow \mathcal{B}(\ell^2(\Gamma)), \end{aligned}$$

thanks to the universal property of the maximal tensor product. It is natural to ask for which groups the homomorphisms $(\lambda \cdot \rho)_r$ or $\lambda \cdot \rho$ can be factorized through minimal tensor products:

$$\begin{array}{ccc} C_\lambda^*(\Gamma) \otimes_{\max} C_\lambda^*(\Gamma) & & C^*(\Gamma) \otimes_{\max} C^*(\Gamma) \\ \begin{array}{c} \downarrow P_r \\ C_\lambda^*(\Gamma) \otimes_{\min} C_\lambda^*(\Gamma) \end{array} & \xrightarrow{(\lambda \cdot \rho)_r} & \begin{array}{c} \downarrow P \\ C^*(\Gamma) \otimes_{\min} C^*(\Gamma) \end{array} \\ & \cdots \cdots \cdots & \xrightarrow{\lambda \cdot \rho} \mathcal{B}(\ell^2(\Gamma)). \end{array}$$

In the reduced setting the answer is simple. This is recalled in section 2. *The three following conditions are equivalent:*

- (i) Γ is amenable;
- (ii) P_r is an isomorphism;
- (iii) $(\lambda \cdot \rho)_r$ passes to quotient.

As always, it is much more difficult to deal with full C^* -algebras. Section 3 is devoted to this situation which has been studied by Kirchberg. He defined Γ to have the factorization property (\mathcal{F}) if $\lambda \cdot \rho$ passes to quotient. Among his many deep results are the following ones:

- Every residually finite² group has property (\mathcal{F}) .
- The converse is true for groups having the Kazhdan property (T) .

The family of groups such that P is an isomorphism is contained in the class of property (\mathcal{F}) groups. It is still mysterious. Kirchberg proved that P is an isomorphism if and only if $C^*(\Gamma)$ has Lance's weak expectation property (WEP). One of the most important open problem is whether the full C^* -algebra of the free

¹Unless stated otherwise, in this paper all groups are assumed to be infinite countable.

²We refer with the next sections for notions used and not defined in the introduction.

group with infinitely many generators has the WEP. The answer to this question would have important consequences in operator algebras theory.

Another important open problem concerning full C^* -algebras is *whether the exactness of $C^*(\Gamma)$ implies the amenability of Γ* . Kirchberg gave a *positive answer when assuming that Γ has property (\mathcal{F})* .

For this question also, the situation is nicer for reduced C^* -algebras: $C_\lambda^*(\Gamma)$ is exact if and only if Γ is boundary amenable. We introduce this latter notion in Section 4. Our main purpose is to provide tools for the study of the Akemann-Ostrand property in Section 5.

Let Γ be the free group with $n \geq 2$ generators. Akemann and Ostrand proved that, although $(\lambda \cdot \rho)_r$ does not factorize via $C_\lambda^*(\Gamma) \otimes_{\min} C_\lambda^*(\Gamma)$, this is however the case for its composition with the projection Q from $\mathcal{B}(\ell^2(\Gamma))$ onto the Calkin algebra $\mathcal{Q}(\ell^2(\Gamma))$:

$$\begin{array}{ccc}
 C_\lambda^*(\Gamma) \otimes_{\max} C_\lambda^*(\Gamma) & & \\
 \downarrow P_r & \searrow^{Q \circ (\lambda \cdot \rho)_r} & \\
 C_\lambda^*(\Gamma) \otimes_{\min} C_\lambda^*(\Gamma) & \dashrightarrow & \mathcal{Q}(\ell^2(\Gamma)),
 \end{array}$$

When such a phenomenon occurs, one says that Γ has the Akemann-Ostrand property (\mathcal{AO}) . We introduce two boundary amenability properties which imply the (\mathcal{AO}) property: property (\mathcal{H}) due to Higson and the weaker condition (\mathcal{S}) due to Skandalis. Among the groups with property (\mathcal{AO}) are the amenable groups and Gromov hyperbolic groups.

Non-amenable groups with property (\mathcal{AO}) have very remarkable features. We mention two of them in Section 6. First the application due to Skandalis giving *examples of C^* -algebras not nuclear in K -theory*. Second, the more recent result of Ozawa who took advantage of property (\mathcal{AO}) to give a simple proof that *the von Neumann algebra of the free group \mathbb{F}_n , $n \geq 2$, (and of many other groups) is a prime factor*.

Section 2 is devoted to basic preliminary results and definitions that we have tried to make as condensed and complete as possible, for the reader's convenience.

All along the text, our objective is to give precise definitions, sufficiently many examples, and a choice of proofs we think to be representative and not too technical. We have tried to provide references for assertions stated without proof.

2. Preliminary background and notations

For fundamentals of C^* -algebras we refer to [20, 17, 49, 66].

2.1. Maximal and minimal tensor products. Let A and B be two C^* -algebras, and denote by $A \odot B$ their algebraic tensor product. A C^* -norm on $A \odot B$ is a norm of involutive algebra such that $\|x^*x\| = \|x\|^2$ for $x \in A \odot B$. There are

two natural ways to define C^* -norms on the $*$ -algebra $A \odot B$. Recall first that a representation of a C^* -algebra in a Hilbert space \mathcal{H} is a homomorphism³ from A into the C^* -algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on \mathcal{H} . An important fact to keep in mind is that homomorphisms of C^* -algebras are automatically contractive maps.

The *maximal C^* -norm* is defined by

$$\|x\|_{\max} = \sup \|\pi(x)\|$$

where π runs over all homomorphisms from $A \odot B$ into some $\mathcal{B}(\mathcal{H})$ ⁴. The *maximal tensor product* is the completion $A \otimes_{\max} B$ of $A \odot B$ for this C^* -norm.

The *minimal tensor product* is defined by taking specific homomorphisms, namely of the form $\pi_1 \otimes \pi_2$ where π_1 is a representation of A in some Hilbert space \mathcal{H}_1 and π_2 is a representation of B in some \mathcal{H}_2 . Thus we define

$$\|x\|_{\min} = \sup \|(\pi_1 \otimes \pi_2)(x)\|$$

where π_1, π_2 run over all representations of A and B respectively. The *minimal tensor product* is the completion $A \otimes_{\min} B$ of $A \odot B$ for this C^* -norm. It is shown that when π_1 and π_2 are faithful, we have

$$\|x\|_{\min} = \|(\pi_1 \otimes \pi_2)(x)\|.$$

Therefore, if A and B are concretely represented as C^* -subalgebras of $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$ respectively, then $A \otimes_{\min} B$ is the closure of $A \odot B$, viewed as a subalgebra of $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. It is why $\|\cdot\|_{\min}$ is also called the *spatial tensor product*.

Proposition 2.1 ([66], Th. IV.4.19). *Given two C^* -algebras A and B , then $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$ are respectively the largest and the smallest C^* -norm on $A \odot B$. Moreover, they are both cross-norms (i.e. the norm of $a \otimes b$ is the product $\|a\|\|b\|$ for $a \in A$ and $b \in B$).*

The possible existence of more than one C^* -norm on the algebraic tensor product of two C^* -algebras was discovered by Takesaki in [65] (see Remark 2.16 below). This motivated the introduction of the following well-behaved class of C^* -algebras.

2.2. Nuclear C^* -algebras.

Definition 2.2. A C^* -algebra A is said to be *nuclear* if for every C^* -algebra B , there is only one C^* -norm on $A \odot B$, that is $A \otimes_{\max} B = A \otimes_{\min} B$.

This class of algebras has several important characterizations that we describe now. First, let us observe that, given a C^* -algebra $A \subset \mathcal{B}(\mathcal{H})$, the $*$ -algebra $M_n(A) = M_n(\mathbb{C}) \odot A$ of $n \times n$ matrices with entries in A is closed in $\mathcal{B}(\mathbb{C}^n \otimes \mathcal{H})$. It follows that there is only one C^* -norm on $M_n(A)$. In other terms we have

$$M_n(A) = M_n(\mathbb{C}) \odot A = M_n(\mathbb{C}) \otimes_{\min} A = M_n(\mathbb{C}) \otimes_{\max} A.$$

³Homomorphisms of involutive algebras are always assumed to preserve the involutions.

⁴It is easily shown that this supremum is actually finite.

Given a bounded linear map $T : A \rightarrow B$ between C^* -algebras, let us denote by $T_n : M_n(A) \rightarrow M_n(B)$ the map $I_n \otimes T$, that is $T_n([a_{i,j}]) = [T(a_{i,j})]$ for $[a_{i,j}] \in M_n(A)$. Recall that a linear map $T : A \rightarrow B$ is said to be positive if $T(A^+) \subset B^+$. It is not true that T_n is then positive for all n . A good reference for this subject is the book [56]. In particular, it is observed at the end of its first chapter that the transposition on $M_2(\mathbb{C})$ is positive but T_2 is not positive.

Definition 2.3. A linear map $T : A \rightarrow B$ is said to be *completely positive* if T_n is positive for all $n \geq 1$.

Remarks 2.4. (1) Given $a_1, \dots, a_n \in A$, the matrix $[a_i^* a_j]$ is a positive element in $M_n(A)$ and it is easily shown that every positive element in $M_n(A)$ is a finite sum of such matrices. It follows that $T : A \rightarrow B$ is completely positive if and only if for every $a_1, \dots, a_n \in A$, then $[T(a_i^* a_j)] \in M_n(B)^+$.

(2) Every homomorphism is completely positive. Also, for $a \in A$, the map $x \mapsto a^* x a$ is completely positive.

As it is shown now, completely positive maps have a very simple structure. We only consider the case where A has a unit, for simplicity. We shall write u.c.p. for unital completely positive.

Proposition 2.5 (Stinespring theorem, [56], Th. 4.1). *Let T be a u.c.p. map from a unital C^* -algebra A into $\mathcal{B}(\mathcal{H})$. There exist a Hilbert space \mathcal{K} , a representation $\pi : A \rightarrow \mathcal{B}(\mathcal{K})$ and an isometry $V : \mathcal{H} \rightarrow \mathcal{K}$ such that $T(a) = V^* \pi(a) V$ for all $a \in A$.*

Sketch of proof. We define on $A \odot \mathcal{H}$ the inner product

$$\langle a_1 \otimes v_1, a_2 \otimes v_2 \rangle = \langle v_1, T(a_1^* a_2) v_2 \rangle.$$

Let \mathcal{K} be the Hilbert space obtained from $A \odot \mathcal{H}$ by separation and completion. For $x \in A \odot \mathcal{H}$, we denote by $[x]$ its class in \mathcal{K} . Then $V : v \mapsto [I_A \otimes v]$ is an isometry from \mathcal{H} into \mathcal{K} . Let π be the representation from A into $\mathcal{B}(\mathcal{K})$ defined by $\pi(a)[a_1 \otimes v] = [aa_1 \otimes v]$ for $a, a_1 \in A$ and $v \in \mathcal{H}$. Then π and V fulfill the required properties. \square

Proposition 2.6 ([66], Prop. IV.4.23). *Let $T_i : A_i \rightarrow B_i$, $i = 1, 2$, be completely positive maps between C^* -algebras.*

- (1) $T_1 \odot T_2$ extends to a completely positive map $T_1 \otimes_{\min} T_2 : A_1 \otimes_{\min} A_2 \rightarrow B_1 \otimes_{\min} B_2$ and $\|T_1 \otimes_{\min} T_2\| \leq \|T_1\| \|T_2\|$.
- (2) The same result holds for the maximal tensor norm. More generally, if T_1 and T_2 are two completely positive maps into a C^* -algebra B such that $T_1(A_1)$ and $T_2(A_2)$ commute, then the map from $A_1 \odot A_2$ into B sending $a_1 \otimes a_2$ to $T_1(a_1)T_2(a_2)$ extends to a completely positive map from $A_1 \otimes_{\max} A_2$ into B .

Theorem 2.7 (Arveson's extension theorem, [56], Th. 7.5). *Let A be a C^* -subalgebra of $\mathcal{B}(\mathcal{H})$ and $T : A \rightarrow \mathcal{B}(\mathcal{K})$ a completely positive map. There exists a completely positive map $\bar{T} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ which extends T .*

Definition 2.8. A completely positive contraction $T : A \rightarrow B$ is said to be *nuclear* if there exist a net (n_i) of positive integers and nets of completely positive contractions⁵ $\phi_i : A \rightarrow M_{n_i}(\mathbb{C})$, $\tau_i : M_{n_i}(\mathbb{C}) \rightarrow B$, such that for every $a \in A$,

$$\lim_i \|\tau_i \circ \phi_i(a) - T(a)\| = 0.$$

Theorem 2.9. *Let A be a C^* -algebra. The following conditions are equivalent:*

- (i) *The identity map of A is nuclear.*
- (ii) *A is a nuclear C^* -algebra.*
- (iii) *There exists a net (ϕ_i) of finite rank completely positive maps $\phi_i : A \rightarrow A$ converging to the identity map of A in the topology of pointwise convergence in norm.*

This is the Choi-Effros theorem 3.1 in [12]. The implication (ii) \Rightarrow (iii) is due to Choi-Effros [12] and Kirchberg [41] independently. For a short proof of the equivalence between (i) and (ii) we refer to [63, Prop. 1.2]. Let us just show the easy direction.

Proof of (i) \Rightarrow (ii). Let $T : A \rightarrow A$ be a completely positive contraction of the form $T = \tau \circ \phi$ where $\phi : A \rightarrow M_n(\mathbb{C})$ and $\tau : M_n(\mathbb{C}) \rightarrow A$ are completely positive contractions. Then the map $T \otimes \text{Id}_B : A \otimes_{\max} B \rightarrow A \otimes_{\max} B$ admits the factorization

$$A \otimes_{\max} B \xrightarrow{\theta} A \otimes_{\min} B \xrightarrow{\phi \otimes \text{Id}_B} M_n(\mathbb{C}) \otimes_{\min} B = M_n(\mathbb{C}) \otimes_{\max} B \xrightarrow{\tau \otimes \text{Id}_B} A \otimes_{\max} B,$$

where θ is the canonical homomorphism from $A \otimes_{\max} B$ onto $A \otimes_{\min} B$. In particular the kernel of $T \otimes \text{Id}_B : A \otimes_{\max} B \rightarrow A \otimes_{\max} B$ contains the kernel of θ . Let now $(T_i = \tau_i \circ \phi_i)$ be a net of completely positive contractions factorizable through matrix algebras and converging to Id_A . Then $\text{Id}_{A \otimes_{\max} B}$ is the norm pointwise limit of $T_i \otimes \text{Id}_B : A \otimes_{\max} B \rightarrow A \otimes_{\max} B$, and therefore its kernel contains the kernel of θ . It follows that $\ker \theta = \{0\}$. \square

2.3. The weak expectation property. We now introduce a notion weaker than nuclearity which also plays an important role.

Definition 2.10 (Lance). A C^* -algebra A has the *weak expectation property* (*WEP* in short) if for any (or some) embedding $A \subset \mathcal{B}(\mathcal{H})$ there is a completely positive contraction T from $\mathcal{B}(\mathcal{H})$ onto the weak closure of A such that $T(a) = a$ for all $a \in A$. One says that T is a *weak expectation*.

Proposition 2.11. *A nuclear C^* -algebra A has the WEP.*

⁵When A and B have a unit and T is unital, one may chose ϕ_i and τ_i to be unital.

Proof. Using the nuclearity of Id_A , and the Arveson extension property for the completely positive maps ϕ_i of definition 2.8, we see that there exists a net (T_i) of completely positive contractions from $\mathcal{B}(\mathcal{H})$ into A such that $\lim_i T_i(a) = a$ for every $a \in A$. We define T to be any limit point of (T_i) in the σ -weak pointwise topology. \square

For a C^* -algebra A , the *opposite* C^* -algebra A^0 has the same involutive Banach space structure but the reverse product. We mention the following nice result of Kirchberg [43] in the “if” direction and Haagerup (unpublished) in the other direction. A proof is available in [58, Th. 15.6].

Theorem 2.12. *A C^* -algebra A has the WEP if and only if the canonical map from $A \otimes_{\max} A^0 \rightarrow A \otimes_{\min} A^0$ is injective.*

Remark 2.13. A von Neumann M is said to be *injective* if for some (or any) normal faithful representation in a Hilbert space \mathcal{H} there exists a norm one projection⁶ from $\mathcal{B}(\mathcal{H})$ onto M . This property is equivalent to the WEP (see [58, Rem 15.2]). On the other hand there exist separable C^* -algebras which have the WEP but are not nuclear (see the remark following Corollary 3.5 in [43]). This is to be compared with the equivalence (2) \Leftrightarrow (3) in Proposition 3.5.

2.4. Exact C^* -algebras. The class of nuclear C^* -algebras is stable by extension and quotient. However a C^* -subalgebra of nuclear C^* algebra need not be nuclear (see Remark 5.6).

Definition 2.14. A C^* -algebra A is said to be *exact* (or *nuclearly embeddable*) if there exists a nuclear embedding $A \hookrightarrow D$ into some C^* -algebra D . Since D may be concretely embedded into some $\mathcal{B}(\mathcal{H})$, we may take $D = \mathcal{B}(\mathcal{H})$ if we wish.

Obviously, every C^* -subalgebra of an exact C^* -algebra is exact and in particular every C^* -subalgebra of a nuclear C^* -algebra is exact. A celebrated (and deep) result of Kirchberg-Phillips [47] says that, conversely, every separable exact C^* -algebra is a C^* -subalgebra of a nuclear one.

The terminology comes from the following fundamental result, saying that A is exact if and only if the functor $B \mapsto B \otimes_{\min} A$ preserves short exact sequences. Indeed this was the original definition of exactness by Kirchberg.

Theorem 2.15 (Kirchberg). *Let A be a C^* -algebra. The following conditions are equivalent:*

- (i) *A is nuclearly embeddable (or exact).*
- (ii) *For every short exact sequence $0 \rightarrow I \rightarrow B \rightarrow C \rightarrow 0$ of C^* -algebras, the sequence*

$$0 \rightarrow I \otimes_{\min} A \rightarrow B \otimes_{\min} A \rightarrow C \otimes_{\min} A \rightarrow 0$$

is exact.

⁶It is automatically completely positive (see [67, 68]).

Proof. (i) \Rightarrow (ii). Denote by q the canonical homomorphism from B onto C . The only point is to show that $I \otimes_{\min} A$ is the kernel of $q \otimes \text{Id}_A : B \otimes_{\min} A \rightarrow C \otimes_{\min} A$. Let $A \hookrightarrow D$ be a nuclear embedding and let (T_i) be net of finite rank completely positive contractions $T_i : A \rightarrow D$ such that $\lim_i \|T_i(a) - a\| = 0$ for every $a \in A$. It is enough to prove that for $x \in \ker(q \otimes \text{Id}_A)$, we have $(\text{Id}_B \otimes T_i)(x) \in I \otimes_{\min} D$. Indeed, considering $B \otimes_{\min} A$ as a C^* -subalgebra of $B \otimes_{\min} D$, we shall obtain

$$x \in (B \otimes_{\min} A) \cap (I \otimes_{\min} D) = I \otimes_{\min} A,$$

since $\lim_i \|x - (\text{Id}_B \otimes T_i)(x)\| = 0$.

Hence, what is left to prove is that for any finite rank completely positive map $T : A \rightarrow D$ we have $(\text{Id}_B \otimes T)(x) \in I \otimes_{\min} D$. Since T has a finite rank, there exist elements $d_1, \dots, d_n \in D$ and bounded linear forms f_1, \dots, f_n on A such that $T(a) = \sum_{k=1}^n f_k(a)d_k$ for $a \in A$. For $k = 1, \dots, n$, denote by R_k the bounded linear map from $B \otimes_{\min} A$ to B such that $R_k(b \otimes a) = f_k(a)b$ for $b \otimes a \in B \otimes A$. It is easily checked that $R_k \circ (q \otimes \text{Id}_A) = q \circ R_k$ and therefore we have $R_k(\ker(q \otimes \text{Id}_A)) \subset I$. It follows that $(\text{Id}_B \otimes T)(x) = \sum_{k=1}^n R_k(x) \otimes d_k \in I \otimes_{\min} D$ and this ends the proof.

(ii) \Rightarrow (i) is a hard result due to Kirchberg (see [45, Th. 4.1] or [73, Theorem 7.3]). \square

In the sequel, we shall study the properties just introduced, and some others, for C^* -algebras arising from discrete groups.

2.5. Groups and operator algebras.

2.5.1. *The reduced and full group C^* -algebras.* Let Γ be a discrete group, and θ be a unitary representation of Γ in a Hilbert space \mathcal{H} . For $\mathbf{c} = \sum_{t \in \Gamma} c_t t$ in the group algebra⁷ $\mathbb{C}[\Gamma]$ we set $\theta(\mathbf{c}) = \sum c_t \theta(t)$. We get in such a way a homomorphism from $\mathbb{C}[\Gamma]$ into $\mathcal{B}(\mathcal{H})$ such that $\|\theta(\mathbf{c})\| \leq \sum |c_t|$. We shall denote by $C_\theta^*(\Gamma)$ the C^* -subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\theta(\Gamma)$.

The most important representation of Γ is the *left regular representation*, that is the representation λ in $\ell^2(\Gamma)$ by left translations : $\lambda(s)\delta_t = \delta_{st}$ (where δ_t is the Dirac function at t). One may equivalently consider the *right regular representation* ρ . We shall denote by ι the trivial representation⁸ of Γ .

Observe that λ defines an injective homomorphism from $\mathbb{C}[\Gamma]$ into $\mathcal{B}(\ell^2(\Gamma))$. Thus we may view any element \mathbf{c} of $\mathbb{C}[\Gamma]$ as the convolution operator on $\ell^2(\Gamma)$ by the function $t \mapsto c_t$.

The *reduced C^* -algebra* $C_\lambda^*(\Gamma)$ is the completion of $\mathbb{C}[\Gamma]$ for the norm $\|\mathbf{c}\|_r = \|\lambda(\mathbf{c})\|$. Equivalently, it is the closure of $\mathbb{C}[\Gamma]$ in $\mathcal{B}(\ell^2(\Gamma))$, when $\mathbb{C}[\Gamma]$ is identified with its image under the left regular representation. Then $\mathbf{c} = \sum c_t t \in \mathbb{C}[\Gamma]$ will also be written $\mathbf{c} = \sum c_t \lambda(t) \in C_\lambda^*(\Gamma)$.

⁷i.e. the $*$ -algebra of formal sums $\sum_{t \in \Gamma} c_t t$, where $t \mapsto c_t$ is a finitely supported function from Γ into \mathbb{C} .

⁸In case of ambiguity, we shall write λ_Γ instead of λ and ι_Γ for ι .

The *full C^* -algebra* $C^*(\Gamma)$ is the completion of $\mathbb{C}[\Gamma]$ for the norm $\|\mathbf{c}\| = \sup_{\theta} \|\theta(\mathbf{c})\|$, where θ ranges over all unitary representations of Γ . Obviously, there is a canonical correspondence between unitary representations of Γ and representations of $C^*(\Gamma)$. We shall always use the same symbol to denote both of them.

Note that Γ is a subgroup of the unitary group of $C^*(\Gamma)$. We shall denote by $s \mapsto s$ this embedding and by $s \mapsto \lambda(s)$ the embedding of Γ into the unitary group of $C_{\lambda}^*(\Gamma)$. The surjective map from $C^*(\Gamma)$ onto $C_{\lambda}^*(\Gamma)$ is also denoted by λ .

Observe that if J^c is the complex conjugacy, then $a \mapsto J^c a^* J^c$ is an isomorphism from $C_{\lambda}^*(\Gamma)$ onto its opposite algebra $C_{\lambda}^*(\Gamma)^0$. Note also that the unitary operator \check{J} defined on $\ell^2(\Gamma)$ by $\check{J}\xi(t) = \xi(t^{-1})$ exchanges the left and right regular representations, so that $\lambda(t) \mapsto \rho(t) = \check{J}\lambda(t)\check{J}$ extends to an isomorphism from $C_{\lambda}^*(\Gamma)$ onto the C^* -algebra $C_{\rho}^*(\Gamma)$ generated by $\rho(\Gamma)$.

Let Γ_1, Γ_2 be two discrete groups. Then it is easily checked that

$$\begin{aligned} C^*(\Gamma_1) \otimes_{\max} C^*(\Gamma_2) &= C^*(\Gamma_1 \times \Gamma_2), \\ C_{\lambda}^*(\Gamma_1) \otimes_{\min} C_{\lambda}^*(\Gamma_2) &= C_{\lambda}^*(\Gamma_1 \times \Gamma_2). \end{aligned}$$

We shall denote by $\lambda \cdot \rho$ the *biregular representation* $(s, t) \mapsto \lambda(s)\rho(t)$ of $\Gamma \times \Gamma$ in $\ell^2(\Gamma)$ as well as its extension to $C^*(\Gamma) \otimes_{\max} C^*(\Gamma)$. Due to the universal property of the maximal tensor product, we also see that the canonical homomorphism $h : C_{\lambda}^*(\Gamma) \odot C_{\lambda}^*(\Gamma) \rightarrow \mathcal{B}(\ell^2(\Gamma))$ defined by

$$h(\lambda(s) \otimes \lambda(t)) = \lambda(s)\check{J}\lambda(t)\check{J} = \lambda(s)\rho(t)$$

has a continuous extension to $C_{\lambda}^*(\Gamma) \otimes_{\max} C_{\lambda}^*(\Gamma)$. It is denoted by $(\lambda \cdot \rho)_r$. We get the following commutative diagram

$$\begin{array}{ccc} C^*(\Gamma) \otimes_{\max} C^*(\Gamma) & & \\ \downarrow & \searrow^{\lambda \cdot \rho} & \\ C_{\lambda}^*(\Gamma) \otimes_{\max} C_{\lambda}^*(\Gamma) & \xrightarrow{(\lambda \cdot \rho)_r} & \mathcal{B}(\ell^2(\Gamma)). \end{array}$$

Remark 2.16. The situation is quite different for the minimal tensor product. In [65], Takesaki has proved that for the free group \mathbb{F}_2 with two generators, $h : C_{\lambda}^*(\mathbb{F}_2) \odot C_{\lambda}^*(\mathbb{F}_2) \rightarrow \mathcal{B}(\ell^2(\mathbb{F}_2))$ is not continuous for the C^* -norm $\|\cdot\|_{\min}$ therefore giving an example where the minimal and maximal tensor products are not the same. The continuity problem of $h : C_{\lambda}^*(\Gamma) \odot C_{\lambda}^*(\Gamma) \rightarrow \mathcal{B}(\ell^2(\Gamma))$ for $\|\cdot\|_{\min}$ will be studied in Proposition 3.5.

2.5.2. Group von Neumann algebras. Let Γ be a discrete group. The weak closure $L(\Gamma)$ of $C_{\lambda}^*(\Gamma)$ in $\mathcal{B}(\ell^2(\Gamma))$ is called the (left) *von Neumann algebra* of Γ . Its commutant $L(\Gamma)'$ is the von Neumann algebra $R(\Gamma)$ generated by $\rho(\Gamma)$. The algebra $L(\Gamma)$ has a natural faithful trace tr defined by $\text{tr}(a) = \langle \delta_e, a\delta_e \rangle$.

This is a particular example of a von Neumann algebra M equipped with a faithful tracial state tr . Let $L^2(M)$ be the completion of M for the scalar product

$\langle m, n \rangle = \text{tr}(m^*n)$ and let $m \mapsto \widehat{m}$ be the embedding of M into its completion. As usual, let J be the antilinear isometric involution of $L^2(M)$ defined by $J\widehat{m} = \widehat{m}^*$. Recall that $JMJ = M'$ and that $L^2(M)$ is a M - M -bimodule for the following actions : $m_1\widehat{m}m_2 = \widehat{m_1m}m_2$, with $m_1, m, m_2 \in M$ ⁹.

2.5.3. Weak containment. Recall that a (diagonal) *coefficient of a representation* (θ, \mathcal{H}) of Γ is a function of the form $s \mapsto \langle \xi, \theta(s)\xi \rangle$, with $\xi \in \mathcal{H}$.

Definition 2.17. Let π_1 and π_2 be two unitary representations of Γ . We say that π_1 is *weakly contained* into π_2 , and we write $\pi_1 \prec \pi_2$, if every coefficient of π_1 is the pointwise limit of sums of coefficients of π_2 .

If ξ_1 is a cyclic vector for the representation π_1 , it is enough to consider the coefficient $s \mapsto \langle \xi_1, \pi_1(s)\xi_1 \rangle$ of π_1 .

A useful equivalent definition is that $\pi_1 \prec \pi_2$ if and only if the kernels of π_1, π_2 , viewed as representations of $C^*(\Gamma)$, satisfy $\ker \pi_2 \subset \ker \pi_1$ (see [20, Prop. 18.1.4]).

Let us recall here the *Fell absorption principle* : for any unitary representation (θ, \mathcal{H}) of Γ , the tensor product representation $\lambda \otimes \theta$ is unitary equivalent to the multiple $(\dim \mathcal{H})\lambda$ of λ (via the unitary of $\ell^2(\Gamma) \otimes \mathcal{H}$ sending $f \otimes \xi$ to the function $s \mapsto f(s)\theta(s^{-1})\xi$). This allows to extend the map $\lambda(s) \mapsto \lambda(s) \otimes \theta(s)$ to a homomorphism from $C_\lambda^*(\Gamma)$ to $C_\lambda^*(\Gamma) \otimes_{\min} C_\theta^*(\Gamma)$.

Finally, we shall also need the following construction of C^* -algebras associated with dynamical systems.

2.6. Crossed products. Let $\alpha : \Gamma \curvearrowright A$ be an action of Γ on a C^* -algebra A . That is, α is a homomorphism from the group Γ into the group $\text{Aut}(A)$ of automorphisms of A . We denote by $A[\Gamma]$ the $*$ -algebra of formal sums $\mathbf{a} = \sum a_t t$ where $t \mapsto a_t$ is a map from Γ into A with finite support and where the operations are given by the following rules:

$$(at)(bs) = a\alpha_t(b)ts, \quad (at)^* = \alpha_{t^{-1}}(a)t^{-1},$$

for $a, b \in A$ and $s, t \in \Gamma$.

For such a non-commutative dynamical system, the notion of unitary representation is replaced by that of covariant representation. A *covariant representation* of $\alpha : \Gamma \curvearrowright A$ is a pair (π, θ) where π and θ are respectively a representation of A and a unitary representation of Γ in the same Hilbert space \mathcal{H} satisfying the covariance rule

$$\forall a \in A, t \in \Gamma, \quad \theta(t)\pi(a)\theta(t)^* = \pi(\alpha_t(a)).$$

We define a homomorphism $\pi \times \theta$ from $A[\Gamma]$ into $\mathcal{B}(\mathcal{H})$ by

$$(\pi \times \theta)\left(\sum a_t t\right) = \sum \pi(a_t)\theta(t).$$

⁹When $M = L(\Gamma)$ we have $L^2(M) = \ell^2(\Gamma)$ and $(J\xi)(t) = \overline{\xi(t^{-1})}$.

The *full crossed product* $A \rtimes \Gamma$ is the C^* -algebra obtained as the completion of $A[\Gamma]$ for the norm

$$\|\mathbf{a}\| = \sup \|(\pi \times \theta)(\mathbf{a})\|$$

where (π, θ) runs over all covariant representations of $\alpha : \Gamma \curvearrowright A$. Every covariant representation (π, θ) extends to a representation of $A \rtimes \Gamma$, denoted by $\pi \times \theta$. Conversely, it is not difficult to see that every representation of $A \rtimes \Gamma$ comes in this way from a covariant representation. In other terms, $A \rtimes \Gamma$ is the universal C^* -algebra describing the covariant representations of $\alpha : \Gamma \curvearrowright A$.

We now describe the analogues of the regular representation, the *induced covariant representations*. To any representation π of A in a Hilbert space \mathcal{H} is associated the following covariant representation $(\tilde{\pi}, \tilde{\lambda})$ of $\alpha : \Gamma \curvearrowright A$, acting on the Hilbert space $\ell^2(\Gamma, \mathcal{H}) = \ell^2(\Gamma) \otimes \mathcal{H}$:

- the representation $\tilde{\pi}$ of A is defined by $\tilde{\pi}(a)\xi(t) = \pi(\alpha_{t^{-1}}(a))\xi(t)$ for $a \in A$ and $\xi \in \ell^2(\Gamma, \mathcal{H})$;
- the unitary representation $\tilde{\lambda}$ of Γ is $t \mapsto \tilde{\lambda}(t) = \lambda(t) \otimes \text{Id}_{\mathcal{H}}$.

The covariant representation $(\tilde{\pi}, \tilde{\lambda})$ is said to be *induced* by π .

The *reduced crossed product* $A \rtimes_r \Gamma$ is the C^* -algebra obtained as the completion of $A[\Gamma]$ for the norm

$$\|\mathbf{a}\|_r = \sup \|(\tilde{\pi} \times \tilde{\lambda})(\mathbf{a})\|$$

where π runs over all representations of A .

When (π, \mathcal{H}) is a faithful representation of A one has $\|\mathbf{a}\|_r = \|(\tilde{\pi} \times \tilde{\lambda})(\mathbf{a})\|$ and therefore $A \rtimes_r \Gamma$ is faithfully represented into $\ell^2(\Gamma) \otimes \mathcal{H}$.

Note that $A \rtimes_r \Gamma$ contains a copy of A , and that whenever A has a unit, then $A \rtimes_r \Gamma$ also contains a copy of $C_\lambda^*(\Gamma)$.

3. Amenable groups and their C^* -algebras

Let Γ be a discrete group. Recall that a complex valued function φ on Γ is said to be *positive definite* (or of *positive type*) if for every $n \geq 1$ and every t_1, \dots, t_n in Γ the matrix $[\varphi(t_i^{-1}t_j)]$ is positive. Given $(\theta, \mathcal{H}, \xi)$ where ξ is a non-zero vector in the Hilbert space \mathcal{H} of the unitary representation θ of Γ , then the coefficient $t \mapsto \langle \xi, \theta(t)\xi \rangle$ of θ is positive definite. The Gelfand-Naimark-Segal (*GNS*) construction asserts that every positive definite function is of this form.

The relations between positive definite functions and completely positive maps on $C_\lambda^*(\Gamma)$ are described in the following lemma.

Lemma 3.1. *Let Γ be a discrete group.*

- (i) *Let φ be a positive definite function on Γ . Then*

$$m_\varphi : \sum c_t \lambda(t) \mapsto \sum \varphi(t) c_t \lambda(t)$$

extends to a completely positive map ϕ from $C_\lambda^(\Gamma)$ into itself.*

- (ii) Let $\phi : C_\lambda^*(\Gamma) \rightarrow C_\lambda^*(\Gamma)$ be a completely positive map. For $t \in \Gamma$ we set $\varphi(t) = \langle \delta_e, \phi(\lambda(t))\lambda(t)^*\delta_e \rangle$. Then φ is a positive definite function on Γ .

Proof. (i) Let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ be given by the GNS construction so that for $t \in \Gamma$ one has $\varphi(t) = \langle \xi_\varphi, \pi_\varphi(t)\xi_\varphi \rangle$. Let S from $\ell^2(\Gamma)$ into $\ell^2(\Gamma, \mathcal{H}_\varphi)$ defined by

$$(Sf)(t) = f(t)\pi_\varphi(t)^*\xi_\varphi.$$

It is a bounded linear map and its adjoint S^* satisfies $S^*(F)(s) = \langle \xi_\varphi, \pi_\varphi(s)F(s) \rangle$ for $F \in \ell^2(\Gamma, \mathcal{H}_\varphi)$. A straightforward computation shows that $m_\varphi \mathbf{c} = S^*(\mathbf{c} \otimes 1)S$ for $\mathbf{c} \in \mathbb{C}[\Gamma]$. It follows that m_φ extends to the completely positive map $a \mapsto S^*(a \otimes 1)S$ from $C_\lambda^*(\Gamma)$ into itself.

(ii) follows from an easy computation. \square

We shall denote by $\ell^1(\Gamma)_1^+$ the space of probability measures on Γ and by $\ell^2(\Gamma)_1$ the unit sphere of $\ell^2(\Gamma)$. For $f : \Gamma \rightarrow \mathbb{C}$ and $t \in \Gamma$ the function tf is defined by $(tf)(s) = f(t^{-1}s)$. Recall that:

Definition 3.2 (-Proposition). Γ is *amenable* if and only if one of the four following equivalent conditions holds:

- (i) There exists an invariant state M on $\ell^\infty(\Gamma)$, that is such that $M(tf) = M(f)$ for $t \in \Gamma$ and $f \in \ell^\infty(\Gamma)$.
(ii) For every $\varepsilon > 0$ and every finite subset F of Γ there exists $f \in \ell^1(\Gamma)_1^+$ such that

$$\max_{t \in F} \|tf - f\|_1 \leq \varepsilon.$$

- (iii) For every $\varepsilon > 0$ and every finite subset F of Γ there exists $\xi \in \ell^2(\Gamma)_1$ such that

$$\max_{t \in F} \|t\xi - \xi\|_2 \leq \varepsilon.$$

- (iv) For every $\varepsilon > 0$ and every finite subset F of Γ there exists a positive definite function φ on Γ with finite support such that

$$\max_{t \in F} |1 - \varphi(t)| \leq \varepsilon.$$

Condition (iii) above means that the left regular representation of Γ almost have invariant vectors, or that the trivial representation ι of Γ is weakly contained into λ .

Theorem 3.3 (Hulanicki). Γ is amenable if and only if $\lambda : C^*(\Gamma) \rightarrow C_\lambda^*(\Gamma)$ is an isomorphism.

Proof. Obviously, if $\lambda : C^*(\Gamma) \rightarrow C_\lambda^*(\Gamma)$ is injective, ι is weakly contained in λ . For the converse we refer to [20, §18] or [57, Th. 7.3.9]. \square

Remark 3.4. In the sequel, we shall denote by σ the *conjugacy representation* $s \mapsto \lambda(s)\rho(s)$ of Γ . We shall use several times the following observation: since σ has a non-zero invariant vector, namely δ_e , the discrete group Γ is amenable if and only if σ is weakly contained into λ .

Proposition 3.5 (Lance, [48]). *Let Γ be a discrete group. The following conditions are equivalent:*

- (1) Γ is amenable.
- (2) $C_\lambda^*(\Gamma)$ is nuclear.
- (3) $C_\lambda^*(\Gamma)$ has the WEP.
- (4) The homomorphism $h : C_\lambda^*(\Gamma) \odot C_\lambda^*(\Gamma) \rightarrow \mathcal{B}(\ell^2(\Gamma))$ defined by

$$h(\lambda(s) \otimes \lambda(t)) = \lambda(s)\rho(t)$$

extends to a continuous homomorphism $\widetilde{(\lambda \cdot \rho)}_r$ on $C_\lambda^*(\Gamma) \otimes_{\min} C_\lambda^*(\Gamma)$. In other terms the representation $(\lambda \cdot \rho)_r$ admits the factorization:

$$\begin{array}{ccc} C_\lambda^*(\Gamma) \otimes_{\max} C_\lambda^*(\Gamma) & & \\ \downarrow P_r & \searrow^{(\lambda \cdot \rho)_r} & \\ C_\lambda^*(\Gamma \times \Gamma) = C_\lambda^*(\Gamma) \otimes_{\min} C_\lambda^*(\Gamma) & \xrightarrow{\widetilde{(\lambda \cdot \rho)}_r} & \mathcal{B}(\ell^2(\Gamma)). \end{array}$$

Proof. (1) \Rightarrow (2). Assume that Γ is amenable. Let (φ_i) be a net of positive definite functions with finite support, normalized by $\varphi_i(e) = 1$, converging pointwise to 1. Denote by m_i the extension of m_{φ_i} to $C_\lambda^*(\Gamma)$. Then (m_i) is a net of finite rank completely positive contractions, converging pointwise to $\text{Id}_{C_\lambda^*(\Gamma)}$. Hence, $C_\lambda^*(\Gamma)$ is nuclear.

(2) \Rightarrow (3) is obvious. Let us show that (3) \Rightarrow (1). Let T be a weak expectation from $\mathcal{B}(\ell^2(\Gamma))$ onto the weak closure of $C_\lambda^*(\Gamma)$. For $f \in \ell^\infty(\Gamma)$, viewed as the multiplication operator by f on $\ell^2(\Gamma)$, we set $M(f) = \langle \delta_e, T(f)\delta_e \rangle$. Since T is the identity on $C_\lambda^*(\Gamma)$ one has $T(axb) = aT(x)b$ for $a, b \in C_\lambda^*(\Gamma)$ and $x \in \mathcal{B}(\ell^2(\Gamma))$ (see [56, Th. 3.18]). It follows that

$$\begin{aligned} M(tf) &= \langle \delta_e, \lambda(t)T(f)\lambda(t)^*\delta_e \rangle \\ &= \langle \delta_{t^{-1}}, T(f)\delta_{t^{-1}} \rangle = \langle \rho(t)\delta_e, T(f)\rho(t)\delta_e \rangle \\ &= \langle \delta_e, T(f)\delta_e \rangle \end{aligned}$$

since the range of T commutes with $\rho(\Gamma)$.

Obviously (2) \Rightarrow (4) because the C^* -norms $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$ are the same for nuclear C^* -algebras.

(4) \Rightarrow (1). Let us assume the existence of a homomorphism

$$\widetilde{(\lambda \cdot \rho)}_r : C_\lambda^*(\Gamma) \otimes_{\min} C_\lambda^*(\Gamma) \rightarrow \mathcal{B}(\ell^2(\Gamma))$$

sending $\lambda(s) \otimes \lambda(t)$ onto $\lambda(s)\rho(t)$. Then the following diagram

$$\begin{array}{ccccc}
 & & C^*(\Gamma) & & \\
 & \swarrow \lambda & & \searrow \sigma & \\
 C_\lambda^*(\Gamma) & \longrightarrow & C_\lambda^*(\Gamma) \otimes_{\min} C_\lambda^*(\Gamma) & \xrightarrow{\widetilde{(\lambda \cdot \rho)}_r} & \mathcal{B}(\ell^2(\Gamma))
 \end{array}$$

is commutative, where the left horizontal arrow is $\lambda(s) \mapsto \lambda(s) \otimes \lambda(s)$. Therefore the representation σ is weakly contained into λ and Γ is amenable. \square

Remark 3.6. In particular, the canonical map

$$P_r : C_\lambda^*(\Gamma) \otimes_{\max} C_\lambda^*(\Gamma) \rightarrow C_\lambda^*(\Gamma) \otimes_{\min} C_\lambda^*(\Gamma)$$

is injective if and only if the group Γ is amenable.

In the von Neumann setting we have that $L(\Gamma)$ is injective (or hyperfinite by Connes' theorem) if and only if Γ is amenable. The previous proposition is to be compared with the following important theorem which combines results of Effros-Lance and Connes.

Theorem 3.7 ([24, 15, 16]). *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and M' its commutant. The natural map $h : M \odot M' \rightarrow \mathcal{B}(\mathcal{H})$ sending $m \otimes m'$ to mm' is continuous for the C^* -norm $\|\cdot\|_{\min}$ if and only if M is an injective von Neumann algebra.*

Remark 3.8. In one of their first papers, Murray and von Neumann showed that, when M is a factor¹⁰, the homomorphism $h : M \odot M' \rightarrow \mathcal{B}(\mathcal{H})$ is injective (see [66, Prop. 4.20]). It follows from the minimality of the norm $\|\cdot\|_{\min}$ that when M is an injective factor, h extends to an isometry from $M \otimes_{\min} M'$ onto the C^* -algebra $C^*(M, M')$ generated by $M \cup M'$.

4. Kirchberg factorization property

In this section we are concerned with full group C^* -algebras and the much more intricate factorization problem of the biregular representation $\lambda \cdot \rho$. For the free group \mathbb{F}_2 , it was proved by S. Wassermann in [72] that the representation

$$\lambda \cdot \rho : C^*(\mathbb{F}_2) \otimes_{\max} C^*(\mathbb{F}_2) \rightarrow \mathcal{B}(\ell^2(\mathbb{F}_2))$$

passes to the quotient through $C^*(\mathbb{F}_2) \otimes_{\min} C^*(\mathbb{F}_2)$. Kirchberg has studied this remarkable property for locally compact groups in a series of papers [42, 43, 44] and named it property (\mathcal{F}) .

¹⁰i.e. its centre is reduced to the scalar operators

Definition 4.1. We say that Γ has the *factorization property* (\mathcal{F}) if the representation $\lambda \cdot \rho$ admits a factorization through $C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$. We denote by $\widetilde{\lambda \cdot \rho}$ the representation of $C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$ obtained by passing to quotient:

$$\begin{array}{ccc} C^*(\Gamma \times \Gamma) = C^*(\Gamma) \otimes_{\max} C^*(\Gamma) & & \\ \downarrow P & \searrow \lambda \cdot \rho & \\ C^*(\Gamma) \otimes_{\min} C^*(\Gamma) & \xrightarrow{\widetilde{\lambda \cdot \rho}} & \mathcal{B}(\ell^2(\Gamma)) \end{array}$$

An important result of Kirchberg is that all residually finite groups have property (\mathcal{F}).

Definition 4.2. We say that Γ is *residually finite* if there exists a decreasing sequence (Γ_n) of normal subgroups with finite index, such that $\bigcap \Gamma_n = \{e\}$.

We say that Γ is *maximally almost periodic* if there is a faithful homomorphism from Γ into a compact group.

Of course a residually finite group is maximally almost periodic. Using the fact that a compact group has sufficiently many finite dimensional unitary representations, we see that any maximally almost periodic group is a subgroup of $\prod_{n \geq 1} U(n)$ where $U(n)$ is the group of unitary $n \times n$ matrices.

We denote, as usually, by R the unique type II_1 hyperfinite (= injective) von Neumann factor. It is easily seen that $\prod_{n \geq 1} U(n)$ is embeddable as a subgroup of the unitary group $\mathcal{U}(R)$ of R .

Theorem 4.3 (Kirchberg, [44]). *Let Γ be a discrete group such that there exists an injective homomorphism i from Γ into the unitary group $\mathcal{U}(R)$ of the type II_1 hyperfinite factor R . Then Γ has property (\mathcal{F}).*

Proof. We use the notations of subsection 2.5.2. Let ϱ and ϱ' be the representations of Γ such that $\varrho(s) = i(s)$ and $\varrho'(s) = Ji(s)J$ respectively, for all $s \in \Gamma$. We define a representation of $\Gamma \times \Gamma$ into the C^* -algebra $C^*(R, R')$ generated by $R \cup R'$ by

$$(\varrho \cdot \varrho')(s, t) = \varrho(s)\varrho'(t).$$

We have

$$\mathrm{tr}(i(st^{-1})) = \langle \hat{1}, (\varrho \cdot \varrho')(s, t)\hat{1} \rangle_{L^2(R)} \quad (4.1)$$

for all s, t .

Setting $\varphi(s, t) = \mathrm{tr}(i(st^{-1}))$, we see that $\varrho \cdot \varrho'$ is the GNS representation of $\Gamma \times \Gamma$ associated with the positive definite function φ . But since R is an injective factor, the map $h : R \odot R' \rightarrow C^*(R, R')$ sending $a \otimes a'$ onto aa' extends to an isomorphism from $R \otimes_{\min} R'$ onto $C^*(R, R')$. It follows that the representation $\varrho \cdot \varrho'$ admits a

factorization through $C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$:

$$\begin{array}{ccc} C^*(\Gamma) \otimes_{\max} C^*(\Gamma) & & \\ \downarrow P & \searrow \varrho \cdot \varrho' & \\ C^*(\Gamma) \otimes_{\min} C^*(\Gamma) & \xrightarrow{\varrho \otimes \varrho'} & R \otimes_{\min} R' \approx C^*(R, R'). \end{array}$$

Given $n \geq 1$, we have a faithful homomorphism $s \mapsto i(s)^{\otimes n}$ from Γ into the unitary group of the von Neumann tensor power $R^{\otimes n}$. Since $R^{\otimes n}$ is isomorphic to R , there is an embedding $s \mapsto i(s)^{\otimes n} \in \mathcal{U}(R)$ and we see similarly that the representation of $\Gamma \times \Gamma$ defined by $\varphi_n(s, t) = [\text{tr}(i(st^{-1}))]^n$ admits a factorization through $C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$.

Another crucial observation is that we can choose i such that $|\text{tr}(i(s))| < 1$ when $s \neq e$. Indeed, for $s \neq e$, the equality $|\text{tr}(i(s))| = 1$ implies $i(s) = c1$ where c is a complex number with $|c| = 1$ and $c \neq 1$ since i is faithful. Therefore, it suffices to replace i by the embedding $s \mapsto \begin{pmatrix} i(s) & 0 \\ 0 & 1 \end{pmatrix}$ into the unitary group of $M_2(R) \approx R$.

It follows that for $s, t \in \Gamma$, we have $\lim_n \varphi_n(s, t) = \delta_{s,t} = \langle \delta_e, \lambda(s)\rho(t)\delta_e \rangle$. We conclude that $\ker P \subset \ker(\lambda \cdot \rho)$ (see 2.5.3). \square

Examples of residually finite groups include:

- ▶ Free groups (see for instance [73, Lemma 3.6])¹¹.
- ▶ Finitely generated subgroups of $GL_n(\mathbb{C})$, by Malcev's theorem (see [4]).

Other examples of groups with property (\mathcal{F}) are:

- ▶ of course all amenable groups¹².
- ▶ all groups admitting an injective homomorphism into an almost connected locally compact group¹³ (see [43, Lemma 7.3 (iii)]).

The following remarkable result of Kirchberg says that for groups with Kazhdan property (T) , having property (\mathcal{F}) is exactly the same as being residually finite. Recall the Γ is said to have *Kazhdan property (T)* if every unitary representation (θ, \mathcal{H}) almost having invariant vectors (in the sense that for every $\varepsilon > 0$ and every finite subset F of Γ there exists a norm one vector $\xi \in \mathcal{H}$ with $\max_{s \in F} \|\theta(s)\xi - \xi\| \leq \varepsilon$) actually has a non-zero invariant vector.

Theorem 4.4 (Kirchberg, [44]). *A discrete group Γ with Kirchberg property (\mathcal{F}) and Kazhdan property (T) is residually finite.*

¹¹It is still an open problem whether all hyperbolic groups are residually finite (see [9, page 512]).

¹²Among them, there are non-residually finite ICC amenable groups (see 6.7 below for the definition of ICC and [44, 60, Remarks] for examples). They have of course a faithful unitary representation in the hyperfinite factor R , since R is isomorphic to $L(\Gamma)$ for any ICC amenable group Γ .

¹³The image need not be closed!

For a short proof, we refer to [73, th. 3.18].

Remarks 4.5. (1) There exist groups with property (T) which are not residually finite. Indeed Gromov [28] has stated the existence of simple infinite property (T) groups. Recently, even finitely presented such groups have been constructed [10].

(2) Let us mention the following related result of Robertson [60] stating that a Kazhdan group Γ having a faithful unitary representation into the factor $L(\mathbb{F}_\infty)$ of the free group on countably many generators, is residually finite. Note that the hyperfinite II_1 factor R is a subfactor of any type II_1 factor and therefore of $L(\mathbb{F}_\infty)$.

Remark 4.6. Of course, the factorization property holds whenever the canonical surjection from $C^*(\Gamma) \otimes_{\max} C^*(\Gamma)$ onto $C^*(\Gamma) \otimes_{\min} C^*(\Gamma)$ is injective. Very few is known about the class of groups satisfying this latter property. In particular, one of the most important open problems is whether \mathbb{F}_∞ is in this class. Kirchberg has proved the equivalence between a positive answer to this question and positive answers to several other fundamental questions, and among them

- *Connes' problem asking whether every separable type II_1 factor is a subfactor of the ultraproduct R_ω of the hyperfinite II_1 factor R ;*
- *whether $C^*(\mathbb{F}_\infty)$ has the WEP (see [43, §8]).*

Proposition 4.7 (Kirchberg, [43], Prop. 7.1). *Let Γ be a discrete group with the factorization property (\mathcal{F}) . Then Γ is amenable if and only if $C^*(\Gamma)$ is exact.*

Proof. Assume that $C^*(\Gamma)$ is exact. Let I be the kernel of $\lambda : C^*(\Gamma) \rightarrow C^*_\lambda(\Gamma)$. We consider the following commutative diagram, where the second row is an exact sequence:

$$\begin{array}{ccccccc}
 & & C^*(\Gamma) & \xrightarrow{\lambda} & C^*_\lambda(\Gamma) & & \\
 & & \downarrow \Delta_{\min} & & \downarrow \psi & & \\
 0 & \longrightarrow & I \otimes_{\min} C^*(\Gamma) & \longrightarrow & C^*(\Gamma) \otimes_{\min} C^*(\Gamma) & \xrightarrow{\lambda \otimes Id} & C^*_\lambda(\Gamma) \otimes_{\min} C^*(\Gamma) \longrightarrow 0 \\
 & & \downarrow \widetilde{\lambda \cdot \rho} & & \swarrow \pi & & \\
 & & \mathcal{B}(\ell^2(\Gamma)) & & & &
 \end{array}$$

The vertical map Δ_{\min} is given by $s \mapsto s \otimes s$ which, after composition with $\widetilde{\lambda \cdot \rho}$, gives the conjugacy representation $s \mapsto \sigma(s) = \lambda(s)\rho(s)$. On the other hand, $\psi : C^*_\lambda(\Gamma) \rightarrow C^*_\lambda(\Gamma) \otimes_{\min} C^*(\Gamma)$ is given by $\lambda(s) \mapsto \lambda(s) \otimes s$. The homomorphism π is obtained from $\widetilde{\lambda \cdot \rho}$ by passing to quotient, since $I \otimes_{\min} C^*(\Gamma) \subset \ker \widetilde{\lambda \cdot \rho}$. From the commutativity of the diagram, we see that the representation σ is weakly contained into the regular representation λ of Γ . The trivial representation is therefore weakly contained into λ and thus, Γ is amenable. \square

Problem : *Are there non-amenable discrete groups Γ for which the full C^* -algebra $C^*(\Gamma)$ is exact ?*

We shall now see that the reduced C^* -algebras of most of the usual discrete groups are known to be exact.

5. Amenable dynamical systems

Let X be a locally compact space and $(t, x) \mapsto tx$ be a left action of Γ on X by homeomorphisms. The space $\ell^1(\Gamma)_1^+$ of probability measures on Γ is equipped with the topology of pointwise convergence, which is the same here that the norm topology.

5.1. Definition and basic results.

Definition 5.1. We say that the action $\Gamma \curvearrowright X$ is amenable if there exists a net (f_i) of continuous maps $x \mapsto f_i^x$ from X into $\ell^1(\Gamma)_1^+$ such that for every $t \in \Gamma$:

$$\lim_i \|tf_i^x - f_i^{tx}\|_1 = 0$$

uniformly on compact subsets of X .

Precisely, we have $\sum_{s \in \Gamma} f_i^x(s) = 1$ for $x \in X$ and every i and

$$\forall t, \lim_i \sum_{s \in \Gamma} |f_i^{tx}(ts) - f_i^x(s)| = 0, \text{ uniformly on compact subsets of } X.$$

Such a net (f_i) is called an *approximate invariant continuous mean* (a.i.c.m. for short).

This is the version with parameter of (ii) in definition 3.2. One has also the analogue of statements (iii) and (iv) of definition 3.2 as equivalent definitions for amenable actions (see [7, Prop. 2.2 and Prop. 2.5] for details).

Examples 5.2. (1) If Γ is an amenable group, every action $\Gamma \curvearrowright X$ is amenable. Indeed, given an approximate invariant net $(f_i)_{i \in I}$ in $\ell^1(\Gamma)_1^+$, arising from definition 3.2 (ii), we define \tilde{f}_i to be the constant map on X whose value is f_i . Then $(\tilde{f}_i)_{i \in I}$ is an a.i.c.m.

Note that if $\Gamma \curvearrowright X$ is amenable and if X has a Γ -invariant probability measure μ , then Γ is an amenable group. Indeed, let us consider an a.i.c.m. $(f_i)_{i \in I}$ and set $k_i(t) = \int f_i^x(t) d\mu(x)$. Then $(k_i)_{i \in I}$ is an approximate invariant net in $\ell^1(\Gamma)_1^+$ and therefore Γ is amenable.

(2) For every group Γ , its left action on itself is amenable. Indeed, $x \in \Gamma \mapsto f^x = \delta_x$ is invariant : $tf^x = f^{tx}$ for every $t, x \in \Gamma$. Such an action, having a continuous invariant system of probability measures, is called *proper*. This is equivalent to the usual properness of the map $(t, x) \mapsto (tx, x)$ from $\Gamma \times X$ into $X \times X$ (see [6, Cor. 2.1.17]).

(3) Let \mathbb{F}_2 be the free group with two generators a and b . The boundary $\partial\mathbb{F}_2$ is the set of all infinite reduced words $\omega = a_1 a_2 \dots a_n \dots$ in the alphabet $S = \{a, a^{-1}, b, b^{-1}\}$. It is equipped with the topology induced by the product topology on $S^{\mathbb{N}^*}$. The group \mathbb{F}_2 acts continuously to the left by concatenation on the Cantor discontinuum $\partial\mathbb{F}_2$. This action is amenable. Indeed, for $n \geq 1$ and $\omega = a_1 a_2 \dots a_n \dots$, define

$$f_n^\omega(t) = \begin{cases} \frac{1}{n} & \text{if } t = a_1 \dots a_k, \quad k \leq \frac{1}{n}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(f_n)_{n \geq 1}$ is an a.i.c.m.

(4) Another convenient way to show that group actions are amenable is to use the invariance of this notion by Morita equivalence [6, Th. 2.2.17]. Let us consider for instance a locally compact group G , an amenable closed subgroup H and a discrete subgroup Γ . Then the left Γ -action on G/H is amenable.

Theorem 5.3. *Let $\Gamma \curvearrowright X$ be an action of a discrete group on a locally compact space X . The two following conditions are equivalent:*

- (i) *The action is amenable.*
- (ii) *The reduced cross product $C_0(X) \rtimes_r \Gamma$ is a nuclear C^* algebra.*

Proof. For (i) \Rightarrow (ii), one may proceed in two different ways. It is possible to construct explicitly a net of finite rank completely positive contractions from $C_0(X) \rtimes_r \Gamma$ into itself, approximating the identity map (see [7, Prop. 8.2]). It is also not too difficult to show directly the statement of definition 2.2 (see [5, Th. 3.4]). We also refer to [5, Th. 3.4] for the converse. \square

Proposition 5.4. *Let $\Gamma \curvearrowright X$ be an amenable action. The canonical surjection from $C_0(X) \rtimes \Gamma$ onto $C_0(X) \rtimes_r \Gamma$ is an isomorphism.*

For the proof, see [5, Th. 3.4] or [7, Th. 5.3].

Problem: *Is the converse true?*

5.2. Boundary amenability.

Definition 5.5. We say that a discrete group Γ is *boundary amenable* or *amenable at infinity* if it has an amenable action on a *compact space*.

Remark 5.6. This is a very useful property. Indeed, when X is compact, $C_\lambda^*(\Gamma)$ is a subalgebra of $C(X) \rtimes_r \Gamma$. Therefore, if $\Gamma \curvearrowright X$ is amenable, $C_\lambda^*(\Gamma)$ is a subalgebra of the nuclear C^* -algebra $C(X) \rtimes \Gamma$ and therefore is exact. This applies for instance to $\Gamma = \mathbb{F}_2$ and $X = \partial\mathbb{F}_2$.

The converse is true:

Theorem 5.7 ([51, 34, 5, 7]). *Let Γ be a discrete group. The following conditions are equivalent:*

- (i) *There exists an amenable action $\Gamma \curvearrowright X$ on a compact space X .*
- (ii) *$C_\lambda^*(\Gamma)$ is exact.*

For that reason, boundary amenable groups are also called *exact groups*. We shall need later the following lemma.

Lemma 5.8. *The following conditions are equivalent:*

- (i) *Γ is exact.*
- (ii) *The natural left action of Γ on the Stone-Ćech compactification $\beta\Gamma$ of Γ is amenable¹⁴.*
- (iii) *The natural left action of Γ on the state space S_Γ of $\ell^\infty(\Gamma)$, equipped with weak*-topology is amenable.*

Proof. (i) \Rightarrow (ii). Let X be a compact space on which Γ acts amenably. We choose $x_0 \in X$. The map $s \mapsto sx_0$ from Γ to X extends to a continuous map $p : \beta\Gamma \rightarrow X$, by the universal property of the Stone-Ćech compactification. Since p is Γ -equivariant, given an a.i.c.m. (f_i) for $\Gamma \curvearrowright X$, the net of maps $y \mapsto f_i^{p(y)}$ defines an a.i.c.m. for $\Gamma \curvearrowright \beta\Gamma$.

(ii) \Rightarrow (iii). Let (f_i) be an a.i.c.m. for $\Gamma \curvearrowright \beta\Gamma$. For $t \in \Gamma$, the function $f_i^x(t) : x \in \Gamma \mapsto f_i^x(t)$ is in $\ell^\infty(\Gamma)$. Given $\varphi \in S_\Gamma$, we define $f_i^\varphi \in \ell^1(\Gamma)_1^+$ by

$$f_i^\varphi(t) = \langle \varphi, f_i^\bullet(t) \rangle.$$

A straightforward computation shows that (f_i^φ) is an a.i.c.m. for $\Gamma \curvearrowright S_\Gamma$.

Finally, (iii) \Rightarrow (i) is obvious since S_Γ is compact. \square

Let us mention that exact groups have characterizations similar to the characterizations of amenable groups given in proposition 3.2.

Proposition 5.9. *Let Γ be a discrete group. The following conditions are equivalent:*

- (i) *Γ is boundary amenable.*
- (ii) *For every $\varepsilon > 0$ and every finite subset $F \subset \Gamma$, there exists a function $f : s \mapsto f_s$ from Γ to $\ell^1(\Gamma)_1^+$ and a finite subset F' of Γ such that*
 - (a) *$\|f_s - f_t\|_1 \leq \varepsilon$ whenever $s^{-1}t \in F$;*
 - (b) *$\text{supp}(f_s) \subset sF'$ for all $s \in \Gamma$.*
- (iii) *For every $\varepsilon > 0$ and every finite subset $F \subset \Gamma$, there exists a function $\xi : s \mapsto \xi_s$ from Γ to $\ell^2(\Gamma)_1$ and a finite subset F' of Γ such that*
 - (a) *$\|\xi_s - \xi_t\|_2 \leq \varepsilon$ whenever $s^{-1}t \in F$;*
 - (b) *$\text{supp}(\xi_s) \subset sF'$ for all $s \in \Gamma$.*
- (iv) *For every $\varepsilon > 0$ and every finite subset $F \subset \Gamma$, there exists a bounded positive definite kernel k on $\Gamma \times \Gamma$ and a finite subset F' of Γ such that*

¹⁴We endow the universal compactification $\beta\Gamma$ of the discrete space Γ with the continuous extension of the left action $\Gamma \curvearrowright \Gamma$. Recall that $\beta\Gamma$ is the spectrum of the C^* -algebra $\ell^\infty(\Gamma)$, so that $\ell^\infty(\Gamma) = C(\beta\Gamma)$

- (a) $|1 - k(s, t)| \leq \varepsilon$ whenever $s^{-1}t \in F$;
- (b) $\text{supp}(k) \subset \{(s, t), s^{-1}t \in F'\}$.

For the proof, we refer to [5, Prop. 4.4].

If Γ is finitely generated and ℓ is the length function associated to any choice of generators, one infers from the above proposition that boundary amenability is a metric property. It is very striking that it is equivalent to the analytic property described in theorem 5.7. One easily shows (see [5, Prop. 4.9] for a proof) that the above condition (iii) implies that Γ , equipped with the metric $d(s, t) = \ell(s^{-1}t)$, is *uniformly* (or *coarsely*) *embeddable* into a Hilbert space in the sense of Gromov [29].

Remark 5.10. A Følner type version of condition (ii) in proposition 5.9 was introduced by Yu in [74], under the name of property (A), for any discrete metric space. Higson and Roe proved [39] that for the length metric on finitely generated groups, this condition is equivalent to the boundary amenability of the group.

A remarkable result of Yu's [74] states that a group Γ with such a simple condition as uniform embeddability into a Hilbert space, and in particular boundary amenability (with a slight restriction removed later in [38, 64]), satisfies the Novikov conjecture. This raised the question whether every finitely generated group is boundary amenable, or at least can be embedded uniformly into a Hilbert space.

Boundary amenability has now been established for a long list of groups (see [55] for more details). Among them we mention

- ▶ amenable groups;
- ▶ hyperbolic groups [1, 27], hyperbolic groups relative to a family of exact subgroups ([54, 18]);
- ▶ Coxeter groups [21];
- ▶ linear groups [33];
- ▶ countable subgroups of almost connected Lie groups [33];
- ▶ discrete subgroups of almost connected groups (use (4) in Examples 5.2).

This class of boundary amenable groups is stable by extension [46], amalgamated free products and *HNN*-extensions [22, 69].

There exist finitely presented groups which are not uniformly embeddable into a Hilbert space, and a fortiori not boundary amenable [31].

Problem: *Does uniform embeddability into a Hilbert space implies boundary amenability ?*

The Thompson group F is uniformly embeddable into a Hilbert space [25], but at the time of writing, it is an open question whether this group is boundary amenable.

5.3. Amenable actions on universal compactifications. Let Γ be a discrete group acting on an infinite countable set D . This action extends to an action on the Stone-Ćech compactification βD . We denote by $\partial^\beta D$ the corona $\beta D \setminus D$. It is very useful to have criteria ensuring that $\Gamma \curvearrowright \partial^\beta D$ is an amenable action. A necessary condition is of course the exactness of Γ .

Proposition 5.11 (Ozawa, [53]). *Let $\Gamma \curvearrowright D$ as above and let α_D be the induced unitary representation in $\ell^2(D)$. The following conditions are equivalent:*

(1) *there exists a map $x \mapsto \xi_x$ from D into the unit sphere of $\ell^2(\Gamma)$ such that*

$$\forall s \in \Gamma, \quad \lim_{x \rightarrow \infty} \|s\xi_x - \xi_{sx}\|_2 = 0;$$

(2) *there exists an isometry $U : \ell^2(D) \rightarrow \ell^2(\Gamma)$ such that*

$$\forall s \in \Gamma, \quad U^* \lambda(s) U - \alpha_D(s) \in \mathcal{K}(\ell^2(D));$$

(3) *there exists a completely positive map $\Phi : C_\lambda^*(\Gamma) \rightarrow \mathcal{B}(\ell^2(D))$ such that*

$$\forall s \in \Gamma, \quad \Phi(\lambda(s)) - \alpha_D(s) \in \mathcal{K}(\ell^2(D)).$$

Proof. (1) \Rightarrow (3). Assume the existence of ξ . We define an isometry V and operators $T(s)$, $s \in \Gamma$, from $\ell^2(D)$ into $\ell^2(\Gamma) \otimes \ell^2(D)$ by

$$\begin{aligned} \forall x \in D, \quad V(\delta_x) &= \xi_x \otimes \delta_x, \\ T(s) &= (\lambda(s) \otimes \alpha_D(s))V - V\alpha_D(s). \end{aligned}$$

We have

$$T(s)\delta_x = (s\xi_x - \xi_{sx}) \otimes \delta_{sx}$$

Since $T(s)^*T(s)\delta_x = \|s\xi_x - \xi_{sx}\|_2^2 \delta_x$ and $\lim_{x \rightarrow \infty} \|s\xi_x - \xi_{sx}\|_2 = 0$, we see that $T(s)$ is a compact operator.

If we define the unitary operator W on $\ell^2(\Gamma) \otimes \ell^2(D)$ by $WF(t, x) = f(t, tx)$, we have

$$W(\lambda(s) \otimes \alpha_D(s))W^* = \lambda(s) \otimes \text{Id}_{\ell^2(D)}.$$

For $a \in C_\lambda^*(\Gamma)$ we set $\Phi(a) = V^*W^*(a \otimes \text{Id}_{\ell^2(D)})WV$. Obviously, Φ is a u.c.p. map satisfying the required condition.

(3) \Rightarrow (2). Let Φ be a u.c.p. map as in (3). Observe that $C_\lambda^*(\Gamma) \cap \mathcal{K}(\ell^2(\Gamma)) = \{0\}$. Indeed, otherwise there would exist a non-zero finite rank projection in $C_\lambda^*(\Gamma)$, and also such a projection in the C^* -algebra $C_\rho^*(\Gamma)$ of the right regular representation. Therefore λ would have a finite dimensional invariant subspace, in contradiction with the fact that Γ is infinite.

Now the conclusion is a consequence of the following result of Voiculescu.

Theorem 5.12 (Voiculescu, [71]). *Let H and K be two Hilbert spaces, and Φ be a u.c.p. map from a separable C^* -subalgebra A of $\mathcal{B}(H)$ into $\mathcal{B}(K)$ such that $\Phi(A \cap \mathcal{K}(H)) = 0$. There exists an isometry $U : K \rightarrow H$ such that $\Phi(a) - U^*aU \in \mathcal{K}(\ell^2(D))$ for every $a \in A$.*

For a proof, we refer to [17, Th. II.5.3]. In fact Voiculescu's result is stronger: it asserts the existence of a sequence (U_n) of isometries from H into K such that $\Phi(a) - U_n^* a U_n \in \mathcal{K}(\ell^2(D))$ for every n and $a \in A$, and $\lim_n \|\Phi(a) - U_n^* a U_n\| = 0$ for all a .

(2) \Rightarrow (1). We set $\xi_x = U\delta_x$. Then we have

$$\begin{aligned} \|s\xi_x - \xi_{sx}\|_2^2 &= \|\lambda(s)U\delta_x - U\delta_{sx}\|_2^2 \\ &= 2(1 - \operatorname{Re}\langle U^*\lambda(s)U\delta_x, \delta_{sx} \rangle) = 2\operatorname{Re}\langle k\delta_x, \delta_{sx} \rangle, \end{aligned}$$

where $k = \alpha_D(s) - U^*\lambda(s)U \in \mathcal{K}(\ell^2(D))$. It follows that $\lim_{x \rightarrow \infty} \|s\xi_x - \xi_{sx}\|_2 = 0$. \square

In the sequel Q will be the canonical homomorphism from $\mathcal{B}(\ell^2(\Gamma))$ onto the Calkin algebra $\mathcal{Q}(\ell^2(D)) = \mathcal{B}(\ell^2(D))/\mathcal{K}(\ell^2(D))$.

Proposition 5.13 (Ozawa, [53], Prop. 4.1). *Let Γ and D as above. The following conditions are equivalent:*

- (1) *The action $\Gamma \curvearrowright \partial^\beta D$ is amenable.*
- (2) *Γ is exact and there exists a nuclear homomorphism $\kappa : C_\lambda^*(\Gamma) \rightarrow \mathcal{Q}(\ell^2(D))$ such that $\kappa(\lambda(s)) = Q(\alpha_D(s))$ for all $s \in \Gamma$.*
- (3) *Γ is exact and there exists a map $x \mapsto \xi_x$ from D into the unit sphere of $\ell^2(\Gamma)$ such that*

$$\forall s \in \Gamma, \quad \lim_{x \rightarrow \infty} \|s\xi_x - \xi_{sx}\|_2 = 0.$$

Proof. (3) \Rightarrow (1). Setting $\mu_x = |\xi_x|^2$, we have

$$\forall s \in \Gamma, \quad \lim_{x \rightarrow \infty} \|s\mu_x - \mu_{sx}\|_1 = 0. \quad (5.1)$$

We extend μ to a continuous map from βD into the state space S_Γ of $\ell^\infty(\Gamma)$, and we denote by $\tilde{\mu}$ its restriction to $\partial^\beta D$. This map is Γ -equivariant, as a consequence of (5.1). Since Γ is exact, the action $\Gamma \curvearrowright S_\Gamma$ is amenable by Lemma 5.8. It follows that $\Gamma \curvearrowright \partial^\beta D$ is also amenable, because any a.i.c.m. for $\Gamma \curvearrowright S_\Gamma$ may be lifted to an a.i.c.m. for $\Gamma \curvearrowright \partial^\beta D$, via $\tilde{\mu}$.

(1) \Rightarrow (2). Assume that $\Gamma \curvearrowright \partial^\beta D$ is an amenable action. Since $\partial^\beta D$ is compact, the group Γ is exact. Observe first that $C(\partial^\beta D) \rtimes \Gamma = C(\partial^\beta D) \rtimes_r \Gamma$ and that it is a nuclear C^* -algebra. We define a covariant representation (π, θ) of $\Gamma \curvearrowright \partial^\beta D$ into $\mathcal{Q}(\ell^2(D))$ as follows. For $f \in \ell^\infty(D) = C(\beta D)$, let m_f be the operator of multiplication by f into $\ell^2(D)$. Then $\pi([f])$ is $Q(m_f)$, where $[f]$ is the class of f in $C(\partial^\beta D) = \ell^\infty(D)/c_0(D)$. We set $\theta = Q \circ \alpha_D$. The universal property of the full crossed product gives a homomorphism from $C(\partial^\beta D) \rtimes \Gamma$ into $\mathcal{Q}(\ell^2(D))$ and, by restriction, since the full and reduced crossed products are the same here, a homomorphism κ from $C_\lambda^*(\Gamma)$ into $\mathcal{Q}(\ell^2(D))$. This homomorphism κ is nuclear since the crossed product is nuclear and moreover $\kappa(\lambda(s)) = \theta(s) = Q(\alpha_D(s))$.

(2) \Rightarrow (3). Given κ as in the statement, the Choi-Effros lifting theorem for u.c.p. nuclear maps ([11, Th. 3.10], or [73, §6.3] for a short proof) implies the existence of a u.c.p. map Φ from $C_\lambda^*(\Gamma)$ into $\mathcal{B}(\ell^2(\Gamma))$ such that $Q \circ \Phi = \kappa$. We conclude by observing that Φ satisfies condition (3) of Proposition 5.11. \square

6. The Akemann-Ostrand property (\mathcal{AO}) and related notions

We have seen in section 2 that

$$(\lambda \cdot \rho)_r : C_\lambda^*(\Gamma) \otimes_{\max} C_\lambda^*(\Gamma) \rightarrow \mathcal{B}(\ell^2(\Gamma))$$

defined by

$$(\lambda \cdot \rho)_r \left(\sum a_i \otimes b_i \right) = \sum a_i \check{J} b_i \check{J}$$

passes to quotient through $C_\lambda^*(\Gamma) \otimes_{\min} C_\lambda^*(\Gamma)$ if and only if Γ is amenable. However, for free groups, Akemann and Ostrand have discovered in [2] the following very striking property: the homomorphism

$$\sum a_i \otimes b_i \mapsto Q \left(\sum a_i \check{J} b_i \check{J} \right)$$

from $C_\lambda^*(\mathbb{F}_2) \odot C_\lambda^*(\mathbb{F}_2)$ into the Calkin algebra $\mathcal{Q}(\ell^2(\mathbb{F}_2))$ is continuous for the minimal tensor norm.

Definition 6.1. We say that a discrete group Γ has the *Akemann-Ostrand property* (\mathcal{AO}) if $\sum a_i \otimes b_i \mapsto Q(\sum a_i \check{J} b_i \check{J})$ from $C_\lambda^*(\Gamma) \odot C_\lambda^*(\Gamma)$ into $\mathcal{Q}(\ell^2(\Gamma))$ is continuous for the minimal tensor norm. Hence there is a homomorphism κ which makes the following diagram commutative (where P_r is the canonical surjection):

$$\begin{array}{ccc} C_\lambda^*(\Gamma) \otimes_{\max} C_\lambda^*(\Gamma) & \xrightarrow{(\lambda \cdot \rho)_r} & \mathcal{B}(\ell^2(\Gamma)) \\ \downarrow P_r & & \downarrow Q \\ C_\lambda^*(\Gamma) \otimes_{\min} C_\lambda^*(\Gamma) & \xrightarrow{\kappa} & \mathcal{Q}(\ell^2(\Gamma)) \end{array}$$

Amenable actions on appropriate boundaries are very useful tools to establish the (\mathcal{AO}) property. We now introduce two boundary amenability conditions which imply this property.

6.1. Property (\mathcal{S}). This property was used by G. Skandalis [63] to show that discrete subgroups of rank one connected simple Lie groups have property (\mathcal{AO}).

We let $\Gamma \times \Gamma$ act on Γ by $(s, t) \cdot x = sxt^{-1}$. The corresponding representation of $\Gamma \times \Gamma$ on $\ell^2(\Gamma)$ is $\lambda \cdot \rho$.

Definition 6.2. We say that Γ has *property (\mathcal{S})* if $\Gamma \times \Gamma \curvearrowright \partial^\beta \Gamma$ is amenable.

Proposition 6.3. *Property (\mathcal{S}) implies property (\mathcal{AO}).*

Proof. This is a particular case of Proposition 5.13, where $D = \Gamma$ with the above action of the group $\Gamma \times \Gamma$, so that $\alpha_D = \lambda \cdot \rho$. \square

Remark 6.4. Observe that here the homomorphism κ of definition 6.1 is nuclear.

The class of groups with property (\mathcal{S}) is stable under free products with amalgamation on finite subgroups [53] but it is not stable under ordinary products. Indeed the following proposition shows in particular that the product $\Gamma_1 \times \Gamma_2$ of an infinite group by a non-amenable group has not the Akemann-Ostrand property. In particular we see that property (\mathcal{S}) is strictly stronger than exactness: for instance $\mathbb{F}_2 \times \mathbb{F}_2$ is exact but has not property (\mathcal{S}) .

Proposition 6.5 (Skandalis, [63], Rem. 4.6). *Let Γ be a discrete group with the (\mathcal{AO}) property. Let Γ_1 be an infinite subgroup of Γ . Then its centraliser Γ_2 is an amenable group.*

Proof. Let us introduce first some notation. We denote by $\Delta_{2,\max}$ the homomorphism from $C^*(\Gamma_2)$ into $C_\lambda^*(\Gamma_2) \otimes_{\max} C_\lambda^*(\Gamma_2)$ sending s onto $\lambda_{\Gamma_2}(s) \otimes \lambda_{\Gamma_2}(s)$. Similarly we define $\Delta_{2,\min} : C^*(\Gamma_2) \rightarrow C_\lambda^*(\Gamma_2) \otimes_{\min} C_\lambda^*(\Gamma_2)$.

Let σ_2 be the representation of Γ_2 into $\ell^2(\Gamma)$ defined by $\sigma_2(s)\xi(t) = \xi(s^{-1}ts)$.

We consider the following commutative diagram (where $\alpha, P_\Gamma, P_{\Gamma_2}, Q_\Gamma$ are the canonical maps):

$$\begin{array}{ccc}
 C^*(\Gamma_2) & & \\
 \Delta_{2,\max} \downarrow & \searrow \Delta_{2,\min} & \\
 C_\lambda^*(\Gamma_2) \otimes_{\max} C_\lambda^*(\Gamma_2) & \xrightarrow{P_{\Gamma_2}} & C_\lambda^*(\Gamma_2) \otimes_{\min} C_\lambda^*(\Gamma_2) \\
 \alpha \downarrow & & \downarrow \\
 C_\lambda^*(\Gamma) \otimes_{\max} C_\lambda^*(\Gamma) & \xrightarrow{P_\Gamma} & C_\lambda^*(\Gamma) \otimes_{\min} C_\lambda^*(\Gamma) \\
 (\lambda \cdot \rho)_r \downarrow & & \downarrow \kappa \\
 \mathcal{B}(\ell^2(\Gamma)) & \xrightarrow{Q_\Gamma} & \mathcal{Q}(\ell^2(\Gamma))
 \end{array}$$

Assume that Γ_2 is not amenable. Then, there exists $x \in C^*(\Gamma_2)$ with $\lambda_{\Gamma_2}(x) = 0$ and $\iota_{\Gamma_2}(x) = 1$. Since the restriction of $\sigma_2(x)$ to the infinite dimensional Hilbert space $\ell^2(\Gamma_1)$ is the identity operator, we see that $\sigma_2(x) \notin \mathcal{K}(\ell^2(\Gamma))$. The representation $\Delta_{2,\min}$ is equivalent to a multiple of the regular representation λ_{Γ_2} by Fell absorption principle. Thus, we have $P_{\Gamma_2} \circ \Delta_{2,\max}(x) = \Delta_{2,\min}(x) = 0$ and therefore $P_\Gamma \circ \alpha \circ \Delta_{2,\max}(x) = 0$. Observe now that

$$\sigma_2(x) = (\lambda \cdot \rho)_r \circ \alpha \circ \Delta_{2,\max}(x).$$

Set $y = \alpha \circ \Delta_{2,\max}(x)$. We have $Q_\Gamma \circ (\lambda \cdot \rho)_r(y) = Q_\Gamma(\sigma_2(x)) \neq 0$ on one hand and $Q_\Gamma \circ (\lambda \cdot \rho)_r(y) = \kappa \circ P_\Gamma(y) = 0$ on the other hand, hence the contradiction. \square

Remark 6.6. As already mentioned, Skandalis has proved in [63] that discrete subgroups of rank one connected simple Lie groups have property (\mathcal{S}) . He also

observed that this is not true in higher rank by giving the example of $SL_4(\mathbb{Z})$. One deduces from the above proposition that this group has not property (\mathcal{AO}) . Indeed $\Gamma_1 = \begin{pmatrix} SL_2(\mathbb{Z}) & 0 \\ 0 & 1 \end{pmatrix}$ has $\begin{pmatrix} 1 & 0 \\ 0 & SL_2(\mathbb{Z}) \end{pmatrix}$ in its centralizer.

We end this section by an observation, due to Ozawa, relating inner amenability and property (\mathcal{S}) .

Definition 6.7. We say that a group Γ has *infinite conjugacy classes* (or is *ICC*) if it has only infinite conjugacy classes except the trivial one.

We say that Γ is *inner amenable* if there exists a state m on $\ell^\infty(\Gamma)$ such that $m(\delta_e) = 0$ and $m \circ \sigma(s) = m$ for every $s \in \Gamma$, where $\sigma(s)(f)(x) = f(s^{-1}xs)$.

For example the group of all finite permutations of an infinite countable set is amenable and *ICC*. Free groups of two or more generators are *ICC*. More generally torsion free non-elementary¹⁵ hyperbolic groups are *ICC*. For more examples we refer to [37, Cor. 12].

Proposition 6.8. *Let Γ be an inner amenable ICC group satisfying the property (\mathcal{S}) . Then Γ is amenable.*

Proof. Indeed, since Γ is *ICC*, the state m has $c_0(\Gamma)$ in its kernel, and therefore there is a probability measure ν on $\partial^\beta\Gamma$ such as $\nu \circ \sigma(s) = \nu$, $s \in \Gamma$. Moreover, since $\Gamma \times \Gamma \curvearrowright \partial^\beta\Gamma$ is amenable, its restriction σ to Γ (embedded diagonally) is still amenable. Now the conclusion follows from 5.2 (1). \square

In particular, we recover the fact that torsion free non-elementary hyperbolic groups are not inner amenable [36] since they are *ICC*, non-amenable with property (\mathcal{S}) (see Examples 6.16).

6.2. Property (\mathcal{H}) . The possibility for a group to act amenably on some specific compact spaces may contain very useful additional informations, as we already saw above. We now consider compactifications K of Γ such that the left action of Γ onto itself extends to an amenable action on $\partial^K\Gamma = K \setminus \Gamma$. As emphasized by Ozawa in [55], the smaller is the compactification, the stronger is the property inherited by Γ .

Definition 6.9. A *compactification* of a discrete group Γ is a compact (Hausdorff) topological space K which contains Γ as a discrete open dense subset. The corona $K \setminus \Gamma$ will be denoted by $\partial^K\Gamma$.

We say that the *compactification is left* (resp. *right*) *equivariant* if the left (resp. right) translations on Γ extend continuously as homeomorphisms of K .

Given two compactifications K_1 and K_2 of Γ , we say that K_1 is smaller than K_2 if the identical map of Γ extends to a continuous map from K_2 onto K_1 .

¹⁵i.e. not almost cyclic

The Stone-Ćech compactification $\beta\Gamma$ is the largest compactification. The smallest one is the one-point compactification $\hat{\Gamma}$. Both of them are left and right equivariant. The group Γ is exact if and only if $\Gamma \curvearrowright \partial^\beta\Gamma$ is amenable, and Γ is amenable if and only if $\Gamma \curvearrowright \hat{\Gamma} \setminus \Gamma$ is so.

An interesting intermediate compactification is the Higson compactification that we describe below.

Remark 6.10. Using the Gelfand representation theorem, these definitions may be expressed in terms of abelian C^* -algebras:

- K is a compactification of Γ if and only if

$$c_0(\Gamma) \subset C(K) \subset \ell^\infty(\Gamma) = C(\beta\Gamma).$$

- K is left equivariant if and only if $C(K)$ is invariant under left translations, that is ${}^s f \in C(K)$ for all $s \in \Gamma$ and $f \in C(K)$, where ${}^s f(x) = f(s^{-1}x)$. Similarly, K is right equivariant if and only if $f^s \in C(K)$ for all $s \in \Gamma$ and $f \in C(K)$, where $f^s(x) = f(xs^{-1})$

The subset of $\ell^\infty(\Gamma)$ consisting of all f such that $f - f^s \in c_0(\Gamma)$ for $s \in \Gamma$ is a C^* -subalgebra containing $c_0(\Gamma)$ and invariant under left and right translations. So it is of the form $C(\mathcal{H}(\Gamma))$.

Definition 6.11. The compact space $\mathcal{H}(\Gamma)$ is called the *Higson compactification* of Γ . The Higson corona is $\partial^{\mathcal{H}}\Gamma = \mathcal{H}(\Gamma) \setminus \Gamma$.

Definition 6.12. We say that a compactification K of Γ is *small at infinity* if it is left and right equivariant and if the restriction to $\partial^K\Gamma$ of the right action is trivial.

This means that for every net (s_i) in Γ such that $\lim_i s_i = x \in \partial^K\Gamma$, then $\lim_i s_i t = x$ for every $t \in \Gamma$.

Remark 6.13. One easily checks that the Higson compactification $\mathcal{H}(\Gamma)$ is small at infinity and that it is the largest one with this property.

This compactification was introduced for metric spaces by Higson in connection with Roe's index theorem for non-compact Riemannian manifolds. For a proper metric space¹⁶ X , it was reformulated by Roe [61] in the following way. Given a continuous function $f : X \rightarrow \mathbb{C}$ and $r > 0$, define

$$V_r(f)(x) = \sup \{|f(y) - f(x)| : y \in B_r(x)\}.$$

Let $C_h(X)$ be the space of all bounded continuous functions $f : X \rightarrow \mathbb{C}$ such that for each $r > 0$, $V_r(f)$ goes to 0 at infinity. Then $C_h(X)$ is a C^* -algebra containing $C_0(X)$. Its spectrum is, by definition, the Higson compactification of X . When X is a finitely generated group Γ , endowed with a length metric, one immediately sees that $C_h(\Gamma) = C(\mathcal{H}(\Gamma))$.

¹⁶i.e. the balls $B_r(x)$ are relatively compact

Definition 6.14. We say that Γ has *property* (\mathcal{H}) if $\Gamma \curvearrowright \partial^{\mathcal{H}}\Gamma$ is amenable. Equivalently, Γ has property (\mathcal{H}) if there exists a compactification K of Γ which is small at infinity and such that $\Gamma \curvearrowright \partial^K\Gamma$ is amenable.

Lemma 6.15. *Property* (\mathcal{H}) *implies property* (\mathcal{S}) .

Proof. Let K be a small at infinity compactification of Γ such $\Gamma \curvearrowright \partial^K\Gamma$ is amenable. We let $\Gamma \times \Gamma$ act on $K \times K$ by

$$(s_1, s_2) \cdot (k_1, k_2) = (s_1 k_1 s_2^{-1}, s_2 k_2 s_1^{-1}).$$

Note that this action is amenable on $\partial^K\Gamma \times \partial^K\Gamma$ since, for $k_1, k_2 \in \partial^K\Gamma$,

$$(s_1, s_2) \cdot (k_1, k_2) = (s_1 k_1, s_2 k_2).$$

The embedding of Γ into $\Gamma \times \Gamma$ by $s \mapsto (s, s^{-1})$ is $\Gamma \times \Gamma$ -equivariant when Γ is equipped with the bilateral action $(s_1, s_2) \cdot x = s_1 x s_2^{-1}$ and $\Gamma \times \Gamma$ with the above defined action. It provides a $\Gamma \times \Gamma$ -equivariant embedding of Γ into $K \times K$.

Let $\bar{\Gamma}^{K \times K}$ denote the closure of Γ into $K \times K$. The $\Gamma \times \Gamma$ action on $\bar{\Gamma}^{K \times K} \setminus \Gamma$ is still amenable, being the restriction of its action on $\partial^K\Gamma \times \partial^K\Gamma$.

By the universal property of the compactification $\beta\Gamma$ we get a $\Gamma \times \Gamma$ -equivariant commutative diagram

$$\begin{array}{ccc} & & \beta\Gamma \\ & \nearrow & \vdots \\ \Gamma & \longrightarrow & \bar{\Gamma}^{K \times K} \\ & & \downarrow q \end{array}$$

Since q sends $\partial\beta\Gamma$ into $\bar{\Gamma}^{K \times K} \setminus \Gamma$, we see that Γ has property (\mathcal{S}) □

Examples 6.16. We have already mentioned that the origin of property (\mathcal{AO}) is the Akemann-Ostrand result stating that free groups \mathbb{F}_n have this property. Their proof uses explicit computations of norms in $C_{\lambda, \rho}^*(\mathbb{F}_n \times \mathbb{F}_n)$. Another nice way to show this result is to observe that free groups have property (\mathcal{H}) .

Indeed, we have seen in 5.2 (3) that the action $\mathbb{F}_n \curvearrowright \partial\mathbb{F}_n$ is amenable. Moreover the compactification $\mathbb{F}_n \cup \partial\mathbb{F}_n$ is small at infinity since, given a sequence (x_k) in \mathbb{F}_n such that $\lim_k x_k = \omega \in \partial\mathbb{F}_n$, then, obviously, for any $t \in \mathbb{F}_n$ we still have $\lim_k x_k t = \omega$.

Similarly, one shows that hyperbolic groups have property (\mathcal{H}) . Ozawa has extended this result to the class of groups that are hyperbolic relative to a family of amenable subgroups [54, Prop.12].

6.3. The C^* -algebra of the biregular representation. Let Γ be a discrete group. The C^* -algebra of the biregular representation is the C^* -subalgebra of $\mathcal{B}(\ell^2(\Gamma))$ generated by $\lambda(\Gamma) \cup \rho(\Gamma)$. We denote it by $C_{\lambda, \rho}^*(\Gamma)$ instead of $C_{\lambda, \rho}^*(\Gamma \times \Gamma)$. We have

$$C_{\sigma}^*(\Gamma) \subset C_{\lambda, \rho}^*(\Gamma) \subset C^*(L(\Gamma), R(\Gamma))$$

where $C_\sigma^*(\Gamma)$ is the C^* -algebra generated by the conjugacy representation σ , and $C^*(L(\Gamma), R(\Gamma))$ is the C^* -algebra generated by $L(\Gamma) \cup R(\Gamma)$. We discuss now the position of these three algebras with respect to $\mathcal{K}(\ell^2(\Gamma))$.

Recall first that in their study of type II_1 factors [50], Murray and von Neumann have introduced an invariant to distinguish between them, they named property *Gamma*. They proved that the hyperfinite II_1 factor has this property but not $L(\mathbb{F}_n)$, $n \geq 2$. We shall not recall Murray-von Neumann's original definition (see [19, p. 283]) but instead give two remarkable equivalent characterizations. In the next theorem, part (1) was established by Connes and Sakai [14, 62], and part (2) by Connes [15, Th. 2.1].

Let M be a type II_1 factor. We provide the group $\text{Aut}(M)$ of automorphisms of M with the topology of pointwise convergence in norm relative to its action on the predual M_* of M . We denote by $C^*(M, M')$ the C^* -subalgebra of $\mathcal{B}(L^2(M))$ generated by M and its commutant.

Theorem 6.17. *M has property Gamma if and only if one of the two following equivalent conditions hold:*

- (1) *The subgroup of inner automorphisms is not closed in $\text{Aut}(M)$.*
- (2) *$C^*(M, M') \cap \mathcal{K}(L^2(M)) = \{0\}$.*

Since M is a factor, the only operators commuting with $C^*(M, M')$ are the scalar ones. It follows from [20, Cor. 4.1.10] that we have either $\mathcal{K}(\ell^2(\Gamma)) \subset C^*(M, M')$ or $C^*(M, M') \cap \mathcal{K}(\ell^2(\Gamma)) = \{0\}$.

The hyperfinite type II_1 factor R has property *Gamma* and therefore

$$C^*(R, R') \cap \mathcal{K}(\ell^2(R)) = \{0\}.$$

Note that by Remark 3.8, the C^* -algebra $C^*(R, R')$ and $R \otimes_{\min} R'$ are canonically isomorphic. In addition, since a type II_1 factor is a simple algebra (see [40, Prop. A.3.1]) and since the minimal tensor product of two simple C^* -algebras is simple (see [66, Cor. 4.21]), we see that $C^*(R, R')$ is simple.

Let us come back to the case of $M = L(\Gamma)$. It is easily seen that Γ is *ICC* if and only if $L(\Gamma)$ is a von Neumann factor (see for instance [40, Prop. 1.4.1]). In this situation, although $L(\Gamma)$ is simple, it is not always the case for its weak* dense C^* -subalgebra $C_\lambda^*(\Gamma)$. The simplicity of $C_\lambda^*(\Gamma)$ implies that Γ is *ICC* and non-amenable (see [37]). Powers proved [59] that $C_\lambda^*(\mathbb{F}_n)$, $n \geq 2$, is simple. This is still true for torsion free non-elementary hyperbolic groups and for large classes of groups appearing naturally in geometry. We refer to the survey [37].

Let Γ be an *ICC* amenable group. We have

$$C_\lambda^*(\Gamma) \otimes_{\min} C_\rho^*(\Gamma) \subset L(\Gamma) \otimes_{\min} L(\Gamma)' = R \otimes_{\min} R'.$$

After identification of $C_\rho^*(\Gamma)$ with $C_\lambda^*(\Gamma)$, and using the above remarks, the restriction to $C_\lambda^*(\Gamma) \otimes_{\min} C_\rho^*(\Gamma)$ of the isomorphism from $R \otimes_{\min} R'$ onto $C^*(R, R')$ gives:

Proposition 6.18. *Let Γ be an amenable ICC group. Then $\widetilde{(\lambda \cdot \rho)_r}$ is an isomorphism from $C_\lambda^*(\Gamma) \otimes_{\min} C_\lambda^*(\Gamma)$ onto $C_{\lambda,\rho}^*(\Gamma)$. Moreover, $C_{\lambda,\rho}^*(\Gamma)$ is not simple and $C_{\lambda,\rho}^*(\Gamma) \cap \mathcal{K}(\ell^2(\Gamma)) = \{0\}$*

Note that when Γ is ICC, $C_{\lambda,\rho}^*(\Gamma)$ behaves as $C^*(L(\Gamma), R(\Gamma))$, since its commutant is still reduced to the scalar operators: either $C_{\lambda,\rho}^*(\Gamma) \cap \mathcal{K}(\ell^2(\Gamma)) = \{0\}$ or $\mathcal{K}(\ell^2(\Gamma)) \subset C_{\lambda,\rho}^*(\Gamma)$. In their paper [2], Akemann and Ostrand proved that the latter inclusion holds for free groups with $n \geq 2$ generators. This is still true for any ICC non-amenable group with property (\mathcal{AO}) :

Proposition 6.19 (Skandalis, [63]). *Let Γ be a non-amenable group with the (\mathcal{AO}) property. Then $\mathcal{K}(\ell^2(\Gamma)) \cap C_{\lambda,\rho}^*(\Gamma) \neq \{0\}$. If moreover Γ is ICC, then $\mathcal{K}(\ell^2(\Gamma))$ is an ideal of $C_{\lambda,\rho}^*(\Gamma)$ and $\lambda(s) \otimes \lambda(t) \mapsto Q(\lambda(s)\rho(t))$ induces an isomorphism κ between $C_\lambda^*(\Gamma) \otimes_{\min} C_\lambda^*(\Gamma)$ and $C_{\lambda,\rho}^*(\Gamma)/\mathcal{K}(\ell^2(\Gamma))$.*

Proof. Since Γ is not amenable, the conjugacy representation σ is not weakly contained into the regular representation λ . Therefore there exists $x \in C^*(\Gamma)$ such that $\sigma(x) \neq 0$ and $\lambda(x) = 0$.

Let us consider the commutative diagram:

$$\begin{array}{ccccccc}
& & & C^*(\Gamma) & & & \\
& & & \downarrow \Delta_{\max} & & & \\
0 & \longrightarrow & I & \longrightarrow & C_\lambda^*(\Gamma) \otimes_{\max} C_\lambda^*(\Gamma) & \xrightarrow{P_r} & C_\lambda^*(\Gamma) \otimes_{\min} C_\lambda^*(\Gamma) \longrightarrow 0 \\
& & \downarrow & & \downarrow (\lambda \cdot \rho)_r & & \downarrow \kappa \\
0 & \longrightarrow & \mathcal{K}(\ell^2(\Gamma)) & \longrightarrow & \mathcal{B}(\ell^2(\Gamma)) & \xrightarrow{Q} & \mathcal{Q}(\ell^2(\Gamma)) \longrightarrow 0
\end{array}$$

where $I = \ker P_r$.

For $s \in \Gamma$, we have $P_r \circ \Delta_{\max}(s) = \lambda(s) \otimes \lambda(s)$, and since the representation $\lambda \otimes \lambda$ is equivalent to a multiple of λ , we see that $P_r \circ \Delta_{\max}(x) = 0$.

Then

$$(\lambda \cdot \rho)_r \circ \Delta_{\max}(x) = \sigma(x) \neq 0 \quad \text{and} \quad Q \circ \sigma(x) = \kappa \circ P_r \circ \Delta_{\max}(x) = 0,$$

and therefore, $\sigma(x)$ is a nonzero element in $C_{\lambda,\rho}^*(\Gamma) \cap \mathcal{K}(\ell^2(\Gamma))$.

Assuming now, in addition, that Γ is ICC, we get the inclusion $\mathcal{K}(\ell^2(\Gamma)) \subset C_{\lambda,\rho}^*(\Gamma)$. Let us show that $\kappa : C_\lambda^*(\Gamma) \otimes_{\min} C_\lambda^*(\Gamma) \rightarrow C_{\lambda,\rho}^*(\Gamma)/\mathcal{K}(\ell^2(\Gamma))$ is injective. Set $M = L(\Gamma)$. For $a_i, b_i \in M$, $i = 1, \dots, n$, define

$$\mathcal{N}\left(\sum a_i \otimes b_i\right) = \left\| Q\left(\sum a_i \check{J} b_i \check{J}\right) \right\|.$$

Observe that \mathcal{N} is a C^* -norm on $M \odot M$. Indeed, $I_M = \{x \in M \odot M : \mathcal{N}(x) = 0\}$ is an ideal of $M \odot M$, and its norm closure is an ideal of $M \otimes_{\min} M$. But this

C^* -algebra is simple since M is simple. Therefore $I_M = 0$. It follows that for $a_i, b_i \in C_{\lambda}^*(\Gamma)$, $i = 1, \dots, n$, we have

$$\left\| \sum a_i \otimes b_i \right\|_{\min} \leq \left\| Q(\sum a_i \check{J} b_i \check{J}) \right\|,$$

and we conclude that $\sum a_i \otimes b_i \mapsto Q(\sum a_i \check{J} b_i \check{J})$ is isometric. \square

Corollary 6.20. *Let Γ be a non-amenable group with property (\mathcal{AO}) , such that $C_{\lambda}^*(\Gamma)$ is simple (for instance a torsion free non-elementary hyperbolic group). Then $\mathcal{K}(\ell^2(\Gamma))$ is the only closed ideal of $C_{\lambda, \rho}^*(\Gamma)$.*

For free groups this is a result of Akemann-Ostrand [2].

Corollary 6.21. *Let Γ be a non-amenable ICC group with property (\mathcal{AO}) . Then $L(\Gamma)$ does not have Murray-von Neumann property Γ .*

Proof. Combine Connes' theorem 6.17 (2) and Proposition 6.19. \square

Let us come back now to inner amenability. As for amenability, there are several equivalent definitions of this notion (see [8, Th. 1]). Among them is the weak containment of the conjugacy representation σ into the regular one λ and the following characterization: if P_{δ_e} denotes the rank one projection on the line generated by $\delta_e \in \ell^2(\Gamma)$, then Γ is inner amenable if and only if $P_{\delta_e} \notin C_{\sigma}^*(\Gamma)$. In particular, if Γ is ICC and if $L(\Gamma)$ has property Γ , then $C^*(L(\Gamma), R(\Gamma)) \cap \mathcal{K}(\ell^2(\Gamma)) = \{0\}$ and therefore Γ is inner amenable. An elementary proof of this fact is due to Effros [23]. The notion of inner amenability was introduced in Effros' paper. The motivation was to give a characterization of property Γ uniquely in terms of group properties, and without the use of its associated operator algebras. However the following question is still open:

Problem : *If Γ is ICC and inner amenable, does it imply that $L(\Gamma)$ has property Γ ?*

Problem : *Let Γ be ICC, non-amenable with property (\mathcal{AO}) . Can we conclude that Γ is not inner amenable ?*

We remind the reader that an ICC, non-amenable group with property (\mathcal{S}) is not inner amenable (Proposition 6.8). It seems difficult to find examples of groups having property (\mathcal{AO}) but not property (\mathcal{S}) . In view of the following proposition, it is conceivable that every boundary amenable group with property (\mathcal{AO}) has property (\mathcal{S}) .

Proposition 6.22. *Let Γ be an ICC group. The two following conditions are equivalent:*

- (1) Γ has property (\mathcal{S}) .
- (2) Γ has property (\mathcal{AO}) , and the quotient map κ is nuclear.

Proof. For (1) \Rightarrow (2) see Proposition 6.3 and Remark 6.4. Let us prove the converse. Since κ is injective and nuclear, we see that $C_\lambda^*(\Gamma \times \Gamma) = C_\lambda^*(\Gamma) \otimes_{\min} C_\lambda^*(\Gamma)$ is exact. The conclusion follows from Propositions 5.13 applied to the group $\Gamma \times \Gamma$ acting on $D = \Gamma$. \square

7. Applications of property (\mathcal{AO})

Non-amenable groups with property (\mathcal{AO}) give rise to very interesting operator algebras. We mention below two remarkable applications due to Skandalis and Ozawa respectively.

7.1. Non-nuclearity in K -theory. When Γ is a non-amenable group, we have seen that the canonical map $P_r : C_\lambda^*(\Gamma) \otimes_{\max} C_\lambda^*(\Gamma) \rightarrow C_\lambda^*(\Gamma) \otimes_{\min} C_\lambda^*(\Gamma)$ is not injective. If Γ is an infinite group and has Kazhdan property (T) it is easy to find a non-zero element in $\text{Ker } P_r$.

Recall that one of the characterizations of property (T) is the existence of a central projection $p \in C^*(\Gamma)$ such that $\varrho(p)$ is the projection on the space of $\varrho(\Gamma)$ -invariant vectors, for every representation ϱ (see [3, Lemma 2] or for instance [70, Prop. 2]).

Assume that Γ has property (T) . Since $\Delta_{\max} : C^*(\Gamma) \rightarrow C_\lambda^* \otimes_{\max} C_\lambda^*$ is injective (see [58, Th. 8.2]), $q = \Delta_{\max}(p)$ is a non-zero projection. On the other hand, the representation $s \mapsto P_r \circ \Delta_{\max}(s)$, which is equivalent to a multiple of the regular representation λ , has no non-trivial invariant vector. It follows that $P_r(q) = 0$.

When, in addition Γ has the Akemann-Ostrand property, then q is even non-trivial in K -theory.

Proposition 7.1 (Skandalis, [63], Cor. 4.5). *Let Γ be an infinite discrete group with both properties (T) and (\mathcal{AO}) . The K -theory class $[q]$ of the above projection $q \in C_\lambda^*(\Gamma) \otimes_{\max} C_\lambda^*(\Gamma)$ is a non-zero element of the kernel of*

$$(P_r)_* : K_0(C_\lambda^*(\Gamma) \otimes_{\max} C_\lambda^*(\Gamma)) \rightarrow K_0(C_\lambda^*(\Gamma) \otimes_{\min} C_\lambda^*(\Gamma)).$$

In particular, the homomorphism P_r is not invertible in K -theory.

Proof. Since $\sigma = (\lambda \cdot \rho)_r \circ \Delta_{\max}$ has the non-zero invariant vector δ_e , we see that $\sigma(p) = (\lambda \cdot \rho)_r(q) \neq 0$, whereas $Q \circ (\lambda \cdot \rho)_r(q) = \kappa \circ P_r(q) = 0$ (see the commutative diagram introduced in the proof of proposition 6.19). It follows that $(\lambda \cdot \rho)_r(q)$ is a non-zero finite rank projection. It is still non-zero in $K_0(\mathcal{K}(\ell^2(\Gamma)))$, and therefore we also have $[q] \neq 0$. \square

Remark 7.2. G. Skandalis proved this proposition in [63] in order to obtain examples of C^* -algebras which are not nuclear in K -theory (see [63], Definition 3.1). Indeed, by [63, Prop. 3.5], the reduced C^* -algebra of any infinite dimensional discrete group having properties (T) and (\mathcal{AO}) is not nuclear in K -theory.

7.2. Solid von Neumann algebras. Ozawa has proved in [52] the analogue of Proposition 6.5 for type II_1 factors $L(\Gamma)$ associated with ICC exact groups Γ having the (\mathcal{AO}) property (and in particular ICC groups with property (\mathcal{S})).

Definition 7.3. A von Neumann algebra M is *solid* if for any diffuse¹⁷ von Neumann subalgebra N of $L(\Gamma)$, the relative commutant $N' \cap L(\Gamma)$ is an injective von Neumann algebra.

Theorem 7.4 (Ozawa, [52], Th. 1). $L(\Gamma)$ is solid whenever Γ is an ICC exact group with property (\mathcal{AO}) .

As a consequence, when in addition $L(\Gamma)$ is not injective (i.e. Γ is not amenable), it follows that $L(\Gamma)$ is prime, that is $L(\Gamma)$ cannot be decomposed as a tensor product of two infinite dimensional type II_1 factors. This gives a wealth of examples of prime type II_1 factors, including free group factors. This latter result was first obtained by Ge [26] as an application of free entropy.

More generally, Ozawa has defined the Akemann-Ostrand property for any type II_1 factor (and even, any finite von Neumann algebra) in such a way that $L(\Gamma)$ has this property whenever Γ is exact and has the (\mathcal{AO}) property. He proved that such finite von Neumann algebras are solid.

REFERENCES

- [1] S. Adams: *Boundary amenability for word hyperbolic groups and an amplification to smooth dynamics of simple groups*, *Topology*, **33** (1994), 765-783.
- [2] C.A. Akemann and P.A. Ostrand: *On a tensor product C^* -algebra associated with the free group on two generators*, *J. Math. Soc. Japan*, **4** (4) (1975), 589-599.
- [3] C.A. Akemann and M. Walter: *Unbounded negative definite functions*, *Canad. J. Math.*, **33** (1981), 862-871.
- [4] R.C. Alperin: *An elementary account of Selberg's lemma*, *L'Enseignement Mathématique*, **33** (1987), 269-273.
- [5] C. Anantharaman-Delaroche: *Amenability and exactness for dynamical systems and their C^* -algebras*, preprint MAPMO no 00-08, Orléans, 2000, available at <http://www.univ-orleans.fr/mapmo/publications/index.php>.
- [6] C. Anantharaman-Delaroche and J. Renault: *Amenable groupoids*, *Monographies de l'Enseignement Mathématique* **36**, Geneva 2000.
- [7] C. Anantharaman-Delaroche: *Amenability and exactness for dynamical systems and their C^* -algebras*, *Trans. Amer. Math. Soc.*, **354** (2002), 4153-4178 (extended version of [5]).
- [8] E. Bédos and P. de la Harpe: *Moyennabilité intérieure des groupes: définitions et exemples*, *L'Enseignement Math.*, **32** (1986), 139-157.
- [9] M. R. Bridson and A. Haefliger: *Metric spaces of non-positive curvature*, *Grundlehren der math. Wissenschaften* **319**, Springer, 1999.
- [10] P-E. Caprace and B. Rémy: *Simplicity and superrigidity of twin buildings*, [arXiv:math/0607664v2](https://arxiv.org/abs/math/0607664v2).
- [11] M.D. Choi and E.G. Effros: *The completely positive lifting problem for C^* algebras*, *Ann. of Math.*, **104** (1976), 585-609.

¹⁷i.e. without non-zero minimal projections

- [12] M.D. Choi and E.G. Effros: *Nuclear C^* -algebras and the approximation property*, Amer. J. Math., **100** (1978), 67-79.
- [13] M.D. Choi and E.G. Effros: *Nuclear C^* -algebras and injectivity : the general case*, Indiana Univ. Math. J., **26** (1977), 443-446.
- [14] A. Connes: *Almost periodic states and factors of type III_1* , J. Funct. Anal., **16** no 4 (1974), 415-445.
- [15] A. Connes: *Classification of injective factors*, Ann. Math., **104** (1976), 73-115.
- [16] A. Connes: *On the equivalence between injectivity and semi-discreteness for operator algebras*, Algèbres d'opérateurs et leurs applications en physique mathématique, Colloques Internationaux du C.N.R.S., No **274** (Marseille 1977), 107-112.
- [17] K.R. Davidson: *C^* -algebras by example*, Fields Institute Monographs, Amer. Math. Soc., Providence, R.I., 1996.
- [18] M. Dadarlat and E. Guentner: *Uniform embeddability of relatively hyperbolic groups*, [arXiv:math.GR/0501495](https://arxiv.org/abs/math/0501495)
- [19] J. Dixmier: *Les algèbres d'opérateurs dans l'espace hilbertien (Algèbres de von Neumann)*, Gauthier-Villars, Paris, 1969.
- [20] J. Dixmier: *Les C^* -algèbres et leurs représentations*, Gauthier-Villars, Paris, 1969.
- [21] A. Dranishnikov and T. Januszkiewicz: *Every Coxeter group acts amenably on a compact space*, In Proceedings of the 1999 Topology and Dynamics Conference (Salt Lake City, UT), Topology Proc. **24** (1999), 135-141.
- [22] K. Dykema: *Exactness of reduced amalgamated free products C^* -algebras*, Forum Math., **16** (2004), 161-180.
- [23] E.G. Effros: *Property Γ and inner amenability*, Proc. Amer. Math. Soc., **47** (1975), 483-486.
- [24] E.G. Effros and C. Lance: *Tensor products of operator algebras*, Adv. in Math., **25** (1977), 1-34.
- [25] D. Farley: *Finiteness and CAT(0) properties of diagram groups*, Topology, **42** (2003), no 5, 1065-1082.
- [26] L. Ge: *Applications of free entropy to finite von Neumann algebras II*, Ann. of Math., **147** (1998), no 1, 143-147.
- [27] E. Germain: *Approximate invariant means for boundary actions of hyperbolic groups*, Appendix B of the book [6].
- [28] M. Gromov: *Hyperbolic groups*, In Essays in group theory, S.M. Gersten (ed.), Springer 1987, 75-263.
- [29] M. Gromov: *Asymptotic invariants of infinite groups*, in Geometric group theory, London Math. Soc. Lecture Note Series **182**, pp. 1-295, Cambridge university Press, 1993.
- [30] M. Gromov: *Spaces and questions*, Geom. Funct. Anal., (2000), 118-161.
- [31] M. Gromov: *Random walks in random groups*, Geom. Funct. Anal., **13** (2003), 73-146.
- [32] E. Guentner and N. Higson: *Group C^* -algebras and K -theory*, In Noncommutative Geometry, Lectures Notes in Math., **1831**, Springer, Berlin, 2004, 137-251.
- [33] E. Guentner, N. Higson and S. Weinberger: *The Novikov conjecture for linear groups*, Publ. Math. I. H. E. S., **101** (2005), 243-268.
- [34] E. Guentner and J. Kaminker: *Exactness and the Novikov conjecture*, Topology, **41**(2) (2002), 411-418.
- [35] E. Guentner and J. Kaminker: *Exactness and uniform embeddability of discrete groups*, J. London Math. Soc., **70** (2004), 703-718.
- [36] P. de la Harpe: *Groupes hyperboliques, algèbres d'opérateurs et un théorème de Jolissaint*, C.R.A.S., Série I, **307** (1988), no 14, 771-774.
- [37] P. de la Harpe: *On simplicity of reduced C^* -algebras of groups*, Bull. London Math. Soc., **39** (2007), 1-26.

- [38] N. Higson: *Bivariant K -theory and the Novikov conjecture*, *Geom. Funct. Anal.*, **10** (2000), 563-581.
- [39] N. Higson and J. Roe: *Amenable groupe actions and the Novikov conjecture*, *J. Reine Angew. Math.*, **519** (2000), 143-153.
- [40] V. Jones and V.S. Sunder: *Introduction to subfactors*, London Math. Soc. Lecture Note Series, **234**, Cambridge University Press, 1997.
- [41] E. Kirchberg: *C^* -nuclearity implies CPAP*, *Math. Nachr.*, **76** (1977), 203-212.
- [42] E. Kirchberg: *Positive maps and C^* -nuclear algebras*, in *Proc. Inter. Conf on Operator Algebras, Ideals and their Applications in Theoretical Physics*, Leipzig, 1977, 255-257, Leipzig, Teubner, 1978.
- [43] E. Kirchberg: *On non-semisplit extensions, tensor products and exactness of group C^* -algebras*, *Invent. Math.*, **112** (1993), 449-489.
- [44] E. Kirchberg: *Discrete groups with Kazhdan's property (T) and factorization property are residually finite*, *Math. Ann.*, **299** (1994), 551-563.
- [45] E. Kirchberg: *On subalgebras of the CAR-algebra*, *J. Functional Analysis*, **129** (1995), 35-63.
- [46] E. Kirchberg and S. Wassermann: *Permanence properties of C^* -exact groups*, *Documenta Mathematica*, **4** (1999), 513-518.
- [47] E. Kirchberg and N.C. Phillips: *Embedding of exact C^* -algebras in the Cuntz algebra \mathcal{O}_2* , *J. Reine Angew. Math.*, **525** (2000), 17-53.
- [48] E.C. Lance : *On nuclear C^* -algebras*, *J. Funct. Anal.*, **12** (1973), 157-176.
- [49] G.J. Murphy: *C^* -algebras and operator theory*, Academic Press, London, 1990.
- [50] F.J. Murray and J. von Neumann: *On rings of operators IV*, *Ann. of Math.*, **44** (1943), 716-808.
- [51] N. Ozawa: *Amenable actions and exactness for discrete groups*, *C.R. Acad. Sci. Paris Sér. I Math.*, **330** (2000), 691-695.
- [52] N. Ozawa: *Solid von Neumann algebras*, *Acta Math.*, **192** (2004), no 1, 11-117.
- [53] N. Ozawa: *A Kurosh-type theorem for type II_1 factors*, *Int. Math. Res. Notices*, (2006), 1-21.
- [54] N. Ozawa: *Boundary amenability of relatively hyperbolic groups*, *Topology Appl.*, **153** (2006), no 14, 2624-2630.
- [55] N. Ozawa: *Amenable actions and applications*, in *International Congress of Mathematicians, Madrid 2006, Vol II*, 1563-1580.
- [56] V. Paulsen: *Completely bounded maps and dilations*, *Cambridge Studies in Advanced Mathematics* **78**, 2003.
- [57] G.K. Pedersen: *C^* -algebras and their automorphism groups*, Academic Press, London, 1979.
- [58] G. Pisier: *Introduction to Operator Space Theory*, London Math. Soc. Lecture Notes Series **294**, Cambridge University Press, 2003.
- [59] R.T. Powers: *Simplicity of the C^* -algebra associated with the free group on two generators*, *Duke Math. J.*, **42** (1975), 151-156.
- [60] A.G. Robertson: *Property (T) for II_1 factors and unitary representations of Kazhdan groups*, *Math. Ann.*, **296** (1993), 547-555.
- [61] J. Roe: *Coarse cohomology and index theory for complete Riemannian manifolds*, *Memoirs Amer. Math. Soc.* no **497**, 1993.
- [62] S. Sakai: *On automorphism groups of II_1 factors*, *Tohoku Math. J.*, **26** no 3 (1974), 423-430.
- [63] G. Skandalis: *Une notion de nucléarité en K -théorie*, *K -theory*, **1** (1988), 549-573.
- [64] G. Skandalis, J-L. Tu and G. Yu: *The coarse Baum-Connes conjecture and groupoids*, *Topology*, **41** (2002), 807-834.
- [65] M. Takesaki: *On the cross-norm of the direct product of C^* -algebras*, *Tôhoku Math. J.*, **16** (1964), 111-122.
- [66] M. Takesaki: *Theory of Operator Algebras I*, Springer, 2001.

- [67] J. Tomiyama: *On the projection of norm one in W^* -algebras*, Proc. Japan Acad., **33** (1957), 608-612.
- [68] J. Tomiyama: *Tensor products and projections of norm one in von Neumann algebras*, Seminar notes, University of Copenhagen, 1970.
- [69] J-L. Tu: *Remarks on Yu's "Property A" for discrete metric spaces and groups*, Bull. Soc. Math. France, **129** (1) (2001), 115-139.
- [70] A. Valette: *Projections in full C^* -algebras of semisimple Lie groups*, Math. Ann., **294** (1992), 277-287.
- [71] D.V. Voiculescu: *A non-commutative Weyl-von Neumann theorem*, Rev. Roum. Pures Appl., **21** (1976), 97-113.
- [72] S. Wassermann: *On tensor products of certain group C^* -algebras*, J. Funct. Anal., **23** (1976), 239-254.
- [73] S. Wassermann: *Exact C^* -algebras and related topics*, Res. Inst. Math., Global Anal. Res. Center, Lecture Notes Series **19**, Seoul, 1994.
- [74] G. Yu: *The Coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Invent. Math., **139** (1) (2000), 201-240.

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ D'ORLÉANS, UMR CNRS 6628, FÉDÉRATION
DENIS POISSON, B.P. 6759, F-45067 ORLÉANS CEDEX 2, FRANCE
E-mail address: `claire.anantharaman@univ-orleans.fr`