ON SPECTRAL CHARACTERIZATIONS OF AMENABILITY

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Abstract. We show that a measured G-space (X, μ), where G is a locally compact group, is amenable in the sense of Zimmer if and only if the following two conditions are satisfied: the associated unitary representation π_X of G into $L^2(X, \mu)$ is weakly contained into the regular representation λ_G and there exists a G-equivariant norm one projection from $L^\infty(X \times X)$ onto $L^\infty(X)$. We give examples of ergodic discrete group actions which are not amenable, although π_X is weakly contained into λ_G.

1. Introduction

To every measured G-space (X, μ), where μ is a quasi-invariant probability measure, is associated a unitary representation π_X of G into the Hilbert space $L^2(X)$. Going back to the works of von Neumann and Halmos [17], it is well known that the analysis of the spectral structure of π_X gives important informations about the dynamics of (X, G, μ).

A particularly fruitful study in the case of transitive actions is due to Kesten [21]. When $X = G/H$, the representation π_X is the quasi-regular representation $\lambda_{G/H}$. Given a probability measure m on G, we have $\|\lambda_G(m)\| \le \|\lambda_{G/H}(m)\|$, where λ_G is the regular representation of G, and where, for every unitary representation π of G, we set $\pi(m) = \int_G \pi(s) dm(s)$. Kesten proved that for any discrete group G, any normal subgroup H, and any adapted symmetric probability measure m on G (i.e. the support of m generates G), one has $\|\lambda_G(m)\| = \|\lambda_{G/H}(m)\|$ if and only if H is amenable.

Zimmer introduced in [32] the notion of amenable G-space (see 3.1.7, 3.1.8). It includes a wide range of interesting examples, and for a transitive G-space G/H, it is equivalent to the amenability of H. An amenable ergodic G-space can be viewed as an amenable "virtual" subgroup of G. Therefore a natural question arises: is the amenability of an ergodic G-space X equivalent to the existence of an adapted probability measure m on G such that $\|\lambda_G(m)\| = \|\pi_X(m)\|$? or such that $r(\lambda_G(m)) = r(\pi_X(m))$? (where $r(T)$ denotes the spectral radius of any operator T acting on a Hilbert space).

G. Kuhn proved [23] that for an ergodic amenable G-space X with G discrete, the representation π_X is weakly contained into λ_G ($\pi_X \prec \lambda_G$), so that for every probability measure m on G we have $\|\lambda_G(m)\| = \|\pi_X(m)\|$. Using Kuhn's result, A. Nevo showed recently [24] that $\pi_X \prec \lambda_G$ for every amenable ergodic G-space when G

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is a locally compact group containing a lattice. One of the motivations of the present paper is to give a direct proof of this fact for any amenable action of any locally compact group. Let us mention that Nevo has found nice applications of this result, obtaining for instance lower bounds for geometric or dynamical invariants, such as the exponent of convergence associated with a discrete subgroup of a connected non compact semi-simple Lie group with finite centre [24]. Conversely, the weak containment of π_X into λ_G is not enough to insure the amenability of the G-space X, even in the transitive case (see Section 4.2). However, A. Nevo has provided geometric examples where the converse holds ([24, Theorem 4.5]). Another of our motivations is the general study of the converse problem.

We work in the following more general context. Let $q : (Y, \nu) \to (X, \mu)$ be a G-equivariant Borel map between measured G-spaces, with $q_* \nu = \mu$. We say that the pair (Y, X) of measured G-spaces is amenable if there exists a G-invariant mean $M : L^\infty(Y) \to L^\infty(X)$ (see Definitions 3.1.1 and 3.1.3). If (Y, X) is amenable, we show (Theorem 3.2.1) that π_X is weakly contained into π_Y, that is $\|\pi_X(m)\| \leq \|\pi_Y(m)\|$ for every bounded measure m on G. By definition, the amenability of the G-space X is the amenability of the pair $(X \times G, X)$; then the weak equivalence between $\pi_{X\times G}$ and λ_G yields $\pi_X \prec \lambda_G$. We are now interested in the following reverse problems:

- (A) Does $\pi_X \prec \pi_Y$ implies that (Y, X) is amenable?
- (A_1) Does $\pi_X \prec \lambda_G$ implies that X is an amenable G-space?
- (B) Let H be a closed subgroup of a locally compact group G. Does the existence of an adapted probability measure m on G such that $r(\lambda_{G/H}(m)) = r(\lambda_G(m))$ imply the amenability of H?

Note that (A_1) is the particular case of (A) where $Y = X \times G$ and that the assumption made in (B) is much weaker than the weak containment $\lambda_{G/H} \prec \lambda_G$.

We give a complete answer to problem (A_1), and examples or counterexamples to illustrate problems (A) and (B). More precisely, our paper is organized as follows.

Section 2 contains basic definitions and results related to cocycle representations. One of the main results is Corollary 2.3.4 which says that for every probability measure m on G and every pair (Y, X) of measured G-spaces we have $\|\pi_Y(m)\| \leq \|\pi_X(m)\|$.

In Section 3, we recall several useful facts concerning amenable pairs. They are mostly taken from the Monograph [4], but written here in the group action framework, instead of the general groupoid context. As already mentioned, when the pair is amenable we show π_X is weakly contained into π_Y.

Section 4 is devoted to the study of the converse. We begin by recalling the result due independently to Berg-Christensen [5] and Derriennic-Guivarc'h [8] when Y is an homogeneous G-space, and extended by Guivarc'h [16] to the case of any G-space, which solves completely the case where X is reduced to a point (Proposition 4.1.1). As for the pair $(X \times G, X)$ we show (Theorem 4.3.1) that it is amenable (i. e. X is an amenable G-space) if and only if π_X is weakly contained in λ_G and in addition there exists an invariant mean $M : L^\infty(X \times X) \to L^\infty(X)$. Although one would be tempted to take for M the restriction to the diagonal, it does not have any meaning in general. Of course, for an homogeneous G-space $X = G/H$, such a mean exists when H is an open or a normal subgroup, but the example, given in Section 4.2, of $H = SL(2,\mathbb{R}) \subset G = SL(3,\mathbb{R})$ shows that there does not always exist an invariant mean from $L^\infty(G/H \times G/H)$ onto $L^\infty(G/H)$. We show in Section 4.4 that the existence of such an invariant mean is equivalent to the weak containment of the trivial representation of H into the restriction to H of the quasi-regular representation $\lambda_{G/H}$. In Section 4.5 we remark that any lattice Γ of $SL(3,\mathbb{R})$ has an ergodic non amenable action on a measured space with $\pi_X \prec \lambda_G$. In fact this remark can be extended to any lattice of a semi-simple Lie group of real rank ≥ 2.

Finally, in Section 5 we end our paper with a few comments on problem (B).We give a positive answer, essentially due to Kesten, when H is an open normal subgroup of G and m is symmetric and adapted to the pair (G,H) (Theorem 5.1.3). Using a deep result of Grigorchuk [15], we observe that the normality assumption cannot be removed. We also point out that the symmetry of the measure m is essential.

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2. Cocycle representations

Let us first recall a few basic definitions, and fix some notations. Let G denote a locally compact second countable group. A *Borel G-space* is a standard Borel space X with a Borel left G-action $(s,x) \in G \times X \mapsto sx \in X$. If X is equipped with a G-quasi-invariant measure μ, then (X,G,μ) (or X for short) is called a (non-singular) *measured G-space*. We denote by r (or r_X in case of ambiguity) the *Radon-Nikodym derivative* defined by

$$\forall s \in G, \forall f \in L^1(X,\mu), \quad \int_X f(s^{-1}x) r(x,s) d\mu(x) = \int_X f(x) d\mu(x).$$

Recall that r can be chosen to be a strict Borel cocycle from $X \times G$ into $\mathbb{R}^+$, that is

$$\forall (x,s,t) \in X \times G \times G, \quad r(x,st) = r(x,s) r(s^{-1}x,t).$$

Observe that every left G-space X can be viewed as a right G-space by setting $x.s = s^{-1}x$. Conversely every right G-space can be viewed as a left G-space. Without further mention, our G-spaces will be left G-spaces.

2.1. Cocycle representations.

Let $\mathcal{H} = \{\mathcal{H}(x) : x \in X\}$ be a μ-measurable field of Hilbert spaces on X (see [10, Chap. II]). We denote by $L^2(\mathcal{H}) = \int_X^\oplus \mathcal{H}(x) d\mu(x)$ the direct integral. For $x, y \in X$, the set of bounded linear maps from $\mathcal{H}(x)$ to $\mathcal{H}(y)$ will be denoted by $\mathcal{B}(\mathcal{H}(x),\mathcal{H}(y))$ and its subset of Hilbert space isomorphisms will be denoted by $\mathrm{Iso}(\mathcal{H}(x),\mathcal{H}(y))$.

Definition 2.1.1. A *(unitary) cocycle representation* of (X,G,μ), acting on $\mathcal{H}$, is a map $\alpha : (x,s) \in X \times G \mapsto \alpha(x,s) \in \mathcal{B}(\mathcal{H}(s^{-1}x),\mathcal{H}(x))$ such that

(a) for each $s \in G$, $\alpha(x,s) \in \mathrm{Iso}(\mathcal{H}(s^{-1}x),\mathcal{H}(x))$ for μ-almost every x;

(b) for each $(s,t) \in G \times G$, $\alpha(x, st) = \alpha(x,s)\alpha(s^{-1}x, t)$ for μ-almost every $x \in X$;
(c) for every pair of measurable sections ξ, η of $\mathcal{H}$, and every $s \in G$ the map $x \mapsto \langle \eta(x), \alpha(x,s)\xi(s^{-1}x)\rangle$ is measurable.

The cocycle is called *strict* if the properties in (a) and (b) hold everywhere.

Observe that α is, equivalently, a unitary representation of the measured semi-direct product groupoid $X \rtimes G$ (see [28, p. 52]).

Definition 2.1.2. Two cocycle representations α and β, acting on $\mathcal{H}$ and $\mathcal{K}$ respectively are said to be *equivalent* ($\simeq$) if there exists a map $\varphi : x \mapsto \varphi(x) \in \mathcal{B}(\mathcal{H}(x), \mathcal{K}(x))$ such that

(a) $\varphi(x) \in \mathrm{Iso}(\mathcal{H}(x), \mathcal{K}(x))$ for μ-almost all $x \in X$;
(b) for each $s \in G$, $\varphi(x)\alpha(x,s) = \beta(x,s)\varphi(s^{-1}x)$ for μ-almost all $x \in X$;
(c) for every pair of measurable sections ξ of $\mathcal{H}$ and η of $\mathcal{K}$ the map $x \mapsto \langle \eta(x), \varphi(x)\xi(x)\rangle$ is measurable.

There is also an obvious notion of strict equivalence.

We denote by $\mathrm{Rep}(X \rtimes G, \mu)$ the set of (equivalence classes of) cocycle representations on (X, G, μ). When X is reduced to a point, $\mathrm{Rep}(X \rtimes G, \mu)$ is the space $\mathrm{Rep}(G)$ of (equivalence classes of) unitary representations of G.

There is a well known map from $\mathrm{Rep}(X \rtimes G, \mu)$ into $\mathrm{Rep}(G)$, called the *induction map.* According to Mackey's point of view, considering measured G-spaces as "virtual subgroups" of G, this map corresponds to the usual induction map in representation group theory. Let us recall its definition. Let $\alpha \in \mathrm{Rep}(X \rtimes G, \mu)$ be a cocycle representation acting on $\mathcal{H}$. The induced representation $\mathrm{Ind}\,\alpha$ (or $\mathrm{Ind}_X\alpha$ in case of ambiguity) is the representation into $L^2(\mathcal{H})$ defined by

$$(\mathrm{Ind}\alpha(t)\xi)(x) = \sqrt{r(x,t)}\alpha(x,t)\xi(t^{-1}x)$$

for $\xi \in L^2(\mathcal{H})$ and $(x,t) \in X \times G$.

We shall also consider the *restriction* map Res (or Res_X in case of ambiguity) from $\mathrm{Rep}(G)$ into $\mathrm{Rep}(X \rtimes G, \mu)$, which sends $\pi \in \mathrm{Rep}(G)$ acting on $\mathcal{H}_\pi$ to $\mathrm{Res}\,\pi : (x,t) \mapsto \pi(t)$ acting on the constant field with fiber $\mathcal{H}_\pi$.

The restriction of the trivial representation ι_G of G will be denoted by $\iota_{X \rtimes G}$ and called the *trivial cocycle.* It is the constant function $(x,t) \mapsto 1$. Similarly, the restriction $\lambda_{X \rtimes G}$ of the (left) regular representation λ_G of G will be called the (left) *regular cocycle.*

The representation $\mathrm{Ind}\,\iota_{X \rtimes G}$ induced by $\iota_{X \rtimes G}$ plays an important role in the study of the measured G-space (X, G, μ). We call it the *canonical representation of G associated with* (X, G, μ) and denote it by π_X. It acts into $L^2(X, \mu)$ by

$$\pi_X(t)\xi(x) = \sqrt{r(x,t)}\xi(t^{-1}x)$$

for $\xi \in L^2(X, \mu)$ and $(x,t) \in X \times G$. Note that given two measured G-spaces X and X', we have

$$\pi_{X \times X'} = \pi_X \otimes \pi_{X'}.$$

Let us observe for further use that

$$\operatorname{Ind} \circ \operatorname{Res} \lambda_G = \operatorname{Ind} \lambda_{X \rtimes G} = \pi_X \otimes \lambda_G \simeq \infty.\lambda_G. \tag{2.1}$$

Here $\infty.\lambda_G$ denotes the direct sum of countably many copies of λ_G and we use the well known observation that $\pi \otimes \lambda_G$ is equivalent to $(\dim \pi).\lambda_G$ for every unitary representation π of G (see [12]). More generally, for every representation U of G we have

$$\operatorname{Ind} \circ \operatorname{Res} U = \pi_X \otimes U. \tag{2.2}$$

As a consequence of (2.1), note that $\operatorname{Res} \circ \operatorname{Ind} \lambda_{X \rtimes G} \simeq \infty.\lambda_{X \rtimes G}$.

The analysis of $\operatorname{Res} \circ \operatorname{Ind} \iota_{X \rtimes G}$ is more subtle. Following Zimmer [31], we say that the *measured G-space (X, G, μ) is normal* if $\operatorname{Res} \circ \operatorname{Ind} \iota_{X \rtimes G} \simeq \infty.\iota_{X \rtimes G}$.

We shall sometimes view G as a left G-space. Then we have $\operatorname{Ind} \iota_{G \rtimes G} = \lambda_G$. On the other hand, when $X = \{pt\}$ is reduced to a point, we have $\operatorname{Ind} \iota_{\{pt\} \rtimes G} = \iota_G$.

For transitive G-spaces, the well known correspondence between cocycles and homomorphisms of the stability subgroups, due to Mackey, motivates the above definitions. More precisely, let H be a closed subgroup of G and let μ be a quasi-invariant measure on $X = G/H$. Given a strict cocycle $\alpha : G/H \times G \to \operatorname{Iso}(\mathcal{H})$, then $\pi_\alpha : h \mapsto \alpha([e], h)$ is a representation of H in the Hilbert space $\mathcal{H}$ (where $[e]$ is the equivalence class of the unit of G). On the other hand, let $\gamma : G/H \to G$ be a Borel section of the natural projection, with $\gamma([e]) = e$. Then $\beta(x, s) := \gamma(x)^{-1} s \gamma(s^{-1}x)$ is a Borel cocycle from $X \times G$ into H. To every $\pi \in \operatorname{Rep}(H)$ is associated the cocycle $\alpha_\pi : (x, s) \mapsto \pi(\beta(x, s))$. In this way we define a bijection between $\operatorname{Rep}(H)$ and $\operatorname{Rep}(X \rtimes G, \mu)$ whose converse is $\alpha \mapsto \pi_\alpha$. The notions of induction and restriction defined above correspond to the usual ones in group representation theory, when $\operatorname{Rep}(H)$ and $\operatorname{Rep}((G/H) \rtimes G, \mu)$ are identified. One can also check that $(G/H, G, \mu)$ is a normal measured G-space if and only if H is a normal subgroup of G (see [31, Prop. 5.3]). For more details we refer to [31] or [36].

2.2. **Pairs of measured G-spaces.** An interesting family of cocycle representations of (X, G, μ) comes from extensions of measured G-spaces. Let Y be another Borel G-space and let $q : Y \to X$ be a Borel G-equivariant map. A *G-quasi-invariant Borel q-system of measures* is a family $\rho = \{\rho^x : x \in X\}$ of measures ρ^x on $q^{-1}(x)$ such that

(i) for every nonnegative Borel function f on Y, the function $\rho(f) : x \mapsto \rho^x(f)$ is Borel;
(ii) there exists a Borel cocycle $c : Y \times G \to [0, +\infty]$ satisfying

$$\int f(s^{-1}y) c(y, s) d\rho^x(y) = \int f(y) d\rho^{s^{-1}x}(y), \tag{2.3}$$

for every Borel nonnegative function f and every $(x, s) \in X \times G$.

We say that ρ is *proper* if there exists a positive Borel function f on Y such that $\rho^x(f) = 1$ for every $x \in X$. We shall always assume that ρ is proper.

Let ρ be a quasi-invariant Borel q-system of measures. We set $\nu = \mu \circ \rho$, that is

$$\int f d\nu = \int f(y) d\rho^x(y) d\mu(x)$$

for every Borel nonnegative function on Y. Obviously, ν is a quasi-invariant measure on Y and

$$r_Y(y,s) = c(y,s) r_X(q(y),s), \quad \forall (y,s) \in Y \times G.$$

Note that $q_*\nu = \mu$, so that the measured G-space (Y,ν) is an *extension* of the measured G-space (X,μ). Conversely, using the disintegration theorem of measures, we see that every extension (Y,ν) of (X,μ) can be expressed as above. The datum of (Y,q,ρ,X,μ) will be called a *pair of measured G-spaces*, and denoted by (Y,X) for simplicity. In particular, the measures μ and ν will always be implicitely included in the structures of X and Y respectively.

Given such a pair, we set $\mathcal{H}(x) = L^2(q^{-1}(x),\rho^x)$. Then $\mathcal{H} = \{\mathcal{H}(x) : x \in X\}$ has a natural structure of measurable field of Hilbert spaces and $L^2(\mathcal{H}) = L^2(Y,\nu)$. For $(y,s) \in Y \times G$ and $\xi \in L^2(q^{-1}(s^{-1}x), \rho^{s^{-1}x})$ let us set

$$(\alpha(x,s)\xi)(y) = \sqrt{c(y,s)}\,\xi(s^{-1}y), \tag{2.4}$$

where $x = q(y)$. Then α is a cocycle representation of (X,G,μ), which will be called the *canonical cocycle representation of the pair* (Y,X) and denoted by $\alpha_{(Y,X)}$. The analogue of the classical theorem of Induction in Stages states here that

$$\mathrm{Ind}_X\, \alpha_{(Y,X)} = \mathrm{Ind}_Y\, \iota_{Y\rtimes G} = \pi_Y, \tag{2.5}$$

where for $\xi \in L^2(Y,\nu)$,

$$\pi_Y(s)\xi(y) = \sqrt{r_Y(y,s)}\,\xi(s^{-1}y).$$

We shall have to consider the case $q : (Y = X \times X', \mu \otimes \mu') \to (X,\mu)$ where (X',μ') is another measured G-space, G acts diagonally on Y, and q is the first projection. Then, $\alpha_{(X\times X',X)}$ acts on the constant field of Hilbert spaces with fiber $L^2(X')$ over X, and for $\xi \in L^2(X')$ we have

$$(\alpha_{(X\times X',X)}(x,s)\xi)(x') = \sqrt{r_{X'}(x',s)}\,\xi(s^{-1}x').$$

Therefore

$$\alpha_{(X\times X',X)} = \mathrm{Res}_X \circ \mathrm{Ind}\, \iota_{X'\rtimes G} = \mathrm{Res}_X(\pi_{X'}). \tag{2.6}$$

In particular, we have

$$\begin{aligned} \alpha_{(X\times G,X)} &= \mathrm{Res}_X(\lambda_G) = \lambda_{X\rtimes G}, \\ \alpha_{(X',\{pt\})} &= \mathrm{Ind}\, \iota_{X'\rtimes G} = \pi_{X'}. \end{aligned}$$

2.3. **Herz majoration principle.**

Given a unitary representation π of G in a Hilbert space $\mathcal{H}$, and a bounded measure m on G we define the operator $\pi(m) \in \mathcal{B}(\mathcal{H})$ by

$$\pi(m)(\xi) = \int_G \pi(s)\,\xi dm(s), \quad \text{for} \quad \xi \in \mathcal{H}.$$

In his study of the Kunze-Stein phenomenon [18], Herz observed that for every homogeneous space G/H and every bounded positive measure m on G we have $\|\lambda_G(m)\| \le \|\lambda_{G/H}(m)\|$, where $\lambda_{G/H}$ is the quasi-regular representation of G into $L^2(G/H)$. In this section, we study some generalizations of this inequality.

Proposition 2.3.1. *Let (X, G, μ) be a measured G-space, and let α be a cocycle representation acting on a measurable field $\mathcal{H}$ of Hilbert spaces. Then, for every bounded positive measure m on G we have $\|Ind\,\alpha(m)\| \le \|\pi_X(m)\|$.*

Proof. Given $\xi, \eta \in \mathcal{H}$, we have

$$\begin{aligned}|\langle \mathrm{Ind}\alpha(s)\xi, \eta\rangle| &= |\int \sqrt{r_X(x,s)}\langle \alpha(x,s)\xi(s^{-1}x), \eta(x)\rangle d\mu(x)| \\ &\le \int \sqrt{r_X(x,s)}\|\xi(s^{-1}x)\|\|\eta(x)\| d\mu(x).\end{aligned}$$

It follows that

$$\begin{aligned}|\int \langle \mathrm{Ind}\alpha(s)\xi, \eta\rangle dm(s)| &\le \int |\langle \mathrm{Ind}\alpha(s)\xi, \eta\rangle| dm(s) \\ &\le \int \sqrt{r_X(x,s)}\|\xi(s^{-1}x)\|\|\eta(x)\| d\mu(x) dm(s).\end{aligned}$$

Let us define $\tilde{\xi}$ by $\tilde{\xi}(x) = \|\xi(x)\|$. Then we have $\tilde{\xi} \in L^2(X, \mu)$, with $\|\tilde{\xi}\|_2 = \|\xi\|$. Similarly, we define $\tilde{\eta}$. Then we get

$$\begin{aligned}|\langle \mathrm{Ind}\alpha(m)\xi, \eta\rangle| &\le \langle \pi_X(m)\tilde{\xi}, \tilde{\eta}\rangle \\ &\le \|\pi_X(m)\|\|\xi\|\|\eta\|.\end{aligned}$$

□

Corollary 2.3.2. *Let (X, G, μ) be a measured G-space and let U be a unitary representation of G. Then, for every bounded positive measure m on G we have $\|(\pi_X \otimes U)(m)\| \le \|\pi_X(m)\|$.*

Proof. We apply the previous proposition to the cocycle $\mathrm{Res}\, U : (x, t) \mapsto U(t)$. □

Corollary 2.3.3. *Let (X, G, μ) be a measured G-space. Then, for every bounded positive measure m on G we have $\|\lambda_G(m)\| \le \|\pi_X(m)\|$.*

Proof. This is the particular case of Corollary 2.3.2 where $U = \lambda_G$. □

Corollary 2.3.4. *Let (Y, X) be a pair of measured G-spaces. Let π_X and π_Y be the unitary representations of G associated with the measured G-spaces X and Y respectively. Then, for every bounded positive measure m on G we have $\|\pi_Y(m)\| \le \|\pi_X(m)\|$.*

Proof. We apply Proposition 2.3.1 to the cocycle representation $\alpha_{(Y,X)}$. □

Remarks 2.3.5. (a) The classical Herz majoration principle is the particular case of Corollary 2.3.3 where X is the transitive G-space G/H.

(b) By applying the above inequalities to the convolution powers m^{*k} instead of m, we get at once the corresponding inequalities for the spectral radii.

Remark 2.3.6. Note that Corollary 2.3.2 is obviously not true when π_X is replaced by any representation of G. Consider for instance the group G of complex numbers of module 1, the representation $\pi : z \mapsto z^n$ where n is a given nonzero integer, and

$U : z \mapsto z^{-n}$. Let m be the Lebesgue measure on G. Then we have $\|\pi(m)\| = 0$ and $\|\pi \otimes U(m)\| = 1$.

On the other hand, Shalom has proved ([29, Lemma 2.3]) that Corollary 2.3.3 is true when π_X is replaced by any representation π having a nonzero vector ξ, positive in the sense that $\langle \pi(s)\xi, \xi \rangle \geq 0$ for every $s \in G$. Of course, the representation π_X has this property. Is Corollary 2.3.2 still true when π_X is replaced by any representation having a nonzero positive vector?

3. Amenable pairs of measured spaces

3.1. **Basic facts.** Let (X, G, μ) be a measured G-space. The algebra $L^1(G)$ acts on $L^\infty(X)$ according to the formula

$$(f * \varphi)(x) = \int f(s)\varphi(s^{-1}x)ds, \quad \forall f \in L^1(G), \forall \varphi \in L^\infty(X).$$

As usually, we let G act on $L^\infty(X)$ by $s.\varphi(x) = \varphi(s^{-1}x)$.

Let us consider a pair (Y, X) of measured G-spaces. Observe that $L^\infty(X)$ is canonically embedded into $L^\infty(Y)$. A *mean* will be a positive unital $L^\infty(X)$-linear map $M : L^\infty(Y) \to L^\infty(X)$. It can be defined equivalently as a norm one projection from $L^\infty(Y)$ onto $L^\infty(X)$. Let us recall two notions of invariance for such a mean.

Definition 3.1.1. We say that a mean M is *invariant* (or G*-equivariant*) if $m(s.\varphi) = s.m(\varphi)$ for every $s \in G$ and every $\varphi \in L^\infty(Y)$. A *topologically invariant mean* is a mean M such that $M(f * \varphi) = f * M(\varphi)$ for every $f \in L^1(G)$ and every $\varphi \in L^\infty(Y)$.

Let us denote by $L^\infty(X)^c$ the subspace of all functions $\varphi \in L^\infty(X)$ such that $s \mapsto s.\varphi$ is norm continuous. We recall first the following classical result (see [3], [16], or [26] for instance):

Proposition 3.1.2. *The following conditions are equivalent:*

(i) *there exists an invariant mean* $M : L^\infty(Y) \to L^\infty(X)$*;*
(ii) *there exists an invariant mean* $M : L^\infty(Y)^c \to L^\infty(X)^c$*;*
(iii) *there exists a topologically invariant mean* $M : L^\infty(Y) \to L^\infty(X)$*.*

Definition 3.1.3. When these equivalent definitions are fulfilled, we say that the pair (Y, X) of measured G-spaces is *amenable*.

We recall, for further use the following result.

Proposition 3.1.4. (see [2, Lemme 2.1] or [4, Prop. 5.3.3]) *Let* (Y_1, X_1)*,* (Y_2, X_2) *be two amenable pairs of measured* G*-spaces. Then* $(Y_1 \times Y_2, X_1 \times X_2)$*, with the diagonal actions, is also an amenable pair.*

It is useful to have characterizations of amenability in terms of L^1 or L^2 approximation properties. We need first to introduce some notations. For $1 \leq p < +\infty$, we define $L^\infty(X, L^p(Y, \rho))$ as the Banach space of $\mu \circ \rho$-measurable functions $g : Y \to \mathbb{C}$ such that $x \mapsto \int |g|^p d\rho^x$ is μ-essentially bounded, normed by $\|g\|_{\infty,p} = (\|\rho(|g|^p)\|_\infty)^{1/p}$. We denote by $B_{L^\infty(X)}(L^\infty(Y), L^\infty(X))$ the Banach space of all bounded $L^\infty(X)$-linear maps from $L^\infty(Y)$ into $L^\infty(X)$. There is an isometric embedding of $L^\infty(X, L^1(Y,\rho))$ into $B_{L^\infty(X)}(L^\infty(Y), L^\infty(X))$ by $g \mapsto m_g$ where $m_g(\varphi) = \rho(g\varphi)$ for $\varphi \in L^\infty(Y)$. Recall ([4, Lemma 1.2.6]) that the convex set formed by the m_g, with g nonnegative in $L^\infty(X, L^1(Y,\rho))$ and such that $\rho(g) \le 1$, is dense in the unit ball of $B_{L^\infty(X)}(L^\infty(Y), L^\infty(X))$ equipped with the weak*-topology.

We define an isometric left action L of G on $L^\infty(X, L^1(Y,\rho))$ by

$$L_s g(y) = c(y,s)g(s^{-1}y), \quad \forall (y,s) \in Y \times G, \forall g \in L^\infty(X, L^1(Y,\rho)).$$

Its integrated form defines the following action of $L^1(G)$:

$$(f * g)(y) = \int_Y f(s)c(y,s)g(s^{-1}y)ds.$$

Note that $L_s(f*g) = (s.f)*g$ for every $s \in G$.

Finally, we shall denote by $L_1^\infty(X, L^p(Y,\rho))$ the set of all elements $g \in L^\infty(X, L^p(Y,\rho))$ with $\rho(|g|^p) = 1$, and by $L_1^\infty(X, L^p(Y,\rho))^+$ the subset of its positive elements.

Theorem 3.1.5. *Let (Y,X) be a pair of measured G-spaces. The following conditions are equivalent:*

(i) *The pair (Y,X) is amenable.*

(ii) *There exists a net (g_i) in $L_1^\infty(X, L^1(Y,\rho))^+$ such that for every $f \in L^1(G)$ satisfying $\int f(t)dt = 1$, we have $\lim_i \rho(|f*g_i - g_i|) = 0$ in the weak*-topology of $L^\infty(X)$.*

(iii) *For every compact subset C of G, every finite subset F of $L^1(X)^+$ and every $\varepsilon > 0$, there exists $g \in L_1^\infty(X, L^1(Y,\rho))^+$ such that*

$$\sup_{(s,h)\in C\times F} \int h(x)\Big(\int |(L_s g)(y) - g(y)| d\rho^x(y)\Big) d\mu(x) \le \varepsilon.$$

Proof. (i) $\Rightarrow$ (ii). Let $M \in B_{L^\infty(X)}(L^\infty(Y), L^\infty(X))$ be a topological invariant mean, and let (g_i) be a net in $L_1^\infty(X, L^1(Y,\rho))^+$ such that $\lim_i m_{g_i} = M$ in the weak*-topology. It follows from the invariance of M that

$$\lim_i \langle m_{g_i}(f*\varphi) - f * m_{g_i}(\varphi), h\rangle = 0$$

for every $\varphi \in L^\infty(Y), f \in L^1(G), h \in L^1(X)$, that is

$$\lim_i \int f(t)h(x)\varphi(t^{-1}y)[c(y,t)g_i(t^{-1}y) - g_i(y)]d\rho^x(y)dt d\mu(x) = 0.$$

We have

$$\int \varphi(t^{-1}y)[c(y,t)g_i(t^{-1}y) - g_i(y)]d\rho^x(y)$$
$$= \int \varphi(y)[c(ty,t)g_i(y) - g_i(ty)]c(y,t^{-1})d\rho^{t^{-1}x}(y)$$

by (2.3), and therefore

$$\begin{aligned}
&\int f(t)h(x)\varphi(t^{-1}y)[c(y,t)g_i(t^{-1}y) - g_i(y)]d\rho^x(y)dtd\mu(x)\\
&\quad = \int f(t)h(x)\varphi(y)[g_i(y) - c(y,t^{-1})g_i(ty)]d\rho^{t^{-1}x}(y)d\mu(x)dt\\
&\quad = \int f(t)h(tx)\varphi(y)[g_i(y) - c(y,t^{-1})g_i(ty)]r_X(x,t^{-1})d\rho^x(y)d\mu(x)dt\\
&\quad = \int \frac{f(t^{-1})}{\Delta(t)}h(t^{-1}x)r_X(x,t)\Big(\int \varphi(y)[g_i(y) - c(y,t)g_i(t^{-1}y)]d\rho^x(y)\Big)d\mu(x)dt,
\end{aligned}$$

where Δ is the modular function of G.

Let us remark that $(x,t)\mapsto \tilde f(x,t) = \dfrac{f(t^{-1})}{\Delta(t)}h(t^{-1}x)r_X(x,t)$ belongs to $L^1(X\times G)$. Hence, we see that

$$\lim_i \int k(x,t)\Big(\int \varphi(y)[g_i(y) - c(y,t)g_i(t^{-1}y)]d\rho^x(y)\Big)d\mu(x)dt = 0$$

for every $k\in L^1(X\times G)$ and every $\varphi\in L^\infty(Y)$.

Now, we use the classical Day-Namioka argument. Let $f_1,\dots,f_k$ be fixed elements in $L^1(X\times G)$. For $1\le i\le k$ and $g\in L^\infty_1(X,L^1(Y,\rho))^+$ we set

$$b_i(g)(y) = \int f_i(q(y),t)[c(y,t)g(t^{-1}y) - g(y)]dt.$$

Let us denote by $\mathcal{C}$ the range of $L^\infty_1(X,L^1(Y,\rho))^+$ in $L^1(Y,\mu\circ\rho)^k$ by the map $g\mapsto (b_1(g),\dots,b_k(g))$. We know that $(0,\dots,0)$ belongs to the closure of $\mathcal{C}$ in $L^1(Y,\mu\circ\rho)^k$ equipped with the product topology, where we consider the weak topology on $L^1(Y,\mu\circ\rho)$. Since $\mathcal{C}$ is convex, we may replace this latter topology by the norm topology, using the Hahn-Banach separation theorem.

Therefore, there exists a net (g_i) in $L^\infty_1(X,L^1(Y,\rho))^+$ such that for every $f\in L^1(G)$ with $\int f(t)dt = 1$ and every $h\in L^1(X)^+$,

$$\begin{aligned}
&\lim_i \int h(x)[\int |f*g_i - g_i|d\rho^x]d\mu(x)\\
&\qquad = \lim_i \int |h\circ q(y)f(t)[c(y,t)g_i(t^{-1}y) - g_i(y)]dt|d\mu\circ\rho(y) = 0.
\end{aligned}$$

(ii) $\Rightarrow$ (i) is obvious since every limit point of (m_{g_i}) is an invariant mean.

(ii) $\Rightarrow$ (iii). Let us consider C and F as in (iii), and $\eta>0$. Let $f\in L^1(G)^+$ such that $\int f(t)dt = 1$, and let V be a neighbourhood of e such that $\|s.f - f\|_1\le\eta$ for every $s\in V$. Then we can find a finite number of elements $s_1,\dots,s_n$ in G such that $C\subset\cup_{i=1}^n s_iV$. We set $s_0 = e$ and we choose $g\in L^\infty_1(X,L^1(Y,\rho))^+$ such that

$$\int h(x)|(s_i.f)*g(y) - g(y)|d\rho^x(y)d\mu(x)\le\eta$$

for $h\in F$ and $0\le i\le n$.

Let $s \in C$ and choose s_i such that $s \in s_i V$. We have

$$\begin{aligned}
&\int h(x)|L_s(f * g)(y) - f * g(y)|d\rho^x(y)d\mu(x)\\
&\leq \int h(x)|(s.f) * g(y) - g(y)|d\rho^x(y)d\mu(x) + \int h(x)|f * g(y) - g(y)|d\rho^x(y)d\mu(x)\\
&\leq \int h(x)|(s.f - s_i.f) * g|d\rho^x(y)d\mu(x) + \int h(x)|s_i.f * g(y) - g(y)|d\rho^x(y)d\mu(x) + \eta\\
&\leq \|s_i^{-1}s.f - f\|_1\|h\|_1 + 2\eta\\
&\leq \eta\|h\|_1 + 2\eta.
\end{aligned}$$

To conclude, it suffices to choose η small enough.

(iii) $\Rightarrow$ (ii) is straightforward. □

Theorem 3.1.6. *Let (Y, X) be a pair of measured G-spaces. The following conditions are equivalent:*

(i) *The pair (Y, X) is amenable.*

(ii) *There exists a net (ξ_i) in $L_1^\infty(X, L^2(Y, \rho))$ such that for every $f \in L^1(X \times G)$, we have*

$$\lim_i \int f(x, s)|\sqrt{c(y, s)}\xi_i(s^{-1}y) - \xi_i(y)|^2 d\rho^x(y)d\mu(x)ds = 0.$$

(iii) *For every compact subset C of G, every finite subset F of $L^1(X)^+$ and every $\varepsilon > 0$, there exists $\xi \in L_1^\infty(X, L^2(Y, \rho))$ such that*

$$\sup_{(s,h)\in C\times F} \int h(x)\Big(\int |(\sqrt{c(y, s)}\xi(s^{-1}y - \xi(y)|^2 d\rho^x(y)\Big)d\mu(x) \leq \varepsilon.$$

Proof. This change from L^1-conditions to L^2-conditions is standard. Given (g_i) as in Theorem 3.1.5 (ii), we set $\xi_i = \sqrt{g_i}$ and we check that (ξ_i) satisfies condition (ii) of Theorem 3.1.6. Conversely, given (ξ_i) we set $g_i = |\xi_i|^2$. □

Definition 3.1.7. Let (X, G, μ) be a measured G-space. We equip $X \times G$ with the diagonal action and the measure $\mu \otimes \lambda$ where λ is the left Haar measure. Note that $X \times G$ is an extension of X with invariant q-system of measures $\{\delta_x \otimes \lambda : x \in X\}$, q being here the first projection. We say that (X, G, μ) is *amenable (in Zimmer's sense)* (or that X is an *amenable measured G-space*) if the pair $(X \times G, X)$ is amenable.

Remark 3.1.8. Amenable measured G-spaces were first introduced by Zimmer in [34], in terms of a conditional fixed point condition. The equivalence of his definition with the existence of an invariant mean $M : L^\infty(X \times G) \to L^\infty(X)$ is established in [32], [33] for G discrete, and later on for any separable locally compact group in [1].

By Theorem 3.1.6, the amenability of the measured G-space X is equivalent to the existence of a net (ξ_i) of Borel fonctions on $X \times G$ such that

(a) $\int |\xi_i(x, t)|^2 dt = 1$ for μ-almost every x;

(b) $\lim_i \int f(x,s)|\xi_i(s^{-1}x, s^{-1}t) - \xi_i(x,t)|^2 dt d\mu(x) ds = 0$ for every $f \in L^1(X \times G)$.

Definition 3.1.9. We say that a measured G-space (X, G, μ) is *amenable in the sense of Greenleaf* if the pair $(X, \{pt\})$ of G-spaces is amenable.

This notion was introduced and studied by Greenleaf in [14]. The particular case where X is a homogeneous space G/H was studied in details by Eymard in [11]. Amenable G-spaces, unless otherwise specified, will always be amenable in Zimmer's sense.

3.2. **Norm estimates for amenable pairs.** Given two unitary representations π_1, π_2 of G we say that π_1 is *weakly contained into* π_2, and we write $\pi_1 \prec \pi_2$, if for every finite subset F of $L^1(G)$, every $\xi \in \mathcal{H}_{\pi_1}$ (Hilbert space of the representation π_1) and every $\varepsilon > 0$, there exist a finite number of vectors $\eta_1, \dots, \eta_p$ in $\mathcal{H}_{\pi_2}$ such that

$$\sup_{f \in F} |\langle \xi, \pi_1(f)\xi \rangle - \sum_{i=1}^{p} \langle \eta_i, \pi_2(f)\eta_i \rangle| \le \varepsilon.$$

A fundamental observation is the equivalence of this definition with the fact that $\|\pi_1(f)\| \le \|\pi_2(f)\|$ for every $f \in L^1(G)$ (see [9, Theorem 3.4.4]). By a standard regularisation argument (see the proof of the following theorem) it is also equivalent to $\|\pi_1(m)\| \le \|\pi_2(m)\|$ for every bounded measure m on G. For further details about this important notion in group representation theory, we refer to [9] and [12].

Given an operator T acting on a Hilbert space H, we shall denote by $r(T)$ its spectral radius.

Theorem 3.2.1. *Let (Y, X) be an amenable pair of measured G-spaces. Then the representation π_X is weakly contained into the representation π_Y. In other words, for every $f \in L^1(G)$ we have*

$$\|\pi_X(f)\| \le \|\pi_Y(f)\|.$$

Moreover, for every bounded positive measure m on G, we have $\|\pi_X(m)\| = \|\pi_Y(m)\|$ and $r(\pi_X(m)) = r(\pi_Y(m))$.

Proof. Let $v \in L^2(X)$ and let us consider a net (ξ_i) as in Theorem 3.1.6 (ii). Let us introduce $v_i = \xi_i(v \circ q)$. Then v_i belongs to $L^2(Y)$ and $\|v_i\|_2 = \|v\|_2$. For $t \in G$, we have

$$\begin{aligned}\langle v_i, \pi_Y(t) v_i \rangle &= \int \overline{v_i(y)} \sqrt{c(y,t)} \sqrt{r_X(q(y),t)}\, v_i(t^{-1}y) d\rho^x(y) d\mu(x) \\ &= \int \overline{v(x)} v(t^{-1}x) \sqrt{r_X(x,t)} [\int \overline{\xi_i(y)} \sqrt{c(y,t)} \xi_i(t^{-1}y) d\rho^x(y)] d\mu(x)\end{aligned}$$

and, for $f \in L^1(G)$,

$$\langle v_i, \pi_Y(f) v_i \rangle = \int f(t) \overline{v(x)} v(t^{-1}x) \sqrt{r_X(x,t)} [\int \overline{\xi_i(y)} \sqrt{c(y,t)} \xi_i(t^{-1}y) d\rho^x(y)] d\mu(x) dt.$$

Observe that $(x,t) \mapsto h(x,t) := f(t)\overline{v(x)}v(t^{-1}x)\sqrt{r_X(x,t)}$ belongs to $L^1(X \times G)$ so that, using condition (ii) of Theorem 3.1.6, we get

$$\lim_i \int h(x,t)[\int \overline{\xi_i(y)}\sqrt{c(y,t)}\xi_i(t^{-1}y)d\rho^x(y)]d\mu(x)dt = \int h(x,t)d\mu(x)dt,$$

that is

$$\lim_i \langle v_i, \pi_Y(f)v_i\rangle = \langle v, \pi_X(f)v\rangle.$$

This implies the weak containment of π_X into π_Y, that is $\|\pi_X(f)\| \le \|\pi_Y(f)\|$ for every $f \in L^1(G)$.

Remark that the weak containment of π_X into π_Y also implies that $\|\pi_X(m)\| \le \|\pi_Y(m)\|$ for every bounded measure m on G. Indeed, given $\varepsilon > 0$ we may choose $\xi \in L^2(X)$ and $f \in L^1(G)$ with $\|\xi\|_2 = 1$ and $\|f\|_1 = 1$ such that

$$\|\pi_X(m)\pi_X(f)\xi\| \ge \|\pi_X(m)\| - \varepsilon.$$

It follows that

$$\|\pi_Y(m)\| \ge \|\pi_Y(m * f)\| \ge \|\pi_X(m * f)\| \ge \|\pi_X(m)\pi_X(f)\xi\| \ge \|\pi_X(m)\| - \varepsilon.$$

Therefore, we have $\|\pi_Y(m)\| \ge \|\pi_X(m)\|$.

When m is positive the equality $\|\pi_X(m)\| = \|\pi_Y(m)\|$ is then a consequence of Corollary 2.3.4.

Finally, recall that $r(\pi_X(m)) = \lim_k \|\pi_X(m)^k\|^{1/k} = \lim_k \|\pi_X(m^{*k})\|^{1/k}$, Then the equality of the spectral radii follows from the previous paragraph applied to the bounded positive measure m^{*k}. □

Corollary 3.2.2. *Let X be an amenable measured G-space. Then for every $f \in L^1(G)$ we have $\|\pi_X(f)\| \le \|\lambda_G(f)\|$. Moreover, for every bounded positive measure m on G, we have $\|\pi_X(m)\| = \|\lambda_G(m)\|$ and $r(\pi_X(m)) = r(\lambda_G(m))$.*

Proof. We apply Theorem 3.2.1 to the pair $(Y = X \times G, X)$. Then

$$\pi_Y = \pi_X \otimes \lambda_G \simeq \infty.\lambda_G$$

and therefore for every $f \in L^1(G)$ we have $\|\pi_X(f)\| \le \|\pi_Y(f)\| = \|\lambda_G(f)\|$. □

This result extends to every cocycle representation of (X, G, μ).

Theorem 3.2.3. *Let X be an amenable measured G-space. Then for every $\alpha \in Rep(X \rtimes G, \mu)$ and every $f \in L^1(G)$ we have $\|Ind\,\alpha(f)\| \le \|\lambda_G(f)\|$, with equality if $f \ge 0$. In particular, for every representation U of G and every $f \in L^1(G)$ we have $\|(\pi_X \otimes U)(f)\| \le \|\lambda_G(f)\|$.*

Proof. We follow the same pattern as in the proof of theorem 3.2.1. Let us choose a net (ξ_i) as in Remark 3.1.8. We shall write $\xi_i(x)(t) = \xi_i(x,t)$. Note that $\xi_i(x) \in L^2(G)$ and $\|\xi_i(x)\|_2 = 1$ almost everywhere. Assume that α acts on $\mathcal{H} = \{\mathcal{H}(x) : x \in X\}$. We consider a norm one vector $v \in L^2(\mathcal{H}) = \int_X^{\oplus} \mathcal{H}(x)d\mu(x)$ and define

$$v_i : x \mapsto v_i(x) = \xi_i(x) \otimes v(x) \in L^2(G) \otimes \mathcal{H}(x).$$

Then $v_i \in L^2(G) \otimes L^2(\mathcal{H}) = \int_X^{\oplus} L^2(G) \otimes \mathcal{H}(x) d\mu(x)$ and

$$\|v_i\|^2 = \int \|\xi_i(x)\|_2^2 \|v(x)\|^2 d\mu(x) = 1.$$

Exactly as in the proof of Theorem 3.2.1 we show that for every $f \in L^2(G)$,

$$\lim_i \langle v_i, (\lambda_G \otimes \operatorname{Ind} \alpha)(f) v_i \rangle = \langle v, \operatorname{Ind} \alpha(f) v \rangle.$$

Therefore $\operatorname{Ind} \alpha$ is weakly contained into $\lambda_G \otimes \operatorname{Ind} \alpha$, which is weakly equivalent to λ_G.

Finally, if we apply this result to the cocycle representation $\operatorname{Res} U$, we get that $\pi_X \otimes U$ is weakly contained into λ_G. □

Remark 3.2.4. The previous result was first proved by G. Kuhn, for discrete group actions, in [23].

4. Characterization of amenabiliy by norm estimates

In this section, we study the converse of Theorem 3.2.1.

4.1. **Characterization of amenability in Greenleaf's sense.** Let us recall below that the converse of theorem 3.2.1 is true when X is reduced to a point. We shall say that a measure m on G is *adapted* if the closed subgroup of G generated by the support of m is G itself.

Proposition 4.1.1. ([5], [8], [16]) *Let (Y, ν) be a measured G-space. Then the following conditions are equivalent:*

- (i) *There exists a G-invariant mean on $L^\infty(Y)$.*
- (ii) *The trivial representation of G is weakly contained in π_Y.*
- (iii) *There exists an adapted probability measure m on G such that the spectral radius $r(\pi_Y(m))$ is equal to 1.*

Observe that the equivalence between (i) and (ii) is the particular case of Theorem 3.1.6 where X is reduced to a point. When m is a symmetric adapted probability measure on a countable discrete group G and $(Y, \nu) = (G, \lambda)$, condition (iii) is Kesten's characterisation of amenability for G (see [22] and also [7]).

In view of the observation following Theorem 5.3 below, it is remarkable that in Proposition 4.1.1 the implication (iii) $\Rightarrow$ (i) is true without assuming m to be symmetric. For the reader's convenience, we recall briefly its proof given in [16].

Proof of (iii) $\Rightarrow$ (i). Let α be an element of the spectrum of $\pi_Y(m)$ such that $|\alpha| = 1$. There exists a sequence (ξ_n) in $L^2(Y)$ of norm one elements with

$$\lim_n \|\pi_Y(m)\xi_n - \alpha\xi_n\|_2 = 0;$$

therefore we have $\lim_n \langle \pi_Y(m)\xi_n, \xi_n \rangle = \alpha$. Since

$$|\langle \pi_Y(m)\xi_n, \xi_n \rangle| \leq \int \langle \pi_Y(t)|\xi_n|, |\xi_n| \rangle dm(t) \leq 1,$$

we see that $\lim_n \int \langle \pi_Y(t)|\xi_n|, |\xi_n|\rangle dm(t) = 1$. It follows that

$$\lim_n \int \|(\pi_Y(t)|\xi_n|)^2 - |\xi_n|^2\|_1 dm(t) = 0.$$

By taking a subsequence of $(|\xi_n|)$, we find a sequence (h_n) of norm one non-negative elements in $L^1(Y)$ such that $\lim_n \|\pi_Y(t)h_n - h_n\|_1 = 0$ for every t in a subset E of G whose complement has m-measure zero.

Let us denote by M_n the state defined on $L^\infty(Y)^c$ by $M_n(f) = \int f(y)h_n(y)d\nu(y)$, and let M be an accumulation point of the sequence (M_n) in $(L^\infty(Y)^c)^*$ endowed with the weak*-topology. We have $M(\pi_Y(t)f) = M(f)$ for every $f \in L^\infty(Y)^c$ and $t \in E^{-1}$. The set F of elements $t \in G$ such that $M(\pi_Y(t)f) = M(f)$ for every $f \in L^\infty(Y)^c$ is a closed subgroup of G, with $m(G \setminus F) = 0$. It follows that $F = G$ since m is an adapted probability measure. To conclude we use Proposition 3.1.2. □

We shall see now that even the equivalence between (i) and (ii) in Proposition 4.1.1 does not extend to any pair (Y, X) of measured G-spaces.

4.2. **A negative answer to problem (A_1) for a transitive action.** Let H be a closed subgroup of a locally compact group G. The homogeneous space G/H carries only one class of quasi-invariant measures. Zimmer proved in [34] that the measured G-space G/H is amenable if and only if H is an amenable group. Corollary 3.2.2 implies that when H is amenable the quasi-regular representation $\lambda_{G/H}$ is weakly contained into the regular representation λ_G. This fact follows also immediately from the continuity of the induction map.

However the converse is not true, as it is shown by the following example that B. Bekka told me. Let us consider the groups

$$H = SL(2,\mathbb{R}) \subset H' = SL(2,\mathbb{R}) \rtimes \mathbb{R}^2 \subset G = SL(3,\mathbb{R}).$$

The action of $SL(2,\mathbb{R})$ on (the dual of) $\mathbb{R}^2$ has two orbits: $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$. The stabilizer of 0 is $SL(2,\mathbb{R})$ and the stabilizer of $(0,1)$ is the group of upper triangular matrices with diagonal entries equal to 1. It follows from Mackey analysis of representations of such semi-direct products that the unitary dual $\hat{H}'$ of H' is a disjoint union

$$\hat{H}' = \widehat{SL(2,\mathbb{R})} \cup E$$

where the elements of E are weakly contained into $\lambda_{H'}$.

Let π be a representation of H into $\mathcal{H}_\pi$. Its induced representation $\mathrm{Ind}_H^{H'}(\pi)$ to H' is the direct sum of a representation π_1 which is trivial on $\mathbb{R}^2$ and a representation π_2 which is weakly contained into $\lambda_{H'}$. Assume that $\pi_1 \neq 0$. Let f be a non-zero vector in the Hilbert space of the subrepresentation π_1 of $\mathrm{Ind}_H^{H'}(\pi)$. Recall that f is a function from H' to $\mathcal{H}_\pi$ such that $f(xh) = \pi(h^{-1})f(x)$ for every $h \in H$ and $\int_{H'/H} \|f(\dot{x})\|^2 d\dot{x} < +\infty$, where $d\dot{x}$ is the Lebesgue measure on on $H'/H = \mathbb{R}^2$. Since π_1 is trivial on $\mathbb{R}^2$ we see that $f(tx) = f(x)$ almost everywhere for every $t \in \mathbb{R}^2$, but this is impossible because $\dot{x} \mapsto \|f(\dot{x})\|$ is non-zero and belongs to $L^2(\mathbb{R}^2)$. Therefore we have $\pi_1 = 0$. It follows that $\mathrm{Ind}_H^{H'}(\pi)$ is weakly contained into $\lambda_{H'}$. Now, using the theorem of induction in stages and the continuity of the induction map, we see that $\mathrm{Ind}_H^G(\pi)$ is weakly contained into λ_G. This gives an example where every induced representation from H to G is weakly contained into λ_G although H is not amenable.

What is missing here is that the trivial representation of H is not weakly contained into the restriction of the quasi-regular representation $\lambda_{G/H}$ of G to H.

Proposition 4.2.1. *Let H be a closed subgroup of a locally compact group G. The following conditions are equivalent:*

(i) *H is an amenable group;*
(ii) *the trivial representation ι_H of H is weakly contained into the restriction to H of the quasi-regular representation $\lambda_{G/H}$, and $\lambda_{G/H}$ is weakly contained into λ_G.*

Proof. Assume first that H is amenable. Then the trivial representation ι_H of H is weakly contained into its regular representation λ_H. Inducing these representations to G we see that $\lambda_{G/H}$ is weakly contained into λ_G (see [12]). Moreover, since H is amenable, there exists an H-invariant state on $L^\infty(G/H)$. Then we use Proposition 4.1.1 to conclude that ι_H of H is weakly contained into the restriction to H of the quasi-regular representation $\lambda_{G/H}$.

Conversely, assume that assertion (ii) is true. In particular, the restriction to H of the quasi-regular representation $\lambda_{G/H}$ is weakly contained into the restriction of λ_G to H. Since the latter is weakly equivalent to λ_H, we get immediately that ι_H is weakly contained into λ_H. □

This characterization of the amenability of the homogeneous G-space G/H will be extended in the next two subsections to any measured G-space.

4.3. **Characterization of amenability in Zimmer's sense.** Let (X, G, μ) be a measured G-space. Then $X \times X$ is equipped with the diagonal G-action and the measure $\mu \otimes \mu$. The first projection turns $(X \times X, X)$ into a pair of measured G-spaces.

Theorem 4.3.1. *Let (X, G, μ) be a measured G-space. The following conditions are equivalent:*

(i) *(X, G, μ) is amenable in Zimmer's sense.*
(ii) *The representation π_X is weakly contained into the regular representation of G and moreover there exists an invariant mean $M : L^\infty(X \times X) \to L^\infty(X)$.*

Before giving the proof, let us recall first the following result.

Proposition 4.3.2. ([2], [35]). *Let (Y, X) be a pair of measured G-spaces. If X is an amenable G-space in Zimmer's sense, then the pair (Y, X) is amenable.*

Proof. Let us recall briefly the proof for the reader's convenience. Let $E : L^\infty(Y) \to L^\infty(X)$ be a norm one projection. Consider $\varphi \in L^\infty(Y)^c$. For $f \in L^1(X)$, the map $s \mapsto \langle s.(E(s^{-1}.\varphi)), f\rangle$ is continuous and bounded on G and therefore $s \mapsto s.(E(s^{-1}.\varphi))$ belongs to $L^\infty(X \times G) = L^\infty(G, L^\infty(X))$. Let us denote by Φ the map from $L^\infty(Y)^c$ to $L^\infty(X \times G)$ so defined. Observe that $\Phi(\varphi) = \varphi$ for every $\varphi \in L^\infty(X)^c$ and that Φ is G-equivariant, $X \times G$ being endowed with the diagonal left G-action.

Let M be an invariant norm one projection from $L^\infty(X \times G)$ onto $L^\infty(X)$. Then $P = M \circ \Phi$ is a G-equivariant norm one linear map from $L^\infty(Y)^c$ to $L^\infty(X)$ such that $P(\varphi) = \varphi$ for every $\varphi \in L^\infty(X)^c$. Obviously $P(L^\infty(Y)^c) = L^\infty(X)^c$ and therefore condition (ii) of Proposition 3.1.2 is fulfilled. □

Proof of (i) ⇒ (ii) in Theorem 4.3.1. It follows immediately from Corollary 3.2.2 and the previous Proposition. □

Now to prove (ii) ⇒ (i) in Theorem 4.3.1, we need some tedious preliminaries. First, we shall define a topology on Rep $(X \rtimes G, \mu)$. Consider a cocycle representation α acting on the field $\mathcal{H} = \{\mathcal{H}(x) : x \in X\}$ of Hilbert spaces. We denote by $L^\infty(X, \mathcal{H})$ the Banach space of essentially bounded measurable sections of $\mathcal{H}$, and by $L^\infty_1(X, \mathcal{H})$ the subset of sections ξ such that $\|\xi(x)\| = 1$ for μ-almost every x.

Let $\varepsilon > 0$, $\Xi = \{\xi_1, \dots, \xi_n\}$ a finite subset of $L^\infty_1(X, \mathcal{H})$ and F a finite subset of $L^1(X \times G)$ be given. We denote by $V_{\varepsilon,\Xi,F}$ the set of $\alpha' \in$ Rep $(X \rtimes G, \mu)$ such that there exist $\xi'_i \in L^\infty_1(X, \mathcal{H}')$, $1 \le i \le n$ with

$$|\int f(x,t)\big(\langle \xi_j(x), \alpha(x,t)\xi_i(t^{-1}x)\rangle - \langle \xi'_j(x), \alpha'(x,t)\xi'_i(t^{-1}x)\rangle\big) d\mu(x)dt| \le \varepsilon$$

for $f \in F$ and $1 \le i, j \le n$.

We equip Rep $(X \rtimes G, \mu)$ with the topology for which every α admits these $V_{\varepsilon,\Xi,F}$'s as a basis of neighbourhoods. When X is reduced to a point, this gives th Fell topology on Rep(G) ([12]).

The trivial cocycle $\iota_{X \rtimes G}$ has a nicer basis of neighbourhoods that we describe now. Given $\varepsilon > 0$, and a finite subset F of $L^1(X \times G)$, let us denote by $W_{\varepsilon,F}$ the set of $\alpha' \in$ Rep $(X \rtimes G, \mu)$ such that there exists $\xi' \in L^\infty_1(X, \mathcal{H}')$ with

$$|\int f(x,t)\|\alpha'(x,t)\xi'(t^{-1}x) - \xi'(x)\|^2 d\mu(x)dt| \le \varepsilon$$

for all $f \in F$.

Lemma 4.3.3. *The family of sets $W_{\varepsilon,F}$ is a basis of neighbourhoods of the trivial cocycle $\iota_{X \rtimes G}$.*

Proof. Given $\varepsilon > 0$, and a finite subset F of $L^1(X \times G)$, let us denote by $V_{\varepsilon,F}$ the set of $\alpha' \in$ Rep$(X \rtimes G, \mu)$ such that there exists $\xi' \in L^\infty_1(X, \mathcal{H}')$ with

$$|\int f(x,t)\Big(1 - \langle \xi'(x), \alpha'(x,t)\xi'(t^{-1}x)\rangle\Big) d\mu(x)dt| \le \varepsilon$$

for every $f \in F$. These sets $V_{\varepsilon,F}$ form obviously a basis of neighbourhoods of $\iota_{X \rtimes G}$. The lemma is then an easy consequence of the equality

$$\|\alpha'(x,t)\xi'(t^{-1}x) - \xi'(x)\|^2 = 2\mathrm{Re}\Big(1 - \langle \xi'(x), \alpha'(x,t)\xi'(t^{-1}x)\rangle\Big),$$

and of the inequality

$$|1 - \langle \xi'(x), \alpha'(x,t)\xi'(t^{-1}x)\rangle| \leq \|\alpha'(x,t)\xi'(t^{-1}x) - \xi'(x)\|.$$

□

Remark 4.3.4. Thus Theorem 3.1.6 says that a pair (Y, X) of measured G-spaces is amenable if and only if the trivial cocycle $\iota_{X\rtimes G}$ belongs to the closure of the canonical cocycle representation $\alpha_{(Y,X)}$ associated with the pair.

In particular, X is an amenable measured G space if and only if $\iota_{X\rtimes G}$ belongs to the closure of $\mathrm{Res}\,\lambda_G$, and the pair $(X \times X, X)$ is amenable if and only if $\iota_{X\rtimes G}$ belongs to the closure of $\alpha_{(X\times X,X)} = \mathrm{Res}\,\pi_X$.

Obviously every cocycle representation belongs to the closure of any of its multiple. The converse is not true but the following result will be useful.

Lemma 4.3.5. *Let (Y, X) be a pair of measured G-spaces. Then the trivial cocycle $\iota_{X\rtimes G}$ belongs to the closure of $\infty.\alpha_{(Y,X)}$ if and only if it belongs the closure of $\alpha_{(Y,X)}$.*

Proof. Let K be a Hilbert space and assume that $\iota_{X\rtimes G}$ belongs to the closure of $\alpha_{(Y,X)} \otimes \mathrm{Id}_K$. There exist a net (ξ_i), where $\xi_i \in L^2(Y) \otimes K$ satisfies

$$\int \|\xi_i(y)\|_K^2 d\rho^x(y) = 1$$

almost everywhere, such that

$$\lim_i \int \|\sqrt{c(y,t)}\xi_i(t^{-1}y) - \xi_(y)\|_K^2 d\rho^x(y) = 0$$

in the weak*-topology of $L^\infty(X \times G)$.

Let us set $\tilde{\xi}_i(y) = \|\xi_i(y)\|_K$. We have

$$\begin{aligned}
\int |\sqrt{(c(y,t)}\tilde{\xi}_i(t^{-1}y) - \tilde{\xi}_i(y)|^2 d\rho^x(y) \\
&\leq \int |c(y,t)\tilde{\xi}_i(t^{-1}y)^2 - \tilde{\xi}_i(y)^2| d\rho^x(y) \\
&\leq \int |\sqrt{(c(y,t)}\tilde{\xi}_i(t^{-1}y) + \tilde{\xi}_i(y)||\sqrt{(c(y,t)}\tilde{\xi}_i(t^{-1}y) - \tilde{\xi}_i(y)| d\rho^x(y) \\
&\leq 2\Big(\int |\sqrt{(c(y,t)}\tilde{\xi}_i(t^{-1}y) - \tilde{\xi}_i(y)|^2 d\rho^x(y)\Big)^{1/2} \\
&\leq 2\Big(\int \|\sqrt{(c(y,t)}\xi_i(t^{-1}y) - \xi_i(y)\|_K^2 d\rho^x(y)\Big)^{1/2}.
\end{aligned}$$

It follows that

$$\lim_i \int |\sqrt{(c(y,t)}\tilde{\xi}_i(t^{-1}y) - \tilde{\xi}_i(y)|^2 d\rho^x(y) = 0$$

in the weak*-topology of $L^\infty(X \times G)$, and that $\iota_{X\rtimes G}$ belongs the the closure of $\alpha_{(Y,X)}$. □

Lemma 4.3.6. *The restriction map $Rep(G) \to Rep(X \rtimes G, \mu)$ is continuous.*

Proof. Let π_0 be a representation of G in a Hilbert space $\mathcal{H}_0$ and set $\alpha_0 = \mathrm{Res}\pi_0$. We consider a neighbourhood $V_{\varepsilon,\Xi,,F}$ of α_0 where $\Xi = \{\xi_1, \dots, \xi_n\}$. Let $(\epsilon_n)_{n\geq 1}$ be an orthonormal basis of $\mathcal{H}_0$. Without loss of generality we may assume the existence of an integer N such that for $1 \leq i \leq n$, and μ-almost every $x \in X$,

$$\xi_i(x) = \sum_{k=1}^{N} c_{k,i}(x)\epsilon_k.$$

We may also assume that $F = \{g_1, \dots, g_m\}$ where every g_k is of the form $g_k(x,t) = f_k(x)h_k(t)$ with $f_k \in L^1(X)$ and h_k continuous and compactly supported. Define $K = \cup_{k=1}^{n} \mathrm{Supp}\, h_k \cup \{e\}$.

Given $\eta > 0$, we consider then the neighbourhood V_η of π_0 in $\mathrm{Rep}(G)$ consisting of all $\pi' \in \mathrm{Rep}(G)$ such that there exist $\epsilon'_1, \dots, \epsilon'_N$ in $\mathcal{H}_\pi$ with

$$\sup_{t\in K} |\langle \pi_0(t)\epsilon_r, \epsilon_s\rangle - \langle \pi'(t)\epsilon'_r, \epsilon'_s\rangle| < \eta$$

for $1 \leq r, s \leq N$. Observe that we can modify $(\epsilon'_i)_{1\leq i\leq N}$ so that it becomes orthonormal. Define

$$\xi'_i(x) = \sum_{k=1}^{N} c_{k,i}(x)\epsilon'_k.$$

Then, with $\alpha' := \mathrm{Res}\pi'$ we get

$$
\begin{aligned}
&|\int f_k(x)h_k(t)\Big(\langle \xi_j(x), \alpha_0(x,t)\xi_i(t^{-1}x)\rangle - \langle \xi'_j(x), \alpha'(x,t)\xi'_i(t^{-1}x)\rangle\Big)d\mu(x)dt| \\
&= |\int f_k(x)h_k(t)\Big(\sum_{r,s=1}^{N} \overline{c_{r,j}(x)}c_{s,i}(t^{-1}x)(\langle \epsilon_r, \pi_0(t)\epsilon_s\rangle - \langle \epsilon'_r, \pi'(t)\epsilon'_s\rangle)\Big)d\mu(x)dt| \\
&\leq \int |f_k(x)h_k(t)|\Big(\sum_{r,s=1}^{N} |c_{r,j}(x)||c_{s,i}(t^{-1}x)||(\langle \epsilon_r, \pi_0(t)\epsilon_s\rangle - \langle \epsilon'_r, \pi'(t)\epsilon'_s\rangle)|\Big)d\mu(x)dt \\
&\leq \eta \int |f_k(x)h_k(t)|(\sum_{r,s=1}^{N} |c_{r,j}(x)||c_{s,i}(t^{-1}x)|)d\mu(x)dt \\
&\leq N^2\eta \int |f_k(x)h_k(t)|d\mu(x)dt.
\end{aligned}
$$

If we choose η small enough such that $N^2\eta \int |f_k(x)h_k(t)|d\mu(x)dt \leq \varepsilon$ for $k = 1, \dots, m$ we see that $\mathrm{Res}(V_\eta) \subset V_{\varepsilon,\Xi,F}$. □

Remark 4.3.7. One can also show that the induction map is continuous, so that Theorem 3.2.1 is then an immediate consequence of Remark 4.3.4 and of the equalities $\mathrm{Ind}\, \iota_{X\rtimes G} = \pi_X$, $\mathrm{Ind}\, \alpha_{(Y,X)} = \pi_Y$.

Proof of (ii) ⇒ (i) in Theorem 4.3.1. Assume that (ii) is fulfilled. Since the representation π_X is weakly contained into λ_G, it belongs to the closure of $\infty.\lambda_G$. It follows from Lemma 4.3.6 that $\mathrm{Res}\, \pi_X$ belongs to the closure of $\mathrm{Res}\, \infty.\lambda_G = \infty.\mathrm{Res}\, \lambda_G$.

Moreover, using Remark 4.3.4, we see that $\iota_{X\rtimes G}$ belongs to the closure of the cocycle representation associated with the pair $(X\times X, X)$, that is $\mathrm{Res}\,\pi_X$ (see (2.6)). Finally, by Lemma 4.3.5, we get that $\iota_{X\rtimes G}$ is in the closure of $\mathrm{Res}\,\lambda_G$. To conclude, we observe that $\mathrm{Res}\,\lambda_G = \lambda_{X\rtimes G} = \alpha_{(X\times G, X)}$ and we use again Remark 4.3.4. $\square$

4.4. **Existence of invariant means from $L^\infty(X\times X)$ onto $L^\infty(X)$.** Let (X,μ) be a measured G-space. The existence of an invariant mean from $L^\infty(X\times X)$ onto $L^\infty(X)$ is fulfilled when the G-space X is amenable in the sense of Zimmer (see Proposition 4.3.2) or in the sense of Greenleaf (see Proposition 3.1.4). It is also satisfied when (X,μ) is a discrete space endowed with the counting measure, or when (X, G,μ) is a normal measured space. Indeed, in this last case, we have $\mathrm{Res}\,\pi_X \simeq \infty.\iota_{X\rtimes G}$, and therefore $\iota_{X\rtimes G}$ is a cocycle subrepresentation of $\mathrm{Res}\pi_X$.

In particular, for a transitive G-space G/H, there exists a G-invariant norm one projection from $L^\infty(G/H\times G/H)$ onto $L^\infty(G/H)$ when H is amenable or when H is an open or a normal subgroup of G. Theorem 4.3.1 and Section 4.2 show that there does not exist a G-invariant norm one projection from $L^\infty(G/H\times G/H)$ onto $L^\infty(G/H)$ when $G = SL(3,\mathbb{R})$ and $H = SL(2,\mathbb{R})$.

Let us show now that the existence of a G-invariant mean $M : L^\infty(G/H\times G/H) \to L^\infty(G/H)$ is equivalent to the weak containment of the trivial representation of H into the restriction of $\lambda_{G/H}$ to H. Before, we need to recall useful facts concerning Morita equivalence of group actions. For further explainations, we refer to Section 3.2 and Appendix A.1 of [4].

Definition 4.4.1. (see [4, Def. A.1.11]) Let K, H be two locally compact groups, X a left Borel K-space and Y a right Borel H-space. A *Borel equivalence* (or *Borel Morita equivalence*) between X and Y is a triple (Z, p_X, p_Y) made of a Borel space Z and Borel surjections $p_X : Z \to X$, $p_Y : Z \to Y$, such that:

(i) Z is a free and proper left K-space and p_X is K-equivariant;
(ii) Z is a free and proper right H-space and p_Y is H-equivariant;
(iii) the left K-action and the right H-action on Z commute;
(iv) $p_X : Z \to X$ induces a Borel isomorphism between Z/H and X;
(v) $p_Y : Z \to Y$ induces a Borel isomorphism between $K \setminus Z$ and Y.

We then say that the K-space X and the H-space Y are *Borel equivalent*.

Example 4.4.2. Let G be a locally compact group and let H, K be two closed subgroups of G. Then the left K-space G/H and the right H-space $K\setminus G$ are Borel equivalent. Indeed, we take $Z = G$ and $p_H : G \to G/H$, $p_K : G \to K\setminus G$ are the quotient maps.

In the rest of this section, we shall only consider this example of Borel equivalence of group actions. Let X be a right Borel H-space and let $p : X \to K\setminus G$ be a Borel H-equivariant surjection. Let

$$G * X = \{(s, x) \in G\times X : p_K(s) = p(x)\},$$

on which H acts diagonally to the right. We set $\underline{X} = (G * X)/H$, and for $(s, x) \in G * X$, we denote by $[s, x]$ its equivalence class. Finally, we introduce $\underline{p} : \underline{X} \to G/H$

by $\underline{p}([s,x]) = p_H(s)$. We observe that $\underline{X}$ has the following natural structure of left K-space which makes $\underline{p}$ equivariant with respect to K:

$$k.[s,x] = [ks,x], \quad \text{for} \quad k \in K, [s,x] \in \underline{X}.$$

In this way we get a functorial equivalence between pairs (X,p) of right H-spaces X and H-equivariant Borel surjections $p : X \to K \setminus G$ and pairs $(\underline{X},\underline{p})$ of left K-spaces $\underline{X}$ and K-equivariant Borel surjections $\underline{p} : \underline{X} \to G/H$.

Assume moreover that a quasi-invariant probability measure μ is given on X and denote by $\underline{\mu}$ a pseudo-image of the restriction of $\lambda \otimes \mu$ to $G * X$ by the quotient map $G * X :\to \underline{X}$. Then $(\underline{X},\underline{\mu})$ is a measured K-space.

Finally, let (Y,ν) be another measured right H-space and let $q : Y \to X$ be a Borel H-equivariant surjective map with $q_*\nu = \mu$. Defining $\underline{q} : \underline{Y} \to \underline{X}$ by $\underline{q}([s,y]) = [s,q(y)]$ we easily get a pair $(\underline{Y},\underline{X})$ of measured left K-spaces.

The invariance of amenability by Morita equivalence is a basic and very useful result. We shall need the following particular case of [4, Theorem 3.2.2].

Theorem 4.4.3. *Let K,H be two closed subgroups of a locally compact group G. Let (Y,X) be a measured pair of right H-spaces. We assume that X is equipped with a Borel H-equivariant surjective map $p : X \to K \setminus G$ as above. Let $(\underline{Y},\underline{X})$ be the associated measured pair of left K-spaces. Then the pair (Y,X) is amenable if and only if the pair $(\underline{Y},\underline{X})$ is amenable.*

Corollary 4.4.4. *Let H be a closed subgroup of a locally compact group G and let (Y,X) be a pair of measured right H-spaces. The quotient spaces $(G \times Y)/H$ and $(G \times X)/H$ for the right diagonal H-actions are obviously left measured G-spaces. Then the pair (Y,X) of measured H-spaces is amenable if and only if the pair $((G \times Y)/H, (G \times X)/H)$ of left measured G-spaces is amenable.*

Proof. We apply Theorem 4.4.3 with $K = G$. □

Corollary 4.4.5. *Let H be a closed subgroup of a locally compact group G. Then H is an amenable group if and only if there exists a G-invariant mean from $L^\infty(G)$ onto $L^\infty(G/H)$.*

Proof. We apply the previous corollary with $Y = H$ and $X = \{pt\}$. We just have to remark that $[g,h] \mapsto gh$ induces a Borel G-equivariant isomorphism from $(G \times H)/H$ onto G. □

Corollary 4.4.6. *Let H be a closed subgroup of a locally compact group G and let (Y,X) be a pair of measured left G-spaces. Then (Y,X) is amenable as a pair of measured left H-spaces for the restricted action, if and only if the pair $((G/H) \times Y, (G/H) \times X)$ of G-spaces for the diagonal G-actions is amenable.*

Proof. We may view X and Y as right H-spaces as explained at the beginning of Section 2. We observe that the map $[g,x] \mapsto (gH, gx)$ is a Borel isomorphism from $(G \times X)/H$ onto $(G/H) \times X$ which transforms the left G-action on $(G \times X)/H$ into the diagonal action. The same remark applies to $(G/H) \times Y$. The conclusion then follows from Corollary 4.4.4. □

Proposition 4.4.7. *Let H be a closed subgroup of a locally compact group G. The following conditions are equivalent:*

(i) *There exists a G-invariant mean $M : L^\infty(G/H \times G/H) \to L^\infty(G/H)$;*

(ii) *the trivial representation ι_H of H is weakly contained into the restriction to H of the quasi-regular representation $\lambda_{G/H}$.*

Proof. By Proposition 4.1.1, the trivial representation ι_H of H is weakly contained into the restriction to H of the quasi-regular representation $\lambda_{G/H}$ if and only if there exists an H-invariant state on $L^\infty(G/H)$. Hence, Proposition 4.4.7 follows from Corollary 4.4.6 applied with $Y = G/H$ and $X = \{pt\}$. □

4.5. **A negative answer to problem (A_1) for an ergodic discrete group action.** As a Corollary to Proposition 4.2.1, we get the following result, which can easily been checked directly: for G discrete, the measured G-space $X = G/H$ is amenable (i.e. H is an amenable subgroup) if and only if the quasi-regular representation is weakly contained into the regular representation of G. We have already observed that the result is not always true in the non discrete case. We shall now see that problem (A_1) may also have a negative answer when we replace transitivity by ergodicity, even if the group is discrete.

Let Γ be a lattice of $SL(3,\mathbb{R})$, acting on $X = SL(3,\mathbb{R})/SL(2,\mathbb{R})$ in the obvious way. We denote by π_X^Γ the corresponding representation of Γ in $L^2(X)$. Observe that π_X^Γ is the restriction to Γ of the quasi-regular representation $\lambda_{SL(3,\mathbb{R})/SL(2,\mathbb{R})}$ and that the restriction of $\lambda_{SL(3,\mathbb{R})}$ to Γ is a multiple of λ_Γ. Since $\lambda_{SL(3,\mathbb{R})/SL(2,\mathbb{R})}$ is weakly contained into $\lambda_{SL(3,\mathbb{R})}$, we have $\pi_X^\Gamma \prec \lambda_\Gamma$.

By C. Moore's ergodicity theorem ([36, Section 2.2]), we see that X is an ergodic Γ-space. However it is not amenable. Otherwise, as a consequence of the following lemma, X would also be an amenable $SL(3,\mathbb{R})$-space, in contradiction with Section 4.2.

Lemma 4.5.1. *Let (X,μ) be a G-space and let H be a closed subgroup of G. Assume that (X,μ) is an amenable H-space for the restricted action, and that there exists a G-invariant state on $L^\infty(G/H)$. Then (X,μ) is an amenable G-space.*

Proof. We endow the measured spaces $(G/H) \times X$ and $(G/H) \times X \times G$ with the diagonal G-actions. Note that the existence of a G-invariant state on $L^\infty(G/H)$ implies the existence of a G-equivariant norm one projection from $L^\infty((G/H) \times X)$ onto $L^\infty(X)$ (see Proposition 3.1.4). On the other hand, Proposition 4.3.2 shows that the pair $(X \times G, X)$ of left H-spaces, where H acts diagonally on $X \times G$, is amenable. The amenability of the pair $((G/H) \times X \times G, (G/H) \times X)$ of G-spaces is then a consequence of Corollary 4.4.6. By composition we construct a G-equivariant mean M from $L^\infty((G/H) \times X \times G)$ onto $L^\infty(X)$. Finally the restriction of M to $L^\infty(X \times G)$ gives a G-equivariant mean from $L^\infty(X \times G)$ onto $L^\infty(X)$. □

5. Study of transitive actions

5.1. In this section, we consider a closed subgroup H of a locally compact group G. As we saw in 4.2, it is not true in general that H is amenable whenever $\|\lambda_G(m)\| =$

$\|\lambda_{G/H}(m)\|$ holds for every bounded measure m on G. We shall now analyse in some details the result of Kesten mentioned in the introduction, dealing with open normal subgroups.

The following two lemmas are taken from the seminal paper [21], although expressed in a different language. We denote by $\mathrm{Sp}(T)$ the spectrum of any operator T and by $\mathrm{Supp}(m)$ the support of any Radon measure m.

Lemma 5.1.1. *Let (X, μ) be a measured G-space and let m be a symmetric probability measure on G. Then*

$$|Min\, Sp(\pi_X(m))| \leq Max\, Sp(\pi_X(m)).$$

Proof. Let f be a nonnegative function in $L^2(X)$ and let us denote by σ_f the spectral measure associated with $\pi_X(m)$ and f. Define $\alpha_f = \mathrm{Min}\,\mathrm{Supp}(\sigma_f)$ and $\beta_f = \mathrm{Max}\,\mathrm{Supp}(\sigma_f)$. Since $\langle \pi_X(m)^k f, f\rangle \geq 0$ for every k, every moment of σ_f is nonnegative. It follows that $|\alpha_f| \leq \beta_f$ (see for instance [19, Lemma 9]). Then we conclude by observing that $\mathrm{Min}\,\mathrm{Sp}(\pi_X(m)) = \mathrm{Min}\,\alpha_f$ and $\mathrm{Max}\,\mathrm{Sp}(\pi_X(m)) = \mathrm{Max}\,\alpha_f$, where f ranges over all nonnegative non-zero functions in $L^2(X)$. □

Lemma 5.1.2. *Let G be a locally compact group and H a normal closed subgroup of G. Let ν and m be symmetric probability measures on H and G respectively. Assume that there exists an integer $l \geq 1$ such that ν is absolutely continuous with respect to $\sum_{k=0}^{l} m^{*k}$. If $r(\lambda_H(\nu)) < 1$ then we have $r(\lambda_G(m)) < r(\lambda_{G/H}(m))$.*

Proof. Let us set $m_1 = (1/2)\delta_e + (1/2)m$, where δ_e is the Dirac measure at the origine. It follows immediately from Lemma 5.1.1 that $r(\lambda_G(m_1)) = 1/2 + 1/2 r(\lambda_G(m))$, and the same observation applies when replacing λ_G by $\lambda_{G/H}$. Therefore it suffices to show that $r(\lambda_G(m_1)) < r(\lambda_{G/H}(m_1))$, or equivalently that $r(\lambda_G(m_1^{*l})) < r(\lambda_{G/H}(m_1^{*l}))$. Since the measures m_1^{*l} and $\sum_{k=0}^{l} m^{*k}$ are equivalent, ν is also absolutely continuous with respect to m_1^{*l}.

Thus we are reduced to prove the following fact: if ν is absolutely continuous with respect to a probability measure m, and if $r(\lambda_H(\nu)) < 1$, then $r(\lambda_G(m)) < r(\lambda_{G/H}(m))$. Under these assumptions, denote by f the density of ν with respect to m, and for $n \geq 1$ define $f_n = \mathrm{Min}(f, n)$, $\nu_n = \dfrac{f_n.m}{\|f_n.m\|_1}$. Observe that $\|\lambda_H(\nu)\| = \lim_{n\to+\infty} \|\lambda_H(\nu_n\|$, so that for n large enough we have $\|\lambda_H(\nu_n\| < 1$. Hence, we may assume that f is bounded. Choosing $a > 0$ such that $1 + a - af \geq 0$, we see that $(1 + a)m - a\nu$ is a probability measure on G. Now, for $c \in [0, 1]$ we define $\mu_c = (1 - c)[(1 + a)m - a\nu] + c\nu$.

Since that ν is a probability measure supported on the normal subgroup H, we have $\lambda_{G/H}(\nu) = \mathrm{Id}_{L^2(G/H)}$. Moreover, using the weak equivalence of λ_H with the restriction of λ_G to H, we get $\|\lambda_G(\nu)\| = \|\lambda_H(\nu)\|$. It follows immediately that

$$\begin{aligned}\|\lambda_G(\mu_c)\| &\leq (1 - c)\|\lambda_G(\mu_0)\| + c\|\lambda_G(\nu)\| \\ &< (1 - c)\|\lambda_G(\mu_0)\| + c,\end{aligned}$$

and

$$\lambda_{G/H}(\mu_c) = (1-c)\lambda_{G/H}(\mu_0) + c\mathrm{Id}_{L^2(G/H)},$$

so that

$$\begin{aligned}\|\lambda_{G/H}(\mu_c)\| &= (1-c)\|\lambda_{G/H}(\mu_0)\| + c \\ &> \|\lambda_G(\mu_c)\|.\end{aligned}$$

To conclude, we observe that $\mu_c = m$ for $c = a/(1+a)$. □

Let us state now two immediate consequences of this lemma.

Theorem 5.1.3. *Let G be a locally compact group, and H an open normal subgroup. Then H is amenable if and only if there exists a symmetric probability measure m on G such that $r(\lambda_{G/H}(m)) = r(\lambda_G(m))$, which is adapted to (G,H) in the following sense: there is an integer $l \geq 1$ such that $H \cap \cup_{k=0}^{l}(Supp(m))^k$ generates H as a closed subgroup.*

Proof. Let m be as above and denote by ν the restriction of $\sum_{k=0}^{l} m^{*k}$ to H. It follows from Lemma 5.1.2 that $r(\lambda_H(\nu)) = \nu(H)$. The support of ν generates H since it contains $H \cap \cup_{k=0}^{l}(\mathrm{Supp}(m))^k$, and therefore H is amenable (see Proposition 4.1.1). The converse is a consequence of Proposition 4.2.1. □

Theorem 5.1.4. [21, Theorem 2] *Let G be a countable discrete group and H a normal subgroup of G. Then H is amenable if and only if there exists an adapted symmetric probability measure m on G such that $r(\lambda_{G/H}(m)) = r(\lambda_G(m))$.*

Proof. Let m be an adapted probability measure on G, and assume that H is not amenable. There exists a symmetric probability measure ν on H with $r(\lambda_H(\nu)) < 1$. Moreover, approximating ν by finitely supported probability measures, we may suppose that the support of ν is finite. Let $A = \{e\} \cup \mathrm{Supp}(m)$. There exists $l \geq 1$ such that $\mathrm{Supp}(\nu) \subset A^l = \mathrm{Supp}(\sum_{k=0}^{l} m^{*k})$ since G is generated by $\mathrm{Supp}(m)$ and $\mathrm{Supp}(\nu)$ is finite. Then it follows from Lemma 5.1.2 that $r(\lambda_{G/H}(m)) > r(\lambda_G(m))$.

The converse is a consequence of Proposition 4.2.1. □

We shall now end this section with several comments on the previous theorem. First, in contrast with Proposition 4.1.1, the existence of an adapted probability measure m such that $r(\lambda_{G/H}(m)) = r(\lambda_G(m))$ is not enough to get the conclusion of Theorem 5.1.4. Indeed let Γ be either the free group $\mathbb{F}_{2k}$ generated by $S = \{a_1, \dots, a_{2k}\}$ or the surface group

$$\Gamma_k = \langle a_1, b_1, \dots, a_k, b_k | \prod_{i=1}^{k} a_i b_i a_i^{-1} b_i^{-1} = 1 \rangle$$

with $S = \{a_1, b_1, \dots, a_k, b_k\}$. Set $m == \dfrac{1}{2k}\sum_{s \in S} \delta_s$. Then de la Harpe, Robertson and Valette proved in [20, Prop. 9] that

$$\mathrm{Sp}(\lambda_\Gamma(m)) = \{z \in \mathbb{C} : |z| \leq \frac{1}{\sqrt{2k}}\}.$$

We can write $\Gamma_k = \mathbb{F}_{2k}/H$ in an obvious way. So we have

$$\mathrm{Sp}(\lambda_{\mathbb{F}_{2k}}(m)) = \mathrm{Sp}(\lambda_{\mathbb{F}_{2k}/H}(m)),$$

although H is not amenable.

Second, Theorem 5.1.4 is not true for non normal subgroups. In order to produce a counterexample, we need to recall some results of Grigorchuk (see [15]). Let H be a subgroup of the free group $\mathbb{F}_k$. We denote by H_n the set of elements in H of length n, and by $|H_n|$ its cardinal. Grigorchuk defined the *relative growth* (or growth exponent) α_H of H to be

$$\alpha_H = \limsup_{n\to+\infty} |H_n|^{1/n}.$$

Let

$$m = \frac{1}{2k}\sum_{i=1}^{k}(\delta_{a_i} + \delta_{a_i^{-1}})$$

where $a_1, \ldots, a_k$ are free generators of $\mathbb{F}_k$. Then Grigorchuk proved the following relations between $\|\lambda_{\mathbb{F}_k/H}(m)\|$ and α_H:

$$\|\lambda_{\mathbb{F}_k/H}(m)\| = \frac{\sqrt{2k-1}}{k} \quad \text{if} 1 \leq \alpha_H \leq \sqrt{2k-1},$$

$$\|\lambda_{\mathbb{F}_k/H}(m)\| = \frac{\sqrt{2k-1}}{2k}\left(\frac{\sqrt{2k-1}}{\alpha_H} + \frac{\alpha_H}{\sqrt{2k-1}}\right) \quad \text{if} \sqrt{2k-1} < \alpha_H \leq 2k-1.$$

Moreover, we have $\sqrt{2k-1} < \alpha_H \leq 2k-1$ when $H \neq \{e\}$ is a normal subgroup of $\mathbb{F}_k$.

Let H be a subgroup of $\mathbb{F}_k$ with relative growth in $]1, \sqrt{2k-1}]$. We see that

$$\|\lambda_{\mathbb{F}_k}(m)\| = \frac{\sqrt{2k-1}}{k} = \|\lambda_{\mathbb{F}_k/H}(m)\|$$

even if H is not amenable.

References

[1] S. Adams, G. Elliot, T. Giordano: *Amenable actions of groups*, Trans. Amer. Math. Soc., **344** (1994), 803-822.

[2] C. Anantharaman-Delaroche: *Action moyennable d'un groupe localement compact sur une algèbre de von Neumann*, Math. Scand., **45** (1979), 289-304.

[3] C. Anantharaman-Delaroche: *Action moyennable d'un groupe localement compact sur une algèbre de von Neumann II*, Math. Scand., **50** (1982), 251-268.

[4] C. Anantharaman-Delaroche, J. Renault: Amenable groupoids, Monographie de L'Enseignement Mathématique No **36**, Genève, 2000.

[5] C. Berg, J. P. R. Christensen: *On the relation between amenability of locally compact groups and the norms of convolution operators*, Math. Ann., **208** (1974), 149-153.

[6] J. M. Cohen: *Cogrowth and amenability of discrete groups*, J. Funct. Anal., **48** (1982), 301-309.

[7] M. M. Day: *Convolutions, means and spectra*, Illinois J. Math., **8** (1964), 100-111.

[8] Y. Derriennic, Y. Guivarc'h: *Théorème de renouvellement pour les groupes non moyennables*, C. R. Acad. Sci. Paris Série A, **277** (1973), 613-615.

[9] J. Dixmier: Les C^*-algèbres et leurs représentations, Gauthier-Villars, Paris, 1964.

[10] J. Dixmier: Les algèbres d'opérateurs dans l'espace hilbertien, Gauthier-Villars, Paris, 1969.

[11] P. Eymard: Moyennes et représentations unitaires, Lecture Notes in Math., **300**, Springer, Berlin, 1972.
[12] J. M. G. Fell: *Weak containment and induced representations of groups*, Can. J. Math., **14** (1962), 237-268.
[13] F. P. Greenleaf: Invariant means on topological groups, Van Nostrand, New-York, 1969.
[14] F. P. Greenleaf: *Amenable actions of locally compact groups*, J. Funct. Anal., **4** (1969), 295-315.
[15] R. I. Grigorchuk: *Symmetrical random walks on discrete groups*, In Multicomponent random systems, ed. R. L. Dobrushin, Ya. G. Sinai, Advances in Probability and related topics, vol. **6** (1980), Marcel Dekker, New-York, 285-325.
[16] Y. Guivarc'h: *Quelques propriétés asymptotiques des produits de matrices aléatoires*, in Ecole d'Eté de Probabilités de Saint-Flour VIII-1978, Lectures Notes in Math., **774** (1980), Springer, 177-250.
[17] P. R. Halmos, J. von Neumann: *Operator methods in classical mechanics II*, Ann. of Math., **43** (1942), 332-350.
[18] C. Herz: *Sur le phénomène de Kunze-Stein*, C. R. Acad. Sci. Paris Série A, **271** (1970), 491-493.
[19] P. de la Harpe, A. G. Robertson, A. Valette: *On the spectrum of the sum of generators for a finitely generated group*, Israel J. Math., **81** (1993), 65-96.
[20] P. de la Harpe, A. G. Robertson, A. Valette: *On the spectrum of the sum of generators for a finitely generated group II*, Coll. Math., **65** (1993), 87-102.
[21] H. Kesten: *Symmetric random walks on groups*, Trans. Amer. Math. Soc., **92** (1959), 336-354.
[22] H. Kesten: *Full Banach mean values on countable groups*, Math. Scand., **7** (1959), 146-156.
[23] G. Kuhn: *Amenable actions and weak containment of certain representations of discrete groups*, Proc. Amer. Math. Soc., **122** (1994), 751-757.
[24] A. Nevo: *The spectral theory of amenable actions and invariants of discrete groups*, in preparation.
[25] A. L. .T. Paterson: Amenability, Mathematical Surveys and Monographs No **29**, A.M.S., 1988.
[26] G. K. Pedersen: C^*-algebras and their automorphism groups, Academic Press, 1979.
[27] A. Ramsay: *Virtual groups and group actions*, Adv. in Math., **6** (1971), 253–322.
[28] J. Renault: A groupoid approach to C^*-algebras, Lecture Notes in Math., **793**, Springer, 1980.
[29] Y. Shalom: *Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation group*, Ann. Math., **152** (2000), 113-182.
[30] J. von Neumann: *Zur Operatorenmethode in der Klassichen Mechanik*, Ann. of Math., **33** (1932), 587-642.
[31] R.J. Zimmer: *Extensions of ergodic group actions*, Illinois J. Math., **20** (1976), 373-409.
[32] R.J. Zimmer: *Hyperfinite factors and amenable ergodic actions*, Inv. Math., **41** (1977), 23-31.
[33] R.J. Zimmer: *On the von Neumann algebra of an ergodic group action*, Proc. Amer. Math. Soc., **66** (1977), 289-293.
[34] R.J. Zimmer: *Amenable ergodic group actions and an application to Poisson boundaries of random walks*, J. Funct. Anal., **27** (1978), 350–372.
[35] R.J. Zimmer: *Amenable pairs of groups and ergodic actions and the associated von Neumann algebras*, Trans. Amer. Math. Soc., **243** (1978), 271–286.
[36] R.J. Zimmer: Ergodic theory and semisimple groups, Birkhäuser, 1984.

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