ERGODIC THEORY AND VON NEUMANN ALGEBRAS : AN INTRODUCTION

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Contents

1. **Finite von Neumann algebras : examples and some basics**

This section presents the basic constructions of von Neumann algebras coming from measure theory, group theory, group actions and equivalence relations. All this examples are naturally equipped with a faithful trace and are naturally represented on a Hilbert space. This provides a plentiful source of tracial von Neumann algebras to play with.

1.1. <u>Notation and preliminaries</u>. Let $\mathcal{H}$ be a complex Hilbert space[1] with inner-product $\langle \cdot, \cdot \rangle$ (always assumed to be antilinear in the first variable), and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. Equipped with the involution $x \mapsto x^*$ (adjoint of x) and with the operator norm, $\mathcal{B}(\mathcal{H})$ is a Banach $*$-algebra with unit $\mathrm{Id}_{\mathcal{H}}$. We shall denote by $\|x\|$, or sometimes $\|x\|_\infty$, the operator norm of $x \in \mathcal{B}(\mathcal{H})$. Throughout this text, we shall consider the two following weaker topologies on $\mathcal{B}(\mathcal{H})$:

- the *strong operator topology* (s.o. topology), that is, the locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by the seminorms

$$p_\xi(x) = \|x\xi\|, \qquad \xi \in \mathcal{H},$$

- the *weak operator topology* (w.o. topology), that is, the locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by the seminorms

$$p_{\xi,\eta}(x) = |\omega_{\xi,\eta}(x)|, \quad \xi, \eta \in \mathcal{H},$$

where $\omega_{\xi,\eta}$ is the linear functional $x \mapsto \langle \xi, x\eta \rangle$ on $\mathcal{B}(\mathcal{H})$.

This latter topology is weaker than the s.o. topology. One of its important properties is that the unit ball of $\mathcal{B}(\mathcal{H})$ is w.o. compact. This is an immediate consequence of Tychonoff's theorem.

This unit ball, endowed with the uniform structure associated with the s.o. topology is a complete space. In case $\mathcal{H}$ is separable, both topologies on the unit ball are metrizable and second-countable. On the other hand, when $\mathcal{H}$ is infinite dimensional, this unit ball is not separable with respect to the operator norm (Exercise 1.1).

A *von Neumann algebra M on $\mathcal{H}$* is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed in the s.o. topology and contains $\mathrm{Id}_{\mathcal{H}}$.[2] We shall sometimes write $(M, \mathcal{H})$ to specify the Hilbert space on which M acts. The unit $\mathrm{Id}_{\mathcal{H}}$ of M will also be denoted 1_M or simply 1.

Given a subset S of $\mathcal{B}(\mathcal{H})$, we denote by S' its *commutant* in $\mathcal{B}(\mathcal{H})$:

$$S' = \{x \in \mathcal{B}(\mathcal{H}) : xy = yx \text{ for all } y \in S\}.$$

[1] In this text, $\mathcal{H}$ is always assumed to be separable.

[2] We shall see in Theorem 1.3 that we may require, equivalently, that M is closed in the w.o. topology.

Then S'' is the commutant of S', that is, the *bicommutant* of S. Note that S' is a s.o. closed unital subalgebra of $\mathcal{B}(\mathcal{H})$; if $S = S^*$, then $S' = (S')^*$ and therefore S' is a von Neumann algebra on $\mathcal{H}$.

The first example of von Neumann algebra coming to mind is of course $M = \mathcal{B}(\mathcal{H})$. Then, $M' = \mathbb{C}\,\mathrm{Id}_{\mathcal{H}}$. When $\mathcal{H} = \mathbb{C}^n$, we get the algebra $M_n(\mathbb{C})$ of $n \times n$ matrices with complex entries, the simplest example of von Neumann algebra.

We recall that a *C^*-algebra on $\mathcal{H}$* is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed in the norm topology. Hence a von Neumann algebra is a C^*-algebra, but the converse is not true. For instance the C^*-algebra $\mathcal{K}(\mathcal{H})$ of compact operators on an infinite dimensional Hilbert space $\mathcal{H}$ is not a von Neumann algebra on $\mathcal{H}$: its s.o. closure is $\mathcal{B}(\mathcal{H})$.

For us, a homomorphism between two C^*-algebras preserves the algebraic operations and the involution. We recall that it is automatically a contraction and a positive map, *i.e.* it preserves the positive cones. For a concise introduction to the theory of C^*-algebras, we refer to [16].

1.1.1. **Von Neumann's bicommutant theorem.** [25]

We begin by showing that, although different (for infinite dimensional Hilbert spaces), the s.o. and w.o. topologies introduced in the first chapter have the same continuous linear functionals. For ξ, η in a Hilbert space $\mathcal{H}$ we denote by $\omega_{\xi,\eta}$ the linear functional $x \mapsto \langle \xi, x\eta \rangle$ on $\mathcal{B}(\mathcal{H})$. We set $\omega_\xi = \omega_{\xi,\xi}$.

Proposition 1.1. *Let ω be a linear functional on $\mathcal{B}(\mathcal{H})$. The following conditions are equivalent :*

- (i) *there exist $\xi_1, \dots, \xi_n, \eta_1, \dots \eta_n \in \mathcal{H}$ such that $\omega(x) = \sum_{i=1}^n \omega_{\eta_i,\xi_i}(x)$ for all $x \in \mathcal{B}(\mathcal{H})$;*
- (ii) *ω is w.o. continuous ;*
- (iii) *ω is s.o. continuous.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is obvious. It remains to show that (iii) $\Rightarrow$ (i). Let ω be a s.o. continuous linear functional. There exist vectors $\xi_1, \dots, \xi_n \in \mathcal{H}$ and $c > 0$ such that, for all $x \in \mathcal{B}(\mathcal{H})$,

$$|\omega(x)| \le c\Big(\sum_{i=1}^n \|x\xi_i\|^2\Big)^{1/2}.$$

Let $\mathcal{H}^{\oplus n} = \overbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}^{n\,\text{times}}$ be the Hilbert direct sum of n copies of $\mathcal{H}$. We set $\xi = (\xi_1, \dots, \xi_n) \in \mathcal{H}^{\oplus n}$ and for $x \in \mathcal{B}(\mathcal{H})$,

$$\theta(x)\xi = (x\xi_1, \dots, x\xi_n).$$

The linear functional $\psi : \theta(x)\xi \mapsto \omega(x)$ is continuous on the vector subspace $\theta(\mathcal{B}(\mathcal{H}))\xi$ of $\mathcal{H}^{\oplus n}$. Therefore it extends to a linear continuous functional on the norm closure $\mathcal{K}$ of $\theta(\mathcal{B}(\mathcal{H}))\xi$. It follows that there exists

$\eta = (\eta_1, \dots, \eta_n) \in \mathcal{K}$ such that, for $x \in \mathcal{B}(\mathcal{H})$,

$$\omega(x) = \psi(\theta(x)\xi) = \langle \eta, \theta(x)\xi \rangle_{\mathcal{H}^{\oplus n}} = \sum_{i=1}^{n} \langle \eta_i, x\xi_i \rangle.$$

□

Corollary 1.2. *The above proposition remains true when $\mathcal{B}(\mathcal{H})$ is replaced by any von Neumann subalgebra M.*

Proof. Immediate, since by the Hahn-Banach theorem, continuous w.o. (resp. s.o.) linear functionals on M extend to linear functionals on $\mathcal{B}(\mathcal{H})$ with the same continuity property. □

In the sequel, the restrictions of the functionals $\omega_{\xi,\eta}$ and $\omega_\xi = \omega_{\xi,\xi}$ to any von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ will be denoted by the same expressions.

Recall that two locally convex topologies for which the continuous linear functionals are the same have the same closed convex subsets. Therefore, the s.o. and w.o. closures of any convex subset of $\mathcal{B}(\mathcal{H})$ coincide.

The following fundamental theorem shows that a von Neumann algebra may also be defined by purely algebraic conditions.

Theorem 1.3 (**von Neumann's bicommutant theorem**). *Let M be a unital self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. The following conditions are equivalent :*

(i) *$M = M''$.*
(ii) *M is weakly closed.*
(iii) *M is strongly closed.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is obvious. Let us show that (iii) $\Rightarrow$ (i). Since the inclusion $M \subset M''$ is trivial, we only have to prove that every $x \in M''$ belongs to the s.o. closure of M (which is M, by assumption (iii)). More precisely, given $\varepsilon > 0$ and $\xi_1, \dots, \xi_n \in \mathcal{H}$, we have to show the existence of $y \in M$ such that

$$\text{for} \quad 1 \le i \le n, \quad \|x\xi_i - y\xi_i\| \le \varepsilon.$$

We consider first the case $n = 1$. Given $\xi \in \mathcal{H}$, we denote by $[M\xi]$ the orthogonal projection from $\mathcal{H}$ onto the norm closure $\overline{M\xi}$ of $M\xi$. Since this vector space is invariant under M, the projection $[M\xi]$ is in the commutant M'. Hence

$$x\xi = x[M\xi]\xi = [M\xi]x\xi,$$

and so we have $x\xi \in \overline{M\xi}$. Therefore, given $\varepsilon > 0$, there exists $y \in M$ such that $\|x\xi - y\xi\| \le \varepsilon$.

We now reduce the general case to the case $n = 1$ thanks to the following very useful and basic matrix trick. We identify the algebra $\mathcal{B}(\mathcal{H}^{\oplus n})$ with the algebra $M_n(\mathcal{B}(\mathcal{H}))$ of n by n matrices with entries in $\mathcal{B}(\mathcal{H})$. We denote by $\theta : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}^{\oplus n})$ the diagonal map

$$y \mapsto \begin{pmatrix} y & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & y \end{pmatrix}.$$

We set $N = \theta(M)$. A straightforward computation shows that the commutant N' of N is the algebra of $n \times n$ matrices with entries in M'. It follows that for every $x \in M''$, we have $\theta(x) \in N''$. We apply the first part of the proof to $\theta(x)$ and N. Given $\varepsilon > 0$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathcal{H}^{\oplus n}$, we get an element $\theta(y) \in N$ such that $\|\theta(x)\xi - \theta(y)\xi\| \leq \varepsilon$, that is, $\|x\xi_i - y\xi_i\| \leq \varepsilon$ for $i = 1, \dots, n$. □

1.2. Measure space von Neumann algebras. Every probability measure space (X, μ) gives rise in a natural way to an abelian von Neumann algebra.

Proposition 1.4. *Let (X, μ) be a probability measure space. We set $A = L^\infty(X, \mu)$.*

(i) *For $f \in L^\infty(X, \mu)$, denote by M_f the multiplication operator by f on $L^2(X, \mu)$. Then M_f is a bounded operator and $\|M_f\| = \|f\|_\infty$.*
(ii) *If A is identified to a subalgebra of $\mathcal{B}(L^2(X, \mu))$, then $A = A'$. In particular, A is a von Neumann algebra in $L^2(X, \mu)$ and a maximal abelian subalgebra of $\mathcal{B}(L^2(X, \mu))$.*

Proof. (i) Obviously, M_f is a bounded operator with $\|M_f\| \leq \|f\|_\infty$ and it is a classical exercise in measure theory to show that $\|M_f\| = \|f\|_\infty$.

(ii) Since A is abelian, we have $A \subset A'$. Let $T \in A'$ and set $f = T(1)$. Then, for $h \in L^\infty(X, \mu)$, we have $T(h) = TM_h 1 = hf$ and $\|fh\|_2 \leq \|T\|\|h\|_2$. It follows that $f \in L^\infty(X, \mu)$ with $\|f\|_\infty \leq \|T\|$ and so $T = M_f$. □

Recall that $L^\infty(X, \mu)$ is the dual Banach space of $L^1(X, \mu)$. The *weak* topology* on $L^\infty(X, \mu)$ is defined by the family of seminorms $q_g(f) = \left|\int_X fg \,\mathrm{d}\mu\right|$, $g \in L^1(X, \mu)$. Equivalently, it is defined by the family of seminorms

$$f \mapsto p_{\xi,\eta}(f) = \left|\int_X f\bar{\xi}\eta \,\mathrm{d}\mu\right|$$

with $\xi, \eta \in L^2(X, \mu)$. Therefore, the weak* topology coincides with the w.o. topology on $L^\infty(X, \mu)$ acting on $L^2(X, \mu)$.

1.3. Group von Neumann algebras. [18]

Let G be a countable group.[3] We denote by λ (or λ_G in case of ambiguity) and ρ (or ρ_G) the *left*, and respectively *right*, *regular representation* of G in $\ell^2(G)$, *i.e.*, for all $s, t \in G$,

$$\lambda(s)\delta_t = \delta_{st}, \quad \rho(s)\delta_t = \delta_{ts^{-1}},$$

[3]For us, unless otherwise stated, countable means countable infinite

where $(\delta_t)_{t\in G}$ is the natural orthonormal basis of $\ell^2(G)$.[4]

1.3.1. **Definition and first properties.** We denote by $L(G)$ the strong operator closure of the linear span of $\lambda(G)$. This von Neumann algebra is called the (left) *group von Neumann algebra of* G. Similarly, one defines the strong operator closure $R(G)$ of the linear span of $\rho(G)$. Obviously, these two algebras commute : $xy = yx$ for $x \in L(G)$ and $y \in R(G)$.[5] They come equipped with a natural trace, as shown below.

Definition 1.5. A linear functional φ on a von Neumann algebra M is *positive* if $\varphi(x^*x) \geq 0$ for every $x \in M$. Whenever $\varphi(x^*x) = 0$ implies $x = 0$, we say that φ is *faithful*. If φ is positive with $\varphi(1) = 1$ we say that φ is a *state*.

A positive linear functional such $\varphi(xy) = \varphi(yx)$ for every $x, y \in M$ is a *trace*. If moreover it is a state, we call it a *tracial state* .

We recall that a positive linear functional is norm continuous.

Definition 1.6. Given a von Neumann algebra M acting on a Hilbert space $\mathcal{H}$, a vector $\xi \in \mathcal{H}$ is called *cyclic for* M if $M\xi$ is dense in $\mathcal{H}$. It is called *separating for* M if, for $x \in M$, we have $x\xi = 0$ if and only if $x = 0$.

We denote by e the unit of G. One easily checks that δ_e is a cyclic and separating vector for $L(G)$ (and $R(G)$). We define a linear functional on $L(G)$ by

$$\tau(x) = \langle \delta_e, x\delta_e \rangle.$$

Obviously, τ is a faithful state on $L(G)$. For $s_1, s_2 \in G$, we have $\tau(\lambda(s_1)\lambda(s_2)) = 1$ if $s_1 s_2 = e$ and $\tau(\lambda(s_1)\lambda(s_2)) = 0$ otherwise. It follows immediately that τ is a trace. We observe that this trace is continuous with respect to the w.o. topology.

Thus, $L(G)$ and $R(G)$ are examples of tracial von Neumann algebras in the following sense.[6]

Definition 1.7. A *tracial von Neumann algebra* (M, τ) is a von Neumann algebra M equipped with a faithful tracial state whose restriction to the unit ball is continuous with respect to the w.o. topology

Since δ_e is a separating vector for $L(G)$, the map $x \mapsto x\delta_e$ provides a natural identification of $L(G)$ with a dense linear subspace of $\ell^2(G)$.

Remark 1.8. Usually, for $g \in G$, we shall put $u_g = \lambda(g) \in L(G)$ and this unitary operator is identified with the vector $\lambda(g)\delta_e = \delta_g \in \ell^2(G)$. Therefore, every $f \in \ell^2(G)$ is written as $f = \sum_{g\in G} f_g u_g$ and, in particular, every

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[4]Given a set X, we denote by δ_x, both the characteristic function of $\{x\}$ and the Dirac measure at $x \in X$.

[5]We shall see in Section 1.4 that actually $R(G) = L(G)'$.

[6]$L^\infty(X, \mu)$ equipped with the integral $\tau_\mu : f \mapsto \int_X f \,\mathrm{d}\mu$ is of course another example.

$x \in L(G)$ is written as

$$x = \sum_{g \in G} x_g u_g. \tag{1}$$

Observe that $\tau(x^*x) = \sum_{g \in G} |x_g|^2$ and that $x_g = \tau(x u_g^*)$. In analogy with developments in Fourier series, the scalars x_g are called the *Fourier coefficients of* x. The unitaries u_g are called the *canonical unitaries of* $L(G)$. We warn the reader that in (1) the convergence is in ℓ^2-norm and not with respect to the s.o. or w.o. topology.

For $x = \sum_{g \in G} x_g u_g \in L(G)$ and $f = \sum_{g \in G} f_g u_g \in \ell^2(G)$, the function $x(f)$ is computed by using the rule $u_g u_h = u_{gh}$.

Example 1.9. Consider the group $G = \mathbb{Z}$. Since $\mathbb{Z}$ is abelian, $L(\mathbb{Z})$ is an abelian von Neumann algebra which coincides with $R(G) = L(G)'$.

Let $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ be the Fourier transform, where $\mathbb{T}$ is the unit circle in $\mathbb{C}$, equipped with the Lebesgue measure m. Then $\mathcal{F}\delta_n = e_n$ with $e_n(z) = z^n$. It follows that $\mathcal{F}L(\mathbb{Z})\mathcal{F}^{-1}$ is the von Neumann subalgebra of $\mathcal{B}(L^2(\mathbb{T}))$ generated by the multiplication operators by these functions e_n, $n \in \mathbb{Z}$. It can be identified, in a natural way with $L^\infty(\mathbb{T})$.

The canonical tracial state τ on $L(\mathbb{Z})$ becomes, after Fourier transform, the integration with respect to the Lebesgue probability measure on $\mathbb{T}$:

$$\tau(L_f) = \int_{\mathbb{T}} \widehat{f} \, \mathrm{d}m.$$

The same observations hold for any abelian countable group G : the group von Neumann algebra $L(G)$ is abelian and isomorphic to $L^\infty(\widehat{G}, m)$ where $\widehat{G}$ is the dual group and m is the Haar probability measure on this compact group.

However, the most interesting examples for us come from groups such that $L(G)$ has, to the contrary, a center reduced to the scalar operators. A von Neumann algebra with such a trivial center is called *a factor*.

Proposition 1.10. *Let G be a countable group. The following conditions are equivalent:*

(i) *$L(G)$ is a factor;*
(ii) *G is an ICC (infinite conjugacy classes) group, i.e. every non trivial conjugacy class $\{gsg^{-1} : g \in G\}$, $s \neq e$, is infinite.*

Proof. Let x be an element of the center of $L(G)$. For $t \in G$ we have

$$x\delta_e = \lambda(t)x\lambda(t^{-1})\delta_e = \lambda(t)x\rho(t)\delta_e = \lambda(t)\rho(t)(x\delta_e).$$

It follows that $x\delta_e$ is constant on conjugacy classes. Therefore, if G is ICC, since $x\delta_e$ is square summable, we see that $x\delta_e = \alpha\delta_e$ with $\alpha \in \mathbb{C}$, and therefore $x = \alpha \, \mathrm{Id}_{\mathcal{H}}$.

Assume now that G is not ICC and let $C \subset G$ be a finite non-trivial conjugacy class. An easy computation shows that the characteristic function $f = \mathbf{1}_C$ of C defines an element L_f of the center of $L(G)$ which is not a scalar operator. $\square$

There are plenty of countable ICC groups. The simplest examples, easily shown to be ICC (see Exercise 1.5), are

- $S_\infty = \bigcup_{n=1}^{\infty} S_n$, the group of those permutations of $\mathbb{N}$ fixing all but finitely many integers (S_n is the group of all permutations of $\{1, 2, \cdots, n\}$).
- $\mathbb{F}_n$, $n \geq 2$, the free group on n generators.

We have previously met examples of factors, namely the factors $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a finite or infinite dimensional Hilbert space. They are type I factors to be defined below. When G is a countable ICC group, $L(G)$ is another type of factor, called a type II_1 factor.

Definition 1.11. A *type II_1 factor* is an infinite dimensional tracial von Neumann algebra M whose center is reduced to the scalar operators

1.3.2. **Digression about type I factors.** A basic result of linear algebra tells us that the von Neumann algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices has a unique tracial state τ, namely $\tau = (1/n)\mathrm{Tr}$ where Tr is the usual trace of matrices. $M_n(\mathbb{C})$ is a finite dimensional tracial factor.

On the other hand, is easily shown that there is no tracial state on the factor $\mathcal{B}(\mathcal{H})$ when $\mathcal{H}$ is infinite dimensional. Indeed, we may write $\mathcal{H}$ as the orthogonal direct sum of two Hilbert subspaces $\mathcal{H}_1, \mathcal{H}_2$ of the same dimension as the dimension of $\mathcal{H}$. If p_1, p_2 are the orthogonal projections on these subspaces, there exist partial isometries u_1, u_2 with $u_i^* u_i = \mathrm{Id}_{\mathcal{H}}$ and $u_i u_i^* = p_i$, $i = 1, 2$. The existence of a tracial state τ on $\mathcal{B}(\mathcal{H})$ leads to the contradiction

$$1 = \tau(p_1) + \tau(p_2) = \tau(u_1 u_1^*) + \tau(u_2 u_2^*) = \tau(u_1^* u_1) + \tau(u_2^* u_2) = 2.$$

Definition 1.12. A *type I* factor is a factor isomorphic to some $\mathcal{B}(\mathcal{H})$. If $\dim \mathcal{H} = n$, we say that M (which is isomorphic to $M_n(\mathbb{C})$) is of *type I_n* . If $\dim \mathcal{H} = \infty$, we say that M is of *type I_∞* .

Definition 1.13. We say that two von Neumann algebras M_1 and M_2 are *isomorphic*, and we write $M_1 \simeq M_2$, if there exists a bijective homomorphism (*i.e.* an isomorphism) $\alpha : M_1 \to M_2$.

An isomorphism preserves the algebraic structures as well as the involution. We recall that it is automatically an isometry. On the other hand it is not necessarily continuous with respect to the w.o. or s.o. topology (see Exercise 1.2).

Factors of type I (on a separable Hilbert space) are classified, up to isomorphism, by their dimension. On the other hand, the classification of type II_1 factors is out of reach. Already, given two countable ICC groups G_1, G_2, to determine whether the type II_1 factors $L(G_1)$ and $L(G_2)$ are isomorphic or not is a very difficult question.

Since we have defined von Neumann algebras as acting on specified Hilbert spaces, the following stronger notion of isomorphism is also very natural.

Definition 1.14. We say that the von Neumann algebras M_1, M_2, acting on $\mathcal{H}_1, \mathcal{H}_2$ respectively, are *spatially isomorphic* if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $x \mapsto UxU^*$ is an isomorphism (called spatial) from M_1 onto M_2.

Two isomorphic von Neumann algebras need not be spatially isomorphic (see Exercise 1.3). Classification, up to spatial isomorphism, involves in addition a notion of multiplicity.

1.3.3. **A remark about** $L(S_\infty)$**.** We end this section by pointing out a nice important property of $L(S_\infty)$: it the s.o. closure of the union of an increasing sequence of finite dimensional von Neumann algebras[7] namely the von Neumann algebras $L(S_n)$, $n \geq 1$. Indeed, these algebras are finite-dimensional since the groups S_n are finite. Moreover, $L(S_n)$ is naturally isomorphic to the linear span of $\lambda_{S_\infty}(S_n)$ in $L(S_\infty)$, as a consequence of the following proposition.

Proposition 1.15. *Let H be a subgroup of a countable group G. Then the restriction of λ_G to H is a multiple of the left regular representation of H*

Proof. Write G as the disjoint union of its right H-cosets : $G = \cup_{s\in S} Hs$, where S is a set of representatives of $H \setminus G$. Then $\ell^2(G) = \oplus_{s\in S}\ell^2(Hs)$. It is enough to observe that $\ell^2(Hs)$ is invariant under the restriction of λ_G to H, and that this restriction is equivalent to the left regular representation of H. □

1.4. <u>Standard form</u>. Even if it is given as acting on a Hilbert space $\mathcal{H}$, we shall see that a tracial von Neumann algebra (M, τ) has also another privileged representation.

Let (M, τ) be a tracial von Neumann algebra on $\mathcal{H}$. We define on M a sesquilinear form by

$$\langle x, y\rangle_\tau = \tau(x^* y).$$

The completion of M with respect to the corresponding norm $x \mapsto \|x\|_2 = \tau(x^*x)^{1/2}$ is denoted by $L^2(M, \tau)$. In particular M appears as a dense subspace of $L^2(M, \tau)$. Sometimes, we write $\widehat{x}$ instead of x, when we want to insist on the fact that $x \in M$ is viewed as a vector in $L^2(M, \tau)$.

For $x, y \in M$, we put

$$\pi_\tau(x)\hat{y} = \widehat{xy}.$$

We have

$$\begin{aligned}\|\pi_\tau(x)\hat{y}\|_\tau^2 = \|\widehat{xy}\|_\tau^2 &= \tau(y^*x^*xy)\\ &\leq \|x^*x\|\tau(y^*y) = \|x\|^2\|\hat{y}\|_\tau^2.\end{aligned}$$

[7]Such a factor is called *hyperfinite*.

It follows that $\pi_\tau(x)$ extends to an element of $\mathcal{B}(L^2(M,\tau))$, still denoted $\pi_\tau(x)$. It is easy to check that π_τ is a homomorphism from M into $\mathcal{B}(L^2(M,\tau))$. Moreover, if we put $\xi_\tau = \hat{1}$, we have for $x \in M$,

$$\tau(x) = \langle \xi_\tau, \pi_\tau(x)\xi_\tau \rangle_\tau. \tag{2}$$

We say that $(\pi_\tau, \mathcal{H}_\tau, \xi_\tau)$ is the *Gelfand-Naimark-Segal* (GNS) representation associated with τ. Note that the vector ξ_τ is cyclic and separating for $\pi_\tau(M)$ and that π_τ is an injective homomorphism. A not obvious fact is that $\pi_\tau(M)$ is w.o. closed in $\mathcal{B}(L^2(M,\tau))$. We shall identify M with $\pi_\tau(M)$ and write $x\xi$ for $\pi_\tau(x)\xi$.

A nice property of this representation is that the commutant of M in $\mathcal{B}(L^2(M,\tau))$ has the "*same size*" as M. We first observe that the operator $\widehat{x} \mapsto \widehat{x^*}$ extends to an antilinear and surjective isometry $J : L^2(M,\tau) \to L^2(M,\tau)$ called the *canonical conjugation operator* on $L^2(M,\tau)$.

Theorem 1.16. *$JMJ = M'$.*

Proof. It is straightforward to check that for $x, y \in M$, we have $JxJy = yJxJ$ and thus $JMJ \subset M'$. Let us show that $Jz\widehat{1} = z^*\widehat{1}$ for every $z \in M'$. Indeed, given $x \in M$, we have

$$\begin{aligned}\left\langle Jz\widehat{1}, \widehat{x}\right\rangle &= \left\langle J\widehat{x}, z\widehat{1}\right\rangle = \left\langle x^*\widehat{1}, z\widehat{1}\right\rangle \\ &= \left\langle \widehat{1}, xz\widehat{1}\right\rangle = \left\langle \widehat{1}, zx\widehat{1}\right\rangle = \left\langle z^*\widehat{1}, x\widehat{1}\right\rangle.\end{aligned}$$

It follows that, for $x', y' \in M'$,

$$JzJx'y'\widehat{1} = Jzy'^*x'^*\widehat{1} = x'y'z^*\widehat{1} = x'y'JzJ\widehat{1} = x'JzJy'\widehat{1},$$

and so $JzJ \in M'' = M$. The inclusion $M' \subset JMJ$ is thus obtained. □

We say that M is in *standard form* on $L^2(M,\tau)$.

Let (M,τ) be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and assume the existence of a norm one cyclic vector $\xi_0 \in \mathcal{H}$ such that $\omega_{\xi_0} = \tau$. Then the given representation π of M on $\mathcal{H}$ is naturally equivalent to the standard representation. More precisely, let U be the operator from $\pi(M)\xi_0$ into $L^2(M,\tau)$ sending $\pi(x)\xi_0$ onto $x\hat{1}$. Then U extends to a unitary operator, still denoted by U, from $\mathcal{H}$ onto $L^2(M,\tau)$ such that $U\pi(x)U^* = \pi_\tau(x)$ for every $x \in M$. Viewed as acting on $\mathcal{H}$, the canonical conjugation operator is defined by $J\pi(x)\xi_0 = \pi(x^*)\xi_0$.

As a first example, for any countable group G, $L(G)$ is in standard form on $\ell^2(G)$, since the natural tracial state τ on $L(G)$ is defined by the cyclic and separating vector $\delta_e \in \ell^2(G)$. Here J is defined by $J\xi(t) = \overline{\xi(t^{-1})}$ and, for every $t \in G$, we have $J\lambda(t)J = \rho(t)$. It follows that $L(G)' = JL(G)J$ is the von Neumann algebra $R(G)$ generated by the right regular representation.

1.5. Group measure space von Neumann algebras. [17] We shall describe in this section a fundamental construction, associated with an action of a countable group G on a probability measure space (X,μ). Section 1.3 was concerned with the case where X is a reduced to a point.

1.5.1. **Probability measure preserving actions.** A *probability measure preserving (p.m.p.) automorphism* of a probability measure space (X,μ) is a Borel automorphism θ of the Borel space X such that $\theta_*\mu = \mu$, i.e. $(\theta_*\mu)(E) = \mu(\theta^{-1}(E)) = \mu(E)$ for every Borel set E. We denote by $\mathrm{Aut}\,(X,\mu)$ the group of (classes modulo null sets of) p.m.p. automorphisms of (X,μ).

Any element $\theta \in \mathrm{Aut}\,(X,\mu)$ induces an automorphism $f \mapsto f\circ\theta$ of the algebra $L^\infty(X,\mu)$ which preserves the functional $\tau_\mu : f \mapsto \int_X f\,\mathrm{d}\mu$, *i.e.*

$$\forall f \in L^\infty(X,\mu), \quad \int_X f\circ\theta\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu.$$

Definition 1.17. A *probability measure preserving (p.m.p.) action* $G \curvearrowright (X,\mu)$ of a countable group G on a probability measure space (X,μ) is a group homomorphism from G into $\mathrm{Aut}\,(X,\mu)$. The action of $g\in G$ on $w\in X$ will be written gw.

The most classical examples of p.m.p. actions are *Bernoulli shift actions.* Let $X=\{0,1\}^G$ be the compact space of all sequences of 0 and 1 indexed by G, equipped with the product measure $\mu=\nu^{\otimes G}$, where $\nu(\{0\}) = p, \nu(\{1\}) = 1-p$, for a given $p\in]0,1[$. The Bernoulli action $G \curvearrowright (X,\mu)$ is defined by $(gx)_h = x_{g^{-1}h}$ for $x=(x_h)_{h\in G}\in X$ and $g\in G$. We may replace $\{0,1\}$ by any Borel space equipped with a probability measure ν.

1.5.2. **Construction of the group measure space algebra.** Let $G \curvearrowright (X,\mu)$ be a p.m.p. action of G on a probability measure space (X,μ). Let A be the von Neumann algebra $L^\infty(X,\mu)$, acting by mutiplication on $L^2(X,\mu)$. Let σ be the unitary representation of G on $L^2(X,\mu)$ defined by $(\sigma_g f)(w) = f(g^{-1}w)$. By restriction to $L^\infty(X,\mu)\subset L^2(X,\mu)$, this induces an action of G by automorphisms on $L^\infty(X,\mu)$. Note that $\sigma_g M_f \sigma_{g^{-1}} = M_{\sigma_g f}$.

We denote by $A[G]$ the space of formal sums of the form

$$\sum_{g\in G} a_g u_g \tag{3}$$

where $a_g\in A$ and where the set of $g\in G$ with $a_g\neq 0$ is finite. Then $A[G]$ is a vector space in an obvious way. We define a product by

$$(a_1u_g)(a_2u_h) = a_1\sigma_g(a_2)u_{gh},$$

and an involution by

$$(au_g)^* = \sigma_{g^{-1}}(a^*)u_{g^{-1}}, \quad \text{where} \quad a^*(w) = \overline{a(w)}.$$

This turns $A[G]$ into a $*$-algebra. Note that these operations are consistent with the operations on A and G: $a \in A \mapsto au_e$ is a $*$-homomorphism from A into $A[G]$ and $g \in G \mapsto u_g$ is a group homomorphism.

We shall complete $A[G]$ in order to get a von Neumann algebra. The first step is to represent $A[G]$ as a concrete $*$-algebra of operators acting on the Hilbert space $\mathcal{H} = \ell^2(G, L^2(X,\mu))$ of functions $f : g \mapsto f_g$ from G into $L^2(X,\mu)$ such that

$$\sum_{g\in G} \|f_g\|^2_{L^2(X)} < +\infty.$$

It is convenient to set

$$\begin{aligned} u_g &= 1 \otimes \delta_g \in L^2(X,\mu) \otimes \ell^2(G) = \ell^2(G, L^2(X,\mu)) \\ f_g u_g &= f_g \otimes \delta_g, \end{aligned}$$

and to write f as the sum $\sum_{g\in G} f_g u_g$, with coefficients $f_g \in L^2(X,\mu)$. The inner product is given by

$$\langle \sum_{g\in G} f_g u_g, \sum_{g\in G} k_g u_g \rangle = \sum_{g\in G} \int_X \overline{f_g} k_g \,\mathrm{d}\mu.$$

We observe that $A[G]$ is naturally embedded as a subspace of $\ell^2(G, L^2(X,\mu))$. The representation L of $A[G]$ on $\mathcal{H}$ is given by

$$L(au_g)(fu_h) = (au_g)(fu_h) = (au_g)(f \otimes \delta_h) = (a\sigma_g(f)) \otimes \delta_{gh} = a\sigma_g(f)u_{gh}.$$

Easy computations show that this defines indeed an injective $*$-homomorphism from $A[G]$ into $\mathcal{B}(\mathcal{H})$. One has

$$\big(\sum_{g\in G} a_g u_g\big)\big(\sum_{g\in G} f_g u_g\big) = \sum_{g\in G} (a * f)_g u_g$$

where $a * f$ is the *twisted convolution product*

$$(a * f)_g = \sum_{h\in G} a_h \sigma_h(f_{h^{-1}g}).$$

The *group measure space von Neumann algebra associated with* $G \curvearrowright (X,\mu)$, or *crossed product*, is the s.o. closure of $A[G]$ in $\mathcal{B}(\mathcal{H})$. We denote it by $L(A,G)$ or $A \rtimes G$.

Similarly, $A[G]$ acts on $\mathcal{H}$ by right convolution:

$$R(au_g)(fu_h) = (fu_h)(au_g) = f\sigma_h(a)u_{hg},$$

and thus

$$\big(\sum_{g\in G} f_g u_g\big)\big(\sum_{g\in G} a_g u_g\big) = \sum_{g\in G} (f * a)_g u_g.$$

We denote by $R(A,G)$ the von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ generated by this right action R. Obviously, $L(A,G)$ and $R(A,G)$ commute. The vector $u_e \in \mathcal{H}$ is easily seen to be cyclic and separating for $L(A,G)$. In particular, the elements of $L(A,G)$ may be identified to elements of $\ell^2(G, L^2(X,\mu))$ by $x \mapsto xu_e$ and thus are written (in compatibility with (3)) as

$$x = \sum_{g\in G} x_g u_g,$$

with $\sum_{g\in G} \|x_g\|^2_{L^2(X)} < +\infty$. The functions $x_g \in L^2(X,\mu)$ are still called the *Fourier coefficients* of x and the u_g are called the *canonical unitaries of the crossed product*. Again, we warn the reader that the convergence does not occur in general with respect to the s.o. topology.

Let τ be the linear functional on $L(A,G)$ defined by

$$\tau(x) = \langle u_e, xu_e\rangle = \int_X x_e \,\mathrm{d}\mu, \quad \text{for} \quad x = \sum_{g\in G} x_g u_g.$$

Using the invariance of the probability measure μ, it is easily seen that τ is a tracial state. We also remark that τ is faithful, with

$$\tau(x^*x) = \sum_{g\in G} \int_X |x_g|^2 \,\mathrm{d}\mu.$$

Note that the observations made just before Section 1.5 show that $A\rtimes G$ is in standard form on $\ell^2(G, L^2(X,\mu))$. The conjugation operator J is defined by

$$Jfu_g = \sigma_{g^{-1}}(\overline{f})u_{g^{-1}}.$$

and it is also straightforward to check that

$$JL\big(\sum_{g\in G} a_g u_g\big)J = R\big(\sum_{g\in G} u_g^* a_g^*\big) = R\big(\sum_{g\in G} \sigma_g(a^*_{g^{-1}})u_g\big).$$

This shows that $L(A,G)' = R(A,G)$.

When $\ell^2(G, L^2(X,\mu))$ is identified with $L^2(X\times G, \mu\otimes\lambda)$ (where λ is the counting measure on G), we have

$$J\xi(x,t) = \overline{\xi(t^{-1}x, t^{-1})}.$$

We now introduce conditions on the action, under which $A\rtimes G$ turns out to be a factor, and so a type II_1 factor.

Definition 1.18. A p.m.p. action $G \curvearrowright (X,\mu)$ is (essentially) *free* if every $g\in G$, $g\neq e$, acts (essentially) freely, *i.e.* the set $\{w\in X : gw = w\}$ has μ-measure 0.

The action is said to be *ergodic* if every Borel subset E of X such that $\mu(gE\setminus E) = 0$ for every $g\neq e$ is either a null set or a conull set.

We give below equivalent formulations.

Lemma 1.19. *Let $G \curvearrowright (X,\mu)$ be a p.m.p. action. The following conditions are equivalent :*

(i) *the action is ergodic ;*

(ii) *(resp.* (ii')) *the only functions* $f \in L^\infty(X,\mu)$ *(resp.* $f \in L^2(X,\mu)$*) that are fixed under the* G*-action (i.e.* $\sigma_g(f) = f$ *for every* $g \in G$*) are the constant (a.e.) functions ;*

(iii) *the only measurable functions* $f : X \to \mathbb{C}$ *that are fixed under the* G*-action are the constant (a.e.) functions.*

Proof. We only prove that (i) ⇒ (iii), from which the whole lemma follows immediately. Let $f : X \to \mathbb{R}$ be a measurable G-invariant function. For every $r \in \mathbb{R}$, the set $E_r = \{w \in X : f(w) < r\}$ is invariant, so has measure 0 or 1. Set $\alpha = \sup\{r : \mu(E_r) = 0\}$. Then for $r_1 < \alpha < r_2$, we have $\mu(E_{r_1}) = 0$ and $\mu(E_{r_2}) = 1$. Then, we easily infer that $f \equiv \alpha$ (a.e.). □

For the next two results, we shall assume that $(X, \mathcal{B})$ is a *countably separated Borel space*. This means the existence of a sequence (E_n) of Borel subsets such that for $w_1 \neq w_2 \in X$ there is some E_n with $w_1 \in E_n$ and $w_2 \notin E_n$. In the sequel we shall only consider such Borel spaces (see Subsection 1.5.4).

Lemma 1.20. *Let* $(X, \mathcal{B}, \mu)$ *be a probability measure space,* $g \in \mathrm{Aut}\,(X,\mu)$ *and* σ_g *the corresponding automorphism of* $L^\infty(X,\mu)$*. The following conditions are equivalent:*

(i) g *acts freely ;*

(ii) *for every* $Y \in \mathcal{B}$ *with* $\mu(Y) > 0$*, there exists a Borel subset* Z *of* Y *with* $\mu(Z) > 0$ *and* $Z \cap gZ = \emptyset$ *;*

(iii) *if* $a \in L^\infty(X,\mu)$ *is such that* $a\sigma_g(x) = xa$ *for every* $x \in L^\infty(X,\mu)$*, then* $a = 0$*.*

Proof. (i) ⇒ (ii). Let (E_n) be a separating family of Borel subsets as above. Assume that (i) holds and let Y be such that $\mu(Y) > 0$. Since

$$Y = \bigcup \big(Y \cap (E_n \setminus g^{-1}E_n)\big)$$

(up to null sets) there exists n_0 such that $\mu(Y \cap (E_{n_0} \setminus g^{-1}E_{n_0})) \neq 0$ and we take $Z = Y \cap (E_{n_0} \setminus g^{-1}E_{n_0})$.

(ii) ⇒ (iii). Let $a \in L^\infty(X,\mu)$ such that $a\sigma_g(x) = xa$ for every $x \in L^\infty(X,\mu)$. If $a \neq 0$, there exists a Borel subset Y of X with $\mu(Y) > 0$ and $x(g^{-1}w) = x(w)$ for almost every $w \in Y$ and for every x. Taking $x = \mathbf{1}_Z$ with Z as in (ii) leads to a contradiction.

Finally, the easy proof of (iii) ⇒ (i) is left to the reader. □

Proposition 1.21. *Let* $G \curvearrowright (X,\mu)$ *be a p.m.p. action and set* $A = L^\infty(X,\mu)$*.*

(i) $A' \cap (A \rtimes G) = A$ *if and only if the action is free.*

(ii) *Assume that the action is free. Then* $A \rtimes G$ *is a factor (and thus a type* II_1 *factor) if and only if the action is ergodic.*

Proof. Recall that A is naturally embedded into $A \rtimes G$ by $a \mapsto au_e$. Let $x = \sum_{g\in G} x_g u_g \in A \rtimes G$. Then for $a \in A$ we have

$$ax = \sum_{g\in G} a x_g u_g, \quad \text{and} \quad xa = \sum_{g\in G} x_g \sigma_g(a) u_g.$$

It follows that x belongs to $A' \cap (A \rtimes G)$ if and only if $ax_g = x_g \sigma_g(a)$ for every $g \in G$ and $a \in A$. Assertion (i) is then immediate.

To prove (ii), we remark that x belongs to the center of $A \rtimes G$ if and only if it commutes with A and with the u_g, $g \in G$. Assuming the freeness of the action, we know that the center of $A \rtimes G$ is contained into A. Moreover, an element $a \in A$ commutes with u_g if and only if $\sigma_g(a) = a$. Hence, the only elements of A commuting with u_g for every g are the scalar operators if and only if the action is ergodic. This concludes the proof □

1.5.3. **Examples.** Examples of such free and ergodic p.m.p. actions are plentiful. We mention below the most basic ones.

First, let G be a countable dense subgroup of a compact group X. Denote by μ the Haar probability measure on X. The left action of G onto X by left multiplication is of course measure preserving. It is obviously free. It is ergodic since any function in $L^2(X)$ which is G-invariant is invariant under the action of the whole group X (using the density of G) and therefore is constant.

The simplest such example is $X = \mathbb{T}$ and $G = \exp(i2\pi\mathbb{Z}\alpha)$ with α irrational. For another nice example, consider $X = (\mathbb{Z}/(2\mathbb{Z}))^{\mathbb{N}}$, the group operation being the coordinate-wise addition, and take for G the subgroup of sequences having only finitely many non-zero coordinates.

Second, let G be any countable group and $X = \{0,1\}^G$ be the compact space of all sequences of 0 and 1 indexed by G, equipped with the product measure $\mu = \nu^{\otimes G}$, where $\nu(\{0\}) = p, \nu(\{1\}) = 1-p$, $0 < p < 1$.

Proposition 1.22. *The Bernoulli action $G \curvearrowright X$ is free and ergodic.*

Proof. We begin by showing that the action is free. Let $g \neq e$ and choose an infinite subset I of G such that $gI \cap I = \emptyset$. Then we have

$$\begin{aligned}\mu(\{x : gx = x\}) &\leq \mu(\big\{x : x_{g^{-1}h} = x_h, \forall h \in I\big\}) \\ &= \prod_{h\in I} \mu(\big\{x : x_{g^{-1}h} = x_h\big\}) = 0.\end{aligned}$$

We now prove a stronger property than ergodicity, that is the *mixing property* : for every Borel subsets A, B we have $\lim_{g\to\infty} \mu(A \cap gB) = \mu(A)\mu(B)$. Using basic arguments appealing to monotone classes, it suffices to prove this property when A, B are both of the form $\prod_{g\in G} Y_g$ wher $Y_g = \{0,1\}$ for all except finitely many g. But then, obviously there is a finite subset $F \subset G$ such that $\mu(A \cap gB) = \mu(A)\mu(B)$ for $g \notin F$. □

Remark 1.23. In the previous proposition we may of course replace $(\{0,1\}, \nu)$ by any non-trivial probability measure space (Y, ν).

It is also interesting to deal with *generalized Bernoulli actions*. We let G act on a countable set Z and we set $X = Y^Z$, endowed with the product measure $\mu = \nu^{\otimes Z}$. This gives rise to the following p.m.p. action on (X, μ), called a generalized Bernoulli action:

$$\forall x \in X, \forall g \in G, \quad (gx)_z = x_{g^{-1}z}.$$

This action is ergodic if and only if every orbit of the action $G \curvearrowright Z$ is infinite (see [20, Proposition 2.3]). If for every $g \neq e$ the set $\{z \in Z : gz \neq z\}$ is infinite, the action is free.

1.5.4. *Digression about standard spaces.* The above fundamental examples of probability spaces are Lebesgue probability spaces. We recall now their nice properties. For our purpose, *they are the only probability spaces we shall consider from now on.*

A *standard Borel space* is a Borel space isomorphic to some Borel space $(X, \mathcal{B})$, where $\mathcal{B}$ is the collection of Borel subsets of a separable topological space X admitting a compatible complete metric. These Borel spaces have a simple classification : they are either finite, or isomorphic to $\mathbb{Z}$, or to $[0,1]$ (see for instance [14, Chapter II, Theorem 15.6]).

A *standard probability measure space* (X, μ) is a standard Borel space X equipped with a probability measure μ. All these spaces are measure preserving isomorphic to $[0,1]$ equipped with its natural Borel structure and a convex combination of the Lebesgue measure and a discrete probability measure. In particular, a standard probability measure space without atoms is isomorphic to $[0,1]$ with the Lebesgue measure. Such a probability space is called a *Lebesgue probability measure space.* For details, see [14, Chapter II, §17].

A nice feature of these probability measure spaces (X, μ) is that the group $\mathrm{Aut}\,(X, \mu)$ may be identified, in a natural way, with the group $\mathrm{Aut}\,(L^\infty(X), \tau_\mu)$ of automorphisms of $L^\infty(X, \mu)$ which preserve the integral τ_μ. Therefore, in this case, p.m.p. actions $G \curvearrowright (X, \mu)$ are the same as group homomorphisms from G to $\mathrm{Aut}\,(L^\infty(X), \tau_\mu)$.

1.6. <u>Von Neumann algebras from equivalence relations</u>. [7, 8]

We now present a construction which allows to obtain factors from non necessarily free group actions.

1.6.1. **Countable measured equivalence relations.**

Definition 1.24. A *countable* or *discrete equivalence relation* is an equivalence relation $\mathcal{R} \subset X \times X$ on a standard Borel space X, which is a Borel subset of $X \times X$ and whose equivalence classes are finite or countable.

Let $G \curvearrowright X$ be an action of a countable group G by Borel automorphisms of the Borel standard space X. The corresponding *orbit equivalence relation* is

$$\mathcal{R}_G = \{(x, gx) : x \in X, g \in G\}.$$

It is an example of countable equivalence relation, and in fact the most general one (see [7, Theorem 1]).

Given a probability measure μ on X, one defines a σ-finite measure ν on $\mathcal{R}$ by

$$\nu(C) = \int_X |C^x| \, \mathrm{d}\mu(x)$$

where $|C^x|$ denotes the cardinal of the set $C^x = \{(x,y) \in C : y\mathcal{R}x\}$. Similarly, we may define the measure $C \mapsto \int_X |C_x| \,\mathrm{d}\mu(x)$ where $|C_x|$ denotes the cardinal of the set $C_x = \{(y,x) \in C : y\mathcal{R}x\}$. When these two measures are the same, we say that $\mathcal{R}$ *preserves the probability measure* μ. In this case, we say that $\mathcal{R}$ is a *countable probability measure preserving* (p.m.p.) *equivalence relation on* (X,μ).

Given two Borel subsets A, B of X, a *partial isomorphism* $\varphi : A \to B$ is a Borel isomorphism from A onto B. We denote by $[[\mathcal{R}]]$ the set of such φ whose graph is contained into $\mathcal{R}$, i.e. $(x, \varphi(x) \in \mathcal{R}$ for every $x \in A$. The *domain* A of φ is written $D(\varphi)$ and its *range* B is written $R(\varphi)$. This family of partial isomorphisms is stable by the natural notions of composition and inverse. It is called the (*full*) *pseudogroup of the equivalence relation*.

Lemma 1.25. *Let $\mathcal{R}$ be a countable equivalence relation on a probability measure space (X,μ). The two following conditions are equivalent:*

(i) *$\mathcal{R}$ preserves the measure μ ;*
(ii) *for every $\varphi : A \to B$ in $[[\mathcal{R}]]$, we have $\varphi_*(\mu_{|A}) = \mu_{|B}$.*

Whenever an action $G \curvearrowright X$ is given and $\mathcal{R} = \mathcal{R}_G$, these conditions are also equivalent to

(iii) *$G \curvearrowright X$ preserves μ.*

Proof. Obviously (i) implies (ii). Conversely, assume that (ii) holds. Let E be a Borel subset of $\mathcal{R}$. Since the two projections from $\mathcal{R}$ onto X are countable to one, there exists a Borel countable partition $E = \cup E_n$ such that that both projections are Borel isomorphisms from E_n onto their respective ranges, as a consequence of a theorem of Lusin-Novikov (see [14, Theorem 18.10]). Each E_n is the graph of an element of $[[\mathcal{R}]]$, and the conclusion (i) follows.

When $\mathcal{R}$ is defined by $G \curvearrowright X$, it suffices to observe that for every $\varphi : A \to B$ in $[[\mathcal{R}]]$, there exists a partition $A = \cup_{g \in G} A_g$ such that $\varphi(x) = gx$ for $x \in A_g$. $\square$

1.6.2. **The von Neumann algebras of a countable measured equivalence relation.** Given a countable p.m.p. equivalence relation $\mathcal{R}$ on (X,μ), we describe now the construction of its associated von Neumann algebra $L(\mathcal{R})$. Let us denote by $\mathcal{M}_b(\mathcal{R})$ the set of bounded Borel functions $F : \mathcal{R} \to \mathbb{C}$ such that there exists a constant $C > 0$ with, for every $x, y \in X$,

$$|\{z \in X : F(z,y) \neq 0\}| \leq C, \quad \text{and} \quad |\{z \in X : F(x,z) \neq 0\}| \leq C.$$

It is easy to see that $\mathcal{M}_b(\mathcal{R})$ is an involutive algebra, where the product $F_1 * F_2$ and the involution are given respectively by the expressions

$$(F_1 * F_2)(x,y) = \sum_{z\mathcal{R}x} F_1(x,z) F_2(z,y),$$
$$F^*(x,y) = \overline{F(y,x)}.$$

Viewing the elements of $\mathcal{M}_b(\mathcal{R})$ as matrices, these operations are respectively the matricial product and adjoint. Note also that $\mathcal{M}_b(\mathcal{R})$ contains the algebra $B_b(X)$ of bounded Borel functions on X: one identifies $f \in B_b(X)$ to the diagonal function $(x, y) \mapsto f(x)\mathbf{1}_\Delta(x, y)$ where $\mathbf{1}_\Delta$ is the characteristic function of the diagonal $\Delta \subset \mathcal{R}$.

Any element $\varphi : A \to B$ of $[[\mathcal{R}]]$ may be identified with the characteristic function S_φ of the set $\{(\varphi(x), x) : x \in A\}$. For $\varphi, \psi \in [[\mathcal{R}]]$, we have $L_{S_\varphi} * L_{S_\psi} = L_{S_{\varphi\circ\psi}}$ and $(L_{S_\varphi})^* = L_{S_{\varphi^{-1}}}$.

Obviously, every finite sum[8]

$$F(x, y) = \sum f_\varphi(x) S_\varphi(x, y), \tag{4}$$

where $\varphi \in [[\mathcal{R}]]$ and $f_\varphi : R(\varphi) \to \mathbb{C}$ is a bounded Borel function, belongs to $\mathcal{M}_b(\mathcal{R})$. Using once again [14, Theorem 18.10], it can be shown that $\mathcal{M}_b(\mathcal{R})$ is exactly the space of such functions (see Exercise 1.9).

Given $F \in \mathcal{M}_b(\mathcal{R})$ and $\xi \in L^2(\mathcal{R}, \nu)$ we define $L_F(\xi)$ by

$$L_F(\xi)(x, y) = (F * \xi)(x, y) = \sum_{z \mathcal{R} x} F(x, z)\xi(z, y).$$

We leave it to the reader to check that $F \mapsto L_F$ is a $*$-homomorphism from the $*$-algebra $\mathcal{M}_b(\mathcal{R})$ into $\mathcal{B}(L^2(\mathcal{R}, \nu))$. Moreover the restriction of L to $B_b(X)$ induces an injective representation of $L^\infty(X, \mu)$, defined by

$$(L_f \xi)(x, y) = f(x)\xi(x, y)$$

for $f \in L^\infty(X, \mu)$ and $\xi \in L^2(\mathcal{R}, \nu)$. Note also that, given $\varphi \in [[\mathcal{R}]]$, the element $u_\varphi = L_{S_\varphi}$ is a partial isometry : $u_\varphi^* u_\varphi$ and $u_\varphi u_\varphi^*$ are projections in $L^\infty(X, \mu) \subset \mathcal{B}(L^2(\mathcal{R}, \nu))$ corresponding to the multiplication by the characteristic functions of the domain of φ and of its range respectively. Whenever F is given by the expression (4), we have

$$L_F = \sum L_{f_\varphi} u_\varphi.$$

The *von Neumann algebra of the countable p.m.p. equivalence relation* $\mathcal{R}$ is the s.o. closure $L(\mathcal{R})$ of $\{L_F : F \in \mathcal{M}_b(\mathcal{R})\}$ into $\mathcal{B}(L^2(\mathcal{R}, \nu))$. Observe that $L^\infty(X, \mu)$ is naturally embedded as a von Neumann subalgebra of $L(\mathcal{R})$. Similarly, we may let $\mathcal{M}_b(\mathcal{R})$ act to the right by

$$R_F(\xi)(x, y) = (\xi * F)(x, y) = \sum_{z \mathcal{R} x} \xi(x, z) F(z, y).$$

We denote by $R(\mathcal{R})$ the von Neumann algebra generated by these operators R_F with $F \in \mathcal{M}_b(\mathcal{R})$. We may proceed as in Sections 1.3 and 1.5 to prove the following facts :

- $\mathbf{1}_\Delta$ is a cyclic and separating vector for $L(\mathcal{R})$. In particular, $T \mapsto T\mathbf{1}_\Delta$ identifies $L(\mathcal{R})$ to a subspace of $L^2(\mathcal{R}, \nu)$. Note that $L_F \mathbf{1}_\Delta = F$ for $F \in \mathcal{M}_b(\mathcal{R})$.

[8] analogous to the expression (3)

- $\tau(L_F) = \langle \mathbf{1}_\Delta, L_F \mathbf{1}_\Delta \rangle = \int_X F(x,x)\,\mathrm{d}\mu(x)$ defines a faithful w.o. continuous tracial state on $L(\mathcal{R})$.

Here too, $L(\mathcal{R})$ is in standard form on $L^2(\mathcal{R},\nu)$ since $\mathbf{1}_\Delta$ is a cyclic vector which defines the canonical trace on $L(\mathcal{R})$. For $\xi \in L^2(\mathcal{R},\nu)$ we have $J\xi(x,y) = \overline{\xi(y,x)}$ and, given $F \in \mathcal{F}(\mathcal{R})$, one easily sees that $JL_F J = R_{F^*}$. Therefore we obtain the equality $L(\mathcal{R})' = R(\mathcal{R})$.

Definition 1.26. Let $\mathcal{R}$ be a countable p.m.p. equivalence relation and let A be a Borel subset of X. We denote by $[A]_\mathcal{R} = p_1(p_2^{-1}(A)) = p_2(p_1^{-1}(A))$ the $\mathcal{R}$-*saturation* of A, where p_1, p_2 are the left and right projections from $\mathcal{R}$ onto X. We say that A is *invariant* (or *saturated*) if $[A]_\mathcal{R} = A$ (up to null sets). The relation $(\mathcal{R},\mu)$ is called *ergodic* if any invariant Borel subset is either null or co-null.

Remark 1.27. The Borel set A is invariant if and only if $\mathbf{1}_A \circ p_1 = \mathbf{1}_A \circ p_2$ ν-a.e. More generally, a Borel function f on X is said to be invariant if $f \circ p_1 = f \circ p_2$ ν-a.e. The equivalence relation is ergodic if and only if the only invariant functions are the constant (up to null sets) ones.

Proposition 1.28. *Let $\mathcal{R}$ be a countable p.m.p. equivalence relation on (X,μ).*

(i) $L^\infty(X,\mu)' \cap L(\mathcal{R}) = L^\infty(X,\mu)$*, that is,* $L^\infty(X,\mu)$ *is a maximal abelian subalgebra of* $L(\mathcal{R})$*.*

(ii) *The center of $L(\mathcal{R})$ is the algebra of invariant functions in $L^\infty(X,\mu)$. In particular, $L(\mathcal{R})$ is a factor (and thus a type II_1 factor) if and only if the equivalence relation is ergodic.*

Proof. (i) Let $T \in L(\mathcal{R}) \cap L^\infty(X,\mu)'$. We set $F = T\mathbf{1}_\Delta \in L^2(\mathcal{R},\nu)$. For every $f \in L^\infty(X,\mu)$ we have

$$L_f T \mathbf{1}_\Delta = T L_f \mathbf{1}_\Delta = T(\mathbf{1}_\Delta * f),$$

where $(\xi * f)(x,y) = \xi(x,y)f(y)$ for $\xi \in L^2(\mathcal{R},\nu)$. Moreover, T commutes with the right convolution $\xi \mapsto \xi * f$ by f, whence $L_f F = F * f$, that is $f(x)F(x,y) = F(x,y)f(y)$ ν-a.e. It follows that F is supported by the diagonal Δ, and belongs to $L^\infty(X,\mu)$ since T is bounded.

(ii) $f \in L^\infty(X,\mu)$ belongs to the center of $L(\mathcal{R})$ if and only if $f(x)F(x,y) = F(x,y)f(y)$ ν-a.e. for every $F \in \mathcal{M}_b(\mathcal{R})$, therefore if and only if $f \circ p_1 = f \circ p_2$ ν-a.e. $\square$

Remark 1.29. When $G \curvearrowright (X,\mu)$ is a free p.m.p. action, the von Neumann algebras $L(\mathcal{R}_G)$ and $L^\infty(X,\mu) \rtimes G$ coincide. Indeed, the map $\phi : (x,g) \mapsto (x, g^{-1}x)$ induces a unitary operator $V : \xi \mapsto \xi \circ \phi$ from $L^2(\mathcal{R},\nu)$ onto $L^2(X \times G, \mu \otimes \lambda) = \ell^2(G, L^2(X,\mu))$, where λ is the counting measure on G. This holds true because the action is free, and therefore ϕ is an isomorphism from $(X \times G, \mu \otimes \lambda)$ onto $(\mathcal{R},\nu)$. We immediately see that $V^*(L^\infty(X,\mu) \rtimes G)V \subset L(\mathcal{R})$. In fact $L^\infty(X,\mu)$ is identically preserved, and we have $V^* u_g V = L_{S_g}$ where S_g is the characteristic function of $\{(gx,x) : x \in X\} \subset \mathcal{R}$. Similarly, we see that the commutant of $L^\infty(X,\mu)\rtimes G$ is sent into the commutant $R(\mathcal{R})$ of $L(\mathcal{R})$, whence $V^*(L^\infty(X,\mu)\rtimes G)V = L(\mathcal{R})$ thanks to the von Neumann bicommutant theorem.

1.6.3. **Isomorphisms of measured equivalence relations.**

Definition 1.30. We say that two countable p.m.p. equivalence relations $\mathcal{R}_1$ and $\mathcal{R}_2$ on (X_1,μ_1) and (X_2,μ_2) respectively are *isomorphic* if there exists an isomorphism $\theta : (X_1,\mu_1) \to (X_2,\mu_2)$ (of probability measure spaces, *i.e.* $\theta_*\mu_1 = \mu_2$) such that $(\theta\times\theta)(\mathcal{R}_1) = \mathcal{R}_2$, up to null sets. We say that θ induces an *isomorphism from* $\mathcal{R}_1$ onto $\mathcal{R}_2$. If this holds when $\mathcal{R}_1 = \mathcal{R}_{G_1}$ and $\mathcal{R}_2 = \mathcal{R}_{G_2}$ we say that the actions $G_1 \curvearrowright (X_1,\mu_1)$ and $G_2 \curvearrowright (X_2,\mu_2)$ are *orbit equivalent*. This means that for a.e. $x\in X_1$, we have $\theta(G_1x) = G_2\theta(x)$.

Let $\theta : (X_1,\mu_1) \to (X_2,\mu_2)$ be an isomorphism of equivalence relations as above. Then $U : \xi \mapsto \xi\circ(\theta\times\theta)$ is a unitary operator from $L^2(\mathcal{R}_2,\nu_2)$ onto $L^2(\mathcal{R}_1,\nu_1)$ such that $UL(\mathcal{R}_2)U^* = L(\mathcal{R}_1)$. Moreover, this spatial isomorphism sends $L^\infty(X_2,\mu_2)$ onto $L^\infty(X_1,\mu_1)$. More precisely, for $f\in L^\infty(X_2,\mu_2)$, we have $UL_fU^* = L_{f\circ\theta}$. We also observe that this isomorphism preserves the canonical traces on $L(\mathcal{R}_1)$ and $L(\mathcal{R}_2)$.

We deduce from Remark 1.29 that when $G_1 \curvearrowright (X_1,\mu_1)$ and $G_2 \curvearrowright (X_2,\mu_2)$ are free p.m.p. actions which are orbit equivalent through $\theta : (X_1,\mu_1)\to(X_2,\mu_2)$, the isomorphism $f\mapsto f\circ\theta$ from $L^\infty(X_2,\mu_2)$ onto $L^\infty(X_1,\mu_1)$ extends to a spatial isomorphism from the crossed product von Neumann algebra $L^\infty(X_2,\mu_2)\rtimes G_2$ onto $L^\infty(X_1,\mu_1)\rtimes G_1$.

1.7. <u>Two non-isomorphic II_1 factors</u>.

Theorem 1.31 ([18]). *$L(S_\infty)$ and $L(\mathbb{F}_2)$ are non-isomorphic.*

The proof, which is not the original one given by Murray and von Neumann, requires some preliminaries.

We first extend to the non-commutative setting the classical notion, in measure theory, of conditional expectation.

Definition 1.32. Let M be a von Neumann algebra and B a von Neumann subalgebra. A *conditional expectation* from M to B is a linear map $E : M\to B$ which satisfies the following properties :

(i) $E(M_+)\subset B_+$;
(ii) $E(b) = b$ for $b\in B$;
(iii) $E(b_1xb_2) = b_1E(x)b_2$ for $b_1,b_2\in B$ and $x\in M$.

Hence E is a positive projection from M onto B, and is left and right B-linear. It is a classical fact that E is *completely positive*, that is, for every $n\geq 1$, the map $[x_{i,j}]\in M_n(M)\mapsto [E(x_{i,j})]\in M_n(B)$ is positive. It is also a well-known fact that a conditional expectation is nothing else than a norm-one projection.

Second, we recall some basics about amenable groups.

Definition 1.33. A countable group G is called *amenable* if there exists a left invariant mean m on G, that is a state m on $\ell^\infty(G)$ such that $m({}_sf) = m(f)$ for every $s \in G$ and $f \in \ell^\infty(G)$, where $({}_sf)(t) = f(s^{-1}t)$ for all $t \in G$.

Examples 1.34. (1) Every finite group G is amenable. Indeed, the uniform probability measure (*i.e.* the Haar measure) m is an invariant mean.

(2) Let $G = \cup_n G_n$ be the union of an increasing sequence of finite subgroups G_n. Then G is amenable. To construct a left invariant mean on G we start with the sequence (m_n) of Haar measures on the subgroups and we take an appropriate limit of the sequence. To this end, we fix a free ultrafilter ω. Recall that, for any bounded sequence (c_n) of complex numbers, $\lim_\omega c_n$ is defined as the value at $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ of this sequence, viewed as a continuous function on the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$. Given $f \in \ell^\infty(G)$, we set

$$m(f) = \lim_\omega m_n(f|_{G_n}).$$

It is easily checked that m is an invariant mean on G.

A basic example is the group S_∞ of all finite permutations of $\mathbb{N}$.

A remarkable fact is that amenability admits many equivalent characterisations. We recall below several of them.

Proposition 1.35. *Let G be a countable group. The following conditions are equivalent:*

(i) *G is amenable ;*

(ii) *there exists a sequence (ξ_n) of unit vectors in $\ell^2(G)$ such that, for every $g \in G$,*

$$\lim_n \|\lambda(g)\xi_n - \xi_n\|_2 = 0;$$

(iii) *there exists a sequence of finitely supported positive definite functions on G which converges pointwise to 1 ;*

(iv) *there exists a sequence (E_n) of finite, non-empty, subsets of G such that, for every $g \in G$,*

$$\lim_n \frac{|gE_n \Delta E_n|}{|E_n|} = 0.$$

For details, see for instance [1, Appendix G]. Condition (ii) means that *the left regular representation of G has a sequence of almost invariant vectors*, or in other terms, that *the trivial representation ι of G is weakly contained in its left regular representation λ*. The notion of positive definite function on a group is recalled in Section 3. A sequence satisfying condition (iv) is called a *Følner sequence*. This condition means that in (ii) we may take normalized characteristic functions.

Proposition 1.36. *Let G be a countable group and $M = L(G)$. Then G is amenable if and only if there exists a conditional expectation E from $\mathcal{B}(L^2(M))$ onto M.*

Proof. As always, u_s, $s \in G$, are the canonical unitaries of $L(G)$. Assume first the existence of E. Given $f \in \ell^\infty(G)$, we denote by M_f the multiplication operator by f on $\ell^2(G)$. We set $m(f) = \tau(E(M_f))$, where τ is the canonical trace on M. Since $u_s M_f u_s^* = M_{sf}$ for every $s \in G$, it is easily seen that the state m is left invariant.

Conversely, assume that G is amenable, and let m be a left invariant mean on $\ell^\infty(G)$. Given $\xi, \eta \in \ell^2(G)$, and $T \in \mathcal{B}(\mathcal{H})$, we introduce the function defined by

$$f^T_{\xi,\eta}(s) = \left\langle \xi, \rho(s)T\rho(s^{-1})\eta \right\rangle$$

where ρ is the right regular representation of G. Obviously, f is a bounded function on G with

$$|f^T_{\xi,\eta}(s)| \leq \|T\|\|\xi\|\|\eta\|.$$

We define a continuous sesquilinear functional on $\ell^2(G)$ by the formula

$$(\xi, \eta) = m(f^T_{\xi,\eta}).$$

It follows that there is a unique operator, denoted by $E(T)$, with

$$\langle \xi, E(T)\eta \rangle = m(f^T_{\xi,\eta})$$

for every $\xi, \eta \in \ell^2(G)$.

The invariance property of m implies that $\rho(g)E(T)\rho(g^{-1}) = E(T)$ for all $g \in G$. Therefore, $E(T)$ commutes with $\rho(G)$, whence $E(T) \in L(G)$. It is easility checked that E is a conditional expectation. $\square$

A type II_1 factor M on a Hilbert space $\mathcal{H}$ such that there exists a conditional expectation $E : \mathcal{B}(\mathcal{H}) \to M$ is said to be injective. One shows that this property is intrinsic (i.e. independent of $\mathcal{H}$).

Proof of Theorem 1.31. Immediate from the previous proposition, since S_∞ is amenable while $\mathbb{F}_2$ is not. $\square$

It is easily proved that a hyperfinite type II_1 factor is injective. A deep result of Connes [6] states that the converse holds. Murray and von Neumann have established [18] the uniqueness of the hyperfinite type II_1 factor. Therefore, the ICC amenable groups give rise to the same factor, namely the hyperfinite one.

J.T. Schwartz has shown [23] that $L(\mathbb{F}_2 \times S_\infty)$ neither isomorphic to $L(S_\infty)$ nor to $L(\mathbb{F}_2)$. We shall see that $L(PSL(2, \mathbb{Z})$ is not isormorphic to any $L(PSL(n, \mathbb{Z})$ with $n \geq 3$. Let us mention two famous and notoriously difficult problems : are $L(\mathbb{F}_m)$ and $L(\mathbb{F}_n)$ isomorphic whenever m, n are distinct and ≥ 2 ? are $L(PSL(m, \mathbb{Z})$ and $L(PSL(n, \mathbb{Z})$ isomorphic whenever m, n are distinct and ≥ 3 ?

Exercises.

Exercise 1.1. Let $\mathcal{H}$ be a separable Hilbert space.

(a) Show that the unit ball $\mathcal{B}(\mathcal{H})_1$ of $\mathcal{B}(\mathcal{H})$ is metrizable and compact (hence second-countable) relative to the w.o. topology.
(b) Show that $\mathcal{B}(\mathcal{H})_1$ is metrizable, complete and second-countable relative to the s.o. topology.
(c) When $\mathcal{H}$ is infinite dimensional, show that $\mathcal{B}(\mathcal{H})_1$ is not separable relative to the operator norm topology (take $\mathcal{H} = L^2([0,1])$ for instance).

Exercise 1.2. Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and let α be the isomorphism sending $x \in \mathcal{B}(\mathcal{H})$ onto $\alpha(x) \in \mathcal{B}(\mathcal{H}^{\oplus\infty})$ with $\alpha(x)((\xi_n)_n) = (x\xi_n)_n$ for every $(\xi_n)_n \in \mathcal{H}^{\oplus\infty}$. Show that $\alpha(\mathcal{B}(\mathcal{H}))$ is a von Neumann algebra on $\mathcal{H}^{\oplus\infty}$, but that α is not continuous with respect to the w.o. (or s.o.) topologies.

Exercise 1.3. Let $\mathcal{H}$ be a separable Hilbert space. Let $k \in \mathbb{N}$ and let α_k be the isomorphism sending $x \in \mathcal{B}(\mathcal{H})$ onto $\alpha_k(x) \in \mathcal{B}(\mathcal{H}^{\oplus k})$ with $\alpha_k(x)((\xi_n)_n) = (x\xi_n)_n$ for every $(\xi_n)_n \in \mathcal{H}^{\oplus k}$. Show that the von Neumann $\alpha_{k_1}(\mathcal{B}(\mathcal{H}))$ and $\alpha_{k_2}(\mathcal{B}(\mathcal{H}))$ are spatially isomorphic if anf only if $k_1 = k_2$.

Exercise 1.4. Let (X,μ) be a probability space and $A = L^\infty(X,\mu) \subset L^2(X,\mu)$. Show that on the unit ball A_1 of A, the s.o. topology coincides with the topology defined by $\|\cdot\|_2$. Show that $(A_1, \|\cdot\|_2)$ is a complete metric space.

Exercise 1.5. Let $S_\infty = \cup_{n=1}^\infty S_n$ be the group of finite permutations of $\mathbb{N}$. Let $\sigma \in S_n$ be a non-trivial permutation and let i be such that $\sigma(i) \neq i$. For $j > n$, denote by s_j the transposition permuting i and j. Show that $\left\{s_j \sigma s_j^{-1} : j > n\right\}$ is infinite.

Exercise 1.6. Show that the free group $\mathbb{F}_n, n \geq 2$, is ICC.

Exercise 1.7. Let $G \curvearrowright (X,\mu)$ be a p.m.p. action of an ICC group G and set $A = L^\infty(X,\mu)$. Show that the commutant of $\{u_g : g \in G\}$ in $A \rtimes G$ is the fixed-point algebra A^G. Conclude that $A \rtimes G$ is a type II_1 factor if and only if the action is ergodic.

Exercise 1.8. Let $\mathbb{F}_2$ be the free group with two generators a, b. Denote by E_a (resp. E_b, $E_{a^{-1}}$, $E_{b^{-1}}$) the subset of $\mathbb{F}_2$ formed by the words beginning by a (resp. b, a^{-1}, b^{-1}). Show that $\mathbb{F}_2 = \{e\} \sqcup E_a \sqcup E_b \sqcup E_{a^{-1}} \sqcup E_{b^{-1}}$, $\mathbb{F}_2 = E_a \sqcup aE_{a^{-1}} = E_b \sqcup bE_{b^{-1}}$. Conclude that $\mathbb{F}_2$ is not amenable.

Exercise 1.9. Let $\mathcal{R}$ be a countable p.m.p. equivalence relation on X.

(i) Let C be a Borel subset of $\mathcal{R}$ with
$$\sup_{x\in X} |C^x| < +\infty, \quad \sup_{x\in X} |C_x| < +\infty.$$

Show that there is a partition $C = \bigsqcup C_n$ into Borel subsets such that the second projection p_2 is injective on each C_n and $p_2(C_m) \supset p_2(C_n)$ for $m < n$. Conclude that there are only finitely many such non-empty subsets. Show that C is the disjoint union of finitely many Borel subsets such that both projections from $X \times X \to X$ are injective when restricted to them.

(ii) Show that every $F \in \mathcal{M}_b(\mathcal{R})$ may be written as a finite sum $F(x, y) = \sum f_\varphi(x) S_\varphi(x, y)$, where $\varphi \in [[\mathcal{R}]]$ and $f_\varphi : R(\varphi) \to \mathbb{C}$ is a bounded Borel function.

2. About factors arising from equivalence relations

2.1. Isomorphisms of equivalence relations vs isomorphisms of their von Neumann algebras.

Theorem 2.1 ([8]). *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two countable p.m.p. equivalence relations, on X_1 and X_2 respectively, and let $\theta : (X_1, \mu_1) \to (X_2, \mu_2)$ be an isomorphism of probability measure spaces. The two following conditions are equivalent:*

(i) *θ induces an isomorphism from $\mathcal{R}_1$ onto $\mathcal{R}_2$;*
(ii) *$\theta_* : L^\infty(X_1) \simeq L^\infty(X_2)$ [9] extends to an isomorphism from the von Neumann algebra $L(\mathcal{R}_1)$ onto $L(\mathcal{R}_2)$.*

Therefore $\mathcal{R}_1$ and $\mathcal{R}_2$ are isomorphic if and only if there exists an isomorphism $\alpha : L(\mathcal{R}_1) \simeq L(\mathcal{R}_2)$ such that $\alpha(L^\infty(X_1)) = L^\infty(X_2)$.

2.2. Cartan subalgebras.

Definition 2.2. Given a von Neumann algebra M and a von Neumann subalgebra A, the *normalizer of A in M* is the group $\mathcal{N}_M(A)$ of unitary operators $u \in M$ such that $uAu^* = A$. Note that $\mathcal{N}_M(A)$ contains the unitary group $\mathcal{U}(A)$ of A as a normal subgroup.

Let (M, τ) be a tracial von Neumann algebra. A *Cartan subalgebra* is a maximal abelian subalgebra A of M such that the normalizer $\mathcal{N}_M(A)$ generates M as a von Neumann algebra.

Consider for instance a free ergodic p.m.p. action $G \curvearrowright (X, \mu)$ of a countable group G. Then $A = L^\infty(X)$ is a Cartan subalgebra of $M = L^\infty(X) \rtimes G$. Indeed, M is generated by $A \cup \{u_g : g \in G\}$ and we have of course $u_g \in \mathcal{N}_M(A)$ for $g \in G$. More generally, given a countable p.m.p. equivalence relation on (X, μ), one shows that $L^\infty(X)$ is a Cartan subalgebra of $L(\mathcal{R})$. Moreover, given a Cartan subalgebra A of a tracial von Neumann algebra M, the pair $A \subset M$ is essentially of the form $L^\infty(X) \subset L(\mathcal{R})$, up to a cocycle (see [8, Theorem 1]).

Given a type II_1 factor, in light of the foregoing, it is therefore a fundamental question to know whether it contains a Cartan subalgebra, and furthermore how many. The first example with two non conjugate Cartan subalgebras[10] is due to Connes and Jones [4]. The first examples of type II_1 factors without Cartan subalgebra are $L(\mathbb{F}_n)$, $n \geq 2$, a result of Voiculescu [24]. Nowadays, many examples of type II_1 factors without, or with only one (up to inner conjugacy), or with uncountably many, Cartan subalgebras has been exhibited.

[9] θ_* is the isomorphism $f \in L^\infty(X_1) \mapsto f \circ \theta^{-1}$.

[10] Two subalgebras A and B of M are *conjugate* if there exists an automorphism α of M such that $\alpha(A) = B$. They are *inner conjugate* if there is a unitary $u \in M$ with $uAu^* = B$.

Assume now that we have an equivalence relation such that any two Cartan subalgebras A and B are conjugate.Then for any other equivalence relation $\mathcal{R}_1$ it follows from Theorem 2.1 that $L(\mathcal{R})$ and $L(\mathcal{R}_1)$ are isomorphic if and only if $\mathcal{R}$ and $\mathcal{R}_1$ are isomorphic. It follows that invariants of $\mathcal{R}$ become invariants of $L(\mathcal{R})$.

2.3. An application: computation of fundamental groups.

2.3.1. *Fundamental group of an equivalence relation.* Let $\mathcal{R}$ be an ergodic countable p.m.p. equivalence relation on (X, μ). Given a measurable subset Y of X with $\mu(Y) > 0$, one defines an equivalence relation $\mathcal{R}_Y$ on (Y, μ_Y), by $\mathcal{R}_Y = \mathcal{R} \cap (Y \times Y)$ and $\mu_Y = \mu(Y)^{-1}\mu_{|_Y}$. One shows that this equivalence relation only depends on $t = \mu(Y)$ and we denote it by $\mathcal{R}^t$.

Gefter and Golodets [10] introduced the *fundamental group* of $\mathcal{R}$ as the subgroup

$$\mathcal{F}(\mathcal{R}) = \left\{ t_1 t_2^{-1} : \mathcal{R}^{t_1} \simeq \mathcal{R}^{t_2} \right\}$$

of $\mathbb{R}_+^*$.

The fundamental group of an equivalence relation provided by an ergodic p.m.p. action of any countable amenable group is $\mathbb{R}_+^*$. Gefter and Golodets gave the first example with $\mathcal{F}(\mathcal{R}) = \{1\}$. Now, one knows a wealth of examples of groups whose essentially free ergodic p.m.p. actions yields equivalence relations withe fundamental group reduced to $\{1\}$. This is in particular the case of free groups with $n \geq 2$ generators, by a result of Gaboriau [9].

2.3.2. *Fundamental group of a type II_1 factor.* The above definition is based on the following definition of the fundamental group of a type II_1 factor M introduced by Murray and von Neumann in 1943 [18]. Let p be a projection in M.[11] Then $pMp = \{pxp : x \in M\}$ is a type II_1 factor, only depending on $t = \tau(p)$, up to isomorphism. We denote it by M^t. The *fundamental group* of M is the subgroup $\mathcal{F}(M)$ of $\mathbb{R}_+^*$ defined as

$$\mathcal{F}(M) = \left\{ t_1 t_2^{-1} : M^{t_1} \simeq M^{t_2} \right\}.$$

Whenever M is the hyperfinite type II_1 factor R, it is easily seen that pRp is hyperfinite for any non-zero projection in R. It follows that $\mathcal{F}(R) = \mathbb{R}_+^*$. It is also known that $\mathcal{F}(\mathbb{F}_\infty) = \mathbb{R}_+^*$ [22]. The computation of the fundamental group of $L(\mathbb{F}_n)$, $2 \leq n < \infty$, is an important open problem: either if is $\mathbb{R}_+^*$ and then the $L(\mathbb{F}_n)$ are all isomorphic, or it is $\{1\}$ and then the $L(\mathbb{F}_n)$ are all distinct.

The first proof of the *existence* of type II_1 factors with countable fundamental group is due to Connes [3]. In his seminal paper [19], Popa provided the first *explicit* computation of a fundamental group strictly contained in $\mathbb{R}_+^*$ (in fact reduced to $\{1\}$).

[11] A basic result is that the set of $\tau(p)$, where p ranges over the projections of a type II_1 factor, is $[0, 1]$. Viewing $\tau(p)$ as the dimension of p, we see that a type II_1 factor has a *continuous geometry*.

Note that whenever $M = L(\mathcal{R})$ we have $\mathcal{F}(\mathcal{R}) \subset \mathcal{F}(L(\mathcal{R}))$ since $L(\mathcal{R}^t) \simeq L(\mathcal{R})^t$. Moreover if M has only one Cartan subalgebra (up to conjugacy) then $\mathcal{F}(\mathcal{R}) = \mathcal{F}(L(\mathcal{R}))$.

2.3.3. *An example.* The group $SL(2,\mathbb{Z})$ acts on $\mathbb{Z}^2$ in the obvious way. It is a classical fact that $\mathbb{F}_2$ is isomorphic to a subgroup of finite index in $SL(2,\mathbb{Z})$, namely to the subgroup generated by the matrices $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. In this way, we let $\mathbb{F}_2$ act on $\mathbb{Z}^2$, and by duality on $\mathbb{T}^2$. We note the following natural identifications :

$$L(\mathbb{Z}^2 \rtimes \mathbb{F}_2) \simeq L(\mathbb{Z}^2) \rtimes \mathbb{F}_2 \simeq L^\infty(\mathbb{T}^2) \rtimes \mathbb{F}_2 \simeq L(\mathcal{R}),$$

where $\mathcal{R}$ is the orbit equivalence relation defined by the $\mathbb{F}_2$-action on $\mathbb{T}^2$.

Theorem 2.3 (Popa 2001, [19])**.** *The fundamental group of $L^\infty(\mathbb{T}^2) \rtimes \mathbb{F}_2$ is $\{1\}$.*

A very recent remarkable result of Popa and Vaes[12] [21] states that *any* free ergodic p.m.p. action of $\mathbb{F}_n$, $n \geq 2$, gives rise to a type II_1 factor with a unique Cartan subalgebra (up to inner conjugacy). However, it will be enough for us (and more accessible) to deal with a previous result of Popa [19] asserting that $L^\infty(\mathbb{T}^2)$ is the unique (up to conjugacy) rigidly embedded Cartan subalgebra (see Definition 3.10 below) of $L^\infty(\mathbb{T}^2) \rtimes \mathbb{F}_2$. Assuming this fact, let us indicate how to prove the above theorem. We set $M = L(\mathcal{R})$, $A = L^\infty(\mathbb{T}^2)$ and for $Y \subset \mathbb{T}^2$ with $\mu(Y) = t$ we introduce $A^t = L^\infty(Y) \subset L(\mathcal{R}^t) = L(\mathcal{R}_Y)$. We have that since A is rigidly embedded in M, similarly A^t is rigidly embedded in M^t. Assuming that M is isomorphic to M^t, Popa's result implies that there is an isomorphism from M onto M^t sending A onto A^t. Therefore $\mathcal{R}$ is isomorphic to $\mathcal{R}^t$. Now, the above mentioned result of Gaboriau implies that $t = 1$.

Exercise. Let M be a type II_1 factor. Show that $M_n(\mathbb{C}) \otimes M$ is isomorphic to M if and only if $1/n \in \mathcal{F}(M)$.

[12] 29/11/2011

3. Study of the inclusion $L^\infty(\mathbb{T}^2) \subset L^\infty(\mathbb{T}^2) \rtimes \mathbb{F}_2$

Recall that a a complex-valued function φ defined on a countable group G is *positive definite* (or of *positive type*) if, for every finite subset $\{s_1, \dots, s_n\}$ of G, the $n \times n$ matrix $[\varphi(s_i^{-1}s_j)]$ is positive definite, that is, for every $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, we have $\sum_{i,j=1}^n \overline{\lambda_i}\lambda_j\varphi(s_i^{-1}s_j) \geq 0$. Obviously, given a unitary representation π on a Hilbert space $\mathcal{H}$, any *coefficient* of π, that is any function $s \mapsto \langle \xi, \pi(s)\xi\rangle$ with $\xi \in \mathcal{H}$, is positive definite. Conversely, given a positive definite function φ on G there is a unique (up to isomorphism) triple $(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi)$ (called the GNS construction) composed of a unitary representation π_φ and a cyclic vector ξ_φ, such that $\varphi(s) = \langle \xi_\varphi, \pi_\varphi(s)\xi_\varphi\rangle$ for all $s \in G$. These two constructions are inverse from each other (see [1, Theorem C.4.10]).

We shall also use below the notion of bimodule.

Definition 3.1. Let M be a von Neumann algebras. A *M-M-bimodule* is a Hilbert space $\mathcal{H}$ which is both a left and a right M-module, and is such that the left and right actions commute.[13] Usually, for $x, y \in M, \xi \in \mathcal{H}$, we write $x\xi y$ instead of $xy^{op}\xi$.

Whenever (M,τ) is a tracial von Neumann algebra, $L^2(M,\tau)$ is the most basic M-M-bimodule, called the *standard* or *identity* or *trivial M-M-bimodule*: for $x, y \in M$ and $\widehat{z} \in L^2(M,\tau)$, we set $x\widehat{z}y = \widehat{xzy} = xJy^*J\widehat{z}$.

Completely positive maps and bimodules are closely related. They can be viewed as generalised morphisms. We shall only need to explain how a M-M-bimodule is canonically constructed, given a completely positive map $\phi : M \to M$.

Let (M,τ) be a tracial von Neumann algebra and let $\phi : M \to M$ be a completely positive map. We define on the linear space $\mathcal{H}_0 = M \odot M$ a sesquilinear functional by

$$\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle_\phi = \tau(y_1^*\phi(x_1^*x_2)y_2), \quad \forall x_1, x_2, y_1, y_2 \in M.$$

The complete positivity of ϕ implies the positivity of this functional. We denote by $\mathcal{H}(\phi)$ the completion of the quotient of $\mathcal{H}_0$ modulo the null space of the sesquilinear functional. We let M act to the left and to the right on $\mathcal{H}_0$ by

$$x(\sum_{i=1}^n x_i \otimes y_i)y = \sum_{i=1}^n xx_i \otimes y_iy.$$

Using again the complete positivity of ϕ, we easily obtain, for $\xi \in \mathcal{H}_0$, and $x, y \in M$ that

$$\langle x\xi, x\xi\rangle_\phi \leq \|x\|^2\langle \xi, \xi\rangle_\phi, \quad \langle \xi y, \xi y\rangle_\phi \leq \|y\|^2\langle \xi, \xi\rangle_\phi.$$

It follows that the actions of M pass to the quotient and extend to representations on $\mathcal{H}(\phi)$.

[13] In other terms, M and its opposite von Neumann algebra M^{op} are commuting von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$.

This bimodule comes equipped with a special vector, namely the class of $1 \otimes 1$, denoted ξ_ϕ. Note that

$$\|x\xi_\phi\|_\phi^2 = \tau(\phi(x^*x)) \text{ and } \|\xi_\phi y\|_\phi^2 = \tau(y^*\phi(1)y) \leq \|\phi(1)\|\tau(y^*y).$$

In particular, if $\tau \circ \phi = \tau$ and $\phi(1) = 1$, we have $\langle \xi_\phi, x\xi_\phi \rangle_\phi = \tau(x) = \langle \xi_\phi, \xi_\phi x \rangle_\phi$ for every $x \in M$.

Proposition 3.2. *Let $G \curvearrowright (X, \mu)$ be a p.m.p. action of a countable group and set $A = L^\infty(X)$, $M = L^\infty(X) \rtimes G$. Let $\varphi : G \to \mathbb{C}$ be a positive definite function. There is a unique completely positive map $\phi : M \to M$, whose restriction to the unit ball of M is w.o. continuous, such that $\phi(au_g) = \varphi(g)au_g$ for every $a \in A = L^\infty(X)$ and $g \in G$.*

Proof. Let $(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi)$ be the GNS construction. Let τ be the canonical trace on M. Recall that $L^2(M, \tau)$ is canonically isomorphic to $\ell^2(G, L^2(X)) = L^2(X) \otimes \ell^2(G)$.

We define a structure of M-M-bimodule on $L^2(M, \tau) \otimes \mathcal{H}_\varphi$ by

$$au_g(\eta \otimes \xi) = (au_g\eta) \otimes \pi_\varphi(g)\xi, \quad (\eta \otimes \xi)m = (\eta m) \otimes \xi$$

for $\eta \in L^2(M, \tau)$, $\xi \in \mathcal{H}_\varphi$, $a \in A, m \in M, g \in G$.

Let $S : L^2(M, \tau) \to L^2(M, \tau) \otimes \mathcal{H}_\varphi$ be the bounded operator defined by $S\widehat{m} = \widehat{m} \otimes \xi_\varphi$. Obviously, S commutes with right M-action and so does S^*mS. Since $M = (JMJ)'$, we get $S^*mS \in M$ for every $m \in M$. The map $\phi : m \mapsto S^*mS$ is clearly completely positive and a straightforward computation shows that $\phi(au_g) = \varphi(g)au_g$ for every $a \in A = L^\infty(X)$ and $g \in G$. $\square$

Remark 3.3. We shall use the following properties of ϕ :

(1) ϕ is A-bilinear : $\phi(a_1 m a_2) = a_1\phi(m)a_2$ for $a_1, a_2 \in A$, $m \in M$ (obvious).

(2) $\tau \circ \phi = \tau$, $\phi(1) = 1$, and $\|\phi(m)\|_2 \leq \|m\|_2$, when assuming $\varphi(e) = 1$ (direct computation plus the inequality $|\varphi(g)| \leq 1$, or use the classical inequality $\phi(m)^*\phi(m) \leq \phi(m^*m)$). It follows that there exists a unique bounded operator $T_\phi : L^2(M, \tau) \to L^2(M, \tau)$ such that $T_\phi(\widehat{m}) = \widehat{\phi(m)}$ for every $m \in M$.

(3) Write $L^2(M, \tau) = L^2(X) \otimes \ell^2(G) = \oplus_{g \in G} L^2(X) \otimes \mathbb{C}\delta_g$. Then T_ϕ is a "diagonal" operator : $T_\phi(f \otimes \delta_g) = \varphi(g) f \otimes \delta_g$ for $f \in L^2(X)$ and $g \in G$.

3.1. The Haagerup property.

Definition 3.4. A countable group G is said to have the *Haagerup property* (or *property* (H)) if there exists a sequence (φ_n) of positive definite functions on G such that $\varphi_n \in c_0(G)$ for every n and $\lim_n \varphi_n = 1$ pointwise.[14]

Examples 3.5. Obviously, amenable groups have property (H).

[14] $c_0(G)$ denotes the algebra of complex-valued functions on G, vanishing to 0 at infinity.

Free groups $\mathbb{F}_k$ with $k \geq 2$ generators are the most basic examples of non amenable groups with property (H). Indeed, let us denote by ℓ the word length function on $\mathbb{F}_k$. Then Haagerup has proved [11] that for $t > 0$ the function $g \mapsto \exp(-t\ell(g))$ is positive definite. Therefore, in order to show property (H), it suffices to take $g \mapsto \varphi_n(g) = \exp(-(1/n)\ell(g))$.

Remark 3.6. Let G and (φ_n) as above. Let $G \curvearrowright (X, \mu)$ be a p.m.p. action. Consider the sequence (ϕ_n) of completely positive maps as defined in Proposition 3.2, from $M = L^\infty(X) \rtimes G$ to itself. Then, for every $m \in M$, we have $\lim_n \|\phi_n(m) - m\|_2 = 0$. This is obvious, since for $a \in L^\infty(X)$ and $g \in G$, we have

$$\|\phi_n(au_g) - au_g\|_2 = \|(\varphi_n(g) - 1)au_g\|_2 = |\varphi_n(g) - 1|\|a\|_2.$$

3.2. Relative property (T).

Definition 3.7. Let H be a subgroup of a countable group G. We say that the pair (G, H) has the *relative property* (T) (or that $H \subset G$ is a *rigid embedding*) if any sequence of positive definite functions on G that converges pointwise to the constant function 1 converges uniformly on H.

Whenever the pair (G, G) has the relative property (T) we say that G has the *property* (T).

Proposition 3.8. *The folllowing conditions are equivalent:*

(i) *the pair (G, H) has the relative property (T) ;*
(ii) *for every $\varepsilon > 0$, there exist a finite subset F of G and $\delta > 0$ such that if $(\pi, \mathcal{H})$ is a unitary representation of G and $\xi \in \mathcal{H}$ is a unit vector satisfying $\max_{g \in F} \|\pi(g)\xi - \xi\| \leq \delta$,*[15] *then there is a H-invariant vector $\eta \in \mathcal{H}$ with $\|\xi - \eta\| \leq \varepsilon$.*

Proof. Given the positive definite function $\varphi : g \mapsto \langle \xi, \pi(g)\xi \rangle$ on G with $\varphi(e) = 1$, we easily see that

$$|\varphi(g) - 1|^2 \leq \|\pi(g)\xi - \xi\|^2 \leq 2|\varphi(g) - 1|.$$

Then the equivalence between (i) and (ii) follows easily (see the proof of (b2) $\Leftrightarrow$ (b3) in [12, Theorem 1.2]). $\square$

Property (T) was introduced by Kazhdan in [13]. Relative property (T) is implicit in this paper, and explicit in [15]. The most basic example of pair with the relative property (T) is $(\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}), \mathbb{Z}^2)$. This is used to show that the groups $SL(n, \mathbb{Z})$, $n \geq 3$, have the property (T).[16] For a nice proof (due to Shalom) of these facts, we refer to the book [2, Theorems 12.1.10, 12.1.14].

Observe that an (infinite) countable group cannot have both properties (H) and (T).

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[15] then we say that ξ is (F, δ)-invariant

[16] $SL(2, \mathbb{Z})$ has the Haagerup property.

Theorem 3.9. *Let H be a subgroup of a countable group G. We set $M = L(G)$ and $A = L(H)$. The following conditions are equivalent :*

(i) *the pair (G, H) has the relative property (T) ;*

(ii) *for every $\varepsilon > 0$, there exist a finite subset F of M and $\delta > 0$ such that for any M-M-bimodule $\mathcal{H}$ and any vector $\xi \in \mathcal{H}$ satisfying*

$$\forall x \in M, \ \langle \xi, x\xi \rangle = \tau(x) = \langle \xi, \xi x \rangle, \quad \text{and} \quad \max_{x \in F} \|x\xi - \xi x\| \leq \delta,$$

there exists a vector $\eta \in \mathcal{H}$ such that $\|\xi - \eta\| \leq \varepsilon$ and $a\eta = \eta a$ for $a \in A$.[17]

(iii) *for every $\varepsilon > 0$, there exist a finite subset F of M and $\delta > 0$ such that for every completely positive map $\phi : M \to M$ satisfying $\tau \circ \phi = \tau$, $\phi(1) = 1$ and $\max_{x \in F} \|\phi(x) - x\|_2 \leq \delta$, then $\sup_{\{a \in A, \|a\| \leq 1\}} \|\phi(a) - a\|_2 \leq \varepsilon$.*

Proof. (i) $\Rightarrow$ (ii). Given $\varepsilon > 0$, we choose F and δ as in Condition (ii) of Proposition 3.8. Let $\mathcal{H}$ be a M-M-bimodule with a vector ξ satisfying

$$\forall x \in M, \ \langle \xi, x\xi \rangle = \tau(x) = \langle \xi, \xi x \rangle \quad \text{and} \quad \max_{g \in F} \|u_g \xi - \xi u_g\| \leq \delta.$$

This vector is (F, δ)-invariant for the the representation π defined by $\pi(g)(\eta) = u_g \eta u_g^*$ for $g \in G$, $\eta \in \mathcal{H}$. Therefore, there is a H-invariant vector η with $\|\xi - \eta\| \leq \varepsilon$ and η is obviously A-central.

(ii) $\Rightarrow$ (iii). Let $\varepsilon' > 0$ be given and set $\varepsilon = (\varepsilon')^2/4$. For this ε, consider F and δ satisfying condition (ii). Let $\phi : M \to M$ be a completely positive map such that $\tau \circ \phi = \tau$, $\phi(1) = 1$ and, for $x \in F$,

$$\|\phi(x) - x\|_2 \leq \delta',$$

with $2\delta' \max_{x \in F} \|x\|_2 = \delta^2$. Let $(\mathcal{H}(\phi), \xi_\phi)$ be the M-M-bimodule associated with ϕ. For $x \in F$, we have

$$\begin{aligned} \|x\xi_\phi - \xi_\phi x\|^2 &= \|x\xi_\phi\|^2 + \|\xi_\phi x\|^2 - 2\Re\langle x\xi_\phi, \xi_\phi x \rangle \\ &= 2\|x\|_2^2 - 2\Re\tau(\phi(x)x^*) \\ &\leq 2\|\phi(x) - x\|_2 \|x\|_2 \leq \delta^2. \end{aligned}$$

Therefore, there exists a A-central vector $\eta \in \mathcal{H}$ with $\|\xi_\phi - \eta\| \leq \varepsilon$. Then, for a in the unit ball of A we get

$$\begin{aligned} \|\phi(a) - a\|_2^2 &= \|\phi(a)\|_2^2 + \|a\|_2^2 - 2\Re\langle \phi(a), a \rangle \\ &\leq 2\Re\langle a\xi_\phi, a\xi_\phi - \xi_\phi a \rangle \\ &\leq 2\|a\xi_\phi\| \|a\xi_\phi - \xi_\phi a\| = 2\|a\xi_\phi\| \|a(\xi_\phi - \eta) - (\xi_\phi - \eta)a\| \\ &\leq 4\|a\|_2 \|a\|_\infty \|\xi_\phi - \eta\| \leq 4\varepsilon = (\varepsilon')^2. \end{aligned}$$

$\square$

[17] We say that η is A-central.

Definition 3.10. We say that A is *rigidly embedded* in M, or that the pair (M, A) has the *relative property* (T) if any of the two equivalent conditions (ii) and (iii) of the previous theorem is satisfied.

It follows from Theorem 3.9 that if G is an ICC group with the Haagerup property (such as $\mathbb{F}_n$, $n \geq 2$, or $PSL(2,\mathbb{Z})$), the type II_1 factor $L(G)$ is never isomorphic two any type II_1 factor M such that (M, M) has the relative property (T).[18] In particular $L(PSL(2,\mathbb{Z}))$ is not isomorphic to $L(PSL(n,\mathbb{Z}))$ for $n \geq 3$.

Property (T) for type II_1 factors was introduced by Connes and Jones [5]. The extension to the relative case is due to Popa [19] as well as the previous theorem and all what follows.

Theorem 3.11 ([19])**.** *In $M = L(\mathbb{Z}^2) \rtimes \mathbb{F}_2$, any two rigidly embedded Cartan subalgebras A_1, A_2 are unitary conjugate, that is, there exists a unitary operator $u \in M$ such that $uA_1u^* = A_2$.*

Sketch of the proof. We set $A = L(\mathbb{Z}^2) = L^\infty(\mathbb{T}^2)$. Let B be a rigidly embedded Cartan subalgebra of M. We want to show the existence of a unitary operator $u \in M$ with $u^*Bu = A$. Assume that this is the case. Then $\mathcal{H} = uL^2(A) = \widehat{uA}$ is a B-A-subbimodule of $L^2(M)$ which has the particularity of being finite dimensional as a right A-module (in fact generated by one element here).

Conversely, Popa has shown that whenever $L^2(M)$ has a B-A-subbimodule which is finite dimensional as a right A-module, then B and A are inner conjugate. We admit this general fact which holds for any pair of Cartan subalgebras.

So, our strategy is to prove the existence of such a B-A-subbimodule $\mathcal{H}$ of $L^2(M)$. Since $\mathbb{F}_2$ has the Haagerup property, there exists a sequence (ϕ_n) of completely positive maps from M to M such that $\lim_n \|\phi_n(x) - x\|_2 = 0$ for every $x \in M$, and such that the ϕ_n possess the properties stated in Remark 3.3. Each ϕ_n comes from a positive definite function φ_n. Since B is rigidly embedded in M, there is such a $\phi = \phi_{n_0}$ associated with the positive definite function $\varphi = \varphi_{n_0}$ (assumed to satisfy $\varphi(e) = 1$) with

$$\sup_{\{b \in B : \|b\| \leq 1\}} \|\phi(b) - b\|_2 \leq 1/3.$$

Let us now look at the contraction $T = T_\phi : L^2(M) \to L^2(M)$ associated with ϕ. Since ϕ is A-bilinear, T commutes with the left and right actions of A on $L^2(M)$. The von Neumann algebra $JA'J$ plays a crucial role in Jones' theory of subfactors. Let us denote by e_A the orthogonal projection from $L^2(M)$ onto $L^2(A) = L^2(X)$. The algebra $JA'J$ carries a trace Tr (not everywhere defined) such that $\mathrm{Tr}(xe_Ay) = \tau(xy)$ for every $x, y \in M$ (when $A = \mathbb{C}1$, then $JA'J = \mathcal{B}(L^2(M))$ and Tr is the usual trace).

[18] One says that such M has the property (T)

We set $F = \{g \in G, \varphi(g) \leq 1/3\}$ and let p be the projection from $L^2(M) = \oplus_{g\in G} L^2(A) \otimes \mathbb{C}\delta_g$ onto $\oplus_{g\in F} L^2(A) \otimes \mathbb{C}\delta_g$. We have $\|pT - T\| \leq 1/3$. Moreover, $p = \sum_{g\in F} u_g e_A u_g^*$, so $p \in JA'J$ and $\mathrm{Tr}(p) = |F| < +\infty$. We want to replace p by a projection $q \in JA'J \cap B'$, with still $\mathrm{Tr} q < \infty$. Then $\mathcal{H} = qL^2(M)$ will be a B-A-subbimodule of $L^2(M)$, finite dimensional as a right A-module.

Let w be a unitary element of B. Then we have

$$\begin{aligned}\|\widehat{w} - p\widehat{w}\|_2 &\leq \|(1-p)T\widehat{w}\|_2 + \|(1-p)(T\widehat{w} - \widehat{w}\|_2 \\ &\leq 1/3 + \|\phi(w) - w\|_2 \leq 2/3.\end{aligned}$$

Thus we have $\left\|\widehat{1} - wpw^*\widehat{1}\right\|_2 \leq 2/3$ for every unitary $w \in B$.

We consider the w.o. closure K of the convex hull of $\{wpw^* : w \in \mathcal{U}(B)\}$. It is w.o. compact and contained in

$$\big\{x \in (JA'J)_+ : \mathrm{Tr}(x) \leq \mathrm{Tr}(p), \|x\| \leq 1\big\}.$$

In the same way as we defined $L^2(M,\tau)$, we define the Hilbert space $L^2(JA'J, \mathrm{Tr})$ as the completion of the space $\{x \in JA'J : \mathrm{Tr}(x^*x) < \infty\}$ with respect to the hilbertian norm $\|x\|_{2,\mathrm{Tr}} = (\mathrm{Tr}(x^*x))^{1/2}$. Then K embeds as a weakly closed, bounded convex subset of $L^2(JA'J, \mathrm{Tr})$. Let $d \in K$ be the element of minimal $\|\cdot\|_{2,\mathrm{Tr}}$ norm. Being unique, we have $wdw^* = d$ for every $w \in \mathcal{U}(B)$, so that $d \in JA'J \cap B'$. Note that $d \neq 0$ since $\left\|\widehat{1} - d\widehat{1}\right\|_2 \leq 2/3$. Finally, we choose for q a non-zero spectral projection of d corresponding to an interval of the form $[c, \|d\|]$ with $c > 0$. $\square$

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