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INVARIANT PROPER METRICS ON COSET SPACES

CLAIRE ANANTHARAMAN-DELAROCHE

Abstract. It is known that for every second countable locally compact group G, there exists a proper G-invariant metric which induces the topology of the group. This is no longer true for coset spaces G/H viewed as G-spaces. We study necessary and sufficient conditions which ensure the existence of such metrics on G/H.

Introduction

Given a Hausdorff topological group G, the study of the existence of a G-invariant[1] metric on G which defines its topology (in which case we say that the metric is compatible) is an old problem which was solved by Birkhoff [2] and Kakutani [9]: a topological group is metrizable if and only if it is Hausdorff and the identity element e of G has a countable basis of neighbourhoods (i.e. G is first countable). Moreover in this case the metric can be taken to be G-invariant (see for instance [3, §3] or [13, §1.22]). More recently, in the last decade, appeared the need for G-invariant compatible metrics which are proper in the sense that the balls are relatively compact. This is for instance the case when one wants to attack the Baum-Connes conjecture by considering proper affine isometric actions of G on various Banach spaces, following an idea of Gromov (see [15], [8]). Actually, this problem had been solved in the seventies by Struble [14]: a locally compact group G has a proper G-invariant compatible metric if and only if it is second countable.

In this paper, we are interested in the same questions on coset spaces. Let H be a closed subgroup of a Hausdorff topological group G. Whenever G is metrizable, it is also the case for the quotient topological space G/H (see [13, §1.23]). However, even if G is a nice second countable locally compact group, there does not always exist a G-invariant compatible metric on G/H (see Example 2.3). Apart from the trivial case where H is a normal subgroup of G, there is another well-known exception: when G is locally compact second countable and H is compact, it is

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[1]We let G act on itself to the left.

easily seen, using Struble's theorem, that there is a proper G-invariant compatible metric on G/H (Proposition 2.4).

We extend this result to give a necessary and sufficient condition (Proposition 2.7) for the existence of such a metric in general. Several simple necessary conditions for this existence follow immediately (Proposition 2.12):

- there must exist a non-zero relatively invariant positive measure on G/H (i.e. the modular function of H extends to a continuous homomorphism from G to $\mathbb{R}_*^+$;
- H must be maximally almost periodic when the G-action on G/H is effective[2];
- H must be almost normal in G in a topological sense (see Definition 2.10).

However, these three conditions together are not enough in general to imply the existence of a proper G-invariant compatible metric on G/H (see Example 2.14). A noteworthy exception is the case where G/H is discrete. Of course, the usual discrete metric on G/H is G-invariant but it is not proper unless G/H is finite. We show that a proper (*i.e.* with finite balls) G-invariant compatible metric exists on the discrete coset space G/H if and only if H is an almost normal subgroup of G (Theorem 2.15).

We also observe that in the opposite case where G/H is connected, the existence of a G-invariant compatible metric and of a proper one are equivalent (Proposition 2.9) and it seems to be rather scarce (in non-trivial situations).

In Section 3, we consider briefly the case of a general G-space X. We show that for a group G acting on a countable (discrete!) space X, there exists a proper G-invariant compatible metric if and only if the orbits of all the stabilizers of the action are finite. The proof uses a recent nice theorem of Abels, Manoussos and Noskov [1], whose arduous proof shows that whenever a locally compact group G acts properly on a second countable locally compact space X, there exists a proper G-invariant compatible metric on X.

Of course, our proposition 2.4 in Section 2 is a direct consequence of this latter result from [1]. However, it seemed to us more accessible to deduce this proposition 2.4 directly from its particular case which is the above mentioned theorem of Struble.

In Section 1, we first introduce some notation and definitions. Since the notion of proper action plays a crucial role in our study, we also recall there, or prove, the related facts that we need in the sequel.

[2]Without lost of generality, we can always make this assumption (see Remark 2.5).

1. Proper actions on metric spaces

1.1. **Preliminaries.** The topological spaces considered in this paper will always be Hausdorff. We recall that a locally compact space X is second countable if and only if it is metrizable and σ-compact (see for instance [3, page 43]). In particular, a locally compact group is second countable if and only if it is first countable and σ-compact.

Let G be a topological group acting continuously on a locally compact space X. The action is said to be *proper* if the map $(g, x) \to (gx, x)$ from $G \times X$ into $X \times X$ is proper. This property is equivalent to the fact that for every pair A, B of compact subsets of X, the set $\{g \in G : gA \cap B \neq \emptyset\}$ is relatively compact, and also to the fact that for every pair x, y of points of X there exist neighbourhoods V_x of x and V_y of y such that $\{g \in G : gV_x \cap V_y \neq \emptyset\}$ is relatively compact

Recall that a topological group G which acts properly on a locally compact space is itself locally compact.

The space of continuous functions from a topological space into itself will always be equipped with the compact-open topology.

Given a metric space (X, d), $x \in X$ and $r > 0$, we shall denote by $B(x, r)$ the closed ball $\{y \in X : d(x, y) \leq r\}$. Whenever these balls are compact, we say the (X, d) is a *proper metric space*[3]. Complete riemannian manifolds and locally finite graphs with the geodesic distance are such examples.

A metric is said to be discrete if it induces the discrete topology. We shall denote by δ the usual discrete metric such that $\delta(x, y) = 1$ whenever $x \neq y$. Note that (X, δ) is a proper metric space only when X is a finite set.

1.2. **Proper actions of** $\mathrm{Iso}(X, d)$**.** Let (X, d) be a metric space. We denote by $\mathrm{Iso}(X, d)$ its group of isometries, equipped with the compact-open topology. We recall that this topology coincides with the topology of pointwise convergence ([4, Théorème 1, page 29]), that $\mathrm{Iso}(X, d)$ is a topological group ([4, Corollaire, page 48]) and that $\mathrm{Iso}(X, d)$ acts continuously on X. However, even if X is locally compact, the group $\mathrm{Iso}(X, d)$ is not always locally compact. It suffices for instance to consider the case of the group of bijections of the set $X = \mathbb{N}$ with its metric δ.

We begin by reviewing some topological features which insure that $\mathrm{Iso}(X, d)$ is locally compact and acts properly on X. The earliest result is the following theorem due to Dantzig and van der Waerden [16].

Theorem 1.1. *Let (X, d) be a locally compact, connected, metric space. The group $\mathrm{Iso}(X, d)$ of isometries of X is locally compact, second countable and acts properly on X.*

[3] A proper and discrete metric space is also said to be *locally finite*.

Proof. For a proof, see [10, Theorem 4.7]. Since it is not explicitely shown in this reference that the action is proper, we give some precisions, for the sake of completeness. We have to show that for every $x, y \in X$, there are neighbourhoods V_x and V_y of x, y respectively such that $\{f \in \mathrm{Iso}(X,d) : f(V_x) \cap V_y \neq \emptyset\}$ is relatively compact. We choose $r > 0$ small enough such that the ball $B(y, 2r)$ is compact. Since

$$\{f \in \mathrm{Iso}(X,d) : f(B(x,r)) \cap B(y,r) \neq \emptyset\} \subset \{f \in \mathrm{Iso}(X,d) : d(f(x), y) \leq 2r\}$$

it suffices to show that the right hand side, denoted by E, is relatively compact. Let (f_n) be a sequence[4] in E. Passing, if necessary to a subsequence, we may assume that $(f_n(x))$ converges. Then, by [10, Lemma 3, page 47] we may assume that (f_n) converges pointwise, and by [10, Lemma 4, page 48] the limit is an isometry. □

This theorem has been recently extended by Gao and Kechris. As a particular case, they obtain in [7, Corollary 5.6] the following result. Its proof is also contained in [1, Theorem 3.2].

Theorem 1.2. *Let (X, d) be a proper metric space. Then the group* $\mathrm{Iso}(X, d)$ *is locally compact and acts properly on X.*

This theorem applies in particular to the group of automorphisms of a locally finite graph X equipped with its geodesic distance.

We shall also need the following result, that we have not met in the literature. Below, X is a discrete space. We denote by $\mathrm{Map}(X)$ the space of maps from X to X endowed with the topology of pointwise convergence. If δ is the usual discrete metric, note that $\mathrm{Iso}(X, \delta)$ is nothing else than the group $\mathrm{Bij}(X)$ of bijections from X to X. It is not closed in $\mathrm{Map}(X)$ when the cardinal of X is infinite.

Theorem 1.3. *Let G be a group acting (to the left) on a space X. Let ρ be the corresponding homomorphism from G into* $\mathrm{Bij}(X)$ *and denote by G' the closure of $\rho(G)$ in* $\mathrm{Map}(X)$. *Then G' is a subgroup of* $\mathrm{Bij}(X)$ *acting properly on the discrete space X if and only if the orbits of all the stabilizers of the G-action on X are finite. Moreover, in this case the group G' is locally compact and totally disconnected.*

Proof. Obviously we may replace G by $\rho(G)$, that is, we may assume that G is a subgroup of $\mathrm{Bij}(X)$.

The condition on the orbits is of course necessary. Assume now that it is fulfilled. We first check that G' is contained into $\mathrm{Bij}(X)$. Obviously, the elements of G' are injective maps, since X is discrete. We now take $f \in G'$ and show that f is surjective. Let (g_i) be a net in G such that $f = \lim_i g_i$ in the topology of the pointwise convergence. We fix $x \in X$ and i_0 such that $f(x) = g_i x$ for $i \geq i_0$. We set $g = g_{i_0}$. So $h_i = g^{-1} g_i$ is in the stabilizer G_x of x whenever $i \geq i_0$.

[4] The use of sequences is justified in [10, page 46].

Observe that $\lim_i h_i = g^{-1}f$. In particular, $f_1 = g^{-1}f$ is an injective map and for $y \in X$, $h \in G_x$, we have $f_1(hy) = (g^{-1}f)(hy) = \lim_i (h_i h)y \subset G_x y$. Hence, $f_1(G_x y) \subset G_x y$. Since $G_x y$ is a finite set, it follows that y is in the range of f_1. Thus, we see that $f_1 \in \mathrm{Bij(X)}$, and $f \in \mathrm{Bij(X)}$ too.

Now, we want to show that G' is a locally compact group. For $x \in X$, we set $G'_x = \{g' \in G' : g'(x) = x\}$. It is easily checked that G'_x is the closure of G_x. Let us prove that G_x is relatively compact. This follows from the fact that G_x is a subset of $\Pi_{y\in X} G_x y$, which is compact, by the Tychonov theorem. Since the finite intersections of such stabilizers G'_x form a basis of neighbourhoods of the identity in G', we see that G' is a locally compact group. Note that the G'_x are both compact and open, and so G' is totally disconnected.

Finally, since X is discrete and the G'-action has compact stabilizers, we see that the G'-action is proper. □

2. Proper G-invariant compatible metrics on coset spaces

Let X be a topological space. A *compatible metric* on X is a distance which defines the topology.

Definition 2.1. Let G be a topological group, acting continuously to the left on a topological space X. A *compatible G-invariant metric* on X is a compatible distance d such that $d(gx, gy) = d(x, y)$ for every $g \in G$ and $x, y \in X$.

Let us remind, for reference purpose, the theorem of Struble mentioned in the introduction.

Theorem 2.2 ([14])**.** *A locally compact group G has a proper (left) G-invariant compatible metric if and only if it is second countable.*

We now consider a topological group G and a closed subgroup H. Whenever G is metrizable, the coset space G/H is still metrizable [13, §1.23]. However, there is not always a G-invariant compatible metric, as shown by the following example.

Example 2.3. Let $G = \mathbb{R}^+_* \ltimes \mathbb{R}$ and $H = \mathbb{R}^+_*$ viewed as a closed subgroup of G. Then $(a, b) \mapsto b$ is a homeomorphism from G/H onto $\mathbb{R}$. Under this identification, the left action of $a \in H$ on $x \in \mathbb{R}$ is the product in $\mathbb{R}$. If d is a G-invariant metric on $G/H = \mathbb{R}$, we have $d(0, x) = d(0, ax)$ for every $a \in H$ and $x \in \mathbb{R}$. If follows that d cannot define the topology of $\mathbb{R}$.

In case H is a compact subgroup of a metrizable topological group G, there is still a G-invariant metric on G/H. Moreover, if G is in addition generated by a compact neighbourhood of the identity (for instance if G is locally compact, metrizable and connected) it has been proved by Kristensen [12] that there is even a proper G-invariant compatible metric on G/H.

We first show that, using Theorem 2.2, it is not difficult to extend the above fact to any locally compact second countable group.

Proposition 2.4. *Let G be a locally compact second countable group and H a compact subgroup. Then there exists a proper G-invariant compatible metric on G/H.*

Proof. Let d be a proper left G-invariant compatible metric on G. We set $\dot{g} = gH$ and define on G/H the Hausdorff distance

$$\begin{aligned}\tilde{d}(\dot{g_1}, \dot{g_2}) &= d(g_1H, g_2H)\\ &= \max(\sup_{h_1\in H}\inf_{h_2\in H} d(g_1h_1, g_2h_2), \sup_{h_2\in H}\inf_{h_1\in H} d(g_1h_1, g_2h_2)).\end{aligned}$$

This distance is obviously G-invariant. Let us show that it is compatible. First, using the fact that H is compact, given $\varepsilon > 0$, there is a neighbourhood W of the unit e such that $d(h, gh) < \varepsilon$ for $h \in H$ and $g \in W$. Therefore, for $g \in W$, we have

$$\sup_{h_1\in H}\inf_{h_2\in H} d(h_1, gh_2) \le \sup_{h\in H} d(h, gh) \le \varepsilon,$$

and similarly $\sup_{h_2\in H}\inf_{h_1\in H} d(h_1, gh_2) \le \varepsilon$. It follows that $\dot{g} \mapsto \tilde{d}(\dot{e}, \dot{g})$ is continuous (for the quotient topology) at $\dot{e}$. Since $\tilde{d}$ is invariant, we deduce that its open balls are open for the quotient topology. Assuming for the moment that the balls $B(r) = \left\{\dot{g} : \tilde{d}(\dot{e}, \dot{g}) \le r\right\}$ are compact for the quotient topology, let us show that every open neighbourhood V of $\dot{e}$ for the quotient topology contains an open ball. This is immediate since $\cap_n B(1/n) \cap \big((G/H) \setminus V\big) = \emptyset$. Then we conclude that $\tilde{d}$ is compatible.

It remains to check that $B(r)$ is compact for $r > 0$. Let $g \in G$ such that $\dot{g} \in B(r)$. For every $h_2 \in H$ there exists $h_1 \in H$ with $d(h_1, gh_2) \le r$. Then we have

$$d(e, gh_2) \le d(e, h_1) + d(h_1, gh_2) \le c + r$$

where $c = \sup_{h\in H} d(e, h)$. It follows that $B(r)$ is contained in the image by the quotient map of the closed ball in G with radius $c + r$, centered in e, which is compact since d is proper. □

Remark 2.5. Let G be a topological group, H a closed subgroup of G and L a normal closed subgroup of G, contained in H. Using the canonical homeomorphism from G/H onto $(G/L)/(H/L)$, we see that there exists a proper G-invariant compatible metric on G/H if and only if there exists a proper G/L-invariant compatible metric on $(G/L)/(H/L)$. In particular, in the previous proposition, it is enough to assume that H contains a closed normal subgroup L of G such that H/L is compact.

An interesting particular case is $L = \cap_{g\in G} gHg^{-1}$, the largest normal subgroup of G contained in H, since the action of G/L onto $(G/L)/(H/L)$ is effective.

In the rest of this section H is a closed subgroup of a locally compact group G and ρ denotes the natural homomorphism from G into the group of homeomorphisms of G/H. Together with the previous proposition, the following lemma will provide a characterization of the coset spaces which carry a proper G-invariant compatible metric.

Lemma 2.6. *Let G be a second countable locally compact group and H a closed subgroup of G. We assume that there is a compatible G-invariant metric d on $X = G/H$ and a locally compact subgroup G' of* $\mathrm{Iso}(G/H, d)$*, which contains $\rho(G)$ and is such that its action on G/H is proper. Then there exists a compact subgroup H' of G' such that*

(a) *$\rho(H) \subset H'$;*
(b) *the map $\tilde{\rho} : G/H \to G'/H'$ induced by ρ is a homeomorphism.*

Proof. We denote by H' the stabilizer of $\dot{e}$ for the action of G' on $X = G/H$. The action of G' on X is transitive and proper: it follows that $g' \mapsto g'\dot{e}$ induces a homeomorphism θ from G'/H' onto X. We immediately check that $\tilde{\rho} = \theta^{-1}$. □

Proposition 2.7. *Let G be a second countable locally compact group and H a closed subgroup of G. The following conditions are equivalent:*

(i) *There exists a proper G-invariant compatible metric on G/H.*
(ii) *There exists a continuous homomorphism φ from G into a metrizable locally compact group G' and a compact subgroup H' of G' such that*
  (a) *$\varphi(H) \subset H'$;*
  (b) *the map $\tilde{\varphi} : G/H \to G'/H'$ induced by φ is a homeomorphism.*

Proof. (ii) $\Rightarrow$ (i). Assume that (ii) holds. Since G'/H' is σ-compact and H' is compact, we see that G' is σ-compact. Being metrizable, it is second countable. Then, applying Proposition 2.4, we see that there exists a proper G'-invariant compatible metric on G'/H'. It gives a metric with the required properties on G/H.

(i) $\Rightarrow$ (ii). Let d be a proper G-invariant compatible distance on $X = G/H$ and let $G' = \mathrm{Iso}(X, d)$ be the group of isometries of X, endowed with the compact-open topology. We take $\varphi = \rho$. Since the metric is proper, by Theorem 1.2, we know that G' is a locally compact group which acts properly on X. Then we apply the previous lemma. Moreover, G' is metrizable since (X, d) is σ-compact. □

Remark 2.8. (a) We observe that in the above statement we may assume that G' is a locally compact group of homeomorphisms of G/H and $\varphi = \rho$. We may also replace G' by the closure of $\varphi(G)$, and thus assume that $\varphi(G)$ is dense in G'.

(b) The fact that $\tilde{\varphi}$ is bijective is equivalent to $G' = \varphi(G)H'$ together with $H = \varphi^{-1}(H')$.

(c) When the action of G over G/H is effective, we may take φ to be injective.

We deduce the following consequence when G/H is connected.

Proposition 2.9. *Let G be a second countable locally compact group and H a closed subgroup of G. Assume that G/H is connected. The two following conditions are equivalent:*

(i) *there exists a G-invariant compatible metric on G/H;*
(ii) *there exists a proper G-invariant compatible metric on G/H.*

Proof. Let d be a G-invariant compatible metric on G/H. It follows from Theorem 1.1 that the group $\mathrm{Iso}(G/H, d)$ is locally compact, second countable, and acts properly on G/H. Then we apply Lemma 2.6 and Proposition 2.7. □

In general, unless there is an obvious candidate for (G', H', φ), Proposition 2.7 is not easy to apply. However, we shall see that it can be used to find obstructions for the existence of a proper G-invariant compatible metric on G/H. We need first to introduce or recall some definitions.

Definition 2.10. Let H be a closed subgroup of a locally compact group G. We say that H is *topologically almost normal* (resp. *almost normal*) in G if for every compact (resp. finite) subset K of G there exists a compact (resp. finite) subset K' of G such that $HK \subset K'H$.

In other terms, H is topologically almost normal if and only if the H-orbits of compact subsets of G/H are relatively compact. It is almost normal if and only if the H-orbits of elements of G/H are finite. Note that compact or co-compact subgroups are topologically almost normal, and that normal subgroups are of course both topologically almost normal and almost normal. Whenever H is an open subgroup of G, then H is almost normal if and only if it is topologically almost normal.

Example 2.11. Let G be a locally compact group acting continuously by isometries on a locally finite metric space (*e.g.* a discrete group of automorphisms of a locally finite graph) and let H be the stabilizer of some element $x \in X$. Then H is an open almost normal subgroup of G (see Theorem 2.15).

Classical exemples of almost normal subgroups are plentiful: $SL_n(\mathbb{Z})$ in $SL_n(\mathbb{Q})$, $\mathbb{Z} \rtimes \{1\}$ in $\mathbb{Q} \rtimes \mathbb{Q}_*^+$, $\langle x \rangle$ in $BS(m,n) = \langle t, x : t^{-1}x^m t = x^n \rangle$, ...

Recall the a group G is *maximally almost periodic* if there exists a continuous injective homomorphism from G into a compact group.

In the next statement, we take $L = \ker \rho = \cap_{g\in G} gHg^{-1}$, where ρ is the canonical homomorphism from G into the group of homeomorphisms of G/H (so that the G/L-action on $(G/L)/(H/L)$ is effective).

Proposition 2.12. *Let G be a second countable locally compact group and H a closed subgroup of G. We assume the the existence of a proper G-invariant compatible metric on G/H. Then*

(i) *there exists a non-zero relatively G-invariant (positive Radon) measure on G/H;*
(ii) *H/L is maximally almost periodic;*
(iii) *H is topologically almost normal.*

Proof. (i) is a consequence of Proposition 2.7, since there is a non-zero relatively G'-invariant measure on G'/H' (see [5, Corollaire 1, page 59]). Again by Proposition 2.7, we get a continuous injective homomorphism from H/L into H' and so H/L is maximally almost periodic. Finally, (iii) is obvious: let d be a proper G-invariant compatible metric, K a compact subset of G/H and set $r = \sup_{k\in K} d(\dot{e}, k)$; then HK is contained in the ball of radius r and center $\dot{e}$. □

Consider for instance a real connected semi-simple, non compact, Lie group G and let $G = KAN$ be a Iwasawa decomposition of G. Then, there does not exist a compatible G-invariant metric on G/AN. Indeed, G/AN does not carry any relatively G-invariant (hence invariant) non-zero measure.

Proposition 2.13. *Let G be a connected locally compact group and H a closed co-compact subgroup such that the the left action of G over G/H is effective. A necessary condition for the existence of a compatible G-invariant metric on G/H is that G is isomorphic to a group of the form $K\times\mathbb{R}^n$ where K is a compact group.*

Proof. Assume that there is a G-invariant compatible metric on G/H. Since G/H is compact, we see, with the notation of Proposition 2.7, that G' is compact. Moreover we may assume that $\varphi : G \to G'$ is injective (see Remark 2.8 (c)). Hence G is maximally almost periodic and therefore of the form $K \times \mathbb{R}^n$ by a theorem of Freudenthal-Weil (see [6, Théorème 16.4.6]). □

Example 2.14. Let G be a connected locally compact second countable group and H a discrete co-compact subgroup. We assume that G has no discrete subgroup N in its center such that G/N is maximally almost periodic. Then, there is no G-invariant compatible metric on G/H. Otherwise, arguing as in the proof of the previous proposition, there will be a continuous homomorphism ρ from G into a compact group G', whose kernel L is a discrete normal subgroup of G, so contained in its center, because G is connected. Moreover, G/L is maximally almost periodic, since its embeds in G', hence a contradiction.

Let us specify the above example to show that the conditions (i), (ii) and (iii) of Proposition 2.12 are not sufficient to insure the existence of a proper G-invariant compatible metric. Let $G = SO_0(2,3)$ be the connected component of the group of matrices in $SL(p+q,\mathbb{R})$ leaving invariant the quadratic form $x_1^2+x_2^2-x_3^2-x_4^2-x_5^2$ and let H be a co-compact lattice in G. Then H is topologically almost normal, residually finite (by a theorem of Malcev, since it is a finitely generated and linear group), and there exists an invariant probability measure on G/H. However, as explained above, G/H does not carry a G-invariant compatible metric.

Non-trivial examples of closed subgroups of G such that G/H is connected and has a G-invariant compatible metric seem to be rather scarce. In case the subgroup H is open in G (for instance when G is a discrete group), the situation is quite different. Of course, in this case Condition (i) of Proposition 2.12 is always satisfied. We shall see that Condition (iii) suffices to imply the existence of a proper G-invariant compatible metric on G/H.

Theorem 2.15. *Let G be a second countable locally compact group and H a closed subgroup. The following conditions are equivalent:*

- (i) *H is open and there exists a proper G-invariant compatible metric d on G/H;*
- (ii) *H is a stabilizer for a continuous action of G by isometries on a locally finite (discrete) metric space;*
- (iii) *H is an open almost normal subgroup of G.*

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). Let $G \curvearrowright (X,d)$ be a continuous action on a proper discrete metric space such that H is the stabilizer of some $x_0 \in X$. Denote by ψ the map $g \mapsto gx_0$. Let $k \in G$ and set $r = d(x_0, kx_0)$. Then $\psi(Hk)$ is contained in the ball of radius r and center x_0, which is finite. Therefore we have $\psi(Hk) = \{g_1x_0, \dots, g_nx_0\}$ and so $Hk = \cup_{i=1}^n g_iH$. Moreover H is open since X is discrete.

(iii) $\Rightarrow$ (i). We assume that H is open and almost normal. So the orbits of the stabilizers of the G action on G/H are finite. By Theorem 1.3, there exists a subgroup G' of $\mathrm{Iso}(G/H,\delta)$, where δ is the usual discrete metric, which contains $\rho(G)$ and acts properly on G/H. Then the conclusion follows from Lemma 2.6 and Proposition 2.7. $\square$

3. Proper G-invariant compatible metrics on G-spaces

We now consider a locally compact group G acting on a locally compact space X and we are interested in the existence of G-invariant compatible metrics on X. In the lecture notes [11], the following result is proved:

Theorem 3.1. ([11, Theorem 3, page 9]) *Let G be a locally compact group acting properly on a second countable locally compact space X. Then there exists a G-invariant compatible metric on X.*

Recently, this result has been improved in [1].

Theorem 3.2 ([1])**.** *Let G be a locally compact group acting properly on a second countable locally compact topological space X. Then there is a proper G-invariant compatible metric on X.*

Of course, the fact that the action is proper is not necessary to obtain the conclusion.

Proposition 3.3. *Let G be a group acting on a countable set X. The following conditions are equivalent:*

(i) *there exists a G-invariant locally finite metric d on X;*
(ii) *the orbits of all the stabilizers of the G-action are finite.*

Proof. Denote by G_x the stabilizer of x. We observe that (i) $\Rightarrow$ (ii) is immediate since for every $x, y \in X$, the set $G_x y$ is contained in the ball of center x and radius $d(x, y)$.

(ii) $\Rightarrow$ (i). We follow the proof of (iii) $\Rightarrow$ (i) in Theorem 2.15. We introduce $\rho : G \to \mathrm{Bij}(X)$ and the group G' as in Theorem 1.3. Since G' is a locally compact group acting properly on X, the conclusion follows from Theorem 3.2. □

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