WEAK CONTAINMENT VS AMENABILITY

CLAIRE ANANTHARAMAN-DELAROCHE

1. Introduction

We say that a locally compact groupoid[a] $\mathcal{G}$ has the *weak containment property* (in short (WCP)) if the canonical surjective map from its full C^*-algebra $C^*(\mathcal{G})$ onto its reduced C^*-algebra $C^*_r(\mathcal{G})$ is injective. Every (topologically) amenable groupoid [2] has the (WCP) (see [14, Theorem 3.6], [2, Proposition 6.1.10]). When $\mathcal{G}$ is a locally compact group it is well known that the converse is true (see for instance [13, Theorem 4.21]). Therefore it is natural to ask whether this converse still holds for locally compact groupoids. Surprisingly, Willett gave an example of an étale groupoid for which this fact fails [16] (see [1] for a related example). In fact, in Willett's example one has the following equivalent properties:

- $\mathcal{G}$ is amenable;
- $\mathcal{G}$ is exact and has the (WCP).

We conjecture that this equivalence applies for any étale groupoid. This is based on the following two results. First, in [12] it is proved that when G is a discrete group acting by homeomorphisms on a compact space X, the corresponding transformation groupoid $\mathcal{G} = X \rtimes G$ is amenable if and only if it has the (WCP) and G is exact. Secondly, Bönicke proved in [9] that if $\mathcal{G}$ is a locally compact groupoid such that the orbit space $\mathcal{G}\backslash X$ equipped with the quotient topology is T_0, then $\mathcal{G}$ is measurewise amenable if and only if it has the (WCP) and is inner exact. Here X denotes the space of units of $\mathcal{G}$ equipped with the natural $\mathcal{G}$-action. Measurewise amenability is a notion weaker than amenability, equivalent to it in many cases and in particular for étale groupoids [2, Theorem 3.3.7]. Inner exactness is a much weaker notion than exactness (see the remark 2.1 below for some details). We also remark that, recently, Matsumura's result in [12] has been extended in [11] as follows: if G is an exact locally compact group acting on a locally compact space X, then $\mathcal{G} = X \rtimes G$ is measurewise amenable if and only if it has the (WCP).

For the notion of locally compact exact groupoid we refer to [6]. In this paper we are only interested in the groupoid $\mathcal{G} = X \rtimes G$, associated with an action of a discrete group on a locally compact space X. In this case the different possible definitions of exactness for $\mathcal{G}$ coincide (see [6, Theorem 8.6]). When X is not compact we observe that $\mathcal{G}$ can be exact,

Date: July 21, 2020.

[a]We implicitely assume that the groupoids we consider are second countable and with Haar system. Moreover locally compact spaces are assumed to be Hausdorff.

although G is not exact (think for instance of the action of G onto itself by translations). On the the other hand, when X is compact, then $X \rtimes G$ is exact if and only if the group G is exact (see [6, Proposition 4.3]). In this paper, we extend Matsumura's result and strengthen our conjecture with the following proposition.

Proposition. *Let G be a discrete group acting by homeomorphisms on a locally compact space X. Then the groupoid $X \rtimes G$ is amenable if and only if it is exact and has the (WCP).*

2. Proof of the above proposition

Let $\alpha : G \curvearrowright X : (g, x) \mapsto gx$ be a left action of a discrete group G on a locally compact space X. In this case, the *weak containment property* (in short (WCP)) means that the canonical surjective map from the full crossed product $C_0(X) \rtimes G$ $(= C^*(X \rtimes G))$ onto the reduced crossed product $C_0(X) \rtimes_r G$ $(= C^*_r(X \rtimes G))$ is injective. Amenable actions of groups are defined for instance in [2], [4, Definition 2.1]. We also refer to [3] for equivalent definitions that will be useful. For reasons explained above we only consider the case where X is not compact.

We set $A = C_0(X)$. The bidual A^{**} of A is a commutative von Neumann algebra. Let us denote by Ω its spectrum. It is a compact, Hausdorff, extremally disconnected space. We denote by $\hat{X}$ the one-point compactification of X. We set $A^+ = C(\hat{X})$. Then, A^+ is a unital C^*-subalgebra of A^{**}. We view $f \in C_0(X)$ as a continuous function on $\hat{X} = X \cup \{\infty\}$, with value 0 at ∞. The canonical action α of G on A extends by bitransposition to an action on A^{**} that we still denote by α, so that $A^+ \subset A^{**}$ is a G-equivariant inclusion. Note that α induces a left action on Ω by setting $f(t\omega) = \alpha_{t^{-1}}(f)(\omega)$ for $f \in C(\Omega)$, $t \in G$ and $\omega \in \Omega$.

Given $\omega \in \Omega$, we denote by $\rho(\omega)$ the character of A^+ sending $f \in A^+$ onto $f(\omega)$. Then $\rho : \Omega \to \hat{X}$ is an equivariant surjective continuous map. In the sequel, when needed, A is canonically identified with an ideal of A^+ and with a C^*-subalgebra of $C(\Omega)$ via the map $f \in A \mapsto f \circ \rho$.

Let us represent A^{**} in standard form on a Hilbert space H. Then there is a localizable Borel probability measure μ on Ω such that the action of A^{**} on H is spatially isomorphic to the action on $L^\infty(\Omega, \mu)$ (identified with $C(\Omega)$) on $L^2(\Omega, \mu)$ by multiplication [8]. We will freely make this identification.

Let $t \mapsto u_t$ be the unitary representation of G on H such that $\mathrm{Ad}\, u_t = \alpha_t$ for every $t \in G$. Note that $(u_t f u_t^*)(\omega) = f(t^{-1}\omega)$ for $f \in A^{**} = C(\Omega)$ and $\omega \in \Omega$.

Any amenable action is obviously exact [6] and has the WCP [4, Theorem 5.3]. We want to prove the converse. Assume that $A \rtimes_r G = A \rtimes G$. Using the universal property of $A \rtimes G$ with respect to covariant representations, there exists a unique representation π of $A \rtimes G$ on H such that $\pi(a\delta_t) = au_t$ for $a \in A$ and $t \in G$ where $a\delta_t$ denotes the function on $X \times G$ with value $a(x)$ on (x, t) and 0 on (x, s) when $s \neq t$. On the other hand, the C^*-algebra $A \rtimes_r G$ is canonically embedded in $\mathcal{B}(L^2(\Omega, \mu) \otimes \ell^2(G))$. In this embedding, $a\delta_t$ is sent onto $au_t \otimes \lambda_t$ where $(\lambda_t \xi)(s) = \xi(t^{-1}s)$ for $\xi \in \ell^2(G)$ and $s, t \in G$. This homomorphism

π extends to a completely positive contraction ϕ from $\mathcal{B}(L^2(\Omega,\mu)\otimes \ell^2(G)$ onto $\mathcal{B}(H))$, by Arveson's extension theorem.

Let $(\beta_r(X\times G), r_\beta)$ be the Stone-Čech fibrewise compactification of $X\times G$ with respect to the projection[b] $r: X\times G\to X$ (see [5], to which we refer for more details and notation). Recall that $\beta_r(X\times G)$ is the spectrum of the ideal $C_0(X\times G, r)$ of $C_b(X\times G)$ of all continuous bounded function f on $X\times G$ such that for every $\varepsilon>0$ there exists a compact subset K on X, with $|f(x,t)|\le\varepsilon$ if $x\notin K$ and $t\in G$.

To every $f\in C_0(X\times G, r)$ we associate a function $\tilde f$ defined on $\Omega\times G$ by

$$\begin{aligned}\tilde f(\omega,t) &= 0 \quad \text{if} \quad \rho(\omega)=\infty\in\hat X\\ &= f(\rho(\omega),t) \quad \text{if} \quad \rho(\omega)\in X.\end{aligned}$$

In fact, $C_0(X\times G, r)$ is canonically contained in the space of continuous bounded functions h on $\hat X\rtimes G$ such that $h(x,t)$ vanishes at infinity on X for every $t\in G$, by setting $h(\infty,t)=0$ for $t\in G$, whenever $h\in C_0(X\times G, r)$. Then $\tilde f = f\circ(\rho\times \mathrm{Id}_G)$.

We denote by ι the injective homomorphism $f\mapsto\tilde f$ from $C_0(X\times G, r)$ into $C_b(\Omega\times G)$.

Note that $C_0(X)$ is embedded into $C_0(X\times G, r)$, thanks to the map $f\mapsto r^*f = f\circ r$. Therefore when viewed as an element of $C_b(\Omega\times G)$, f become the function $(\omega,t)\mapsto f(\rho(\omega))$.

We define a faithful representation Π of $C_0(X\times G, r)$ on the Hilbert space $L^2(\Omega,\mu)\otimes\ell^2(G)$ by setting

$$(\Pi(f)\xi)(\omega,t) = \tilde f(\omega,t)\xi(\omega,g) = \iota(f)(\omega,t)\xi(\omega,t).$$

Therefore $\phi\circ\Pi$ is a completely positive contraction from $C_0(X\times G, r)$ into $\mathcal{B}(H)$.

Recall from [5], that r_β is the unique map from $\beta_r(X\times G)$ onto X that extends the projection $r = X\times G\to X$. Then, $f\mapsto r_\beta^*(f) = f\circ r_\beta \equiv f\circ r = r^*f$ is an inclusion from $C_0(X)$ into

$$C_0(\beta_r(X\times G)) = C_0(X\times G, r).$$

For $f\in C_0(X)$, we have have

$$\big(\Pi(r_\beta^* f)\xi\big)(\omega,t) = \big(\Pi(r^*f)\xi\big)(\omega,t) = (f\circ\rho)(\omega)\xi(\omega,t) = \big((f\otimes\lambda_e)\xi\big)(\omega,t)$$

and therefore $\phi\circ\Pi(r^*(f)) = f$.

Note that $\Pi(r^*f)$ is in the multiplicative domain of ϕ for $f\in C_0(X)$. It follows that for $F\in C_0(X\times G, r)$,

$$\phi\circ\Pi(F)f = \phi(\Pi(F)\Pi(r^*(f))) = \phi(\Pi(r^*(f))\Pi(F)) = f\phi\circ\Pi(F).$$

This implies that $\phi\circ\Pi(F)\in A^{**}$.

The action on $C_b(\hat X\times G)$ provided by the diagonal action of G induces by restriction an action on $C_0(X\times G, r)$ that we denote by θ. For $t\in G$ and $f\in C_0(X\times G, r)\subset C_b(\hat X\times G)$

[b]which is also the range projection of the transformation groupoid $\mathcal{G} = X\rtimes G$

we have

$$\theta_t(f)(x,s) = f(t^{-1}x, t^{-1}s)$$

and since $\rho \times \mathrm{Id}_G$ is G-equivariant we have

$$\iota(\theta_t(f))(\omega, s) = \iota(f)(t^{-1}\omega, t^{-1}s).$$

It follows that for $f \in C_0(X \times G, r)$ and $t \in G$ we have

$$\iota(\theta_t f) = (u_t \otimes \lambda_t)\iota(f)(u_t \otimes \lambda_t)^*,$$

that is

$$\Pi(\theta_t f) = (u_t \otimes \lambda_t)\Pi(f)(u_t \otimes \lambda_t)^*.$$

Let us show now that $\Phi = \phi\circ\Pi$ is equivariant. Let (e_i) be an approximate unit in $C_0(X)$. For $f \in C_0(X \times G, r)$ and $t \in G$ we have

$$\begin{aligned}
&\phi \circ \Pi\Big(\theta_t(r^*(e_i)fr^*(e_j))\Big)\\
&\quad = \phi\Big((u_t \otimes \lambda_t)\Pi\big(r^*(e_i)fr^*(e_j)\big)(u_t \otimes \lambda_t)^*\Big)\\
&\quad = \phi\Big((u_t(e_i \circ \rho) \otimes \lambda_t)\Pi(f)(u_t(e_j \circ \rho) \otimes \lambda_t)^*\Big)\\
&\quad = u_t e_i \phi \circ \Pi(f) e_j u_t^*,
\end{aligned}$$

since $u_t e_k \circ \rho \otimes \lambda_t$ is in the multiplicative domain of ϕ with $\phi(u_t e_k \circ \rho \otimes \lambda_t) = u_t e_k$ for $k = i, j$ (note that we identify $e_k \in A$ with $e_k \circ \rho$ when viewed as an element of A^{**}).

We have $\lim_{i,j} \|r^*(e_i)fr^*(e_j) - f\| = 0$ on one hand and $\lim_{i,j} e_i \phi \circ \Pi(f) e_j = \phi \circ \Pi(f)$ in the weak* topology of A^{**}. This concludes our claim about the equivariance of $\phi \circ \Pi(f)$.

We assume now that the groupoid $\mathcal{G} = X \rtimes G$ is exact. By [6, Theorem 8.6], this means that the Stone-Čech fibrewise compactification $(\beta_r\mathcal{G}, r_\beta)$ of $\mathcal{G}$ is an amenable $\mathcal{G}$-space. Thus, the semidirect product groupoid $\beta_r\mathcal{G} \rtimes \mathcal{G}$ is amenable. Recall that the $\mathcal{G}$-action on $\beta_r\mathcal{G}$ denoted by $\big((x,t), z\big) \mapsto (x,t)z$ requires that $r_\beta(z) = t^{-1}x$.

Note that the group G acts on $\beta_r\mathcal{G}$ by

$$t \cdot z = (tr_\beta(z), t)z$$

where $(tr_\beta(z), t) \in \mathcal{G} = X \rtimes G$. One immediately checks that $(t, z) \mapsto t \cdot z$ defines a left action of G on $\beta_r\mathcal{G}$, since r_β is G-equivariant, due to the fact that $\mathcal{G}$ acts to the left on $\beta_r\mathcal{G}$. Moreover the semidirect product groupoids $\beta_r\mathcal{G} \rtimes G$ and $\beta_r\mathcal{G} \rtimes \mathcal{G}$ are canonically isomorphic via the isomorphism $(z, t) \mapsto (z, (r_\beta(z), t))$. Therefore the left action of G on $\beta_r\mathcal{G}$ is amenable. It follows that there exists a net (h_i) of positive type finitely supported functions $h_i : G \to C_0(X \rtimes G, r)$ with $h_i(e) \le 1$ such that, for every $t \in G$, we have $\lim_i h_i(t) = 1$ in the strict topology of $C_0(X \rtimes G, r)$ (see [3]).

Let us consider the net $(\Phi \circ h_i)$ of functions from $G \to A^{**}$. Since Φ is equivariant and completely positive we see that $\Phi \circ h_i$ is of positive type. Let us show that $\lim_i \Phi \circ h_i(t) = 1$

in the ultraweak topology of A^{**}. Let $\varphi \in A^*$ and $f \in A$ and set $\psi : a \mapsto \varphi(fa)$. We have, since $f = \Phi(r^*f)$, and since $\Pi(r^*f)$ is in the multiplicative domain of ϕ,

$$|\psi(\Phi(h_i(t)) - 1)| = \Big|\varphi\Big(\Phi\big(r^*f(h_i(t) - 1)\big)\Big)\Big|.$$

We have $\lim_i r^*f(h_i(t) - 1) = 0$ in norm and therefore

$$\lim_i |\psi(\Phi(h_i(t)) - 1)| = \lim_i \Big|\varphi\Big(\Phi\big(r^*f(h_i(t) - 1)\big)\Big)\Big| = 0.$$

It follows that $\lim_i \Phi \circ h_i(t) = 1$ in the ultraweak topology of A^{**} because the elements of A^* like ψ are dense in norm into A^*. We conclude that the action $\alpha : G \curvearrowright X$ is amenable by using again results from [3].

Remark 2.1. **About inner exactness**. Let $\mathcal{G}$ be a locally compact groupoid and let Y be a subset of $X = \mathcal{G}^{(0)}$. We set $\mathcal{G}(Y) = r^{-1}(Y) \cap s^{-1}(Y)$.

Definition 2.2. [7] We say that a locally compact groupoid $\mathcal{G}$ with Haar system is *inner exact* if for every $\mathcal{G}$-invariant closed subset F of X, the canonical sequence

$$0 \xrightarrow{i} C_r^*(\mathcal{G}(U)) \longrightarrow C_r^*(\mathcal{G}) \xrightarrow{p} C_r^*(\mathcal{G}(F)) \longrightarrow 0$$

is exact.

Recall that i is always injective and p is always surjective. The term "inner" in the above definition aims to highlight that we only consider short sequences with respect to the left action of the groupoid on its set of units. This notion is much weaker than the KW- exactness that we defined in [6, Definition 7.6], which, when the groupoid is a group is the notion of exactness defined by Kirchberg and Wassermann. Obviously, every locally compact group is inner exact. More generally, locally compact groupoids $\mathcal{G}$ such that $\mathcal{G}$ acts transitively on $\mathcal{G}^{(0)}$, and groupoid group bundles are inner exact.

Problems:

1) Let $G \curvearrowright X$ be an action of a discrete group on a compact space. Is it true that the (WCP) implies that G is exact? If yes, (WCP) would be equivalent to amenability. This question was already asked in [16].

Observe that if G is a Gromov monster group, $G \curvearrowright \beta G$ has not the (WCP) (see [10, Lemma 4.7]). Gromov monsters are not exact either.

2) Another open question seems to be whether the (WCP) for $G \curvearrowright \beta G$ (for the canonical extension of $G \curvearrowright G$) implies the exactness of G.

Observe that for the boundary compact set $X = \partial G = \beta G \setminus G$, equipped with the natural action of G the answer is positive. Indeed, the weak containment property for $G \ltimes \partial G$ implies that the sequence

$$0 \longrightarrow C_r^*(G \ltimes G) \longrightarrow C_r^*(G \ltimes \beta G) \longrightarrow C_r^*(G \ltimes \partial G) \longrightarrow 0$$

is exact. Roe and Willett proved in [15] that this exactness property implies that G has Yu's property A and thus is exact.

References

[1] Vadim Alekseev and Martin Finn-Sell. Non-amenable principal groupoids with weak containment. *Int. Math. Res. Not.*, 8:2332—-2340, 2018.

[2] C. Anantharaman-Delaroche and J. Renault. *Amenable groupoids*, volume 36 of *Monographies de L'Enseignement Mathématique*. L'Enseignement Mathématique, Geneva, 2000. With a foreword by Georges Skandalis and Appendix B by E. Germain.

[3] Claire Anantharaman-Delaroche. Systèmes dynamiques non commutatifs et moyennabilité. *Math. Ann.*, 279:297–315, 1987.

[4] Claire Anantharaman-Delaroche. Amenability and exactness for dynamical systems and their C^*-algebras. *Trans. Amer. Math. Soc.*, 354(10):4153–4178 (electronic), 2002.

[5] Claire Anantharaman-Delaroche. Fibrewise equivariant compactifications under étale groupoid actions. `https://hal.archives-ouvertes.fr/hal-01081807`, 2014.

[6] Claire Anantharaman-Delaroche. Exact groupoids. https://arxiv.org/pdf/1605.05117.pdf, 2016.

[7] Claire Anantharaman-Delaroche. Some remarks about the weak containment property for groupoids and semigroups. https://arxiv.org/pdf/arXiv:1604.01724.pdf, 2016.

[8] Tim Austin. Commutative von neumann algebras and representations of normal hilbert space operators. https://pdfs.semanticscholar.org/e4d5/eb1e3214861f752561008ef4475b7f46b2bd.pdf, 2016.

[9] Christian Bönicke. Amenability and exactness for group bundles. Private communication, 2019.

[10] Alcides Buss, Siegfried Echteroff, and Rufus Willett. The maximal injective crossed product. https://arxiv.org/pdf/1808.06804v1.pdf, 2018.

[11] Alcides Buss, Siegfried Echteroff, and Rufus Willett. Amenability and weak containment for actions of locally compact groups on C*-algebras. https://arxiv.org/pdf/2003.03469v1.pdf, 2020.

[12] Masayoshi Matsumura. A characterization of amenability of group actions on C^*-algebras. *J. Operator Theory*, 72(1):41–47, 2014.

[13] Alan L. T. Paterson. *Amenability*, volume 29 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1988.

[14] Jean Renault. The ideal structure of groupoid crossed product C^*-algebras. *J. Operator Theory*, 25(1):3–36, 1991. With an appendix by Georges Skandalis.

[15] John Roe and Rufus Willett. Ghostbusting and property A. *J. Funct. Anal.*, 266(3):1674–1684, 2014.

[16] Rufus Willett. A non-amenable groupoid whose maximal and reduced C^*-algebras are the same. *Münster J. Math.*, 8:241–252, 2015.

Institut Denis Poisson
Université d'Orléans, Université de Tours et CNRS-UMR 7013,
Route de Chartres, B.P. 6759; F-45067 Orléans Cedex 2, France

Email address: `claire.anantharaman@univ-orleans.fr`