The Haagerup property for discrete measured groupoids

Claire Anantharaman-Delaroche

Abstract We define the Haagerup property in the general context of countable groupoids equipped with a quasi-invariant measure. One of our objectives is to complete an article of Jolissaint devoted to the study of this property for probability measure preserving countable equivalence relations. Our second goal, concerning the general situation, is to provide a definition of this property in purely geometric terms, whereas this notion had been introduced by Ueda in terms of the associated inclusion of von Neumann algebras. Our equivalent definition makes obvious the fact that treeability implies the Haagerup property for such groupoids and that it is not compatible with Kazhdan property (T).

1 Introduction

Since the seminal paper of Haagerup [14], showing that free groups have the (now so-called) Haagerup property, or property (H), this notion plays an increasingly important role in group theory (see the book [10]). A similar property (H) has been introduced for finite von Neumann algebras [12, 11] and it was proved in [11] that a countable group Γ has property (H) if and only if its von Neumann algebra $L(\Gamma)$ has property (H).

Later, given a von Neumann subalgebra A of a finite von Neumann algebra M, a property (H) for M relative to A has been considered [9, 24] and proved to be very useful. It is in particular one of the crucial ingredients used by Popa [24], to provide the first example of a II_1 factor with trivial fundamental group.

A discrete (also called countable) measured groupoid (G,μ) with set of units X (see Sect. 2.1) gives rise to an inclusion $A \subset M$, where $A = L^\infty(X,\mu)$ and $M = L(G,\mu)$ is the von Neumann algebra of the groupoid. This inclusion is canonically equipped with a conditional expectation $E_A : M \to A$. Although M is not always a

Laboratoire MAPMO - UMR6628, Fédération Denis Poisson (FDP - FR2964), CNRS/Université d'Orléans, B. P. 6759, F-45067 Orléans Cedex 2, e-mail: claire.anantharaman@univ-orleans.fr

finite von Neumann algebra, there is still a notion of property (H) relative to A and E_A (see [31]). However, to our knowledge, this property has not been translated in terms only involving (G,μ), as in the group case. A significant exception concerns the case where $G=\mathscr{R}$ is a countable equivalence relation on X, preserving the probability measure μ, *i.e.* a type II_1 equivalence relation [17].

Our first goal is to extend the work of Jolissaint [17] in order to cover the general case of countable measured groupoids, and in particular the case of group actions leaving a probability measure quasi-invariant. Although it is not difficult to guess the right definition of property (H) for (G,μ) (see Definition 6.2), it is more intricate to prove the equivalence of this notion with the fact that $L(G,\mu)$ has property (H) relative to $L^\infty(X,\mu)$.

We begin in Sect. 2 by introducing the basic notions and notation relative to countable measured groupoids. In particular we discuss the Tomita-Takesaki theory for their von Neumann algebras. This is essentially a reformulation of the pioneering results of P. Hahn [15] in a way that fits better for our purpose. In Section 3 we discuss in detail several facts about the von Neumann algebra of the Jones' basic construction for an inclusion $A\subset M$ of von Neumann algebras, assuming that A is abelian. We also recall here the notion of relative property (H) in this setting.

In Sects. 4 and 5, we study the relations between positive definite functions on our groupoids and completely positive maps on the corresponding von Neumann algebras. These results are extensions of well known results for groups and of results obtained by Jolissaint in [17] for equivalence relations, but additional difficulties must be overcome. After this preliminary work, it is immediate (Sect. 6) to show the equivalence of our definition of property (H) for groupoids with the definition involving operator algebras (Theorem 6.1).

Our main motivation originates from the reading of Ueda's paper [31] and concerns treeable groupoids. This notion was introduced by Adams for probability measure preserving countable equivalence relations [1]. Treeable groupoids may be viewed as the groupoid analogue of free groups. So a natural question, raised by C.C. Moore in his survey [21, p. 277] is whether a treeable equivalence relation must have the Haagerup property. In fact, this problem is solved in [31] using operator algebras techniques. In Ueda's paper, the notion of treeing is translated in an operator algebra framework regarding the inclusion $L^\infty(X,\mu)\subset L(G,\mu)$, and it is proved that this condition implies that $L(G,\mu)$ has the Haagerup property relative to $L^\infty(X,\mu)$.

Our approach is opposite. For us, it seems more natural to compare these two notions, treeability and property (H), purely at the level of the groupoid. Indeed, the definition of treeability is more nicely read at this level: roughly speaking, it means that there is a measurable way to endow each fibre of the groupoid with a structure of tree (see Definition 7.2). The direct proof that treeability implies property (H) is given in Sect. 7 (Theorem 7.3).

Using our previous work [6] on groupoids with property (T), we prove in Sect. 8 that, under an assumption of ergodicity, this property is incompatible with the Haagerup property (Theorem 8.2). As a consequence, we recover the result of Jolissaint [17, Proposition 3.2] stating that if Γ is a Kazhdan countable group which acts ergodically on a Lebesgue space (X,μ) and leaves the probability measure μ invariant, then the orbit equivalence relation $(\mathscr{R}_\Gamma,\mu)$ has not the Haagerup property (Corollary 8.4). A fortiori, $(\mathscr{R}_\Gamma,\mu)$ is not treeable, a result due to Adams and Spatzier [2, Theorem 18] and recovered in a different way by Ueda.

This text is an excerpt from the survey [7] which is not intended to publication.

2 The von Neumann algebra of a measured groupoid

2.1 Preliminaries on countable measured groupoids

Our references for measured groupoids are [8, 15, 26]. Let us first introduce some notation. Given a groupoid G, we denote by $G^{(0)}$ its unit space and by $G^{(2)}$ the set of composable pairs. The range and source maps from G to $G^{(0)}$ are denoted respectively by r and s. The corresponding fibres are denoted respectively by $G^x = r^{-1}(x)$ and $G_x = s^{-1}(x)$. Given a subset A of $G^{(0)}$, the *reduction* of G to A is the groupoid $G|_A = r^{-1}(A)\cap s^{-1}(A)$. Two elements x,y of $G^{(0)}$ are said to be equivalent if $G^x\cap G_y \neq \emptyset$. We denote by $\mathscr{R}_G$ this equivalence relation. Given $A\subset G^{(0)}$, its *saturation* $[A]$ is the set $s(r^{-1}(A))$ of all elements in $G^{(0)}$ that are equivalent to some element of A. When $A=[A]$, we say that A is *invariant*.

A *Borel groupoid* is a groupoid G endowed with a standard Borel structure such that the range, source, inverse and product are Borel maps, where $G^{(2)}$ has the Borel structure induced by $G\times G$ and $G^{(0)}$ has the Borel structure induced by G. We say that G is *countable* (or *discrete*) if the fibres G^x (or equivalently G_x) are countable.

In the sequel, we only consider such groupoids. We *always denote by X the set* $G^{(0)}$ *of units of* G. A *bisection S* is a Borel subset of G such that the restrictions of r and s to S are injective. A useful fact, consequence of a theorem of Lusin-Novikov, states that, since r and s are countable-to-one Borel maps between standard Borel spaces, there exists a countable partition of G into bisections (see [19, Theorem 18.10]).

Let μ be a probability measure on on $X=G^{(0)}$. We define a σ-finite measure ν on G by the formula

$$\int_G F\,\mathrm{d}\nu = \int_X \Big(\sum_{s(\gamma)=x} F(\gamma)\Big)\,\mathrm{d}\mu(x).$$

We say that μ is *quasi-invariant* if ν is equivalent to it image ν^{-1} under $\gamma\mapsto\gamma^{-1}$. In other terms, for every bisection S, one has $\mu(s(S))=0$ if and only if $\mu(r(S))=0$. This notion only depends on the measure class of μ. We set $\delta = \frac{\mathrm{d}\nu^{-1}}{\mathrm{d}\nu}$. Whenever $\nu=\nu^{-1}$, we say that μ is *invariant*.

Definition 2.1 A *countable (or discrete) measured groupoid*[1] (G,μ) is a countable Borel groupoid G with a quasi-invariant probability measure μ on $X = G^{(0)}$.

Examples 2.2 (a) Let $\Gamma \curvearrowright X$ be a (right) action of a countable group Γ on a standard Borel space X, and assume that the action preserves the class of a probability measure μ. Let $G = X \rtimes \Gamma$ be the *semi-direct product groupoid*. We have $r(x,t) = x$ and $s(x,t) = xt$. The product is given by the formula $(x,s)(xs,t) = (x,st)$. Equipped with the quasi-invariant measure μ, (G,μ) is a countable measured groupoid. As a particular case, we find the group $G = \Gamma$ when X is reduced to a point..

(b) Another important family of examples concerns *countable measured equivalence relations*. A countable Borel equivalence relation $\mathscr{R} \subset X \times X$ on a standard Borel space X is a Borel subset of $X \times X$ whose equivalence classes are finite or countable. It has an obvious structure of Borel groupoid with $r(x,y) = x$, $s(x,y) = y$ and $(x,y)(x,z) = (x,z)$. When equipped with a quasi-invariant probability measure μ, we say that $(\mathscr{R},\mu)$ is a countable measured equivalence relation. Here, quasi-invariance also means that for every Borel subset $A \subset X$, we have $\mu(A) = 0$ if and only if the measure of the saturation $s(r^{-1}(A))$ of A is still 0.

The orbit equivalence relation associated with an action $\Gamma \curvearrowright (X,\mu)$ is denoted $(\mathscr{R}_\Gamma,\mu)$.

A general groupoid is a combination of an equivalence relation and groups. Indeed, let (G,μ) be a countable measured groupoid. Let $c = (r,s)$ be the map $\gamma \mapsto (r(\gamma),s(\gamma))$ from G into $X \times X$. The range of c is the graph $\mathscr{R}_G$ of the equivalence relation induced on X by G. Moreover $(\mathscr{R}_G,\mu)$ is a countable measured equivalence relation. The kernel of the groupoid homomorphism c is the isotropy bundle.

A reduction $(G_{|U},\mu_{|U})$ such that U is conull in X is called *inessential*. Since we are working in the setting of measured spaces, it will make no difference to replace (G,μ) by any of its inessential reductions.

2.2 The von Neumann algebra of (G,μ)

If $f : G \to \mathbb{C}$ is a Borel function, we set

$$\|f\|_I = \max\left\{ \left\| x \mapsto \sum_{r(\gamma)=x} |f(\gamma)| \right\|_\infty, \left\| x \mapsto \sum_{s(\gamma)=x} |f(\gamma)| \right\|_\infty \right\}.$$

Let $I(G)$ be the set of functions such that $\|f\|_I < +\infty$. It only depends on the measure class of μ. We endow $I(G)$ with the (associative) convolution product

$$(f*g)(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g(\gamma_2) = \sum_{s(\gamma)=s(\gamma_2)} f(\gamma\gamma_2^{-1})g(\gamma_2) = \sum_{r(\gamma_1)=r(\gamma)} f(\gamma_1)g(\gamma_1^{-1}\gamma).$$

[1] In [8], a countable measured groupoid is called r-discrete. Another difference is that we have swapped here the definitions of ν and ν^{-1}.

and the involution $f^*(\gamma) = \overline{f(\gamma^{-1})}$.

We have $I(G) \subset L^1(G,\nu) \cap L^\infty(G,\nu) \subset L^2(G,\nu)$, with $\|f\|_1 \leq \|f\|_I$. Therefore $\|\cdot\|_I$ is a norm on $I(G)$, where two functions which coincide ν-almost everywhere are identified. It is easily checked that $I(G)$ is complete for this norm. Moreover for $f,\, g \in I(G)$ we have $\|f * g\|_I \leq \|f\|_I \|g\|_I$. Therefore $(I(G), \|\cdot\|_I)$ is a Banach $*$-algebra.

This variant of the Banach algebra $I(G)$ introduced by Hahn [15] has been considered by Renault in [28, p. 50]. Its advantage is that it does not involve the Radon-Nikodym derivative δ.

For $f \in I(G)$ we define a bounded operator $L(f)$ on $L^2(G,\nu)$ by

$$(L(f)\xi)(\gamma) = (f * \xi)(\gamma) = \sum_{\gamma_1 \gamma_2 = \gamma} f(\gamma_1)\xi(\gamma_2). \tag{2.1}$$

We have $\|L(f)\| \leq \|f\|_I$, $L(f)^* = L(f^*)$ and $L(f)L(g) = L(f * g)$. Hence, L is a representation of $I(G)$, called the *left regular representation*.

Definition 2.3 The *von Neumann algebra of the countable measured groupoid* (G,μ) is the von Neumann subalgebra $L(G,\mu)$ of $\mathscr{B}(L^2(G,\nu))$ generated by $L(I(G))$. It will also be denoted by M in the rest of the paper.

Note that $L^2(G,\nu)$ is a direct integral of Hilbert spaces :

$$L^2(G,\nu) = \int_X^{\oplus} \ell^2(G_x)\, \mathrm{d}\mu(x).$$

We define on $L^2(G,\nu)$ a structure of $L^\infty(X)$-module by $(f\xi)(\gamma) = f \circ s(\gamma)\xi(\gamma)$, where $f \in L^\infty(X)$ and $\xi \in L^2(G,\nu)$. In fact $L^\infty(X)$ is the algebra of diagonalizable operators with respect to the above disintegration of $L^2(G,\nu)$.

Obviously, the representation L commutes with this action of $L^\infty(X)$. It follows that the elements of $L(G,\mu)$ are decomposable operators ([13, Theorem 1, p. 164]). We have $L(f) = \int_X^{\oplus} L_x(f)\, \mathrm{d}\mu(x)$, where $L_x(f) : \ell^2(G_x) \to \ell^2(G_x)$ is defined as in (2.1), but for $\xi \in \ell^2(G_x)$.

Let $C_n = \{1/n \leq \delta \leq n\}$. Then (C_n) is an increasing sequence of measurable subsets of G with $\cup_n C_n = G$ (up to null sets). We denote by $I_n(G)$ the set of elements in $I(G)$ taking value 0 outside C_n and we set $I_\infty(G) = \cup_n I_n(G)$. Obviously, $I_\infty(G)$ is an involutive subalgebra of $I(G)$. It is easily checked that $I_\infty(G)$ is dense into $L^2(G,\nu)$ and that $L(G,\mu)$ is generated by $L(I_\infty(G))$.

The von Neumann algebra $L^\infty(X)$ is isomorphic to a subalgebra of $I_\infty(G)$, by giving to $f \in L^\infty(X)$ the value 0 outside $X \subset G$. Note that, for $\xi \in L^2(G,\nu)$,

$$(L(f)\xi)(\gamma) = f \circ r(\gamma)\xi(\gamma).$$

In this way, $A = L^\infty(X)$ appears as a von Neumann subalgebra of M.

Obviously, the pair $A \subset M$ only depends on the measure class of μ, up to unitary equivalence.

We view $I(G)$ as a subspace of $L^2(G,\nu)$. The characteristic function $\mathbf{1}_X$ of $X \subset G$ is a norm one vector in $L^2(G,\nu)$. Let φ be the normal state on M defined by

$$\varphi(T) = \langle \mathbf{1}_X, T\mathbf{1}_X \rangle_{L^2(G,\nu)}.$$

For $f \in I(G)$, we have $\varphi(L(f)) = \int_X f(x)\,\mathrm{d}\mu(x)$, and therefore, for $f,g \in I(G)$,

$$\varphi(L(f)^* L(g)) = \langle f, g \rangle_{L^2(G,\nu)}. \tag{2.2}$$

Lemma 2.4 *Let g be a Borel function on G such that $\delta^{-1/2} g = f \in I(G)$ (for instance $g \in I_\infty(G)$). Then $\xi \mapsto \xi * g$ is a bounded operator on $L^2(G,\nu)$. More precisely, we have*

$$\|\xi * g\|_2 \le \|f\|_I \|\xi\|_2.$$

Proof. Straightforward. □

We set $R(g)(\xi) = \xi * g$. We have $L(f) \circ R(g) = R(g) \circ L(f)$ for every $g \in I_\infty(G)$ and $f \in I(G)$. We denote by $R(G,\mu)$ the von Neumann algebra generated by $R(I_\infty(G))$.

Lemma 2.5 *The vector $\mathbf{1}_X$ is cyclic and separating for $L(G,\mu)$, and therefore φ is a faithful state.*

Proof. Immediate from the fact that $L(f)$ and $R(g)$ commute for $f,g \in I_\infty(G)$, with $L(f)\mathbf{1}_X = f$ and $R(g)\mathbf{1}_X = g$, and from the density of $I_\infty(G)$ into $L^2(G,\nu)$. □

The von Neumann algebra $L(G,\mu)$ is on standard form on $L^2(G,\nu)$, canonically identified with $L^2(M,\varphi)$ (see (2.2)). We identify M with a dense subspace of $L^2(G,\nu)$ by $T \mapsto \widehat{T} = T(\mathbf{1}_X)$. The modular conjugation J and the one-parameter modular group σ relative to the vector $\mathbf{1}_X$ (and φ) have been computed in [15]. With our notations, we have

$$\forall \xi \in L^2(G,\nu), \quad (J\xi)(\gamma) = \delta(\gamma)^{1/2}\overline{\xi(\gamma^{-1})}$$

and

$$\forall T \in L(G,\mu), \quad \sigma_t(T) = \delta^{it} T \delta^{-it}.$$

Here, for $t \in \mathbb{R}$, the function δ^{it} acts on $L^2(G,\nu)$ by pointwise multiplication and defines a unitary operator. Note that for $f \in L(G,\mu)$, we have $\delta^{it} L(f)\delta^{-it} = L(\delta^{it} f)$. In particular, σ acts trivially on A. Therefore (see [30]), there exists a unique faithful conditional expectation $E_A : M \to A$ such that $\varphi = \varphi \circ E_A$, and for $T \in M$, we have

$$\widehat{E_A(T)} = e_A(\widehat{T}),$$

where e_A is the orthogonal projection from $L^2(G,\nu)$ onto $L^2(X,\mu)$. If we view the elements of M as functions on G, then E_A is the restriction map to X. The triple (M,A,E_A) only depends on the class of μ, up to equivalence.

For $f \in I(G)$ and $\xi \in L^2(G,\nu)$ we observe that

$$(JL(f)J)\xi = R(g)\xi = \xi * g \quad \text{with} \quad g = \delta^{1/2} f^*. \tag{2.3}$$

3 Basic facts on the module $L^2(M)_A$

We consider, in an abstract setting, the situation we have met above. Let $A \subset M$ be a pair of von Neumann algebras, where $A = L^\infty(X,\mu)$ is abelian. We assume the existence of a normal faithful conditional expectation $E_A : M \to A$ and we set $\varphi = \tau_\mu \circ E_A$, where τ_μ is the state on A defined by the probability measure μ. Recall that M is on standard form on the Hilbert space $L^2(M,\varphi)$ of the Gelfand-Naimark-Segal construction associated with φ. We view $L^2(M,\varphi)$ as a left M-module and a right A-module. Identifying[2] M with a subspace of $L^2(M,\varphi)$, we know that E_A is the restriction to M of the orthogonal projection $e_A : L^2(M,\varphi) \to L^2(A,\tau_\mu)$.

For further use, we make the following observation

$$\forall m \in M, \forall a \in A, \quad \widehat{m}a = Ja^*J\widehat{m} = \widehat{ma} = m\widehat{a}. \tag{3.4}$$

Indeed, if S is the closure of the map $\widehat{m} \mapsto \widehat{m^*}$ and if $S = J\Delta^{1/2} = \Delta^{-1/2}J$ is its polar decomposition, then every $a \in A$ commutes with Δ since it is invariant under σ^φ. Then (3.4) follows easily. Note that (2.3) gives a particular case of this remark.

3.1 The commutant $\langle M, e_A\rangle$ of the right action

The algebra of all operators which commute with the right action of A is the von Neumann algebra of the basic construction for $A \subset M$. It is denoted $\langle M, e_A\rangle$ since it is generated by M and e_A. The linear span of $\{m_1 e_A m_2 : m_1, m_2 \in M\}$ is a $*$-subalgebra which is weak operator dense in $\langle M, e_A\rangle$. Moreover $\langle M, e_A\rangle$ is a semi-finite von Neumann algebra, carrying a canonical normal faithful semi-finite trace Tr_μ (depending on the choice of μ), defined by

$$\mathrm{Tr}_\mu(m_1 e_A m_2) = \int_X E_A(m_2 m_1)\,\mathrm{d}\mu = \varphi(m_2 m_1).$$

(for these classical results, see [18], [23]). We shall give more information on this trace in Lemma 3.4 and its proof. We need some preliminaries.

Definition 3.1 A vector $\xi \in L^2(M,\varphi)$ is A-*bounded* if there exists $c > 0$ such that $\|\xi a\|_2 \le c\tau_\mu(a^*a)^{1/2}$ for every $a \in A$.

We denote by $L^2(M,\varphi)^0$, or $\mathscr{L}^2(M,\varphi)$, the subspace of A-bounded vectors. It contains M. We also recall the obvious fact that $T \mapsto T(1_A)$ is an isomorphism

[2] When necessary, we shall write $\widehat{m}$ the element $m \in M$, when viewed in $L^2(M,\varphi)$, in order to stress this fact.

from the space $\mathscr{B}(L^2(A,\tau_\mu)_A, L^2(M,\varphi)_A)$ of bounded (right) A-linear operators $T : L^2(A,\tau_\mu) \to L^2(M,\varphi)$ onto $\mathscr{L}^2(M,\varphi)$. For $\xi \in \mathscr{L}^2(M,\varphi)$, we denote by L_ξ the corresponding operator from $L^2(A,\tau_\mu)$ into $L^2(M,\varphi)$. In particular, for $m \in M$, we have $L_m = m_{|_{L^2(A,\tau_\mu)}}$. It is easy to see that $\mathscr{L}^2(M,\varphi)$ is stable under the actions of $\langle M, e_A\rangle$ and A, and that $L_{T\xi a} = T \circ L_\xi \circ a$ for $T \in \langle M, e_A\rangle$, $\xi \in \mathscr{L}^2(M,\varphi)$, $a \in A$.

For $\xi,\eta \in \mathscr{L}^2(M,\varphi)$, the operator $L_\xi^* L_\eta \in \mathscr{B}(L^2(A,\tau_\mu))$ is in A, since it commutes with A. We set $\langle \xi,\eta\rangle_A = L_\xi^* L_\eta$. In particular, we have $\langle m_1, m_2\rangle_A = E_A(m_1^* m_2)$ for $m_1, m_2 \in M$. The A-valued inner product $\langle \xi,\eta\rangle_A = L_\xi^* L_\eta$ gives to $\mathscr{L}^2(M,\varphi)$ the structure of a self-dual Hilbert right A-module [22]. It is a normed space with respect to the norm $\|\xi\|_{\mathscr{L}^2(M)} = \|\langle \xi,\xi\rangle_A\|_A^{1/2}$. Note that

$$\|\xi\|_{L^2(M)}^2 = \tau_\mu(\langle \xi,\xi\rangle_A) \le \|\xi\|_{\mathscr{L}^2(M)}^2.$$

On the algebraic tensor product $\mathscr{L}^2(M,\varphi) \odot L^2(A)$ a positive hermitian form is defined by

$$\langle \xi \otimes f, \eta \otimes g\rangle = \int_X \overline{f} g \langle \xi,\eta\rangle_A \,\mathrm{d}\mu.$$

The Hilbert space $\mathscr{L}^2(M,\varphi) \otimes_A L^2(A)$ obtained by separation and completion is isomorphic to $L^2(M,\varphi)$ as a right A-module by $\xi \otimes f \mapsto \xi f$. Moreover the von Neumann algebra $\mathscr{B}(\mathscr{L}^2(M,\varphi)_A)$ of bounded A-linear endomorphisms of $\mathscr{L}^2(M,\varphi)$ is isomorphic to $\langle M, e_A\rangle$ by $T \mapsto T \otimes 1$. We shall identify these two von Neumann algebras (see [22], [29] for details on these facts).

Definition 3.2 An *orthonormal basis* of the A-module $L^2(M,\varphi)$ is a family (ξ_i) of elements of $\mathscr{L}^2(M,\varphi)$ such that $\overline{\sum_i \xi_i A} = L^2(M,\varphi)$ and $\left\langle \xi_i, \xi_j \right\rangle_A = \delta_{i,j} p_j$ for all i, j, where the p_j are projections in A.

It is easily checked that $L_{\xi_i} L_{\xi_i}^*$ is the orthogonal projection on $\overline{\xi_i A}$, and that these projections are mutually orthogonal with $\sum_i L_{\xi_i} L_{\xi_i}^* = 1$.

Using a generalization of the Gram-Schmidt orthonormalization process, one shows the existence of orthonormal bases (see [22]).

Lemma 3.3 *Let (ξ_i) be an orthonormal basis of the A-module $L^2(M,\varphi)$. For every $\xi \in \mathscr{L}^2(M,\varphi)$, we have (weak* convergence)*

$$\langle \xi,\xi\rangle_A = \sum_i \langle \xi,\xi_i\rangle_A \langle \xi_i,\xi\rangle_A. \tag{3.5}$$

Proof. Indeed

$$\langle \xi,\xi\rangle_A = L_\xi^* L_\xi = L_\xi^* (\sum_i L_{\xi_i} L_{\xi_i}^*) L_\xi = \sum_i (L_\xi^* L_{\xi_i})(L_{\xi_i}^* L_\xi) = \sum_i \langle \xi,\xi_i\rangle_A \langle \xi_i,\xi\rangle_A.$$

□

Lemma 3.4 *Let $(\xi_i)_{i\in I}$ be an orthonormal basis of the A-module $L^2(M,\varphi)$.*

(i) For every $x \in \langle M, e_A\rangle_+$ we have

$$\mathrm{Tr}_\mu(x) = \sum_i \tau_\mu(\langle \xi_i, x\xi_i\rangle_A) = \sum_i \langle \xi_i, x\xi_i\rangle_{L^2(M)}. \tag{3.6}$$

(ii) $\mathrm{span}\{L_\xi L_\eta^* : \xi, \eta \in \mathscr{L}^2(M,\varphi)\}$ *is contained in the ideal of definition of* Tr_μ *and we have, for $\xi, \eta \in \mathscr{L}^2(M,\varphi)$,*

$$\mathrm{Tr}_\mu(L_\xi L_\eta^*) = \tau_\mu(L_\eta^* L_\xi) = \tau_\mu(\langle \eta, \xi\rangle_A). \tag{3.7}$$

Proof. (i) The map $U : L^2(M,\varphi) = \oplus_i \overline{\xi_i A} \to \oplus_i p_i L^2(A)$ defined by $U(\xi_i a) = p_i a$ is an isomorphism which identifies $L^2(M,\varphi)$ to the submodule $p(\ell^2(I) \otimes L^2(A))$ of $\ell^2(I) \otimes L^2(A)$, with $p = \oplus_i p_i$. The canonical trace on $\langle M, e_A\rangle$ is transfered to the restriction to $p\big(\mathscr{B}(\ell^2(I))\overline{\otimes} A\big)p$ of the trace $\mathrm{Tr} \otimes \tau_\mu$, defined on $T = [T_{i,j}] \in (\mathscr{B}(\ell^2(I)\overline{\otimes} A)_+$ by $(\mathrm{Tr} \otimes \tau_\mu)(T) = \sum_i \tau_\mu(T_{ii})$. It follows that

$$\mathrm{Tr}_\mu(x) = \sum_i \tau_\mu((UxU^*)_{ii}) = \sum_i \langle \xi_i, x\xi_i\rangle_{L^2(M)} = \sum_i \tau_\mu(\langle \xi_i, x\xi_i\rangle_A).$$

(ii) Taking $x = L_\xi L_\xi^*$ in (i), the equality $\mathrm{Tr}_\mu(L_\xi L_\xi^*) = \tau_\mu(\langle \xi, \xi\rangle_A)$ follows from Equations (3.5) and (3.6). Formula (3.7) is deduced by polarization. □

3.2 Compact operators

In a semi-finite von Neumann algebra N, there is a natural notion of ideal of compact operators, namely the norm-closed ideal $\mathscr{I}(N)$ generated by its finite projections (see [24, Sect. 1.3.2] or [25]).

Concerning $N = \langle M, e_A\rangle$, there is another natural candidate for the space of compact operators. First, we observe that given $\xi, \eta \in \mathscr{L}^2(M,\varphi)$, the operator $L_\xi L_\eta^* \in \langle M, e_A\rangle$ plays the role of a rank one operator in ordinary Hilbert spaces: indeed, if $\alpha \in \mathscr{L}^2(M,\varphi)$, we have $(L_\xi L_\eta^*)(\alpha) = \xi\langle \eta, \alpha\rangle_A$. In particular, for $m_1, m_2 \in M$, we note that $m_1 e_A m_2$ is a "rank one operator" since $m_1 e_A m_2 = L_{m_1} L_{m_2^*}^*$. We denote by $\mathscr{K}(\langle M, e_A\rangle)$ the norm closure into $\langle M, e_A\rangle$ of

$$\mathrm{span}\{L_\xi L_\eta^* : \xi, \eta \in \mathscr{L}^2(M,\varphi)\}.$$

It is a two-sided ideal of $\langle M, e_A\rangle$.

For every $\xi \in \mathscr{L}^2(M,\varphi)$, we have $L_\xi e_A \in \langle M, e_A\rangle$. Since

$$L_\xi L_\eta^* = (L_\xi e_A)(L_\eta e_A)^*$$

we see that $\mathscr{K}(\langle M, e_A\rangle)$ is the norm closed two-sided ideal generated by e_A in $\langle M, e_A\rangle$. The projection e_A being finite (because $\mathrm{Tr}_\mu(e_A) = 1$), we have

$$\mathscr{K}(\langle M, e_A\rangle) \subset \mathscr{I}(\langle M, e_A\rangle).$$

The subtle difference between $\mathscr{K}(\langle M,e_A\rangle)$ and $\mathscr{I}(\langle M,e_A\rangle)$ is studied in [24, Sect. 1.3.2]. We recall in particular that for every $T\in\mathscr{I}(\langle M,e_A\rangle)$ and every $\varepsilon>0$, there is a projection $p\in A$ such that $\tau_\mu(1-p)\leq\varepsilon$ and $TJpJ\in\mathscr{K}(\langle M,e_A\rangle)$ (see [24, Proposition 1.3.3 (3)]).[3]

3.3 The relative Haagerup property

Let Φ be a unital completely positive map from M into M such that $E_A\circ\Phi=E_A$. Then for $m\in M$, we have

$$\|\Phi(m)\|_2^2=\varphi(\Phi(m)^*\Phi(m))\leq\varphi(\Phi(m^*m))=\varphi(m^*m)=\|m\|_2^2.$$

It follows that Φ extends to a contraction $\widehat{\Phi}$ of $L^2(M,\varphi)$. Whenever Φ is A-bimodular, $\widehat{\Phi}$ commutes with the right action of A (due to (3.4)) and so belongs to $\langle M,e_A\rangle$. It also commutes with the left action of A and so belongs to $A'\cap\langle M,e_A\rangle$.

Definition 3.5 We say that M has the *Haagerup property* (*or property* (H)) *relative to A and E_A* if there exists a net (Φ_i) of unital A-bimodular completely positive maps from M to M such that

(i) $E_A\circ\Phi_i=E_A$ for all i ;
(ii) $\widehat{\Phi_i}\in\mathscr{K}(\langle M,e_A\rangle)$ for all i ;
(iii) $\lim_i\|\Phi_i(x)-x\|_2=0$ for every $x\in M$.

This notion is due to Boca [9]. In [24], Popa uses a slightly different formulation.

Lemma 3.6 *In the previous definition, we may equivalently assume that $\widehat{\Phi_i}\in\mathscr{I}(\langle M,e_A\rangle)$ for every i.*

Proof. This fact is explained in [24]. Let Φ be a unital A-bimodular completely positive map from M to M such that $E_A\circ\Phi=E_A$ and $\widehat{\Phi}\in\mathscr{I}(\langle M,e_A\rangle)$. As already said, by [24, Proposition 1.3.3 (3)], for every $\varepsilon>0$, there is a projection p in A with $\tau_\mu(1-p)<\varepsilon$ and $\widehat{\Phi}JpJ\in\mathscr{K}(\langle M,e_A\rangle)$. Thus we have $p\widehat{\Phi}JpJ\in\mathscr{K}(\langle M,e_A\rangle)$. Moreover, this operator is associated with the completely positive map $\Phi_p: m\in M\mapsto\Phi(pmp)$, since

$$(p\widehat{\Phi}JpJ)(\widehat{m})=p\widehat{\Phi}(\widehat{m}p)=p\widehat{\Phi}(\widehat{mp})=p\widehat{\Phi(mp)}=\widehat{\Phi(pmp)}.$$

Then, $\Phi'=\Phi_p+(1-p)E_A$ is unital, satisfies $E_A\circ\Phi'=E_A$ and still provides an element of $\mathscr{K}(\langle M,e_A\rangle)$. This modification allows to prove that if Definition 3.5 holds with $\mathscr{K}(\langle M,e_A\rangle)$ replaced by $\mathscr{I}(\langle M,e_A\rangle)$, then the relative Haagerup property is satisfied (see [24, Proposition 2.2 (1)]). ☐

[3] In [24], $\mathscr{K}(\langle M,e_A\rangle)$ is denoted $\mathscr{I}_0(\langle M,e_A\rangle$.

3.4 Back to $L^2(G,\nu)_A$

We apply the facts just reminded to $M = L(G,\mu)$, which is on standard form on $L^2(G,\nu) = L^2(M,\varphi)$. This Hilbert space is viewed as a right A-module: for $\xi \in L^2(G,\mu)$ and $f \in A$, the action is given by $\xi f \circ s$.

It is easily seen that $\mathscr{L}^2(M,\varphi)$ is the space of $\xi \in L^2(G,\nu)$ such that $x \mapsto \sum_{s(\gamma)=x} |\xi(\gamma)|^2$ is in $L^\infty(X)$. Moreover, for $\xi,\eta \in \mathscr{L}^2(M,\varphi)$ we have

$$\langle \xi,\eta\rangle_A = \sum_{s(\gamma)=x} \overline{\xi(\gamma)}\eta(\gamma).$$

For simplicity of notation, we shall often identify $f \in I(G) \subset L^2(G,\nu)$ with the operator $L(f)$.[4] For instance, for $f,g \in I(G)$, the operator $L(f)\circ L(g)$ is also written $f*g$, and for $T \in \mathscr{B}(L^2(G,\mu))$, we write $T\circ f$ instead of $T\circ L(f)$.

Let $S \subset G$ be a bisection. Its characteristic function $\mathbf{1}_S$ is an element of $I(G)$ and a partial isometry in M since

$$\mathbf{1}_S^* * \mathbf{1}_S = \mathbf{1}_{s(S)}, \quad \text{and} \quad \mathbf{1}_S * \mathbf{1}_S^* = \mathbf{1}_{r(S)}.$$

Let $G = \sqcup S_n$ be a countable partition of G into Borel bisections. Another straightforward computation shows that $(\mathbf{1}_{S_n})_n$ is an orthonormal basis of the right A-module $L^2(M,\varphi)$.

By Lemma 3.4, for $x \in \langle M,e_A\rangle_+$ we have

$$\mathrm{Tr}_\mu(x) = \sum_n \langle \mathbf{1}_{S_n}, x\mathbf{1}_{S_n}\rangle_{L^2(M)}.$$

In particular, whenever x is the multiplication operator $\mathfrak{m}(f)$ by some bounded non-negative Borel function f, we get

$$\mathrm{Tr}_\mu(\mathfrak{m}(f)) = \int_G f\,\mathrm{d}\nu. \tag{3.8}$$

4 From completely positive maps to positive definite functions

Recall that if G is a countable group, and $\Phi : L(G) \to L(G)$ is a completely positive map, then $t \mapsto F_\Phi(t) = \tau(\Phi(u_t)u_t^*)$ is a positive definite function on G, where τ is the canonical trace on $L(G)$ and u_t, $t \in G$, are the canonical unitaries in $L(G)$. We want to extend this classical fact to the groupoid case. This was achieved by Jolissaint [17] for countable probability measure preserving equivalence relations.

Let (G,μ) be a countable measured groupoid and $M = L(G,\mu)$. Let $\Phi : M \to M$ be a normal A-bimodular unital completely positive map. Let $G = \sqcup S_n$ be a partition

[4] The reader should not confuse $L(f) : L^2(G,\nu) \to L^2(G,\nu)$ with its restriction $L_f : L^2(A,\tau_\mu) \to L^2(G,\nu)$.

into Borel bisections. We define $F_\Phi : G \to \mathbb{C}$ by

$$F_\Phi(\gamma) = E_A(\Phi(\mathbf{1}_{S_n}) \circ \mathbf{1}_{S_n}^*) \circ r(\gamma), \tag{4.9}$$

where S_n is the bisection which contains γ.

That F_Φ does not depend (up to null sets) on the choice of the partition is a consequence of the following lemma.

Lemma 4.1 *Let S_1 and S_2 be two Borel bisections. Then*

$$E_A(\Phi(\mathbf{1}_{S_1}) \circ \mathbf{1}_{S_1}^*) = E_A(\Phi(\mathbf{1}_{S_2}) \circ \mathbf{1}_{S_2}^*)$$

almost everywhere on $r(S_1 \cap S_2)$.

Proof. Denote by e the characteristic function of $r(S_1 \cap S_2)$. Then $e * \mathbf{1}_{S_1} = e * \mathbf{1}_{S_2} = \mathbf{1}_{S_1 \cap S_2}$. Thus we have

$$\begin{aligned} eE_A(\Phi(\mathbf{1}_{S_1}) \circ \mathbf{1}_{S_1}^*)e &= E_A\big(\Phi(e * \mathbf{1}_{S_1}) \circ (\mathbf{1}_{S_1}^* * e)\big) \\ &= E_A\big(\Phi(e * \mathbf{1}_{S_2}) \circ (\mathbf{1}_{S_2}^* * e) = eE_A(\Phi(\mathbf{1}_{S_2}) \circ \mathbf{1}_{S_2}^*)e. \end{aligned}$$

□

We now want to show that F_Φ is a positive definite function in the following sense. We shall need some preliminary facts.

Definition 4.2 A Borel function $F : G \to \mathbb{C}$ is said to be *positive definite* if there exists a μ-null subset N of $X = G^{(0)}$ such that for every $x \notin N$, and every $\gamma_1, \dots, \gamma_k \in G^x$, the $k \times k$ matrix $[F(\gamma_i^{-1}\gamma_j)]$ is non-negative.

Definition 4.3 We say that a *Borel bisection S is admissible* if there exists a constant $c > 0$ such that $1/c \le \delta(\gamma) \le c$ almost everywhere on S.

In other terms, $\mathbf{1}_S \in I_\infty(G)$ and so the convolution to the right by $\mathbf{1}_S$ defines a bounded operator $R(\mathbf{1}_S)$ on $L^2(M, \varphi)$, by (2.3).

Lemma 4.4 *Let S be a Borel bisection and let $T \in M$. We have $\widehat{\mathbf{1}_S \circ T} = \mathbf{1}_S * \widehat{T}$. Moreover, if S is admissible, we have $\widehat{T \circ \mathbf{1}_S} = \widehat{T} * \mathbf{1}_S$.*

Proof. First, we observe that $\widehat{\mathbf{1}_S \circ T} = \mathbf{1}_S \circ T(\mathbf{1}_X) = \mathbf{1}_S * \widehat{T}$.

On the other hand, given $f \in I(G)$, we have $L(f)(\widehat{\mathbf{1}_S}) = f * 1_S$. So, if (f_n) is a sequence in $I(G)$ such that $\lim_n L(f_n) = T$ in the strong operator topology, we have

$$\widehat{T \circ \mathbf{1}_S} = T(\widehat{\mathbf{1}_S}) = \lim_n L(f_n)(\widehat{\mathbf{1}_S}) = \lim_n f_n * \mathbf{1}_S$$

in $L^2(G, \nu)$. But, when S is admissible, the convolution to the right by $\mathbf{1}_S$ is the bounded operator $R(\mathbf{1}_S)$. Noticing that $\lim_n \left\| f_n - \widehat{T} \right\|_2 = 0$, it follows that

$$\widehat{T \circ \mathbf{1}_S} = \lim_n f_n * \mathbf{1}_S = \widehat{T} * \mathbf{1}_S.$$

□

Lemma 4.5 *Let $T \in M$, and let S be an admissible bisection. Then*

$$\mathbf{1}_S(\gamma)E_A(T\circ \mathbf{1}_S)(s(\gamma)) = \mathbf{1}_S(\gamma)E_A(\mathbf{1}_S\circ T)(r(\gamma))$$

for almost every γ.

Proof. We have

$$(\widehat{T\circ \mathbf{1}_S})(x) = \sum_{\gamma_1\gamma_2=x} \widehat{T}(\gamma_1)\mathbf{1}_S(\gamma_2) = \widehat{T}(\gamma_2^{-1})$$

whenever $x\in s(S)$, where γ_2 is the unique element of S with $s(\gamma_2)=x$. Otherwise $(T\circ \mathbf{1}_S)(x)=0$.

On the other hand,

$$(\widehat{\mathbf{1}_S\circ T})(x) = \widehat{T}(\gamma_1^{-1})$$

whenever $x\in r(S)$, where γ_1 is the unique element of S with $r(\gamma_1)=x$. Otherwise $(\widehat{\mathbf{1}_S\circ T})(x)=0$. Our statement follows immediately. □

Lemma 4.6 *F_Φ is a positive definite function.*

Proof. We assume that F_Φ is defined by equation (4.9) through a partition under admissible bisections. We set $S_{ij} = S_i^{-1}S_j = \left\{\gamma^{-1}\gamma' : \gamma\in S_i, \gamma'\in S_j\right\}$. Note that $\mathbf{1}_{S_i}^* * \mathbf{1}_{S_j} = \mathbf{1}_{S_{ij}}$. Morever, the S_{ij} are admissible bisections. We set

$$Z_{ijm} = \left\{x\in r(S_{ij}\cap S_m) : E_A(\Phi(\mathbf{1}_{S_{ij}})\circ \mathbf{1}_{S_{ij}}^*)(x) \neq E_A(\Phi(\mathbf{1}_{S_m})\circ \mathbf{1}_{S_m}^*)(x)\right\}$$

and $Z = \cup_{i,j,m} Z_{ijm}$. It is a null set by Lemma 4.1.

By Lemma 4.5, for every i there is a null set $E_i \subset r(S_i)$ such that for $\gamma\in S_i$ with $r(\gamma)\notin E_i$ and for every j, we have

$$E_A(\Phi(\mathbf{1}_{S_{ij}})\circ \mathbf{1}_{S_j}^*\circ \mathbf{1}_{S_i})(s(\gamma)) = E_A(\mathbf{1}_{S_i}\circ\Phi(\mathbf{1}_{S_{ij}})\circ \mathbf{1}_{S_j}^*)(r(\gamma))$$

We set $E=\cup_i E_i$. Let Y be the saturation of $Z\cup E$. It is a null set, since μ is quasi-invariant.

Let $x\notin Y$, and $\gamma_1,\dots,\gamma_k\in G^x$. Assume that $\gamma_i^{-1}\gamma_j \in S_{n_i}^{-1}S_{n_j}\cap S_m$. We have $r(\gamma_i^{-1}\gamma_j) = s(\gamma_i)\notin Y$ since $r(\gamma_i)=x\notin Y$. Therefore,

$$F_\Phi(\gamma_i^{-1}\gamma_j) = E_A(\Phi(\mathbf{1}_{S_{n_in_j}})\circ \mathbf{1}_{S_{n_j}}^*\circ \mathbf{1}_{S_{n_i}})(s(\gamma_i)).$$

But $\gamma_i\in S_{n_i}$ with $r(\gamma_i)=x\notin Y$, so

$$E_A(\Phi(\mathbf{1}_{S_{n_in_j}})\circ \mathbf{1}_{S_{n_j}}^*\circ \mathbf{1}_{S_{n_i}})(s(\gamma_i)) = E_A(\mathbf{1}_{S_{n_i}}\circ\Phi(\mathbf{1}_{S_{n_in_j}})\circ \mathbf{1}_{S_{n_j}}^*)(r(\gamma_i)).$$

Given $\lambda_1,\dots,\lambda_k\in\mathbb{C}$, we have

$$\sum_{i,j=1}^{k} \lambda_i \overline{\lambda_j} F_\Phi(\gamma_i^{-1}\gamma_j) = E_A\Big(\sum_{i=1}^{k}(\lambda_i \mathbf{1}_{S_{n_i}}) \circ \Phi(\mathbf{1}_{S_{n_i}}^* \circ \mathbf{1}_{S_{n_j}}) \circ \sum_{j=1}^{k}(\lambda_j \mathbf{1}_{S_{n_j}})^*\Big)(x) \geq 0.$$

□

Obviously, if Φ is unital, F_Φ takes value 1 almost everywhere on X.

Proposition 4.7 *We now assume that Φ is unital, with $E_A \circ \Phi = E_A$ and $\widehat{\Phi} \in \mathscr{K}(\langle M, e_A\rangle)$. Then, for every $\varepsilon > 0$, we have*

$$\nu(\{|F_\Phi| > \varepsilon\} < +\infty.$$

Proof. Let (S_n) be a partition of G into Borel bisections. Given $\varepsilon > 0$ we choose $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_k \in \mathscr{L}^2(M, \varphi)$ such that

$$\Big\|\widehat{\Phi} - \sum_{i=1}^{k} L_{\xi_i} L_{\eta_i}^*\Big\| \leq \varepsilon/2.$$

We view $\widehat{\Phi} - \sum_{i=1}^{k} L_{\xi_i} L_{\eta_i}^*$ as an element of $\mathscr{B}(\mathscr{L}^2(M,\varphi)_A)$ and we apply it to $\mathbf{1}_{S_n} \in \mathscr{L}^2(M,\varphi)$. Then

$$\Big\|\Phi(\mathbf{1}_{S_n}) - \sum_{i=1}^{k} \xi_i \langle \eta_i, \mathbf{1}_{S_n}\rangle_A\Big\|_{\mathscr{L}^2(M)} \leq \varepsilon/2 \|\mathbf{1}_{S_n}\|_{\mathscr{L}^2(M)} \leq \varepsilon/2.$$

Using the Cauchy-Schwarz inequality $\langle \xi, \eta\rangle_A^* \langle \xi, \eta\rangle_A \leq \|\xi\|^2_{\mathscr{L}^2(M)} \langle \eta, \eta\rangle_A$, we get

$$\Big\|\Big\langle \mathbf{1}_{S_n}, \Phi(\mathbf{1}_{S_n}) - \sum_{i=1}^{k} \xi_i \langle \eta_i, \mathbf{1}_{S_n}\rangle_A \Big\rangle_A\Big\| \leq \Big\|\Phi(\mathbf{1}_{S_n}) - \sum_{i=1}^{k} \xi_i \langle \eta_i, \mathbf{1}_{S_n}\rangle_A\Big\|_{\mathscr{L}^2(M)} \leq \varepsilon/2.$$

We have, for almost every $\gamma \in S_n$ and $x = s(\gamma)$,

$$\begin{aligned}|F_\Phi(\gamma)| &= \big|E_A(\Phi(\mathbf{1}_{S_n}) \circ \mathbf{1}_{S_n}^*)(r(\gamma))\big| = \big|E_A(\mathbf{1}_{S_n}^* \circ \Phi(\mathbf{1}_{S_n}))(x)\big| = \big|\langle \mathbf{1}_{S_n}, \Phi(\mathbf{1}_{S_n})\rangle_A(x)\big| \\ &\leq \Big|\Big\langle \mathbf{1}_{S_n}, \Phi(\mathbf{1}_{S_n}) - \sum_{i=1}^{k} \xi_i \langle \eta_i, \mathbf{1}_{S_n}\rangle_A \Big\rangle_A (x)\Big| + \sum_{i=1}^{k} \big|\langle \mathbf{1}_{S_n}, \xi_i \langle \eta_i, \mathbf{1}_{S_n}\rangle_A\rangle_A(x)\big|.\end{aligned}$$

The first term is $\leq \varepsilon/2$ for almost every $x \in s(S_n)$. As for the second term, we have, almost everywhere,

$$\big|\langle \mathbf{1}_{S_n}, \xi_i\rangle_A(x) \langle \eta_i, \mathbf{1}_{S_n}\rangle_A(x)\big| \leq \|\xi_i\|_{\mathscr{L}^2(M)} \big|\langle \eta_i, \mathbf{1}_{S_n}\rangle_A(x)\big|.$$

Hence, we get

$$|F_\Phi(\gamma)| \leq \varepsilon/2 + \sum_{i=1}^{k} \|\xi_i\|_{\mathscr{L}^2(M)} \big|\langle \eta_i, \mathbf{1}_{S_n}\rangle_A(s(\gamma))\big|$$

for almost every $\gamma \in S_n$.

We want to estimate

$$\nu(\{|F_\Phi| > \varepsilon\}) = \sum_n \nu(\{\gamma \in S_n : |F_\Phi(\gamma)| > \varepsilon\}).$$

For almost every $\gamma \in S_n$ such that $|F_\Phi(\gamma)| > \varepsilon$, we see that

$$\sum_{i=1}^{k} \|\xi_i\|_{\mathscr{L}^2(M)} \big|\langle \eta_i, \mathbf{1}_{S_n}\rangle_A (s(\gamma))\big| > \varepsilon/2.$$

Therefore

$$\begin{aligned}\nu(\{|F_\Phi| > \varepsilon\}) &\le \sum_n \nu\Big(\Big\{\gamma \in S_n : \sum_{i=1}^{k} \|\xi_i\|_{\mathscr{L}^2(M)} \big|\langle \eta_i, \mathbf{1}_{S_n}\rangle_A (s(\gamma))\big| > \varepsilon/2\Big\}\Big)\\ &\le \sum_n \mu\Big(\Big\{x \in s(S_n) : \sum_{i=1}^{k} \|\xi_i\|_{\mathscr{L}^2(M)} \big|\langle \eta_i, \mathbf{1}_{S_n}\rangle_A (x)\big| > \varepsilon/2\Big\}\Big).\end{aligned}$$

Now,

$$\sum_{i=1}^{k} \|\xi_i\|_{\mathscr{L}^2(M)} \big|\langle \eta_i, \mathbf{1}_{S_n}\rangle_A (x)\big| \le \Big(\sum_{i=1}^{k} \|\xi_i\|^2_{\mathscr{L}^2(M)}\Big)^{1/2} \Big(\sum_{i=1}^{k} \big|\langle \eta_i, \mathbf{1}_{S_n}\rangle_A (x)\big|^2\Big)^{1/2}.$$

We set $c = \sum_{i=1}^{k} \|\xi_i\|^2_{\mathscr{L}^2(M)})$ and $f_n(x) = \sum_{i=1}^{k} \big|\langle \eta_i, \mathbf{1}_{S_n}\rangle_A (x)\big|^2$. We have

$$\begin{aligned}\sum_n f_n(x) &= \sum_n \sum_{i=1}^{k} \big|\langle \eta_i, \mathbf{1}_{S_n}\rangle_A (x)\big|^2 = \sum_{i=1}^{k} \sum_n \big|\langle \eta_i, \mathbf{1}_{S_n}\rangle_A (x)\big|^2\\ &= \sum_{i=1}^{k} \langle \eta_i, \eta_i\rangle_A (x) \le \sum_{i=1}^{k} \|\eta_i\|^2_{\mathscr{L}^2(M)},\end{aligned}$$

since, by Lemma 3.3 (or directly here),

$$\langle \eta_i, \eta_i\rangle_A = \sum_k \big\langle \eta_i, \mathbf{1}_{S_k}\big\rangle_A \big\langle \mathbf{1}_{S_k}, \eta_i\big\rangle_A = \sum_k \big|\big\langle \eta_i, \mathbf{1}_{S_k}\big\rangle_A\big|^2.$$

We set $d = \sum_{i=1}^{k} \|\eta_i\|^2_{\mathscr{L}^2(M)}$.

We have

$$\nu(\{|F_\Phi| > \varepsilon\}) \le \sum_n \mu\big(\{x \in s(S_n) : c f_n(x) > (\varepsilon/2)^2\big\}.$$

We set $\alpha = c^{-1}(\varepsilon/2)^2$. Denote by $i(x)$ the number of indices n such that $f_n(x) > \alpha$. Then $i(x) \le N$, where N is the integer part of d/α. We denote by $\mathscr{P} = \{P_n\}$ the set of subsets of $\mathbb{N}$ whose cardinal is $\le N$. Then there is a partition $X = \sqcup_m B_m$ into Borel subsets such that

$$\forall x \in B_m, \quad P_m = \{n \in \mathbb{N} : f_n(x) > \alpha\}.$$

We have

$$\begin{aligned}
\nu(\{|F_\Phi| > \varepsilon\}) &\le \sum_{n,m} \mu(\{x \in B_m \cap s(S_n) : f_n(x) > \alpha\}) \\
&\le \sum_m \big(\sum_n \mu(\{x \in B_m \cap s(S_n) : f_n(x) > \alpha\}) \\
&\le \sum_m \sum_{n \in P_m} \mu(\{x \in B_m \cap s(S_n) : f_n(x) > \alpha\}) \le \sum_m N\mu(B_m) = N
\end{aligned}$$

□

5 From positive type functions to completely positive maps

Again, we want to extend a well known result in the group case, namely that, given a positive definite function F on a countable group G, there is a normal completely positive map $\Phi : L(G) \to L(G)$, well defined by the formula $\Phi(u_t) = F(t)u_t$ for every $t \in G$.

We need some preliminaries. For the notion of representation used below, see for instance [6, Sect. 3.1].

Lemma 5.1 *Let F be a positive definite function on (G,μ). There exists a representation π of G on a measurable field $\mathscr{K} = \{\mathscr{K}(x)\}_{x\in X}$ of Hilbert spaces, and a measurable section $\xi : x \mapsto \xi(x) \in \mathscr{K}(x)$ such that*

$$F(\gamma) = \langle \xi \circ r(\gamma), \pi(\gamma)\xi \circ s(\gamma)\rangle$$

almost everywhere, that is F is the coefficient of the representation π, associated with ξ.

Proof. This classical fact may be found in [27]. The proof is straightforward, and similar to the classical GNS construction in the case of groups. Let $V(x)$ the space of finitely supported complex-valued functions on G^x, endowed with the semi-definite positive hermitian form

$$\langle f, g\rangle_x = \sum_{\gamma_1,\gamma_2 \in G^x} \overline{f(\gamma_1)} g(\gamma_2) F(\gamma_1^{-1}\gamma_2).$$

We denote by $\mathscr{K}(x)$ the Hilbert space obtained by separation and completion of $V(x)$, and $\pi(\gamma) : \mathscr{K}(s(\gamma)) \to \mathscr{K}(r(\gamma))$ is defined by $(\pi(\gamma)f)(\gamma_1) = f(\gamma^{-1}\gamma_1)$. The Borel structure on the field $\{\mathscr{K}(x)\}_{x\in X}$ is provided by the Borel functions on G whose restriction to the fibres G^x are finitely supported. Finally, ξ is the characteristic function of X, viewed as a Borel section. □

Now we assume that $F(x) = 1$ for almost every $x \in X$, and thus ξ is a unit section. We consider the measurable field $\left\{\ell^2(G_x) \otimes \mathscr{K}(x)\right\}_{x\in X}$. Note that

$$\ell^2(G_x) \otimes \mathscr{K}(x) = \ell^2(G_x, \mathscr{K}(x)).$$

Let $f \in \ell^2(G_x)$. We define $S_x(f) \in \ell^2(G_x, \mathscr{K}(x))$ by

$$S_x(f)(\gamma) = f(\gamma)\pi(\gamma)^*\xi \circ r(\gamma)$$

for $\gamma \in G_x$. Then

$$\sum_{s(\gamma)=x} \|S_x(f)(\gamma)\|^2_{\mathscr{K}(x)} = \|f\|_{\ell^2(G_x)}.$$

The field $(S_x)_{x\in X}$ of operators defines an isometry

$$S : L^2(G,\nu) \to \int_X^{\oplus} \ell^2(G_x, \mathscr{K}(x))\, \mathrm{d}\mu(x),$$

by

$$S(f)(\gamma) = f(\gamma)\pi(\gamma)^*\xi \circ r(\gamma).$$

Note that $\int_X^{\oplus} \ell^2(G_x, \mathscr{K}(x))\, \mathrm{d}\mu(x)$ is a right A-module, by

$$(\eta a)_x = \eta_x a(x) : \gamma \in G_x \mapsto \eta(\gamma) a \circ s(\gamma).$$

Of course, S commutes with the right actions of A. We also observe that, as a right A-module, $\mathscr{L}^2(M,\varphi) \otimes_A \int_X^{\oplus} \mathscr{K}(x)\, \mathrm{d}\mu(x)$ and $\int_X^{\oplus} \ell^2(G_x, \mathscr{K}(x))\, \mathrm{d}\mu(x)$ are canonically isomorphic under the map

$$\zeta \otimes \eta \mapsto \zeta \eta \circ s, \quad \forall \zeta \in \mathscr{L}^2(M,\varphi), \forall \eta \in \int_X^{\oplus} \mathscr{K}(x)\, \mathrm{d}\mu(x),$$

where $(\zeta\eta \circ s)_x$ is the function $\gamma \in G_x \mapsto \zeta(\gamma)\eta \circ s(\gamma)$ in $\ell^2(G_x, \mathscr{K}(x))$. It follows that M acts on $\int_X^{\oplus} \ell^2(G_x, \mathscr{K}(x))\, \mathrm{d}\mu(x)$ by $m \mapsto m \otimes \mathrm{Id}$. In particular, for $f \in I(G)$, we see that $L(f) \otimes \mathrm{Id}$, viewed as an operator on $\int_X^{\oplus} \ell^2(G_x, \mathscr{K}(x))\, \mathrm{d}\mu(x)$, is acting as

$$((L(f) \otimes \mathrm{Id})\eta)(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)\eta(\gamma_2) \in \mathscr{K}(s(\gamma)).$$

Lemma 5.2 *For $f \in I(G)$, we have $S^*\big(L(f) \otimes \mathrm{Id}\big)S = L(Ff)$.*

Proof. A straightforward computation shows that for $\eta \in \int_X^{\oplus} \ell^2(G_x, \mathscr{K}(x))\, \mathrm{d}\mu(x)$, we have

$$(S^*\eta)(\gamma) = \langle \pi(\gamma)^*\xi \circ r(\gamma), \eta(\gamma)\rangle_{\mathscr{K}(s(\gamma))}.$$

Moreover, given $h \in L^2(G,\nu)$, we have

$$\big((L(f) \otimes \mathrm{Id})Sh\big)(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)(Sh)(\gamma_2) = \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)h(\gamma_2)\pi(\gamma_2)^*\xi \circ r(\gamma_2).$$

Hence,

$$
\begin{aligned}
\big(S^*(L(f)\otimes \mathrm{Id})Sh\big)(\gamma) &= \Big\langle \pi(\gamma)^*\xi\circ r(\gamma), \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)h(\gamma_2)\pi(\gamma_2)^*\xi\circ r(\gamma_2)\Big\rangle\\
&= \Big\langle \xi\circ r(\gamma), \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)h(\gamma_2)\pi(\gamma_1)\xi\circ r(\gamma_2)\Big\rangle\\
&= \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)h(\gamma_2)\langle \xi\circ r(\gamma_1),\pi(\gamma_1)\xi\circ r(\gamma_2)\rangle\\
&= \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)F(\gamma_1)h(\gamma_2) = (L(Ff)h)(\gamma).
\end{aligned}
$$

□

Proposition 5.3 *Let $F : G\to\mathbb{C}$ be a Borel positive type function on G such that $F_{|X} = 1$. Then there exists a unique normal completely positive map Φ from M into M such that $\Phi(L(f)) = L(Ff)$ for every $f\in I(G)$. Morever, Φ is A-bimodular, unital and $E_A\circ\Phi = E_A$.*

Proof. The uniqueness is a consequence of the normality of Φ, combined with the density of $L(I(G))$ into M. With the notation of the previous lemma, for $m\in M$ we put $\Phi(m) = S^*\big(m\otimes \mathrm{Id}\big)S$. Obviously, Φ satisfies the required conditions. □

Remark 5.4 We keep the notation of the previous proposition. A straightforward computation shows that F is the positive definite function F_Φ constructed from Φ.

Proposition 5.5 *Let F be a Borel positive definite function on G such that $F_{|X} = 1$. We assume that for every $\varepsilon > 0$, we have $\nu(\{|F| > \varepsilon\}) < +\infty$. Let Φ be the completely positive map defined by F. Then $\widehat{\Phi}$ belongs to the norm closed ideal $\mathscr{I}(\langle M, e_A\rangle)$ generated by the finite projections of $\langle M, e_A\rangle$.*

Proof. We observe that $T = \widehat{\Phi}$ is the multiplication operator $\mathfrak{m}(F)$ by F. We need to show that for every $t > 0$, the spectral projection $e_t(|T|)$ of $|T|$ relative to $]t,+\infty[$ is finite. This projection is the multiplication operator by $f_t = \mathbf{1}_{]t,+\infty[}\circ|F|$. By (3.8), we have

$$\mathrm{Tr}_\mu(\mathfrak{m}(f_t)) = \nu(f_t) = \nu(\{|F| > t\} < +\infty.$$

□

6 Characterizations of the relative Haagerup property

We keep the same notation as in the previous section.

Theorem 6.1 *The following conditions are equivalent:*

(1) M has the Haagerup property relative to A and E_A.
(2) There exists a sequence (F_n) of positive definite functions on G such that

(i) $(F_n)_{|X} = 1$ almost everywhere ;

(ii) *for every* $\varepsilon > 0$, $\nu(\{|F_n| > \varepsilon\}) < +\infty$ *;*
(iii) $\lim_n F_n = 1$ *almost everywhere.*

Proof. (1) $\Rightarrow$ (2). Let (Φ_n) a sequence of unital completely positive maps $M \to M$ satisfying conditions (i), (ii), (iii) of Definition 3.5. We set $F_n = F_{\Phi_n}$. By Proposition 4.7 we know that condition (ii) of (2) above is satisfied. It remains to check (iii). For $m \in M$, we have

$$\|\Phi_n(m) - m\|_2^2 = \int_X E_A((\Phi_n(m) - m)^*(\Phi_n(m) - m))(x)\,\mathrm{d}\mu(x).$$

Let $G = \sqcup_n S_n$ be a partition of G by Borel bisections. There is a null subset Y of X such that, for every k and for $\gamma \in S_k \cap r^{-1}(X \setminus Y)$ we have

$$F_n(\gamma) - 1 = E_A(\mathbf{1}_{S_k}^* \circ \Phi_n(\mathbf{1}_{S_k}))(s(\gamma)) - E_A(\mathbf{1}_{S_k}^* \circ \mathbf{1}_{S_k})(s(\gamma)).$$

Thus

$$\begin{aligned}|F_n(\gamma) - 1|^2 &= \left|E_A(\mathbf{1}_{S_k}^* \circ (\Phi_n(\mathbf{1}_{S_k}) - \mathbf{1}_{S_k}))(s(\gamma))\right|^2 \\ &\le E_A((\Phi_n(\mathbf{1}_{S_k}) - \mathbf{1}_{S_k})^*(\Phi_n(\mathbf{1}_{S_k}) - \mathbf{1}_{S_k}))(s(\gamma)).\end{aligned}$$

It follows that

$$\begin{aligned}\int_G |F_n - 1|^2 \mathbf{1}_{S_k}\,\mathrm{d}\nu &\le \int_{s(S_k)} E_A((\Phi_n(\mathbf{1}_{S_k}) - \mathbf{1}_{S_k})^*(\Phi_n(\mathbf{1}_{S_k}) - \mathbf{1}_{S_k}))(x)\,\mathrm{d}\mu(x) \\ &\le \left\|\Phi_n(\mathbf{1}_{S_k}) - \mathbf{1}_{S_k}\right\|_2^2 \to 0.\end{aligned}$$

So there is a subsequence of $(|F_n(\gamma) - 1|)_n$ which goes to 0 almost everywhere on S_k. Using the Cantor diagonal process, we get the existence of a subsequence $(F_{n_k})_k$ of $(F_n)_n$ such that $\lim_k F_{n_k} = 1$ almost everywhere, which is enough for our purpose.

(2) $\Rightarrow$ (1). Assume the existence of a sequence $(F_n)_n$ of positive definite functions on G, satisfying the three conditions of (2). Let Φ_n be the completely positive map defined by F_n. Let us show that for every $m \in M$, we have

$$\lim_n \|\Phi_n(m) - m\|_2 = 0.$$

We first consider the case $m = L(f)$ with $f \in I(G)$. Then we have

$$\|\Phi_n(L(f)) - L(f)\|_2 = \|L((F_n - 1)f)\|_2 = \|(F_n - 1)f\|_2 \to 0$$

by the Lebesgue dominated convergence theorem.

Let now $m \in M$. Then

$$\|\Phi_n(m) - m\|_2 \le \|\Phi_n(m - L(f))\|_2 + \|\Phi_n(L(f)) - L(f)\|_2 + \|L(f) - m\|_2.$$

We conclude by a classical approximation argument, since

$$\|\Phi_n(m - L(f))\|_2 \le \|L(f) - m\|_2.$$

Together with Propositions 5.3, 5.5 and Lemma 3.6, this proves (1). □

This theorem justifies the following definition.

Definition 6.2 We say that a countable measured groupoid (G,μ) has the *Haagerup property* (or has *property* (H)) if there exists a sequence (F_n) of positive definite functions on G such that

(i) $(F_n)_{|X} = 1$ almost everywhere ;
(ii) for every $\varepsilon > 0$, $\nu(\{|F_n| > \varepsilon\}) < +\infty$;
(iii) $\lim_n F_n = 1$ almost everywhere.

We observe that, by Theorem 6.1, this notion only involves the conditional expectation E_A and therefore only depends on the measure class of μ. This fact does not seem to be obvious directly from the above definition 6.2.

Of course, we get back the usual definition for a countable group. The other equivalent definitions for groups also extend to groupoids as we shall see now.

Definition 6.3 A *real conditionally negative definite function* on G is a Borel function $\psi : G \to \mathbb{R}$ such that

(i) $\psi(x) = 0$ for every $x \in G^{(0)}$;
(ii) $\psi(\gamma) = \psi(\gamma^{-1})$ for every $\gamma \in G$;
(iii) for every $x \in G^{(0)}$, every $\gamma_1, \dots, \gamma_n \in G^x$ and every real numbers $\lambda_1, \dots, \lambda_n$ with $\sum_{i=1}^n \lambda_i = 0$, then
$$\sum_{i,j=1}^n \lambda_i \lambda_j \psi(\gamma_i^{-1}\gamma_j) \le 0.$$

Such a function is non-negative.

Definition 6.4 Let (G,μ) be a countable measured groupoid. A *real conditionally negative definite function* on (G,μ) is a Borel function $\psi : G \to \mathbb{R}$ such that there exists a co-null subset U of $G^{(0)}$ with the property that the restriction of ψ to the inessential reduction $G_{|_U}$ satisfies the conditions of the previous definition.

We say that ψ is *proper* if for every $c > 0$, we have $\nu(\{\psi \le c\}) < +\infty$.

Theorem 6.5 *The groupoid (G,μ) has the Haagerup property if and only if there exists a real conditionally negative definite function ψ on (G,μ) such that*

$$\forall c > 0, \quad \nu(\{\psi \le c\}) < +\infty.$$

Proof. We follow the steps of the proof given by Jolissaint [17] for equivalence relations and previously by Akemann-Walter [3] for groups. Let ψ be a proper conditionally negative definite function. We set $F_n = \exp(-\psi/n)$. Then (F_n) is a sequence of positive definite functions which goes to 1 pointwise. Moreover, we have $F_n(\gamma) > c$ if and only if $\psi(\gamma) < -n \ln c$. Therefore (G,μ) has the Haagerup property.

Conversely, let (F_n) be a sequence of positive definite functions on G satisfying conditions (i), (ii), (iii) of Theorem 6.1 (2). We choose sequences (α_n) and (ε_n) of positive numbers such (α_n) is increasing with $\lim_n \alpha_n = +\infty$, (ε_n) is decreasing with $\lim_n \varepsilon_n = 0$, and such that $\sum_n \alpha_n (\varepsilon_n)^{1/2} < +\infty$.

Let $G = \sqcup S_n$ be a partition of G into Borel bisections. Taking if necessary a subsequence of (F_n), we may assume that for every n,

$$\sum_{1\le k\le n} \int_G |1-F_n|^2 \mathbf{1}_{S_k}\, \mathrm{d}\nu \le \varepsilon_n^2.$$

It follows that

$$\begin{aligned}\int_G \left(\Re(1-F_n\right)^2 \mathbf{1}_{\cup_{1\le k\le n} S_k}\, \mathrm{d}\nu &\le \int_G |1-F_n|^2 \mathbf{1}_{\cup_{1\le k\le n} S_k}\, \mathrm{d}\nu \\ &\le \varepsilon_n^2.\end{aligned}$$

We set $E_n = \left\{\gamma \in \cup_{1\le k\le n} S_k : |\Re(1-F_n(\gamma)| \ge (\varepsilon_n)^{1/2}\right\}$ and $E = \cap_l \cup_{n\ge l} E_n$. Since $\nu(E_n) \le \varepsilon_n$ and $\sum_n \varepsilon_n < +\infty$, we see that $\nu(E) = 0$.

Let us set $\psi = \sum_n \alpha_n \Re(1-F_n)$ on $G \setminus E$ and $\psi = 0$ on E. We claim that the series converges pointwise. Indeed, let $\gamma \in G\setminus E$. There exists m such that

$$\gamma \in (\cup_{1\le i\le m} S_i) \cap (\cap_{n\ge m} E_n^c).$$

Thus, $|\Re(1-F_n(\gamma)| \le (\varepsilon_n)^{1/2}$ for $n \ge m$, which shows our claim.

It remains to show that ψ is proper. Let $c > 0$, and let $\gamma \in G\setminus E$ with $\psi(\gamma) \le c$. Then we have $\Re(1-F_n(\gamma)) \le c/\alpha_n$ for every n and therefore $\Re F_n(\gamma) \ge 1 - c/\alpha_n$. Let n be large enough such that $1 - c/\alpha_n \ge 1/2$. It follows that

$$\nu(\{\psi \le c\}) \le \nu(\{|F_n| \ge 1/2)\} < +\infty.$$

□

Definition 6.6 Let π be a representation of (G,μ) on a measurable field $\mathscr{K} = \{\mathscr{K}(x)\}_{x\in X}$ of Hilbert spaces. A π*-cocycle* is a Borel section b of the pull-back bundle $r : r^*\mathscr{K} = \{(\gamma,\xi) : \xi \in \mathscr{K}(r(\gamma))\} \to G$ such that, up to an inessential reduction, we have, for $\gamma_1,\gamma_2 \in G$ with $s(\gamma_1) = r(\gamma_2)$,

$$b(\gamma_1\gamma_2) = b(\gamma_1) + \pi(\gamma_1) b(\gamma_2).$$

We say that b is *proper* if for every $c > 0$, we have $\nu(\{\|b\| \le c\}) < +\infty$.

Let b be a π-cocycle. It is easily seen that $\gamma \mapsto \|b(\gamma)\|^2$ is conditionally negative definite. Moreover, every real conditionally negative definite is of this form (see [6, Proposition 5.21]).

Corollary 6.7 *The groupoid (G,μ) has the Haagerup property if and only if it admits a proper π-cocycle for some representation π.*

Examples 6.8 Let $\Gamma \curvearrowright (X,\mu)$ be an action of a countable group Γ which leaves quasi-invariant the probability measure μ. If Γ has the property (H), then $(X \rtimes \Gamma, \mu)$

inherits this property. Indeed, if $\psi : \Gamma \to \mathbb{R}$ is a proper conditionally negative definite function, then $\widetilde{\psi} : (x,s) \mapsto \psi(s)$ is a proper conditionally negative definite function on $(X \rtimes \Gamma, \mu)$. Conversely, when the action is free, preserves μ and is such that $(X \rtimes \Gamma, \mu)$ has property (H), then Γ has property (H) [17, Proposition 3.3]. However, free non-singular actions of groups not having property (H) can generate semi-direct product groupoids with this property. Such actions can even be amenable (see for instance [8, Examples 5.2.2]).

Interesting examples are provided by treeable groupoids, as we shall see now. For instance, the free product of the type II_1 hyperfinite equivalence relation by itself, being treeable [5, Proposition 2.4], has the Haagerup property. Note also that property (H) passes to subgroupoids.

7 Treeable countable measured groupoids have property (H)

The notion of treeable countable measured equivalence relation has been introduced by Adams in [1]. Its obvious extension to the case of countable measured groupoids is exposed in [6]. We recall here the main definitions. Let Q be a Borel subset of a countable Borel groupoid G. We set $Q^0 = X$ and for $n \geq 1$, we set

$$Q^n = \{\gamma \in G : \exists \gamma_1, \dots, \gamma_n \in Q, \gamma = \gamma_1 \cdots \gamma_n\}.$$

Definition 7.1 A *graphing* of G is a Borel subset Q of G such that $Q = Q^{-1}$, $Q \cap X = \emptyset$ and $\cup_{n\geq 0} Q^n = G$.

A graphing defines a structure of G-bundle of graphs on X: the set of vertices is G and

$$\mathbb{E} = \left\{(\gamma_1, \gamma_2) \in G \times G : r(\gamma_1) = r(\gamma_2), \gamma_1^{-1}\gamma_2 \in Q\right\}$$

is the set of edges. In particular, for every $x \in X$, the fibre G^x is a graph, its set or edges being $\mathbb{E} \cap (G^x \times G^x)$. Moreover, for $\gamma \in G$, the map $\gamma_1 \mapsto \gamma\gamma_1$ induces an isomorphism of graphs from $G^{s(\gamma)}$ onto $G^{r(\gamma)}$. Thus, a graphing is an equivariant Borel way of defining a structure of graph on each fibre G^x. These graphs are connected since $\cup_{n\geq 0} Q^n = G$.

When the graphs G^x are trees for every $x \in X$, the graphing Q is called a *treeing*.

Definition 7.2 A countable Borel groupoid G is said to be *treeable* if there is a graphing which gives to $r : G \to X$ a structure of G-bundle of trees.

A countable measured groupoid (G, μ) is said to be *treeable* if there exists an inessential reduction $G_{|_U}$ which is a treeable Borel groupoid in the above sense.

Equipped with such a structure, (G, μ) is said to be a *treed measured groupoid*.

Consider the case where G is a countable group and Q is a symmetric set of generators. The corresponding graph structure on G is the Cayley graph defined by Q. If $Q = S \cup S^{-1}$ with $S \cap S^{-1} = \emptyset$, then Q is a treeing if and only if S is a free subset of generators of G (and thus G is a free group).

As made precise in [4, Proposition 3.9], treeable groupoids are the analogue of free groups and therefore the following theorem is no surprise.

Theorem 7.3 (Ueda) *Let (G,μ) be a countable measured groupoid which is treeable. Then (G,μ) has the Haagerup property.*

Let Q be a treeing of (G,μ). We endow G^x with the length metric d_x defined by

$$d_x(\gamma_1,\gamma_2) = \min\left\{n \in \mathbb{N} : \gamma_1^{-1}\gamma_2 \in Q^n\right\}.$$

The map $(\gamma_1,\gamma_2) \in \{(\gamma_1,\gamma_2) : r(\gamma_1) = r(\gamma_2)\} \mapsto d_{r(\gamma_1)}(\gamma_1,\gamma_2)$ is Borel.

We set $\psi(\gamma) = d_{r(\gamma)}(r(\gamma),\gamma)$. It is a real conditionally negative definite function on G. Indeed, given $\gamma_1,\dots,\gamma_n \in G^x$ and $\lambda_1,\dots,\lambda_n \in \mathbb{R}$ such that $\sum_{i=1}^n \lambda_i = 0$, we have

$$\sum_{i,j=1}^n \lambda_i\lambda_j\psi(\gamma_i^{-1}\gamma_j) = \sum_{i,j=1}^n \lambda_i\lambda_j d_{s(\gamma_i)}(s(\gamma_i),\gamma_i^{-1}\gamma_j) = \sum_{i,j=1}^n \lambda_i\lambda_j d_x(\gamma_i,\gamma_j) \le 0,$$

since the length metric on a tree is conditionally negative definite (see [16, p. 69] for instance).

We begin by proving Theorem 7.3 in the case where Q is bounded, i.e. there exists $k > 0$ such that $\sharp Q^x \le k$ for almost every $x \in X$.

Lemma 7.4 *Assume that Q is bounded. For every $c > 0$ we have $\nu(\{\psi \le c\}) < +\infty$.*

Proof. We have

$$\nu(\{\psi \le c\}) = \int_X \sharp\left\{\gamma : s(\gamma) = x, d_x(x,\gamma^{-1}) \le c\right\} \mathrm{d}\mu(x).$$

If k is such that $\sharp Q^x \le k$ for almost every $x \in X$, the cardinal of the ball in G^x of center x and radius c is smaller than k^c. It follows that $\nu(\{\psi \le c\}) \le k^c$. □

In view of the proof in the general case, we make a preliminary observation. Whenever Q is bounded, G is the union of the increasing sequence $(\{\psi \le k\})_{k\in\mathbb{N}}$ of Borel subsets with $\nu(\{\psi \le k\}) < +\infty$. Moreover, setting $F_n = \exp(-\psi/n)$, we have $\lim_n F_n = 1$ uniformly on each subset $\{\psi \le k\}$. Indeed, if $\psi(\gamma) \le k$, we get

$$0 \le 1 - F_n(\gamma) \le \sum_{j\ge 1} \frac{1}{n^j}\frac{\psi(\gamma)^j}{j!} \le \frac{k}{n}\exp(k/n).$$

Proof of theorem 7.3. The treeing Q is no longer supposed to be bounded. Let $G = \sqcup S_k$ be a partition of G into Borel bisections. For every integer n we set

$$Q'_n = \cup_{k\le n}(Q \cap S_k) \quad \text{and} \quad Q_n = Q'_n \cup (Q'_n)^{-1}.$$

Note that (Q_n) is an increasing sequence of Borel symmetric and bounded subsets of Q with $\cup_n Q_n = Q$. Let G_n be the subgroupoid of G generated by Q_n, that is $G_n = \cup_{k\ge 0} Q_n^k$, where we put $Q_n^0 = X$.

We observe that Q_n is a treeing for G_n. Denote by ψ_n the associated conditionally negative definite function on G_n. Since $Q_{n-1} \subset Q_n$, we have

$$(\psi_n)_{|G_{n-1}} \leq \psi_{n-1}.$$

Given two integers k and N, we set

$$A_{k,N} = \{\gamma \in G_k : \psi_k(\gamma) \leq N\}.$$

Then, obviously we have

$$A_{k,N} \subset A_{k+1,N} \quad \text{and} \quad A_{k,N} \subset A_{k,N+1}.$$

In particular, $(A_{k,k})_k$is an increasing sequence of Borel subsets of G with $\cup_k A_{k,k} = G$.

We fix k. We set $F_{k,n}(\gamma) = \exp(-\psi_k(\gamma)/n)$ if $\gamma \in G_k$ and $F_{k,n}(\gamma) = 0$ if $\gamma \notin G_k$. By Lemma 7.5 to follow, $F_{k,n}$ is positive definite on G. Since Q_k is bounded, Lemma 7.4 implies that for every $\varepsilon > 0$, and for every n, we have $\nu(\{F_{k,n} \geq \varepsilon\}) < +\infty$. Moreover, $\lim_n F_{k,n} = 1$ uniformly on each $A_{k,N}$, $N \geq 1$, as previously noticed.

We choose, step by step, a strictly increasing sequence $(n_i)_{i\geq 1}$ of integers such that for every k,

$$\sup_{\gamma \in A_{k,k}} 1 - F_{k,n_k}(\gamma) \leq 1/k.$$

Then the sequence $(F_{k,n_k})_k$ of positive definite functions satisfies the required conditions showing that (G,μ) has property (H). □

Lemma 7.5 *Let H be a subgroupoid of a groupoid G with $G^{(0)} = H^{(0)}$. Let F be a positive definite function on H and extend F to G by setting $F(\gamma) = 0$ if $\gamma \notin H$. Then F is positive definite on G.*

Proof. Let $\gamma_1, \dots, \gamma_n \in G^x$ and let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. We want to show that

$$\sum_{i,j=1}^{n} \lambda_i \overline{\lambda_j} F(\gamma_i^{-1}\gamma_j) \geq 0.$$

We assume that this inequality holds for every number $k < n$ of indices. For $k = n$, this inequality is obvious if for every $i \neq j$ we have $\gamma_i^{-1}\gamma_j \notin H$. Otherwise, up to a permutation of indices, we take $j = 1$ and we assume that $2, \dots, l$ are the indices i such that $\gamma_i^{-1}\gamma_1 \in H$. Then, if $1 \leq i, j \leq l$ we have $\gamma_i^{-1}\gamma_j = (\gamma_i^{-1}\gamma_1)(\gamma_1^{-1}\gamma_j) \in H$ and for $i \leq l < j$ we have $\gamma_i^{-1}\gamma_j \notin H$. It follows that

$$\sum_{i,j=1}^{n} \lambda_i \overline{\lambda_j} F(\gamma_i^{-1}\gamma_j) = \sum_{i,j=1}^{l} \lambda_i \overline{\lambda_j} F(\gamma_i^{-1}\gamma_j) + \sum_{i,j>l} \lambda_i \overline{\lambda_j} F(\gamma_i^{-1}\gamma_j),$$

where the first term of the right hand side is ≥ 0. As for the second term, it is also ≥ 0 by the induction assumption. □

8 Properties (T) and (H) are not compatible

Property (T) for group actions and equivalence relations has been introduced by Zimmer in [32]. Its extension to measured groupoids is immediate. We say that (G,μ) has *property (T)* if whenever a representation of (G,μ) almost has unit invariant sections, it actually has a unit invariant section (see [6, Definitions 4.2, 4.3] for details). We have proved in [6, Theorem 5.22] the following characterization of property (T).

Theorem 8.1 *Let (G,μ) be an ergodic countable measured groupoid. The following conditions are equivalent:*

(i) (G,μ) has property (T) ;
(ii) for every real conditionally negative definite function ψ on G, there exists a Borel subset E of X, with $\mu(E)>0$, such that the restriction of ψ to $G_{|_E}$ is bounded.

Theorem 8.2 *Let (G,μ) be an ergodic countable measured groupoid. We assume that $(G^{(0)},\mu)$ is a diffuse standard probability space. Then (G,μ) cannot have simultaneously properties (T) and (H).*

Proof. Assume that (G,μ) has both properties (H) and (T). There exists a Borel conditionally negative definite function ψ such that for every $c>0$, we have $\nu(\{\psi\le c\})<+\infty$. Moreover, there exists a Borel subset E of X, with $\mu(E)>0$, such that the restriction of ψ to $G_{|_E}$ is bounded. Then, we have

$$\int_E \sharp\{\gamma : s(\gamma)=x, r(\gamma)\in E\}\,\mathrm{d}\mu(x)<+\infty.$$

Therefore, for almost every $x\in E$, we have $\sharp\{\gamma : s(\gamma)=x, r(\gamma)\in E\}<+\infty$. Replacing if necessary E by a smaller subset we may assume the existence of $N>0$ such that for every $x\in E$,

$$\sharp\{\gamma : s(\gamma)=x, r(\gamma)\in E\}\le N.$$

Since $(G_{|_E},\mu_{|_E})$ is ergodic, we may assume that all the fibres of this groupoid have the same finite cardinal. Therefore, this groupoid is proper and so the quotient Borel space $E/(G_{|_E})$ is countably separated (see [8, Lemma 2.1.3]). A classical argument (see [33, Proposition 2.1.10]) shows that $\mu_{|_E}$ is supported by an equivalence class, that is by a finite subset of E. But this contradicts the fact that the measure is diffuse. □

In the following corollaries, we always assume that (X,μ) is a diffuse standard probability measure space.

Corollary 8.3 *Let (G,μ) be a countable ergodic measured groupoid such that $(\mathscr{R}_G,\mu)$ has property (H) (e.g. is treeable). Then (G,μ) has not property (T).*

Proof. If (G,μ) had property (T) then $(\mathscr{R}_G,\mu)$ would have the same property by [6, Theorem 5.18]. But this is impossible by Theorem 8.2. □

This allows to retrieve results of Jolissaint [17, Proposition 3.2] and Adams-Spatzier [2, Theorem 1.8].

Corollary 8.4 *Let $\Gamma \curvearrowright (X,\mu)$ be an ergodic probability measure preserving action of a countable group Γ having property (T). Then $(\mathscr{R}_\Gamma,\mu)$ has not property (H) and in particular is not treeable.*

Proof. Indeed, under the assumptions of the corollary, the semi-direct product groupoid $(X \rtimes \Gamma,\mu)$ has property (T) by [32, Proposition 2.4], and we apply the previous corollary. □

Corollary 8.5 *Let $(\mathscr{R},\mu)$ be a type II_1 equivalence relation on X having property (H). Then its full group $[\mathscr{R}]$ does not contain any countable subgroup Γ which acts ergodically on (X,μ) and has property (T).*

Proof. If $[\mathscr{R}]$ contains such a subgroup, then $(\mathscr{R}_\Gamma,\mu)$ has property (T), and also property (H) as a subequivalence relation of $\mathscr{R}$, in contradiction with Corollary 8.3. □

Problem. Since by Dye's theorem $(\mathscr{R},\mu)$ is entirely determined by its full group [20, Theorem 4.1], it would be interesting to characterize property (H) in terms of this full group.

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