AN INTRODUCTION TO DUNKL THEORY AND ITS ANALYTIC ASPECTS

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1. Introduction

Dunkl theory is a far reaching generalization of Fourier analysis and special function theory related to root systems. During the sixties and seventies, it became gradually clear that radial Fourier analysis on rank one symmetric spaces was closely connected with certain classes of special functions in one variable :

- Bessel functions in connection with radial Fourier analysis on Euclidean spaces,
- Jacobi polynomials in connection with radial Fourier analysis on spheres,
- Jacobi functions (i.e. the Gauss hypergeometric function ${}_2\mathrm{F}_1$) in connection with radial Fourier analysis on hyperbolic spaces.

See [51] for a survey. During the eighties, several attempts were made, mainly by the Dutch school (Koornwinder, Heckman, Opdam), to extend these results in higher rank (i.e. in several variables), until the discovery of Dunkl operators in the rational case and Cherednik operators in the trigonometric case. Together with q–special functions introduced by Macdonald, this has led to a beautiful theory, developed by several authors which encompasses in a unified way harmonic analysis on all Riemannian symmetric spaces and spherical functions thereon :

- generalized Bessel functions on flat symmetric spaces, and their asymmetric version, known as the Dunkl kernel,
- Heckman–Opdam hypergeometric functions on positively or negatively curved symmetric spaces, and their asymmetric version, due to Opdam,
- Macdonald polynomials on affine buildings.

Beside Fourier analysis and special functions, this theory has also deep and fruitful interactions with

- algebra (double affine Hecke algebras),
- mathematical physics (Calogero-Moser-Sutherland models, quantum many body problems),
- probability theory (Feller processes with jumps).

There are already several surveys about Dunkl theory available in the literature :

- [67] (see also [27]) about rational Dunkl theory (state of the art in 2002),
- [64] about trigonometric Dunkl theory (state of the art in 1998),
- [29] about integrable systems related to Dunkl theory,
- [54] and [18] about q−Dunkl theory and affine Hecke algebras,
- [40] about probabilistic aspects of Dunkl theory (state of the art in 2006).

These lectures are intended to give an overview of some analytic aspects of Dunkl theory. The topics are indicated in red in Figure 1, where we have tried to summarize relations between several theories of special functions, which were alluded to above, and where arrows mean limits.

Let us describe the content of our notes. In Section 2, we consider several geometric settings (Euclidean spaces, spheres, hyperbolic spaces, homogeneous trees, . . .) where radial Fourier analysis is available and can be applied successfully, for instance to study evolutions equations (heat equation, wave equation, Schrödinger equation, . . .). Section 3 is devoted to the rational Dunkl theory and Section 4 to the trigonometric Dunkl theory. In both cases, we first review the basics and next address some important analytic issues. We conclude with an appendix about root systems and with a comprehensive bibliography. For lack of time and competence, we haven't touched upon other aforementioned aspects of Dunkl theory, for which we refer to the bibliography.

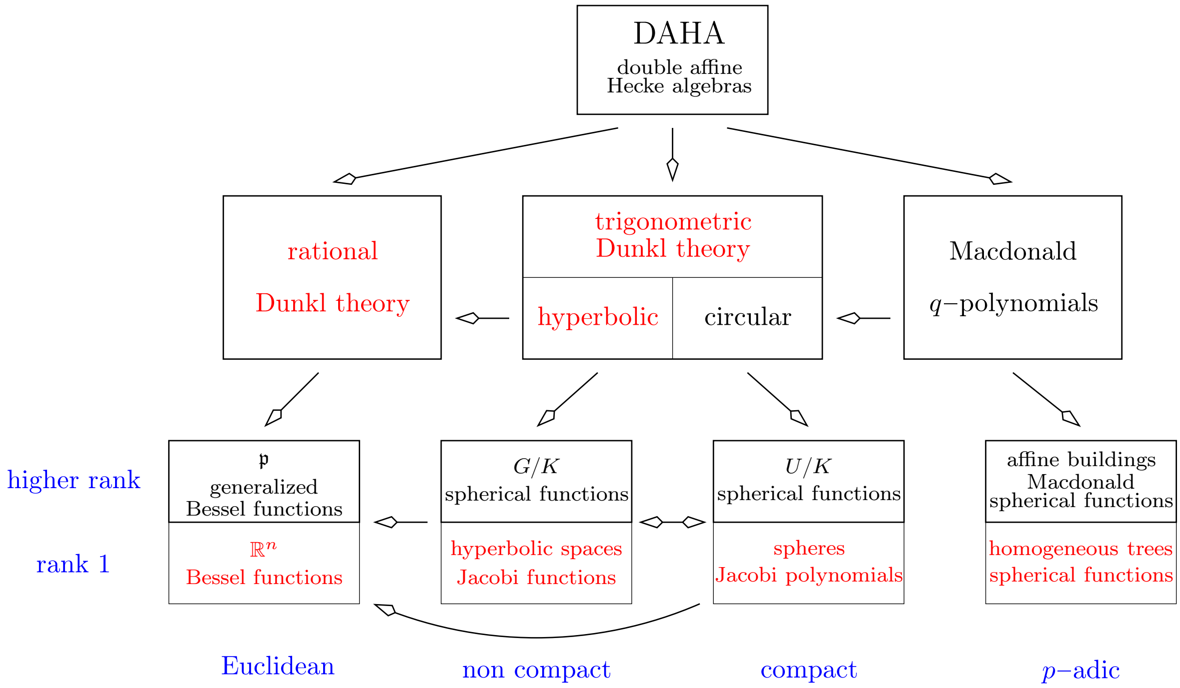

FIGURE 1. Relation between various special function theories

2. Spherical Fourier analysis in rank 1

2.1. **Cosine transform.** Let us start with an elementary example. Within the framework of even functions on the real line $\mathbb{R}$, the Fourier transform is given by

$$\widehat{f}(\lambda)=\int_{\mathbb{R}}dx\,f(x)\,\cos\lambda x$$

and the inverse Fourier transform by

$$f(x)=\tfrac{1}{2\pi}\int_{\mathbb{R}}d\lambda\,\widehat{f}(\lambda)\,\cos\lambda x\,.$$

The cosine functions $\varphi_\lambda(x)=\cos\lambda x$ $(\lambda\in\mathbb{C})$ occurring in these expressions can be characterized in various ways. Let us mention

- *Power series expansion* :
$$\varphi_\lambda(x)=\sum\nolimits_{\ell=0}^{+\infty}\tfrac{(-1)^\ell}{(2\ell)!}\,(\lambda x)^{2\ell}\qquad\forall\;\lambda,x\in\mathbb{C}\,.$$
- *Differential equation* : the functions $\varphi=\varphi_\lambda$ are the smooth eigenfunctions of $(\frac{\partial}{\partial x})^2$, which are even and normalized by $\varphi(0)=1$.
- *Functional equation* : the functions $\varphi=\varphi_\lambda$ are the nonzero continuous functions on $\mathbb{R}$ which satisfy
$$\tfrac{\varphi(x+y)\,+\,\varphi(x-y)}{2}=\varphi(x)\,\varphi(y)\qquad\forall\;x,y\in\mathbb{R}.$$

2.2. **Hankel transform on Euclidean spaces.** The Fourier transform on $\mathbb{R}^n$ and its inverse are given by

$$\widehat{f}(\xi)=\int_{\mathbb{R}^n}dx\,f(x)\,e^{-i\langle\xi,x\rangle}\tag{1}$$

and

$$f(x)=(2\pi)^{-n}\int_{\mathbb{R}^n}d\lambda\,\widehat{f}(\xi)\,e^{i\langle\xi,x\rangle}\tag{2}$$

Notice that the Fourier transform of a radial function $f=f(r)$ on $\mathbb{R}^n$ is again a radial function $\widehat{f}=\widehat{f}(\lambda)$. In this case, (1) and (2) become

$$\widehat{f}(\lambda)\,=\,\tfrac{2\,\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\int_0^{+\infty}\!dr\;r^{n-1}\,f(r)\;j_{\frac{n-2}{2}}(i\lambda r) \tag{3}$$

and

$$f(r)\,=\,\frac{1}{2^{n-1}\,\pi^{\frac{n}{2}}\,\Gamma(\frac{n}{2})}\int_0^{+\infty}\!d\lambda\;\lambda^{n-1}\,\widehat{f}(\lambda)\;j_{\frac{n-2}{2}}(i\lambda r)\,. \tag{4}$$

Instead of the exponential function or the cosine function, (3) and (4) involve now the modified Bessel function $j_{\frac{n-2}{2}}$, which can be characterized again in various ways:

- *Relation with classical special functions and power series expansion*. For every $z\in\mathbb{C}$,

$$\begin{aligned} j_{\frac{n-2}{2}}(z)\,&=\,\Gamma(\tfrac{n}{2})\,\big(\tfrac{iz}{2}\big)^{\frac{2-n}{2}}\,J_{\frac{n-2}{2}}(iz)\\ &=\,\textstyle\sum_{\ell=0}^{+\infty}\,\frac{\Gamma(\frac{n}{2})}{\ell\,!\;\Gamma(\frac{n}{2}+\ell)}\,\big(\tfrac{z}{2}\big)^{2\ell}\\ &=\,{}_0\mathrm{F}_1(\tfrac{n}{2};\tfrac{z^2}{4})\,=\,e^{-z}\,{}_1\mathrm{F}_1(\tfrac{n-1}{2};n-1;2z)\,, \end{aligned}$$

  where J_ν denotes the classical Bessel function of the first kind and

$${}_p\mathrm{F}_q(a_1,\dots,a_p;b_1,\dots,b_q;z)\,=\,\textstyle\sum_{\ell=0}^{+\infty}\frac{(a_1)_\ell\,\cdots\,(a_p)_\ell}{(b_1)_\ell\,\cdots\,(b_q)_\ell}\,\frac{z^\ell}{\ell\,!}$$

  the generalized hypergeometric function.
- *Differential equations*. The function $\varphi_\lambda(r)=j_{\frac{n-2}{2}}(i\lambda r)$ is the unique smooth solution to the differential equation

$$\big(\tfrac{\partial}{\partial r}\big)^2\varphi_\lambda+\tfrac{n-1}{r}\,\big(\tfrac{\partial}{\partial r}\big)\,\varphi_\lambda+\lambda^2\,\varphi_\lambda\,=\,0\,,$$

  which is normalized by $\varphi_\lambda(0)=1$. Equivalently, the function

$$x\longmapsto\varphi_\lambda(|x|)\,=\,j_{\frac{n-2}{2}}(i\lambda|x|) \tag{5}$$

  is the unique smooth radial normalized eigenfunction of the Euclidean Laplacian

$$\Delta_{\mathbb{R}^n}\,=\,\textstyle\sum_{j=1}^{n}\big(\frac{\partial}{\partial x_j}\big)^2\,=\,\big(\frac{\partial}{\partial r}\big)^2+\frac{n-1}{r}\,\big(\frac{\partial}{\partial r}\big)+\frac{1}{r^2}\,\Delta_{\mathbb{S}^{n-1}}$$

  corresponding to the eigenvalue $-\lambda^2$.

Remark 2.1. *The function* (5) *is a matrix coefficient of a continuous unitary representation of the Euclidean motion group* $\mathbb{R}^n\rtimes O(n)$.

The function (5) is a spherical average of plane waves. Specifically,

$$\varphi_\lambda(|x|)\,=\int_{\mathrm{O}(n)}\!dk\;e^{\,i\lambda\langle u,k.x\rangle}\,=\,\frac{\Gamma(\frac{n}{2})}{2\,\pi^{\frac{n}{2}}}\int_{\mathbb{S}^{n-1}}\!dv\;e^{\,i\lambda\langle v,x\rangle}\,,$$

where u is any unit vector in $\mathbb{R}^n$. Hence the integral representation

$$\begin{aligned} \varphi_\lambda(r)\,&=\,\frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\;\Gamma(\frac{n-1}{2})}\int_0^{\pi}\!d\theta\,(\sin\theta)^{n-2}\,e^{\,i\lambda\,r\cos\theta}\\ &=\,\frac{2\,\Gamma(\frac{n}{2})}{\sqrt{\pi}\;\Gamma(\frac{n-1}{2})}\,r^{2-n}\int_0^{r}\!ds\;(r^2-s^2)^{\frac{n-3}{2}}\,\cos\lambda s\,. \end{aligned} \tag{6}$$

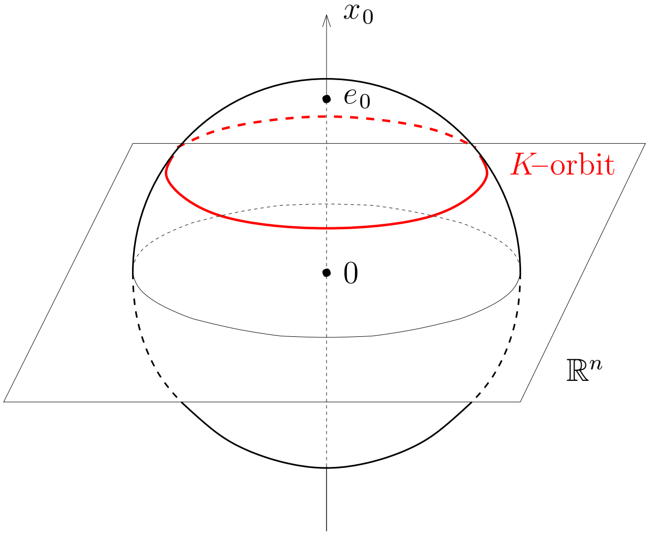

FIGURE 2. Real sphere $\mathbb{S}^n$

2.3. **Spherical Fourier analysis on real spheres.** Real spheres (Figure 2)

$$\mathbb{S}^n = \{\, x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{1+n} \mid |x|^2 = x_0^2 + \ldots + x_n^2 = 1 \,\}$$

of dimension $n \geqslant 2$ are the simplest examples of Riemannian symmetric spaces of compact type. They are simply connected Riemannian manifolds, with constant positive sectional curvature. The Riemannian structure on $\mathbb{S}^n$ is induced by the Euclidean metric in $\mathbb{R}^{1+n}$, restricted to the tangent bundle of $\mathbb{S}^n$, and the Laplacian on $\mathbb{S}^n$ is given by $\Delta f = \widetilde{\Delta} \widetilde{f}\,\big|_{\mathbb{S}^n}$, where $\widetilde{\Delta} = \sum_{j=0}^n \big(\frac{\partial}{\partial x_j}\big)^2$ denotes the Euclidean Laplacian in $\mathbb{R}^{1+n}$ and $\widetilde{f}(x) = f\big(\frac{x}{|x|}\big)$ the homogeneous extension of f to $\mathbb{R}^{1+n} \smallsetminus \{0\}$. In spherical coordinates

$$\begin{cases} x_0 &= \cos\theta_1\,, \\ x_1 &= \sin\theta_1 \cos\theta_2\,, \\ &\vdots \\ x_{n-1} &= \sin\theta_1 \sin\theta_2 \,\ldots\, \sin\theta_{n-1} \cos\theta_n\,, \\ x_n &= \sin\theta_1 \sin\theta_2 \,\ldots\, \sin\theta_{n-1} \sin\theta_n\,, \end{cases}$$

the Riemannian metric, the Riemannian volume and the Laplacian read respectively

$$ds^2 = \sum\nolimits_{j=1}^n (\sin\theta_1)^2 \ldots (\sin\theta_{j-1})^2\, (d\theta_j)^2,$$

$$d\,\mathrm{vol} = (\sin\theta_1)^{n-1} \ldots (\sin\theta_{n-1}) d\theta_1 \ldots d\theta_n$$

and

$$\Delta = \sum\nolimits_{j=1}^n \tfrac{1}{(\sin\theta_1)^2 \ldots (\sin\theta_{j-1})^2} \left\{ \big(\tfrac{\partial}{\partial\theta_j}\big)^2 + (n-j)(\cot\theta_j)\, \tfrac{\partial}{\partial\theta_j} \right\}.$$

Let $G = \mathrm{O}(n+1)$ be the isometry group of $\mathbb{S}^n$ and let $K \approx \mathrm{O}(n)$ be the stabilizer of $e_0 = (1, 0, \dots, 0)$. Then $\mathbb{S}^n$ can be realized as the homogeneous space G/K. As usual, we identify right–K–invariant functions on G with functions on $\mathbb{S}^n$, and bi–K–invariant functions on G with radial functions on $\mathbb{S}^n$ i.e. functions on $\mathbb{S}^n$ which depend only on $x_0 = \cos\theta_1$. For such functions,

$$\int_{\mathbb{S}^n} d\,\mathrm{vol}\; f \;=\; 2\,\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^\pi d\theta_1\, (\sin\theta_1)^{n-1} f(\cos\theta_1)$$

and

$$\Delta f \;=\; \tfrac{\partial^2 f}{\partial\theta_1^2} + (n-1)(\cot\theta_1)\, \tfrac{\partial f}{\partial\theta_1}\,.$$

The spherical functions on $\mathbb{S}^n$ are the smooth normalized radial eigenfunctions of the Laplacian on $\mathbb{S}^n$. Specifically,

$$\begin{cases} \Delta\,\varphi_\ell = -\ell\,(\ell+n-1)\,\varphi_\ell\,, \\ \varphi_\ell(e_0) = 1\,, \end{cases}$$

where $\ell\in\mathbb{N}$. They can be expressed in terms of classical special functions, namely

$$\varphi_\ell(x_0) = \tfrac{\ell\,!\,(n-2)!}{(\ell+n-2)!}\,C_\ell^{(\frac{n-1}{2})}(x_0) = \tfrac{\ell\,!}{(\frac{n}{2})_\ell}\,P_\ell^{(\frac{n}{2}-1,\frac{n}{2}-1)}(x_0)$$

or

$$\varphi_\ell(\cos\theta_1) = {}_2\mathrm{F}_1\big(-\ell,\ell+n-1;\tfrac{n}{2};\sin^2\theta_1\big)\,,$$

where $C_\ell^{(\lambda)}$ are the Gegenbauer or ultraspherical polynomials, $P_\ell^{(\alpha,\beta)}$ the Jacobi polynomials and ${}_2\mathrm{F}_1$ the Gauss hypergeometric function.

Remark 2.2. *We have emphasized the characterization of spherical functions on $\mathbb{S}^n$ by a differential equation. Here are other characterizations:*

- *The spherical functions are the continuous bi–K–invariant functions φ on G which satisfy the functional equation*

$$\int_K dk\;\varphi(xky) = \varphi(x)\,\varphi(y) \qquad \forall\;x,y\in G. \tag{7}$$

- *The spherical functions are the continuous bi–K–invariant functions φ on G such that*

$$f \;\longmapsto \int_G dx\;f(x)\,\varphi(x) \tag{8}$$

defines a character of the (commutative) convolution algebra $C_c(K\backslash G/K)$.

- *The spherical functions are the matrix coefficients*

$$\varphi(x) = \langle\pi(x)v,v\rangle\,,$$

where π is a continuous unitary representation of G, which has nonzero K–fixed vectors and which is irreducible, and v is a K–fixed vector, which is normalized by $|v|=1$.

- Integral representation:

$$\begin{aligned} \varphi_\ell(\cos\theta_1) &= \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}\int_0^\pi d\theta_2\,(\sin\theta_2)^{n-2}\big[\cos\theta_1+i\,(\sin\theta_1)(\cos\theta_2)\big]^\ell \\ &= \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}\,(\sin\theta_1)^{2-n}\int_{-\sin\theta_1}^{\sin\theta_1} ds\,(\sin^2\theta_1-s^2)^{\frac{n-3}{2}}\,(\cos\theta_1+i\,s)^\ell\,. \end{aligned} \tag{9}$$

The spherical Fourier expansion of radial functions on $\mathbb{S}^n$ reads

$$f(x) = \sum\nolimits_{\ell\in\mathbb{N}} d_\ell\,\langle f,\varphi_\ell\rangle\,\varphi_\ell(x)\,,$$

where

$$d_\ell = \tfrac{n\,(n+2\ell-1)\,(n+\ell-2)!}{n\,!\,\ell\,!}$$

and

$$\langle f,\varphi_\ell\rangle = \frac{\Gamma(\frac{n+1}{2})}{2\,\pi^{\frac{n+1}{2}}}\int_{\mathbb{S}^n} dx\,f(x)\,\varphi_\ell(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\,\Gamma(\frac{n}{2})}\int_0^\pi d\theta_1\,(\sin\theta_1)^{n-1} f(\cos\theta_1)\,\varphi_\ell(\cos\theta_1)\,.$$

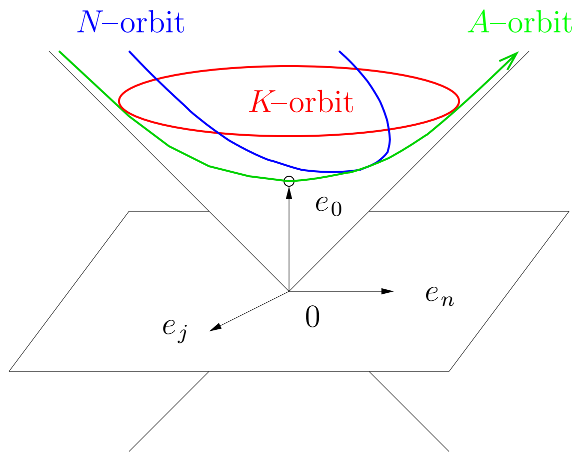

FIGURE 3. Hyperboloid model of $\mathbb{H}^n$

2.4. **Spherical Fourier analysis on real hyperbolic spaces.** Real hyperbolic spaces $\mathbb{H}^n$ are the simplest examples of Riemannian symmetric spaces of noncompact type. They are simply connected Riemannian manifolds, with constant negative sectional curvature. Let us recall the following three models of $\mathbb{H}^n$.

• **Model 1 : Hyperboloid (Figure 3)**

In this model, $\mathbb{H}^n$ consists of the hyperboloid sheed

$$\{\, x=(x_0,x_1,\dots,x_n)\in\mathbb{R}^{1+n}=\mathbb{R}\times\mathbb{R}^n \mid L(x,x)=-1,\ x_0\geqslant 1\,\}$$

defined by the Lorentz quadratic form $L(x,x)=-\,x_0^2+x_1^2+\ldots+x_n^2$. The Riemannian structure is given by the metric $ds^2=L(dx,dx)$, restricted to the tangent bundle of $\mathbb{H}^n$, and the Laplacian by $\Delta f=L\big(\frac{\partial}{\partial x},\frac{\partial}{\partial x}\big)\widetilde{f}\,\big|_{\mathbb{H}^n}$, where $\widetilde{f}(x)=f\big(\frac{x}{\sqrt{-L(x,x)}}\big)$ denotes the homogeneous extension of f to the light cone $\{\,x\in\mathbb{R}^{1+n}\mid L(x,x)<0,\ x_0>0\,\}$.

• **Model 2 : Upper half–space (Figure 4)**

In this model, $\mathbb{H}^n$ consists of the upper half-space $\mathbb{R}^n_+=\{\,y\in\mathbb{R}^n\mid y_n>0\}$ equipped with the Riemannian metric $ds^2=y_n^{-2}\,|dy|^2$. The volume is given by $d\mathrm{vol}=y_n^{-n}\,dy_1\dots dy_n$ and the Laplacian by

$$\Delta=y_n^2\sum\nolimits_{j=1}^{n}\tfrac{\partial^2}{\partial y_j^2}-(n-2)\,y_n\tfrac{\partial}{\partial y_n}\,.$$

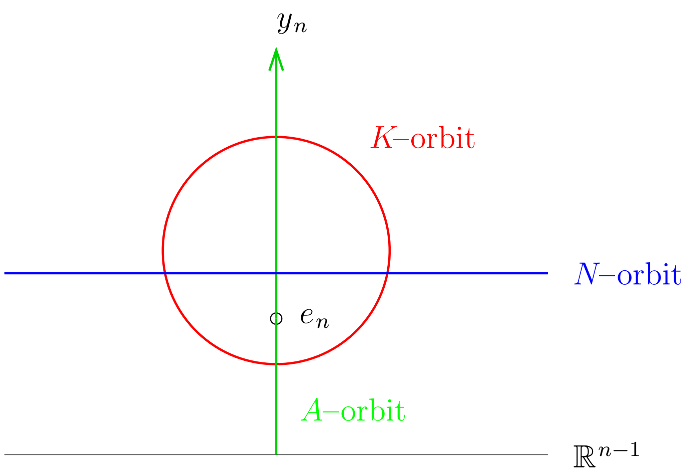

FIGURE 4. Upper half–space model of $\mathbb{H}^n$

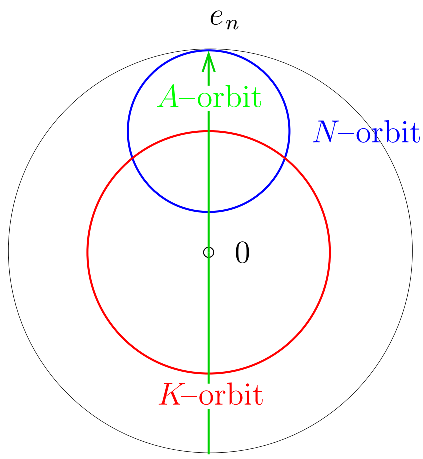

FIGURE 5. Ball model of $\mathbb{H}^n$

• **Model 3 : Ball (Figure 5)**

In this model, $\mathbb{H}^n$ consists of the unit ball $\mathbb{B}^n = \{ z \in \mathbb{R}^n \,|\, |z| < 1 \}$. The Riemannian metric is given by $ds^2 = \big(\frac{1-|z|^2}{2}\big)^{-2} |dz|^2$, the volume by $d\,\mathrm{vol} = \big(\frac{1-|z|^2}{2}\big)^{-n} dz_1 \dots dz_n$, the distance to the origin by $r = 2\,\mathrm{artanh}\,|z| = \log \frac{1+|z|}{1-|z|}$, and the Laplacian by

$$\Delta = \big(\tfrac{1-|z|^2}{2}\big)^2 \sum\nolimits_{j=1}^{n} \tfrac{\partial^2}{\partial z_j^2} + (n-2)\,\tfrac{1-|z|^2}{2} \sum\nolimits_{j=1}^{n} z_j \,\tfrac{\partial}{\partial z_j} \,.$$

Remark 2.3. *Model 1 and Model 3 are mapped onto each other by the stereographic projection with respect to* $-e_0$*, while Model 2 and Model 3 are mapped onto each other by the inversion with respect to the sphere* $S(-e_n,\sqrt{2})$*. This leads to the following formulae*

$$\begin{cases} x_0 = \frac{1+|y|^2}{2y_n} = \frac{1+|z|^2}{1-|z|^2} \\ x_j = \frac{y_j}{y_n} = \frac{2z_j}{1-|z|^2} \qquad (j=1,\dots,n-1) \\ x_n = \frac{1-|y|^2}{2y_n} = \frac{2z_n}{1-|z|^2} \end{cases}$$

$$\begin{cases} y_j = \frac{x_j}{x_0+x_n} = \frac{2z_j}{1+|z|^2+2z_n} \qquad (j=1,\dots,n-1) \\ y_n = \frac{1}{x_0+x_n} = \frac{1-|z|^2}{1+|z|^2+2z_n} \end{cases}$$

$$\begin{cases} z_j = \frac{x_j}{1+x_0} = \frac{2y_j}{1+|y|^2+2y_n} \qquad (j=1,\dots,n-1) \\ z_n = \frac{x_n}{1+x_0} = \frac{1-|y|^2}{1+|y|^2+2y_n} \,. \end{cases}$$

Let G be the isometry group of $\mathbb{H}^n$ and let K be the stabilizer of a base point in $\mathbb{H}^n$. Then $\mathbb{H}^n$ can be realized as the homogeneous space G/K. In Model 1, G is made up of two among the four connected components of the Lorentz group $\mathrm{O}(1,n)$, and the stabilizer of e_0 is $K = \mathrm{O}(n)$. Consider the subgroup $A \approx \mathbb{R}$ in G consisting of

• the matrices

$$a_r = \begin{pmatrix} \cosh r & 0 & \sinh r \\ 0 & I & 0 \\ \sinh r & 0 & \cosh r \end{pmatrix} \qquad (r \in \mathbb{R})$$

in Model 1,

• the dilations $a_r : y \longmapsto e^{-r} y$ in Model 2,

and the subgroup $N \approx \mathbb{R}^{n-1}$ consisting of horizontal translations $n_v : y \longmapsto y + v$ $(v \in \mathbb{R}^{n-1})$ in Model 2. Then we have

• the Cartan decomposition $G = K\overline{A^+}K$, which corresponds to polar coordinates in Model 3,

- the Iwasawa decomposition $G=NAK$, which corresponds to Cartesian coordinates in Model 2.

We shall denote by $a_{r(g)}$ and $a_{h(g)}$ the $\overline{A^+}$ and A components of $g\in G$ in the Cartan and Iwasawa decompositions.

Remark 2.4. *In small dimensions,* $\mathbb{H}^2\approx\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ *and* $\mathbb{H}^3\approx\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$.

As usual, we identify right$-K-$invariant functions on G with functions on $\mathbb{H}^n$, and bi $-K-$invariant functions on G with radial functions on $\mathbb{H}^n$ i.e. functions on $\mathbb{H}^n$ which depend only on the distance r to the origin. For radial functions $f=f(r)$,

$$\int_{\mathbb{H}^n} d\,\mathrm{vol}\;f\;=\;2\,\tfrac{\pi^{\frac n2}}{\Gamma(\frac n2)}\int_0^{+\infty}dr\,(\sinh r)^{n-1}f(r)$$

and

$$\Delta f\,=\,\tfrac{\partial^2 f}{\partial r^2}+(n-1)(\coth r)\,\tfrac{\partial f}{\partial r}\,.$$

The spherical functions φ_λ are the smooth normalized radial eigenfunctions of the Laplacian on $\mathbb{H}^n$. Specifically,

$$\begin{cases}\Delta\,\varphi_\lambda=-\left\{\lambda^2+\rho^2\right\}\varphi_\lambda\,,\\ \varphi_\lambda(0)=1\,,\end{cases}$$

where $\rho=\frac{n-1}{2}$.

Remark 2.5. *The spherical functions on* $\mathbb{H}^n$ *can be characterized again in several other ways. Notably,*

- Differential equation: *the function* $\varphi_\lambda(r)$ *is the unique smooth solution to the differential equation*
$$\left(\tfrac{\partial}{\partial r}\right)^2\varphi_\lambda+(n-1)(\coth r)\left(\tfrac{\partial}{\partial r}\right)\varphi_\lambda+(\lambda^2+\rho^2)\,\varphi_\lambda=0\,,$$
which is normalized by $\varphi_\lambda(0)=1$.
- Relation with classical special functions:
$$\varphi_\lambda(r)=\varphi_\lambda^{\frac{n-2}{2},-\frac12}(r)={}_2\mathrm{F}_1\left(\tfrac{\rho+i\lambda}{2},\tfrac{\rho-i\lambda}{2};\tfrac n2;-\sinh^2 r\right),$$
where $\varphi_\lambda^{\alpha,\beta}$ *are the Jacobi functions and* ${}_2\mathrm{F}_1$ *the Gauss hypergeometric function.*
- *Same functional equations as* (7) *and* (8).
- *The spherical functions are the matrix coefficients*
$$\varphi_\lambda(x)\,=\,\frac{\Gamma(\frac n2)}{2\,\pi^{\frac n2}}\,\langle\pi_\lambda(x)1,1\rangle\,,$$
of the spherical principal series representations of G *on* $L^2(\mathbb{S}^{n-1})$.
- *According to the Harish–Chandra formula*
$$\varphi_\lambda(x)\,=\,\int_K dk\;e^{\,(\rho-i\lambda)\,h(kx)}\,,$$
the function φ_λ *is a spherical average of horocyclic waves. Let us make this integral representation more explicit*:
$$\begin{aligned}\varphi_\lambda(r)\,&=\,\frac{\Gamma(\frac n2)}{2\,\pi^{\frac n2}}\int_{\mathbb{S}^{n-1}}dv\,\langle\,\cosh r-(\sinh r)\,v,e_n\,\rangle^{i\lambda-\rho}\\ &=\,\frac{\Gamma(\frac n2)}{\sqrt\pi\,\Gamma(\frac{n-1}{2})}\int_0^\pi d\theta\,(\sin\theta)^{n-2}\left[\cosh r-(\sinh r)(\cos\theta)\right]^{i\lambda-\rho} &(10)\\ &=\,\frac{2^{\frac{n-1}{2}}\,\Gamma(\frac n2)}{\sqrt\pi\,\Gamma(\frac{n-1}{2})}\,(\sinh r)^{2-n}\int_0^r ds\,(\cosh r-\cosh s)^{\frac{n-3}{2}}\cos\lambda s\,.\end{aligned}$$

Remark 2.6. *The asymptotic behavior of the spherical functions is given by the Harish–Chandra expansion*

$$\varphi_\lambda(r) = \mathbf{c}(\lambda)\,\Phi_\lambda(r) + \mathbf{c}(-\lambda)\,\Phi_{-\lambda}(r)\,,$$

where

$$\mathbf{c}(\lambda) = \tfrac{\Gamma(2\rho)}{\Gamma(\rho)}\,\tfrac{\Gamma(i\lambda)}{\Gamma(i\lambda+\rho)}$$

and

$$\begin{aligned}\Phi_\lambda(r) &= (2\cosh r)^{i\lambda-\rho}\,{}_2\mathrm{F}_1\big(\tfrac{\rho-i\lambda}{2},\tfrac{\rho+1-i\lambda}{2};1-i\lambda;\cosh^{-2}r\big)\\ &= e^{(i\lambda-\rho)r}\sum\nolimits_{\ell=0}^{+\infty}\Gamma_\ell(\lambda)\,e^{-2\ell r}\,,\end{aligned}$$

with $\Gamma_0\equiv 1$.

The spherical Fourier transform (or Harish–Chandra transform) of radial functions on $\mathbb{H}^n$ is defined by

$$\mathcal{H}f(\lambda) = \int_{\mathbb{H}^n} dx\,f(x)\,\varphi_\lambda(x) = \frac{\Gamma(\frac{n}{2})}{2\,\pi^{\frac{n}{2}}}\int_0^{+\infty} dr\,(\sinh r)^{n-1}f(r)\,\varphi_\lambda(r) \tag{11}$$

and the inversion formula reads

$$f(x) = 2^{n-3}\,\pi^{-\frac{n}{2}-1}\,\Gamma(\tfrac{n}{2})\int_0^{+\infty} d\lambda\,|\mathbf{c}(\lambda)|^{-2}\,\mathcal{H}f(\lambda)\,\varphi_\lambda(x)\,. \tag{12}$$

Remark 2.7.

- *The Plancherel density reads*

$$|\mathbf{c}(\lambda)|^{-2} = \tfrac{\pi}{2^{2n-4}\,\Gamma(\frac{n}{2})^2}\prod\nolimits_{j=0}^{\frac{n-3}{2}}(\lambda^2+j^2)$$

in odd dimension, and

$$|\mathbf{c}(\lambda)|^{-2} = \tfrac{\pi}{2^{2n-4}\,\Gamma(\frac{n}{2})^2}\,\lambda\tanh\pi\lambda\prod\nolimits_{j=0}^{\frac{n}{2}-1}\big[\lambda^2+(j+\tfrac{1}{2})^2\big]$$

in even dimension. Notice the different behaviors

$$|\mathbf{c}(\lambda)|^{-2} \sim \tfrac{\pi}{2^{2n-4}\,\Gamma(\frac{n}{2})^2}\,|\lambda|^{n-1}$$

at infinity, and

$$|\mathbf{c}(\lambda)|^{-2} \sim \tfrac{\pi\,\Gamma(\frac{n-1}{2})^2}{2^{2n-4}\,\Gamma(\frac{n}{2})^2}\,\lambda^2$$

at the origin.

- *Observe that* (11) *and* (12) *are not symmetric, unlike* (1) *and* (2), *or* (3) *and* (4).

The spherical Fourier transform (11), which is somewhat abstract, can be bypassed by considering the Abel transform, which is essentially the horocyclic Radon transform restricted to radial functions. Specifically,

$$\begin{aligned}\mathcal{A}f(r) &= e^{-\rho r}\int_N dn\,f(na_r)\\ &= \tfrac{(2\pi)^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\int_{|r|}^{+\infty} ds\,\sinh s\,(\cosh s-\cosh r)^{\frac{n-3}{2}}f(s)\,.\end{aligned}$$

Then the following commutative diagram holds, let say in the Schwartz space setting:

$$\begin{array}{ccc} & \mathcal{S}_{\mathrm{even}}(\mathbb{R}) & \\ \mathcal{H}\nearrow & & \nwarrow\mathcal{F} \\ \mathcal{S}_{\mathrm{rad}}(\mathbb{H}^n) & \underset{\mathcal{A}}{\longrightarrow} & \mathcal{S}_{\mathrm{even}}(\mathbb{R}) \end{array}$$

Here $\mathcal{S}_{\mathrm{rad}}(\mathbb{H}^n)$ denotes the L^2 radial Schwartz space on $\mathbb{H}^n$, which can be identified with $(\cosh r)^{-\rho}\,\mathcal{S}_{\mathrm{even}}(\mathbb{R})$, $\mathcal{F}$ the Euclidean Fourier transform on $\mathbb{R}$ and each arrow is an isomorphism. Thus the inversion of the spherical Fourier transform $\mathcal{H}$ boilds down to the inversion of the Abel transform $\mathcal{A}$. In odd dimension,

$$\mathcal{A}^{-1}g(r)=(2\pi)^{-\frac{n-1}{2}}\left(-\tfrac{1}{\sinh r}\tfrac{\partial}{\partial r}\right)^{\frac{n-1}{2}}g(r)$$

while, in even dimension,

$$\mathcal{A}^{-1}g(r)=\tfrac{1}{2^{\frac{n-1}{2}}\pi^{\frac{n}{2}}}\int_{|r|}^{+\infty}\tfrac{ds}{\sqrt{\cosh s-\cosh r}}\left(-\tfrac{\partial}{\partial s}\right)\left(-\tfrac{1}{\sinh s}\tfrac{\partial}{\partial s}\right)^{\frac{n}{2}-1}g(s)\,.$$

Consider finally the transform

$$\mathcal{A}^{*}g(x)=\int_{K}dk\;e^{\rho\,h(kx)}\,g(h(kx))\,,$$

which is dual to the Abel transform, i.e.,

$$\int_{\mathbb{H}^n}dx\;f(x)\,\mathcal{A}^{*}g(x)\;=\int_{-\infty}^{+\infty}dr\;\mathcal{A}f(r)\,g(r)\,,$$

and which is an isomorphism between $\mathcal{C}^{\infty}_{\mathrm{even}}(\mathbb{R})$ and $\mathcal{C}^{\infty}_{\mathrm{rad}}(\mathbb{H}^n)$. It is given explicitly by

$$\mathcal{A}^{*}g(r)=\tfrac{2^{\frac{n-1}{2}}\,\Gamma(\frac{n}{2})}{\sqrt{\pi}\;\Gamma(\frac{n-1}{2})}\,(\sinh r)^{2-n}\int_{0}^{r}ds\,(\cosh r-\cosh s)^{\frac{n-3}{2}}\,g(s)$$

and its inverse by

$$(\mathcal{A}^{*})^{-1}f(r)=\tfrac{\sqrt{\pi}}{2^{\frac{n-1}{2}}\,\Gamma(\frac{n}{2})}\,\tfrac{\partial}{\partial r}\left(\tfrac{1}{\sinh r}\tfrac{\partial}{\sinh r}\right)^{\frac{n-3}{2}}\left\{(\sinh r)^{n-2}f(r)\right\}$$

in odd dimension and by

$$(\mathcal{A}^{*})^{-1}f(r)=\tfrac{1}{2^{\frac{n-1}{2}}\,(\frac{n}{2}-1)!}\,\tfrac{\partial}{\partial r}\left(\tfrac{1}{\sinh r}\tfrac{\partial}{\partial r}\right)^{\frac{n}{2}-1}\int_{0}^{r}\tfrac{ds}{\sqrt{\cosh r-\cosh s}}\,(\sinh s)^{n-1}f(s)$$

in even dimension.

Remark 2.8. *Notice that the spherical function $\varphi_\lambda(r)$ is the dual Abel transform of the cosine function $\cos\lambda s$.*

Applications. Spherical Fourier analysis is an efficient tool for solving invariant PDEs on $\mathbb{H}^n$. Here are some examples of evolution equations.

- The heat equation

$$\begin{cases}\partial_t u(x,t)=\Delta_x u(x,t)\\ u(x,0)=f(x)\end{cases}$$

can be solved explicitly by means of the inverse Abel transform. Specifically,

$$u(x,t)=f*h_t(x)\,,$$

where the heat kernel is given by

$$h_t(r)=\tfrac{1}{2^{\frac{n+1}{2}}\pi^{\frac{n}{2}}}\,t^{-\frac{1}{2}}\,e^{-\rho^2 t}\left(-\tfrac{1}{\sinh r}\tfrac{\partial}{\partial r}\right)^{\frac{n-1}{2}}e^{-\frac{r^2}{4t}}$$

in odd dimension and by

$$h_t(r)=(2\pi)^{-\frac{n+1}{2}}\,t^{-\frac{1}{2}}\,e^{-\rho^2 t}\int_{|r|}^{+\infty}\tfrac{ds}{\sqrt{\cosh s-\cosh r}}\left(-\tfrac{\partial}{\partial s}\right)\left(-\tfrac{1}{\sinh s}\tfrac{\partial}{\partial s}\right)^{\frac{n}{2}-1}e^{-\frac{s^2}{4t}}$$

in even dimension. Moreover, the following global estimate holds:

$$h_t(r)\asymp e^{-\rho^2 t}\,e^{-\rho r}\,e^{-\frac{r^2}{4t}}\times\begin{cases}t^{-\frac{3}{2}}\,(1+r) & \text{if }\;t\geqslant 1+r\,,\\ t^{-\frac{n}{2}}\,(1+r)^{\frac{n-1}{2}} & \text{if }\;0<t\leqslant 1+r\,.\end{cases}\tag{13}$$

- Similarly for the Schrödinger equation

$$\begin{cases} i\,\partial_t u(x,t) \,=\, \Delta_x u(x,t)\,, \\ u(x,0) \,=\, f(x)\,. \end{cases}$$

In this case $u(x,t)=f*h_{-it}(x)$, where

$$h_{-it}(r) \,=\, \tfrac{1}{2^{\frac{n+1}{2}}\,\pi^{\frac{n}{2}}}\, e^{\,i\frac{\pi}{4}\operatorname{sign}(t)}\,|t|^{-\frac12}\,e^{-i\rho^2 t}\,\bigl(-\tfrac{1}{\sinh r}\tfrac{\partial}{\partial r}\bigr)^{\frac{n-1}{2}}\,e^{-i\frac{r^2}{4t}}$$

in odd dimension,

$$\begin{aligned} h_{-it}(r) \,&=\, (2\pi)^{-\frac{n+1}{2}}\,e^{\,i\frac{\pi}{4}\operatorname{sign}(t)}\,|t|^{-\frac12}\,e^{-i\rho^2 t} \\ &\quad\times \int_{|r|}^{+\infty}\tfrac{ds}{\sqrt{\cosh s-\cosh r}}\,\bigl(-\tfrac{\partial}{\partial s}\bigr)\bigl(-\tfrac{1}{\sinh s}\tfrac{\partial}{\partial s}\bigr)^{\frac{n}{2}-1}e^{-i\frac{s^2}{4t}} \end{aligned}$$

in even dimension, and

$$|h_{-it}(r)| \,\lesssim\, e^{-\rho r}\,\times \begin{cases} |t|^{-\frac32}\,(1+r) & \text{if } \ |t|\geqslant 1+r \\ |t|^{-\frac{n}{2}}\,(1+r)^{\frac{n-1}{2}} & \text{if } \ 0<|t|\leqslant 1+r \end{cases}$$

in all dimensions.

- The shifted wave equation

$$\begin{cases} \partial_t^2 u(x,t) \,=\, \bigl(\Delta_x+\rho^2\bigr)\,u(x,t) \\ u(x,0) \,=\, f(x),\ \partial_t\big|_{t=0}\,u(x,t) \,=\, g(x) \end{cases}$$

can be solved explicitly by means of the inverse dual Abel transform. Specifically,

$$\begin{aligned} u(t,x) \,&=\, \tfrac{1}{2^{\frac{n+1}{2}}\,\pi^{\frac{n-1}{2}}}\,\tfrac{\partial}{\partial t}\,\bigl(\tfrac{1}{\sinh t}\tfrac{\partial}{\partial t}\bigr)^{\frac{n-3}{2}}\Bigl\{\tfrac{1}{\sinh t}\int_{S(x,|t|)}dy\,f(y)\Bigr\} \\ &\quad+\,\tfrac{1}{2^{\frac{n+1}{2}}\,\pi^{\frac{n-1}{2}}}\,\bigl(\tfrac{1}{\sinh t}\tfrac{\partial}{\partial t}\bigr)^{\frac{n-3}{2}}\Bigl\{\tfrac{1}{\sinh t}\int_{S(x,|t|)}dy\;g(y)\Bigr\} \end{aligned}$$

in odd dimension and

$$\begin{aligned} u(t,x) \,&=\, \tfrac{1}{2^{\frac{n+1}{2}}\,\pi^{\frac{n}{2}}}\,\tfrac{\partial}{\partial|t|}\,\bigl(\tfrac{\partial}{\partial(\cosh t)}\bigr)^{\frac{n}{2}-1}\int_{B(x,|t|)}dy\;\tfrac{f(y)}{\sqrt{\cosh t-\cosh d(y,x)}} \\ &\quad+\,\tfrac{1}{2^{\frac{n+1}{2}}\,\pi^{\frac{n}{2}}}\,\operatorname{sign}(t)\,\bigl(\tfrac{\partial}{\partial(\cosh t)}\bigr)^{\frac{n}{2}-1}\int_{B(x,|t|)}dy\;\tfrac{g(y)}{\sqrt{\cosh t-\cosh d(y,x)}} \end{aligned}$$

in even dimension.

2.5. **Spherical Fourier analysis on homogeneous trees.** A homogeneous tree is a connected graph, with no loop and with the same number of edges at each vertex. Let us denote by $\mathbb{T}_q$ the set of vertices of the homogeneous tree with $q+1>2$ edges (see Figure 6). It is equipped with the counting measure and with the geodesic distance, given by the number of edges between two points. The volume of any sphere $S(x,r)$ of radius $r\in\mathbb{N}$ is given by

$$\delta(r) \,=\, \begin{cases} 1 & \text{if } \ r=0, \\ (q+1)\,q^{r-1} & \text{if } \ r\in\mathbb{N}^*. \end{cases}$$

Once we have chosen an origin $0\in\mathbb{T}_q$ and an oriented geodesic $\omega:\mathbb{Z}\to\mathbb{T}_q$ through 0, let us denote by $|x|\in\mathbb{N}$ the distance of a vertex $x\in\mathbb{T}_q$ to the origin and by $h(x)\in\mathbb{Z}$ its horocyclic height (see Figure 7). Let G be the isometry group of $\mathbb{T}_q$ and let K be the stabilizer of 0. Then G is a locally compact group, K is a compact open subgroup, and $\mathbb{T}_q\approx G/K$.

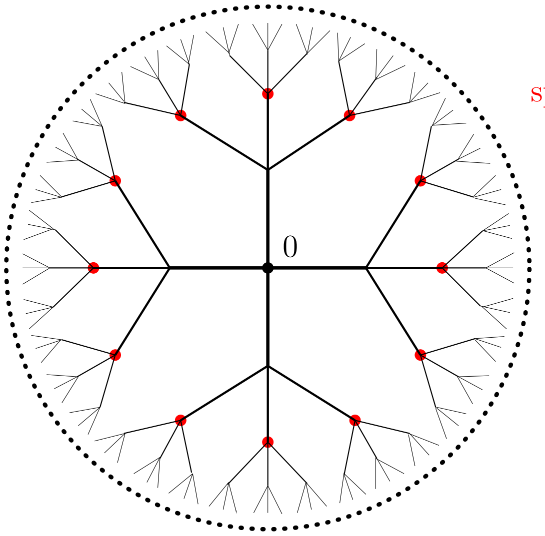

FIGURE 6. Ball picture of $\mathbb{T}_3$

Remark 2.9. *If q is a prime number, then $\mathbb{T}_q \approx \mathrm{PGL}(2,\mathbb{Q}_q)/\mathrm{PGL}(2,\mathbb{Z}_q)$.*

The combinatorial Laplacian Δ on $\mathbb{T}_q$ is defined by $\Delta f = Af - f$, where A denotes the average operator

$$Af(x) = \tfrac{1}{q+1} \sum\nolimits_{y\in S(x,1)} f(y)\,.$$

Remark 2.10. *Notice that f is harmonic, i.e., $\Delta f = 0$ if and only if f has the mean value property, i.e., $f = Af$.*

The spherical functions φ_λ on $\mathbb{T}_q$ are the normalized radial eigenfunctions of Δ, or equivalently A. Specifically,

$$\begin{cases} A\,\varphi_\lambda = \gamma(\lambda)\,\varphi_\lambda\,, \\ \varphi_\lambda(0) = 1\,, \end{cases}$$

where $\gamma(\lambda) = \frac{q^{i\lambda} + q^{-i\lambda}}{q^{1/2} + q^{-1/2}}$.

Remark 2.11. *The spherical functions on $\mathbb{T}_q$ can be characterized again in several other ways. Notably,*

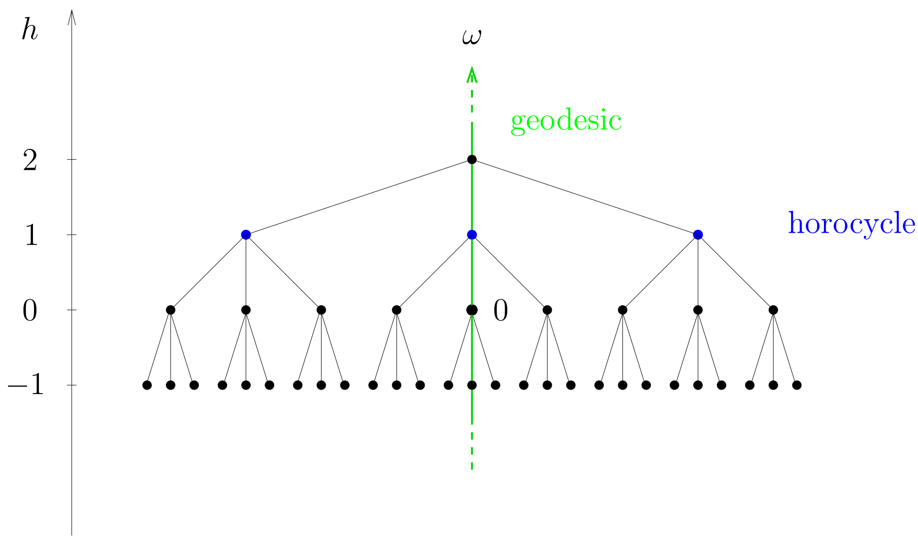

FIGURE 7. Upper half–space picture of $\mathbb{T}_3$

- Explicit expression :
$$\varphi_\lambda(r)=\begin{cases}\mathbf{c}(\lambda)\,q^{(-1/2+i\lambda)r}+\mathbf{c}(-\lambda)\,q^{(-1/2-i\lambda)r} & \textit{if } \lambda\in\mathbb{C}\smallsetminus\frac{\tau}{2}\mathbb{Z},\\ (-1)^{jr}\left(1+\frac{q^{1/2}-q^{-1/2}}{q^{1/2}+q^{-1/2}}\,r\right)q^{-r/2} & \textit{if } \lambda=\frac{\tau}{2}\,j,\end{cases}\tag{14}$$
where $\mathbf{c}(\lambda)=\frac{1}{q^{1/2}+q^{-1/2}}\,\frac{q^{1/2+i\lambda}-q^{-1/2-i\lambda}}{q^{i\lambda}-q^{-i\lambda}}$ *and* $\tau=\frac{2\pi}{\log q}$. *Notice that* (14) *is even and* τ*–periodic in* λ.
- *Same functional equations as* (7) *and* (8).
- *The spherical functions are the bi–K–invariant matrix coefficients of the spherical principal series representations of* G *on* $L^2(\partial\,\mathbb{T}_q)$.

The spherical Fourier transform of radial functions on $\mathbb{T}_q$ is defined by
$$\mathcal{H}f(\lambda)=\sum\nolimits_{x\in\mathbb{T}_q}f(x)\,\varphi_\lambda(x)=f(0)+\tfrac{q^{1/2}+q^{-1/2}}{q^{1/2}}\sum\nolimits_{r=1}^{+\infty}q^r f(r)\,\varphi_\lambda(r)\tag{15}$$
and the inversion formula reads
$$f(r)=\tfrac{q^{1/2}}{q^{1/2}+q^{-1/2}}\,\tfrac{1}{\tau}\int_0^{\tau/2}d\lambda\,|\mathbf{c}(\lambda)|^{-2}\,\mathcal{H}f(\lambda)\,\varphi_\lambda(r)\,.\tag{16}$$
Consider the Abel transform
$$\begin{aligned}\mathcal{A}f(h)&=q^{\frac{h}{2}}\sum\nolimits_{\substack{x\in\mathbb{T}_q\\ h(x)=h}}f(|x|)\\ &=q^{\frac{|h|}{2}}f(|h|)+\tfrac{q-1}{q}\sum\nolimits_{j=1}^{+\infty}q^{\frac{|h|}{2}+j}f(|h|+2j)\,,\end{aligned}$$
which is essentially the horocyclic Radon transform restricted to radial functions. Then the following commutative diagram holds, let say in the Schwartz space setting :
$$\begin{array}{ccc} & C^\infty_{\text{even}}(\mathbb{R}/\tau\mathbb{Z}) & \\ \mathcal{H}\nearrow & & \nwarrow\mathcal{F}\\ \mathcal{S}_{\text{rad}}(\mathbb{T}_q) & \underset{\mathcal{A}}{\longrightarrow} & \mathcal{S}_{\text{even}}(\mathbb{Z})\end{array}$$
Here $\mathcal{S}_{\text{even}}(\mathbb{Z})$ denotes the space of even functions on $\mathbb{Z}$ such that
$$\sup\nolimits_{r\in\mathbb{N}^*}r^m\,|f(r)|<+\infty\qquad\forall\;m\in\mathbb{N},$$
$\mathcal{S}_{\text{rad}}(\mathbb{T}_q)$ the space of radial functions on $\mathbb{T}_q$, whose radial part coincides with $q^{-\frac{r}{2}}\mathcal{S}(\mathbb{N})$,
$$\mathcal{F}f(\lambda)=\sum\nolimits_{h\in\mathbb{Z}}q^{i\lambda h}f(h)$$
is a variant of the Fourier transform on $\mathbb{Z}$, and each arrow is an isomorphism. The inverse Abel transform is given by
$$\mathcal{A}^{-1}g(r)=\sum\nolimits_{j=0}^{+\infty}q^{-\frac{r}{2}-j}\big\{g(r+2j)-g(r+2j+2)\big\}\,.$$
Consider finally the transform
$$\mathcal{A}^*g(r)=\tfrac{1}{\delta(r)}\sum\nolimits_{\substack{x\in\mathbb{T}_q\\ |x|=r}}q^{\frac{h(x)}{2}}g\big(h(x)\big)\,,$$
which is dual to the Abel transform, i.e.,
$$\sum\nolimits_{x\in\mathbb{T}_q}f(x)\,\mathcal{A}^*g(|x|)=\sum\nolimits_{h\in\mathbb{Z}}\mathcal{A}f(h)\,g(h)\,,$$
and which is an isomorphism between the space of all even functions on $\mathbb{Z}$ and the space of all radial functions on $\mathbb{T}_q$. It is given explicitly by $\mathcal{A}^*g(0)=g(0)$, and
$$\mathcal{A}^*g(r)=\tfrac{2q}{q+1}\,q^{-\frac{r}{2}}\,g(r)+\tfrac{q-1}{q+1}\,q^{-\frac{r}{2}}\sum\nolimits_{\substack{-r<h<r\\ h\text{ has same parity as }r}}g(h)$$

if $r\in\mathbb{N}^*$. Its inverse is given by $(\mathcal{A}^*)^{-1}f(0)=f(0)$,

$$\begin{aligned}(\mathcal{A}^*)^{-1}f(h) &= \tfrac{q^{1/2}+q^{-1/2}}{2}\,q^{\frac{h-1}{2}}f(h)\\ &\quad-\tfrac{q-q^{-1}}{2}\,q^{-\frac{h}{2}}\sum\nolimits_{0<r\text{ odd}<h}q^r f(r)\end{aligned}$$

if $h\in\mathbb{N}$ is odd, and

$$\begin{aligned}(\mathcal{A}^*)^{-1}f(h) &= \tfrac{q^{1/2}+q^{-1/2}}{2}\,q^{\frac{h-1}{2}}f(h)-\tfrac{q^{1/2}-q^{-1/2}}{2}\,q^{-\frac{h-1}{2}}f(0)\\ &\quad-\tfrac{q-q^{-1}}{2}\,q^{-\frac{h}{2}}\sum\nolimits_{0<r\text{ even}<h}q^r f(r)\end{aligned}$$

if $h\in\mathbb{N}^*$ is even.

Applications. Let us use spherical Fourier analysis to study discrete evolution equations on $\mathbb{T}_q$, as we did in the differential setting on $\mathbb{H}^n$.

- Consider the heat equation with discrete time $t\in\mathbb{N}$

$$\begin{cases}u(x,t+1)-u(x,t)=\Delta_x u(x,t)\\ u(x,0)=f(x)\end{cases}$$

or, equivalently, the simple random walk

$$u(x,t)=A^t f(x)=f*h_t(x)\,.$$

Its density $h_t(x)$ vanishes unless $|x|\leqslant t$ have the same parity. In this case,

$$h_t(x)\asymp\frac{1+|x|}{(1+t)\sqrt{1+t-|x|}}\,\gamma_0^t\,q^{-\frac{|x|}{2}}\,e^{-t\,\psi\left(\frac{1+|x|}{1+t}\right)}\,,\tag{17}$$

where $\psi(z)=\frac{1+z}{2}\log(1+z)-\frac{1-z}{2}\log(1-z)$ and $\gamma_0=\gamma(0)=\frac{2}{q^{1/2}+q^{-1/2}}<1$ is the spectral radius of A.

- The shifted wave equation with discrete time $t\in\mathbb{Z}$

$$\begin{cases}\gamma_0\,\Delta_t^{\mathbb{Z}}u(x,t)=\big(\Delta_x^{\mathbb{T}_q}+1-\gamma_0\big)\,u(x,t)\\ u(x,0)=f(x),\ \{u(x,1)-u(x,-1)\}/2=g(x)\end{cases}$$

can be solved explicitly by using the inverse dual Abel transform. Specifically,

$$u(x,t)=C_t f(x)+S_t\,g(x)\,,$$

where

$$\begin{cases}C_t=\frac{M_{|t|}-M_{|t|-2}}{2}\\ S_t=\operatorname{sign}(t)\,M_{|t|-1}\end{cases}$$

and

$$M_t f(x)=q^{-\frac{t}{2}}\sum\nolimits_{\substack{d(y,x)\leqslant t\\ t-d(y,x)\text{ even}}}f(y)$$

if $t\geqslant 0$, while $M_t=0$ if $t<0$.

2.6. Comments, references and further results.

- Our main reference for classical special functions is [1].
- The spherical Fourier analysis presented in this section takes place on homogeneous spaces G/K associated with Gelfand pairs (G,K).

- The sphere $\mathbb{S}^n$ and the hyperbolic space $\mathbb{H}^n$ are dual symmetric spaces. By letting their curvature tend to 0, the Euclidean space $\mathbb{R}^n$ is obtained as a limit. At the level of spherical functions, the duality is reflected by the relation
$$\varphi_\ell^{\mathbb{S}^n}(\cos\theta_1)=\varphi_\lambda^{\mathbb{H}^n}(r)$$
between (9) and (10), when we specialize $\lambda=\pm i(\rho+\ell)$ $(\ell\in\mathbb{N})$ and take $r=\pm i\,\theta_1$. And (6) is a limit of (10) and (9):
$$\varphi_\lambda^{\mathbb{R}^n}(r)=\lim_{\varepsilon\to 0}\varphi_{\frac{\lambda}{\varepsilon}}^{\mathbb{H}^n}(\varepsilon r)=\lim_{\ell\to+\infty}\varphi_\ell^{\mathbb{S}^n}(\cos\tfrac{\lambda r}{\ell})\,.$$
Homogeous trees may be considered as discrete analogs of hyperbolic spaces. This is for instance justified by the structural similarity between $\mathbb{H}^2\approx\mathrm{PSL}(2,\mathbb{R})/\mathrm{PSO}(2)$ and $\mathbb{T}_q\approx\mathrm{PGL}(2,\mathbb{Q}_q)/\mathrm{PGL}(2,\mathbb{Z}_q)$ when q is a prime number. At the analytic level, an actual relation is provided by the meta–theory developed by Cherednik [18], which includes as limit cases spherical Fourier analysis on $\mathbb{T}_q$ and on $\mathbb{H}^n$, as well as on $\mathbb{R}^n$ or on $\mathbb{S}^n$.
- The material in Subsection 2.4 generalizes to the class of Riemannian symmetric spaces of noncompact type and of rank 1, which consist of all hyperbolic spaces
$$\mathbb{H}^n=\mathrm{H}^n(\mathbb{R}),\ \mathrm{H}^n(\mathbb{C}),\ \mathrm{H}^n(\mathbb{H}),\ \mathrm{H}^2(\mathbb{O}),$$
and further to the class of Damek–Ricci spaces. One obtains this way a group theoretical interpretation of Jacobi functions $\varphi_\lambda^{\alpha,\beta}$ for infinitely many discrete parameters
$$\alpha=\tfrac{m_1+m_2-1}{2},\ \beta=\tfrac{m_2-1}{2}\,.$$
Our main references are [31], [51], [11], [75] and [12].
- Our main references for Subsection 2.5 are [32], [20], and [12]. Evolution equations (heat, Schrödinger, wave) with continuous time were also considered on homogeneous trees (see [82], [57], [21] and [47]).
- Spherical Fourier analysis generalizes to higher rank (see [45], [39] for Riemannian symmetric spaces and [53], [65], [55] for affine buildings).

  Classification :

type	constant curvature	rank 1	general case
Euclidean	$\mathbb{R}^n$		$\mathfrak{p}\rtimes K/K$
compact	$\mathbb{S}^n=\mathrm{S}(\mathbb{R}^{n+1})$	$\mathrm{S}(\mathbb{F}^n)$	U/K
non compact	$\mathbb{H}^n=\mathrm{H}^n(\mathbb{R})$	$\mathrm{H}^n(\mathbb{F})$	G/K
p–adic		homogeneous trees	affine buildings

  Notation :
  - $\mathfrak{g}_{\mathbb{C}}$ is a complex semisimple Lie algebra,
  - $\mathfrak{g}$ is a noncompact real form of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$,
  - $\mathfrak{u}=\mathfrak{k}\oplus i\mathfrak{p}$ is the compact dual form of $\mathfrak{g}$,
  - $\mathfrak{a}$ is a Cartan subspace of $\mathfrak{p}$,
  - $\mathfrak{g}=\mathfrak{a}\oplus\mathfrak{m}\oplus\left(\oplus_{\alpha\in\mathcal{R}}\,\mathfrak{g}_\alpha\right)$ is the root space decomposition of $(\mathfrak{g},\mathfrak{a})$,
  - $\mathcal{R}^+$ is a positive root subsystem and $\mathfrak{a}^+$ the corresponding positive Weyl chamber in $\mathfrak{a}$,
  - $\mathfrak{n}=\oplus_{\alpha\in\mathcal{R}^+}\mathfrak{g}_\alpha$ is the corresponding nilpotent Lie subalgebra,
  - $m_\alpha=\dim\mathfrak{g}_\alpha$ is the multiplicity of the root α,
  - $\rho=\frac{1}{2}\sum_{\alpha\in\mathcal{R}^+}m_\alpha$,

- ◦ $G_{\mathbb{C}}$ is a complex Lie group with finite center and Lie algebra $\mathfrak{g}_{\mathbb{C}}$,
- ◦ G, U, K and N are the Lie subgroups of $G_{\mathbb{C}}$ corresponding to the Lie subalgebras $\mathfrak{g}$, $\mathfrak{u}$, $\mathfrak{k}$ and $\mathfrak{n}$.

Special functions:

- ◦ Bessel functions on $\mathfrak{p}$:

$$\varphi_\lambda^{\mathfrak{p}}(x) = \int_K dk\, e^{\,i\langle\lambda,(\mathrm{Ad}\,k)x\rangle} \qquad \forall\,\lambda\in\mathfrak{p}_{\mathbb{C}},\ \forall\,x\in\mathfrak{p}_{\mathbb{C}}. \tag{18}$$

- ◦ Spherical functions on G/K:

$$\varphi_\lambda^G(x) = \int_K dk\, e^{\langle i\lambda+\rho,\,\mathfrak{a}(kx)\rangle} \qquad \forall\,\lambda\in\mathfrak{a}_{\mathbb{C}},\ \forall\,x\in G, \tag{19}$$

  where $\mathfrak{a}(y)$ denotes the $\mathfrak{a}$–component of $y\in G$ in the Iwasawa decomposition $G = N(\exp\mathfrak{a})K$.
- ◦ Spherical functions on affine buildings are classical Macdonald polynomials (i.e. Hall–Littlewood polynomials for the type $\widetilde{A}$).

- The global heat kernel estimate (13) generalizes as follows to G/K:

$$\begin{aligned} h_t(x) \asymp\ & t^{-\frac{n}{2}} \Big\{ \prod\nolimits_{\alpha\in R^+} \big(1+\langle\alpha,x^+\rangle\big)\big(1+t+\langle\alpha,x^+\rangle\big)^{\frac{m_\alpha+m_{2\alpha}}{2}} \Big\} \\ & \times\ e^{-|\rho|^2 t - \langle\rho,x^+\rangle - \frac{|x^+|^2}{4t}}\,, \end{aligned} \tag{20}$$

  where R^+ is the set of positive indivisible roots in $\mathcal{R}$ and x^+ denotes the $\overline{\mathfrak{a}^+}$–component of $x\in G$ in the Cartan decomposition $G = K(\exp\overline{\mathfrak{a}^+})K$ (see [13] and the references therein).
- Random walks are harder to analyze on affine buildings. A global estimate similar to (17) and (20) was established in [14] for the simple random walk on affine buildings of type $\widetilde{\mathrm{A}}_2$. In general, the main asymptotics of random walks were obtained in [91].

2.7. **Epilogue.** This section outlines spherical Fourier analysis around 1980 (except for the later applications to evolution equations). In the 1980s, Heckman and Opdam addressed the following problem (which goes back to Koornwinder for the root system BC_2): for any root system R, construct a continuous family of special functions on $\mathfrak{a}$ generalizing spherical functions on the corresponding symmetric spaces G/K, as Jacobi functions (or equivalently the Gauss hypergeometric function) generalize spherical functions on hyperbolic spaces. This problem was solved during the 1990s, mainly by Cherednik, Dunkl, Heckman, Macdonald, and Opdam, and has actually given rise to a large theory of special functions associated to root system, which is nowadays often referred to as Dunkl theory.

Remark 2.12. *A different generalization of hypergeometric functions to Grassmannians was developed by Aomoto and Gelfand at the end of the 20th century.*

3. Rational Dunkl theory

Rational Dunkl theory originates from the seminal paper [24]. This theory of special functions in several variables encompasses

- Euclidean Fourier analysis (which corresponds to the multiplicity $k=0$),
- classical Bessel functions in dimension 1,
- generalized Bessel functions associated with Riemannian symmetric spaces of Euclidean type (which correspond to a discrete set of multiplicities k).

In this subsection, we use [67] (or alternately [40], pp. 1–69) as our primary reference and quote only later works. Our notation goes as follows (see the appendix for more details) :

- $\mathfrak{a}$ is a Euclidean vector space of dimension n, which we identify with its dual space,
- R is a root system, which is reduced but not necessarily crystallographic,
- W is the associated reflection group,
- $\mathfrak{a}^+$ is a positive Weyl chamber in $\mathfrak{a}$, $\overline{\mathfrak{a}^+}$ its closure, R^+ the corresponding positive root subsystem, and S the subset of simple roots,
- $\overline{\mathfrak{a}_+}$ denotes the closed cone generated by R^+, which is the dual cone of $\overline{\mathfrak{a}^+}$,
- for every $x\in\mathfrak{a}$, x^+ denotes the element of the orbit Wx which lies in $\overline{\mathfrak{a}^+}$,
- w_0 denotes the longest element in W, which interchanges $\mathfrak{a}^+$ and $-\mathfrak{a}^+$, respectively R^+ and $R^-=-R^+$,
- k is a multiplicity, which will remain implicit in most formulae and which will be assumed to be nonnegative after a while,
- $\gamma=\sum_{\alpha\in R^+}k_\alpha$,
- $\delta(x)=\prod_{\alpha\in R^+}|\langle\alpha,x\rangle|^{2k_\alpha}$ is the reference density in the case $k\geqslant 0$.

3.1. **Dunkl operators.**

Definition 3.1. *The rational Dunkl operators, which are often simply called Dunkl operators, are the differential–difference operators defined by*

$$D_\xi f(x)=\partial_\xi f(x)+\sum_{\alpha\in R^+}k_\alpha\,\tfrac{\langle\alpha,\xi\rangle}{\langle\alpha,x\rangle}\,\{f(x)-f(r_\alpha x)\}\tag{21}$$

for every $\xi\in\mathfrak{a}$.

Remark 3.2.

- *Notice that Dunkl operators D_ξ reduce to partial derivatives ∂_ξ when $k=0$.*
- *The choice of R^+ plays no role in Definition 3.1, as*

$$\sum_{\alpha\in R^+}k_\alpha\,\tfrac{\langle\alpha,\xi\rangle}{\langle\alpha,x\rangle}\,\{f(x)-f(r_\alpha x)\}=\sum_{\alpha\in R}\tfrac{k_\alpha}{2}\,\tfrac{\langle\alpha,\xi\rangle}{\langle\alpha,x\rangle}\,\{f(x)-f(r_\alpha x)\}\,.$$

- *Dividing by $\langle\alpha,x\rangle$ produces no actual singularity in* (21), *as*

$$\tfrac{f(x)-f(r_\alpha x)}{\langle\alpha,x\rangle}=-\tfrac{1}{\langle\alpha,x\rangle}\int_0^1 dt\,\tfrac{\partial}{\partial t}f(x-t\langle\alpha^\vee,x\rangle\alpha)=\tfrac{2}{|\alpha|^2}\int_0^1 dt\,\partial_\alpha f(x-t\langle\alpha^\vee,x\rangle\alpha)\,.$$

Commutativity is a remarkable property of Dunkl operators.

Theorem 3.3. *Fix a multiplicity k. Then*

$$D_\xi\circ D_\eta=D_\eta\circ D_\xi\qquad\forall\,\xi,\eta\in\mathfrak{a}.$$

This result leads to the notion of Dunkl operators D_p, for every polynomial $p\in\mathcal{P}(\mathfrak{a})$, and of their symmetric part $\widetilde{D}_p$ on symmetric (i.e. W–invariant) functions.

Example 3.4.

- *The Dunkl Laplacian is given by*

$$\begin{aligned}\Delta f(x)=\sum_{j=1}^n D_j^2 f(x)=&\overbrace{\sum_{j=1}^n\partial_j^2 f(x)+\sum_{\alpha\in R^+}\tfrac{2k_\alpha}{\langle\alpha,x\rangle}\,\partial_\alpha f(x)}^{\text{differential part }\widetilde{\Delta}f(x)}\\&\underbrace{-\sum_{\alpha\in R^+}\tfrac{k_\alpha|\alpha|^2}{\langle\alpha,x\rangle^2}\,\{f(x)-f(r_\alpha x)\}}_{\text{difference part}}\,,\end{aligned}$$

where D_j, respectively ∂_j denote the Dunkl operators, respectively the partial derivatives with respect to an orthonormal basis of $\mathfrak{a}$.

- *In dimension 1, the Dunkl operator is given by*
$$Df(x) = \big(\tfrac{\partial}{\partial x}\big) f(x) + \tfrac{k}{x} \big\{ f(x) - f(-x) \big\}$$
and the Dunkl Laplacian by
$$Lf(x) = \big(\tfrac{\partial}{\partial x}\big)^2 f(x) + \tfrac{2k}{x} \big(\tfrac{\partial}{\partial x}\big) f(x) - \tfrac{k}{x^2} \big\{ f(x) - f(-x) \big\}\,.$$

Here are some other properties of Dunkl operators.

Proposition 3.5.

- *The Dunkl operators map the following function spaces into themselves:*
$$\mathcal{P}(\mathfrak{a}),\ \mathcal{C}^\infty(\mathfrak{a}),\ \mathcal{C}^\infty_c(\mathfrak{a}),\ \mathcal{S}(\mathfrak{a}),\ \dots$$
More precisely, the Dunkl operators D_ξ, with $\xi \in \mathfrak{a}$, are homogeneous operators of degree -1 on polynomials.
- W–equivariance: *For every $w \in W$ and $p \in \mathcal{P}(\mathfrak{a})$, we have*
$$w \circ D_p \circ w^{-1} = D_{wp}\,.$$
Hence $\widetilde{D}_{pq} = \widetilde{D}_p \circ \widetilde{D}_q$, for all symmetric (i.e. W–invariant) polynomials $p, q \in \mathcal{P}(\mathfrak{a})^W$.
- Skew–adjointness: *Assume that $k \geqslant 0$. Then, for every $\xi \in \mathfrak{a}$,*
$$\int_{\mathfrak{a}} dx\ \delta(x)\, D_\xi f(x)\, g(x) = - \int_{\mathfrak{a}} dx\ \delta(x)\, f(x)\, D_\xi g(x)\,.$$

3.2. **Dunkl kernel.**

Theorem 3.6. *For generic multiplicities k and for every $\lambda \in \mathfrak{a}_{\mathbb{C}}$, the system*
$$\begin{cases} D_\xi E_\lambda = \langle \lambda, \xi \rangle E_\lambda & \forall\, \xi \in \mathfrak{a}, \\ E_\lambda(0) = 1, \end{cases}$$
has a unique smooth solution on $\mathfrak{a}$, which is called the Dunkl kernel.

Remark 3.7. *In this statement, generic means that k belongs to a dense open subset K_{reg} of K, whose complement is a countable union of algebraic sets. The set K_{reg} is known explicitly and it contains in particular $\{\, k \in K \mid \operatorname{Re} k \geqslant 0 \,\}$.*

Definition 3.8. *The generalized Bessel function is the average*
$$J_\lambda(x) = \tfrac{1}{|W|} \sum\nolimits_{w \in W} E_\lambda(wx) = \tfrac{1}{|W|} \sum\nolimits_{w \in W} E_{w\lambda}(x)\,. \tag{22}$$

Remark 3.9.

- *Mind the possible formal confusion between* (22) *and the classical Bessel function of the first kind J_ν.*
- *Conversely, the Dunkl kernel $E_\lambda(x)$ can be recovered by applying to the generalized Bessel function $J_\lambda(x)$ a linear differential operator in x whose coefficients are rational functions of λ (see* [62, proposition 6.8.(4)]*).*
- *In dimension 1, K_{reg} is the complement of $-\mathbb{N} - \frac{1}{2}$ in $\mathbb{C}$. The generalized Bessel function* (22) *reduces to the modified Bessel function encountered in Subsection 2.2:*
$$J_\lambda(x) = j_{k-\frac{1}{2}}(\lambda x)\,.$$
The Dunkl kernel is a combination of two such functions:
$$E_\lambda(x) = \underbrace{j_{k-\frac{1}{2}}(\lambda x)}_{\text{even}} + \underbrace{\tfrac{\lambda x}{2k+1}\, j_{k+\frac{1}{2}}(\lambda x)}_{\text{odd}}$$

It can be also expressed in terms of the confluent hypergeometric function:

$$E_\lambda(x) = \tfrac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(k)} \int_{-1}^{+1} du\,(1-u)^{k-1}(1+u)^k\,e^{\lambda x u}$$
$$= e^{\lambda x} \underbrace{\tfrac{\Gamma(2k+1)}{\Gamma(k)\,\Gamma(k+1)} \int_0^1 dv\, v^{k-1}(1-v)^k\,e^{-2\lambda x v}}_{{}_1\mathrm{F}_1(k;2k+1;-2\lambda x)}\,.$$

- *When* $k=0$, *the Dunkl kernel* $E_\lambda(x)$ *reduces to the exponential* $e^{\langle\lambda,\,x\rangle}$ *and the generalized Bessel function* $J_\lambda(x)$ *to*

$$\mathrm{Cosh}_\lambda(y) = \tfrac{1}{|W|}\sum\nolimits_{w\in W} e^{\langle w\lambda,\,y\rangle}\,. \tag{23}$$

- *As far as we know, the non–symmetric Dunkl kernel had not occurred previously in special functions, group theory or geometric analysis.*
- *Bessel functions* (18) *on Riemannian symmetric spaces of Euclidean type* $\mathfrak{p}\rtimes K/K$ *are special cases of* (22), *corresponding to crystallographic root systems and to certain discrete sets of mutiplicities. More precisely, if*
  - $\mathfrak{g}=\mathfrak{p}\oplus\mathfrak{k}$ *is the associated semisimple Lie algebra,*
  - $\mathfrak{a}$ *is a Cartan subspace of* $\mathfrak{p}$,
  - $\mathcal{R}$ *is the root system of* $(\mathfrak{g},\mathfrak{a})$,
  - m_α *is the multiplicity of* $\alpha\in\mathcal{R}$,

  and
  - R *is the subsystem of indivisible roots in* $\mathcal{R}$,
  - $k_\alpha = \frac{m_\alpha+m_{2\alpha}}{2}\quad\forall\,\alpha\in R$,

  then

$$\varphi_\lambda^{\mathfrak{p}}(x) = J_{i\lambda}(x) \qquad \forall\,\lambda\in\mathfrak{a}_{\mathbb{C}},\ \forall\,x\in\mathfrak{a}.$$

In next proposition, we collect some properties of the Dunkl kernel.

Proposition 3.10.

- Regularity: $E_\lambda(x)$ *extends to a holomorphic function in* $\lambda\in\mathfrak{a}_{\mathbb{C}}$, $x\in\mathfrak{a}_{\mathbb{C}}$ *and* $k\in K_{\mathrm{reg}}$.
- Symmetries:

$$\begin{cases} E_\lambda(x) = E_x(\lambda), & \\ E_{w\lambda}(wx) = E_\lambda(x) & \forall\,w\in W, \\ E_\lambda(tx) = E_{t\lambda}(x) & \forall\,t\in\mathbb{C}, \\ \overline{E_\lambda(x)} = E_{\overline{\lambda}}(\overline{x}) & \text{when } k\geqslant 0. \end{cases}$$

- Positivity: *Assume that* $k\geqslant 0$. *Then,*

$$0 < E_\lambda(x) \leqslant e^{\langle\lambda^+,\,x^+\rangle} \qquad \forall\,\lambda\in\mathfrak{a},\ \forall\,x\in\mathfrak{a}.$$

- Global estimate: *Assume that* $k\geqslant 0$. *Then, for every* $\xi_1,\dots,\xi_N\in\mathfrak{a}$,

$$\bigl|\partial_{\xi_1}\dots\,\partial_{\xi_N}E_\lambda(x)\bigr| \leqslant |\xi_1|\dots|\xi_N|\,|\lambda|^N\,e^{\langle(\mathrm{Re}\,\lambda)^+,\,(\mathrm{Re}\,x)^+\rangle} \qquad \forall\,\lambda\in\mathfrak{a}_{\mathbb{C}},\ \forall\,x\in\mathfrak{a}_{\mathbb{C}}.$$

3.3. **Dunkl transform.** From now on, we assume that $k\geqslant 0$.

Definition 3.11. *The Dunkl transform is defined by*

$$\mathcal{H}f(\lambda) = \int_{\mathfrak{a}} dx\,\delta(x)\,f(x)\,E_{-i\lambda}(x)\,. \tag{24}$$

In next theorem, we collect the main properties of the Dunkl transform.

Theorem 3.12.

- *The Dunkl transform is an automorphism of the Schwartz space* $\mathcal{S}(\mathfrak{a})$.
- *We have*

$$\begin{cases} \mathcal{H}(D_\xi f)(\lambda) = i\,\langle \xi, \lambda\rangle \mathcal{H}f(\lambda) & \forall\ \xi\in\mathfrak{a}, \\ \mathcal{H}(\langle \xi, .\,\rangle f) = i\,D_\xi \mathcal{H}f & \forall\ \xi\in\mathfrak{a}, \\ \mathcal{H}(wf)(\lambda) = \mathcal{H}f(w\lambda) & \forall\ w\in W, \\ \mathcal{H}\big[f(t\,.\,)\big](\lambda) = |t|^{-n-2\gamma}\,(\mathcal{H}f)(t^{-1}\lambda) & \forall\ t\in\mathbb{R}^*. \end{cases}$$

- Inversion formula:

$$f(x) = c_{\text{rat}}^{-2} \int_{\mathfrak{a}} d\lambda\ \delta(\lambda)\,\mathcal{H}f(\lambda)\,E_{i\lambda}(x)\,, \tag{25}$$

where

$$c_{\text{rat}} = \int_{\mathfrak{a}} dx\ \delta(x)\,e^{-\frac{|x|^2}{2}} \tag{26}$$

is the so–called Mehta–Macdonald integral.

- Plancherel identity: *The Dunkl transform extends to an isometric automorphism of* $L^2(\mathfrak{a},\delta(x)dx)$, *up to a positive constant. Specifically,*

$$\int_{\mathfrak{a}} d\lambda\ \delta(\lambda)\,|\mathcal{H}f(\lambda)|^2 = c_{\text{rat}}^2 \int_{\mathfrak{a}} dx\ \delta(x)\,|f(x)|^2.$$

- Riemann–Lebesgue Lemma: *The Dunkl transform maps* $L^1(\mathfrak{a},\delta(x)dx)$ *into* $\mathcal{C}_0(\mathfrak{a})$.
- Paley–Wiener Theorem: *The Dunkl transform is an isomorphism between* $\mathcal{C}_c^\infty(\mathfrak{a})$ *and the Paley–Wiener space* $\mathcal{PW}(\mathfrak{a}_{\mathbb{C}})$, *which consists of all holomorphic functions* $h:\mathfrak{a}_{\mathbb{C}}\longrightarrow\mathbb{C}$ *such that*

$$\exists\ R>0,\ \forall\ N\in\mathbb{N},\ \sup\nolimits_{\lambda\in\mathfrak{a}_{\mathbb{C}}}(1+|\lambda|)^N e^{-R\,|\operatorname{Im}\lambda|}\,|h(\lambda)| < +\infty\,. \tag{27}$$

More precisely, the support of $f\in\mathcal{C}_c^\infty(\mathfrak{a})$ *is contained in the closed ball* $\overline{B(0,R)}$ *if and only if* $h=\mathcal{H}f$ *satisfies* (27).

Remark 3.13.

- *Notice that* (24) *and* (25) *are symmetric, as the Euclidean Fourier transform* (1) *and its inverse* (2), *or the Hankel transform* (3) *and its inverse* (4).
- *In the* W*–invariant case,* $E_{\pm i\lambda}(x)$ *is replaced by* $J_{\pm i\lambda}(x)$ *in* (24) *and* (25).
- *The Dunkl transform of a radial function is again a radial function.*
- *The following sharper version of the Paley–Wiener Theorem was proved in* [4], *as a consequence of the corresponding result in the trigonometric setting (see Theorem 4.10) and thus under the assumption that* R *is crystallographic. Given a* W*–invariant convex compact neighborhood* C *of the origin in* $\mathfrak{a}$, *consider the gauge* $\chi(\lambda)=\max_{x\in C}\langle\lambda,x\rangle$. *Then the support of* $f\in\mathcal{C}_c^\infty(\mathfrak{a})$ *is contained in* C *if and only if its Dunkl transform* $h=\mathcal{H}f$ *satisfies the condition*

$$\forall\ N\in\mathbb{N},\ \sup\nolimits_{\lambda\in\mathfrak{a}_{\mathbb{C}}}(1+|\lambda|)^N e^{-\chi(\operatorname{Im}\lambda)}\,|h(\lambda)| < +\infty\,. \tag{28}$$

Problem 3.14. *Extend the latter result to the non–crystallographic case.*

3.4. **Heat kernel.** The heat equation

$$\begin{cases} \partial_t u(x,t) = \Delta_x u(x,t) \\ u(x,0) = f(x) \end{cases}$$

can be solved via the Dunkl transform (under suitable assumptions). This way, one obtains

$$u(x,t) = \int_{\mathfrak{a}} dy\ \delta(y)\,f(y)\,h_t(x,y)\,,$$

where the heat kernel is given by

$$h_t(x,y) = c_{\text{rat}}^{-2} \int_{\mathfrak{a}} d\lambda\, \delta(\lambda)\, e^{-t|\lambda|^2} E_{i\lambda}(x)\, E_{-i\lambda}(y) \qquad \forall\, t>0\,,\ \forall\, x,y\in\mathfrak{a}\,.$$

In next proposition, we collect some properties of the heat kernel established by Rösler.

Proposition 3.15.

- $h_t(x,y)$ *is an smooth symmetric probability density. More precisely,*
  - $h_t(x,y)$ *is an analytic function in* $(t,x,y)\in(0,+\infty)\times\mathfrak{a}\times\mathfrak{a}$,
  - $h_t(x,y)=h_t(y,x)$,
  - $h_t(x,y)>0$ *and* $\int_{\mathfrak{a}} dy\,\delta(y)\,h_t(x,y)=1$.
- Semigroup property:

$$h_{s+t}(x,y) = \int_{\mathfrak{a}} dz\; \delta(z)\, h_s(x,z)\, h_t(z,y)\,.$$

- Expression by means of the Dunkl kernel:

$$h_t(x,y) = c_{\text{rat}}^{-1}\,(2t)^{-\frac{n}{2}-\gamma}\, e^{-\frac{|x|^2}{4t}-\frac{|y|^2}{4t}}\, E_{\frac{x}{\sqrt{2t}}}\Big(\tfrac{y}{\sqrt{2t}}\Big) \qquad \forall\, t>0,\ \forall\, x,y\in\mathfrak{a}\,. \tag{29}$$

- Upper estimate:

$$h_t(x,y) \leqslant c_{\text{rat}}^{-1}\,(2t)^{-\frac{n}{2}-\gamma}\, \max_{w\in W} e^{-\frac{|wx-y|^2}{4t}}\,.$$

Remark 3.16.

- *In* [10], *the following sharp heat kernel estimates were obtained in dimension* 1 (*and also in the product case*) :

$$h_t(x,y) \asymp \begin{cases} t^{-k-\frac12}\, e^{-\frac{x^2+y^2}{4t}} & \text{if } |xy|\leqslant t,\\ t^{-\frac12}\,(xy)^{-k}\, e^{-\frac{(x-y)^2}{4t}} & \text{if } xy\geqslant t,\\ t^{\frac12}\,(-xy)^{-k-1}\, e^{-\frac{(x+y)^2}{4t}} & \text{if } -xy\geqslant t. \end{cases} \tag{30}$$

  Notice the lack of Gaussian behavior when $-xy\geqslant t$, *in particular when* $y=-x$ *tends to infinity faster then* $\sqrt{t}$.
- *The Dunkl Laplacian is the infinitesimal generator of a Feller–Markov process on* $\mathfrak{a}$, *which has remarkable features* (*Brownian motion with jumps*) *and which has drawn a lot of attention in the 2000s. We refer to* [40] *and* [23] *for probabilistic aspects of Dunkl theory.*

Problem 3.17. *Prove in general heat kernel estimates similar to* (30).

3.5. **Intertwining operator and (dual) Abel transform.** Consider the Abel transform

$$\mathcal{A} = \mathcal{F}^{-1}\circ\mathcal{H}\,,$$

which is obtained by composing the Dunkl transform $\mathcal{H}$ with the inverse Euclidean Fourier transform $\mathcal{F}^{-1}$ on $\mathfrak{a}$, and the dual Abel transform $\mathcal{A}^*$, which satisfies

$$\int_{\mathfrak{a}} dx\; \delta(x)\, f(x)\, \mathcal{A}^*g(x) \;=\; \int_{\mathfrak{a}} dy\; \mathcal{A}f(y)\, g(y)\,.$$

Theorem 3.18.

- *The dual Abel transform* $\mathcal{A}^*$ *coincides with the intertwining operator* $\mathcal{V}$ *defined on polynomials by Dunkl and extended to smooth functions by Trimèche.*
- Intertwining property: *for every* $\xi\in\mathfrak{a}$,

$$\mathcal{A}\circ D_\xi = \partial_\xi\circ\mathcal{A} \qquad \textit{and} \qquad \mathcal{V}\circ\partial_\xi = D_\xi\circ\mathcal{V}\,. \tag{31}$$

- Symmetries :

$$\mathcal{A}(wf) = w(\mathcal{A}f) \qquad \textit{and} \qquad \mathcal{V}(wg) = w(\mathcal{V}g) \qquad \forall\, w\in W,$$
$$\mathcal{A}\big[f(t\,.)\big](y) = |t|^{-2\gamma}(\mathcal{A}f)(ty) \quad \textit{and} \quad \mathcal{V}\big[g(t\,.)\big](x) = (\mathcal{V}g)(tx) \quad \forall\, t\in\mathbb{R}^*.$$

- *For every $x\in\mathfrak{a}$, there is a unique Borel probability measure μ_x on $\mathfrak{a}$ such that*

$$\mathcal{V}g(x) = \int_{\mathfrak{a}} d\mu_x(y)\, g(y)\,. \tag{32}$$

The support of μ_x is contained in the convex hull of Wx. Moreover, if $k>0$, the support of μ_x is W–invariant and contains Wx.

- *$\mathcal{A}$ is an automorphism of the spaces $\mathcal{C}_c^\infty(\mathfrak{a})$ and $\mathcal{S}(\mathfrak{a})$, while $\mathcal{V}$ is an automorphism of $\mathcal{C}^\infty(\mathfrak{a})$, with*

$$|\mathcal{V}g(x)| \leqslant \max\nolimits_{y\in\operatorname{co}(Wx)} |g(y)| \qquad \forall\, x\in\mathfrak{a}.$$

The following integral representations, which follow from (22), (31) and (32), generalize (6) and (18) in the present setting.

Corollary 3.19. *For every $\lambda\in\mathfrak{a}_{\mathbb{C}}$, we have*

$$E_\lambda(x) = \int_{\mathfrak{a}} d\mu_x(y)\, e^{\langle\lambda,y\rangle}$$

and

$$J_\lambda(x) = \int_{\mathfrak{a}} d\mu_x^W(y)\, \operatorname{Cosh}_\lambda(y)\,,$$

where $\operatorname{Cosh}_\lambda$ *is defined in* (23) *and*

$$\mu_x^W = \tfrac{1}{|W|}\sum\nolimits_{w\in W}\mu_{wx}\,.$$

Remark 3.20.

- *The first three items in Theorem 3.18 hold for all multiplicities $k\in K_{\text{reg}}$.*
- *The following symmetries hold :*

$$\begin{cases} d\mu_{wx}(wy) = d\mu_x(y) & \forall\, w\in W,\\ d\mu_{tx}(ty) = d\mu_x(y) & \forall\, t\in\mathbb{R}^*. \end{cases}$$

- *In* [49], *it is conjectured that the measure μ_x is absolutely continuous with respect to the Lebesgue measure under the following two assumptions :*
  - *x is regular (which means that $\langle\alpha,x\rangle\neq 0$, for every $\alpha\in R$),*
  - *$\mathfrak{a}$ is spanned by the roots α with multiplicity $k_\alpha>0$.*
- *These conjectures hold in dimension 1 (hence in the product case), where*

$$d\mu_x(y) = \tfrac{\Gamma(k+\frac12)}{\sqrt{\pi}\,\Gamma(k)}\, |x|^{-2k}\, \operatorname{sign}(x)\,(x+y)\,(x^2-y^2)^{k-1}\, \mathbb{1}_{(-|x|,+|x|)}(y)\, dy$$

if $x\neq0$, while μ_0 is the Dirac measure at the origin.

3.6. **Generalized translations, convolution and product formula.**

Definition 3.21.

- *The generalized convolution corresponds, via the Dunkl transform, to pointwise multiplication :*

$$(f*g)(x) = c_{\text{rat}}^{-2}\int_{\mathfrak{a}} d\lambda\, \delta(\lambda)\, \mathcal{H}f(\lambda)\, \mathcal{H}g(\lambda)\, E_{i\lambda}(x)\,. \tag{33}$$

- *The generalized translations are defined by*

$$(\tau_y f)(x) = c_{\text{rat}}^{-2}\int_{\mathfrak{a}} d\lambda\, \delta(\lambda)\, \mathcal{H}f(\lambda)\, E_{i\lambda}(x)\, E_{i\lambda}(y) = (\tau_x f)(y)\,. \tag{34}$$

The key objects here are the tempered distributions

$$f \longmapsto \langle\, \nu_{x,y}\,,f\,\rangle, \tag{35}$$

which are defined by (34) and which enter the product formula

$$E_\lambda(x)\,E_\lambda(y) = \langle\, \nu_{x,y}\,,E_\lambda\,\rangle\,. \tag{36}$$

Remark 3.22.

- *When* $k=0$, (33) *reduces to the usual convolution on* $\mathfrak{a}$, (34) *to* $(\tau_y f)(x)=f(x+y)$, *and* $\nu_{x,y}=\delta_{x+y}\,.$
- *In the* W*–invariant case,* (33) *becomes*

  $$(f*g)(x) = c_{\text{rat}}^{-2}\int_{\mathfrak{a}} d\lambda\;\delta(\lambda)\,\mathcal{H}f(\lambda)\,\mathcal{H}g(\lambda)\,J_{i\lambda}(x)$$

  and (36)

  $$J_\lambda(x)\,J_\lambda(y) = \langle\, \nu^W_{x,y}\,,J_\lambda\,\rangle, \tag{37}$$

  where

  $$\nu^W_{x,y} = \tfrac{1}{|W|}\sum\nolimits_{w\in W}\nu_{wx,wy}\,.$$

Lemma 3.23. *The distributions* (35) *are compactly supported.*

- *Specifically,* $\nu_{x,y}$ *is supported in the spherical shell*

  $$\{\, z\in\mathfrak{a} \;|\; \big||x|-|y|\big| \leqslant |z| \leqslant |x|+|y|\,\}\,.$$

- *Assume that* W *is crystallographic. Then* $\nu_{x,y}$ *is actually supported in*

  $$\{\, z\in\mathfrak{a} \;|\; z^+\!\preccurlyeq x^+\!+y^+,\; z^+\succcurlyeq y^+\!+w_0x^+ \text{ and } x^+\!+w_0y^+\,\}\,, \tag{38}$$

  where $\preccurlyeq$ *denotes the partial order on* $\mathfrak{a}$ *associated with the cone* $\overline{\mathfrak{a}_+}$ *(see Figure 8).*

Example 3.24. *In dimension* 1, $\nu_{x,y}$ *is a bounded signed measure. Specifically,*

$$d\nu_{x,y}(z) = \begin{cases} \nu(x,y,z)\,|z|^{2k}\,dz & \textit{if } x,y\in\mathbb{R}^*,\\ d\delta_y(z) & \textit{if } x=0\,,\\ d\delta_x(z) & \textit{if } y=0\,,\end{cases}$$

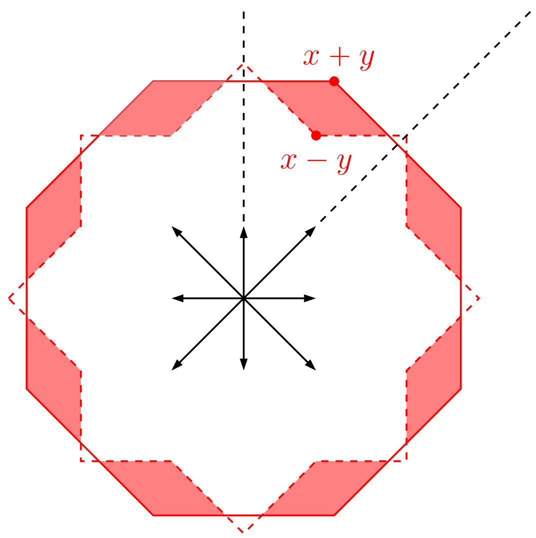

FIGURE 8. Picture of the set (38) for the root system B_2

where

$$\nu(x,y,z) = \frac{\Gamma(k+\frac12)}{\sqrt{\pi}\,\Gamma(k)}\,\frac{(x+y+z)(-x+y+z)(x-y+z)}{2\,xyz}$$
$$\times\,\frac{\{(|x|+|y|+|z|)(-|x|+|y|+|z|)(|x|-|y|+|z|)(|x|+|y|-|z|)\}^{k-1}}{\{2|x||y||z|\}^{2k-1}}$$

if $x,y,z\in\mathbb{R}^*$ *satisfy the triangular inequality* $\big||x|-|y|\big|<|z|<|x|+|y|$ *and* $\nu(x,y,z)=0$ *otherwise. Moreover, let*

$$M = \sup_{x,y\in\mathbb{R}}\int_{\mathbb{R}} d|\nu_{x,y}|(z) = \sqrt{2}\,\frac{\left[\Gamma(k+\frac12)\right]^2}{\Gamma(k+\frac14)\,\Gamma(k+\frac34)}\,.$$

Then $M\geqslant 1$ *and* $M\nearrow\sqrt{2}$ *as* $k\nearrow+\infty$.

In general there is a lack information about (35) and the following facts are conjectured [68].

Problem 3.25.
(a) *The distributions* $\nu_{x,y}$ *are bounded signed Borel measures.*
(b) *They are uniformly bounded in* x *and* y.
(c) *The measures* $\nu_{x,y}^W$ *are positive.*

If $\nu_{x,y}^{(W)}$ is a measure, notice that it is normalized by

$$\int_{\mathfrak{a}} d\nu_{x,y}^{(W)}(z) = 1\,.$$

Problem 3.25 and especially item (b) is important for harmonic analysis. It implies indeed the following facts, for the reference measure $\delta(x)dx$ on $\mathfrak{a}$.

Problem 3.26.
(d) *The generalized translations* (34) *are uniformly bounded on* L^1 *and hence on* L^p, *for every* $1\leqslant p\leqslant\infty$.
(e) Young's inequality: *For all* $1\leqslant p,q,r\leqslant\infty$ *satisfying* $\frac1p+\frac1q-\frac1r=1$, *there exists a constant* $C\geqslant 0$ *such that*

$$\|f*g\|_{L^r}\leqslant C\,\|f\|_{L^p}\,\|g\|_{L^q}\,. \tag{39}$$

Beside the trivial L^2 setting and the one–dimensional case (hence the product case), here are two more situations where Problems 3.25 and 3.26 have been solved.

- **Radial case** [68]. Translations of radial functions are positive. Specifically, for radial functions $f(z)={}'f(|z|)$ and nonzero $y\in\mathfrak{a}$, we have

$$(\tau_y f)(x) = \int_{\mathfrak{a}} d\mu_y(z)\,{}'f\big(\sqrt{|x|^2+|y|^2+2\,\langle x,z\rangle}\,\big)\,. \tag{40}$$

  Hence (39) holds if f or g is radial.

- **Symmetric space case.** Assume that the multiplicity k corresponds to a Riemannian symmetric space of Euclidean type. Then $\nu_{x,y}^W$ is a positive measure and (39) holds for W–invariant functions.

3.7. Comments, references and further results.

- The computation of the integral (26) has a long history. A closed form was conjectured by Mehta for the root systems of type A and by Macdonald for general root systems. For the four infinite families of classical root systems, it can be actually deduced from an earlier integral formula of Selberg. In general, the Mehta–Macdonald formula was proved by Opdam, first for crystallographic root systems [61] and next for all root systems [62]. His proof was simplified by Etingof [30], who removed in particular the computer–assisted calculations used in the last cases.

- We have not discussed the shift operators, which move the multiplicity k by integers and which have proven useful in the W–invariant setting (see [62]).
- The following asymptotics hold for the Dunkl kernel, under the assumption that $k \geqslant 0$ (see [49]) : there exists $\upsilon : W \longrightarrow \mathbb{C}$ such that, for every $w \in W$ and for every $\lambda, x \in \mathfrak{a}^+$,

$$\lim_{t\to+\infty} (it)^{\gamma}\, e^{-it\langle\lambda,wx\rangle}\, E_{it\lambda}(wx) = \upsilon(w)\, \delta(\lambda)^{-\frac{1}{2}}\, \delta(x)^{-\frac{1}{2}}\ . \qquad (41)$$

If $w = \mathrm{I}$, we have $\upsilon(\mathrm{I}) = (2\pi)^{-\frac{n}{2}} c$, where c is defined by (26), and (41) holds more generally when $\widetilde{t} = it$ tends to infinity in the half complex space $\{\, \widetilde{t} \in \mathbb{C} \,|\, \mathrm{Re}\, \widetilde{t} \geqslant 0\}$. If $w \neq \mathrm{I}$, the Dunkl kernel is expected to have a different asymptotic behavior, when $\mathrm{Re}\, \widetilde{t}$ becomes positive. In dimension 1, we have indeed, for any $0 < \varepsilon \leqslant \frac{\pi}{2}$,

$$\lim_{\substack{\widetilde{t}\to+\infty \\ |\arg \widetilde{t}\,| \leqslant \frac{\pi}{2}-\varepsilon}} \widetilde{t}^{\,k+1} e^{-\widetilde{t}\lambda x} E_{\widetilde{t}\lambda}(-x) = \tfrac{2^{k-1}\, k\, \Gamma(k+\frac{1}{2})}{\sqrt{\pi}}\, \lambda^{-k-1}\, x^{-k-1}\,.$$

This discrepancy plays an important role in [10].
- In the fourth item of Theorem 3.18, the sharper results about the support of μ_x when $k > 0$ were obtained in [36].
- Specific information is available in the W–invariant setting for the root systems A_n. In this case, an integral recurrence formula over n was obtained in [2] and [77] for J_λ, by taking rational limits of corresponding formulae in the trigonometric case (see the tenth item in Subsection 4.7). Moreover, an explicit expression of μ_x^W is deduced in [77]. As a consequence, the support of μ_x is shown to be equal to the convex hull of Wx, when $k > 0$.
- The asymmetric setting is harder. Beyond the one–dimensional case (and the product case), explicit expressions of the measure μ_x are presently available in some two–dimensional cases. For the root system A_2, two closely related expressions were obtained, first in [25] and recently in [3]. For the root system B_2, a complicated formula was obtained in [26] and a simpler one recently in [5]. The case of dihedral root systems $\mathrm{I}_2(m)$ is currently investigated (see [22], [19] and the references therein).
- In Lemma 3.23, the sharper result in the crystallographic case was obtained in [4].
- An explicit product formula was obtained in [69] for generalized Bessel functions associated with root systems of type B and for three one–dimensional families of multiplicities (which are two–dimensional in this case). The method consists in computing a product formula in the symmetric space case, which corresponds to a discrete set of multiplicities k, and in extending it holomorphically in k. The resulting measure lives in a matrix cone, which projects continuously onto $\overline{\mathfrak{a}^+}$, and its image $|W|\, \nu_{x,y}^W$ is a probability measure if $k \geqslant 0$.
- Potential theory in the rational Dunkl setting has been studied in [58], [33], [56], [34], [35], [36], [37], [41] (see also [66] and the references therein).
- Many current works deal with generalizations of results in Euclidean harmonic analysis to the rational Dunkl setting. Among others, let us mention
  - [84] about the Hardy–Littlewood and the Poisson maximal functions,
  - [6] and [7] about singular integrals and Calderon–Zygmund theory,
  - [10] and [28] about the Hardy space H^1.
- A further interesting deformation of Euclidean Fourier analysis, encompassing rational Dunkl theory and the Laguerre semigroup, was introduced and studied in [17].

4. Trigonometric Dunkl theory

Trigonometric Dunkl theory was developed in the symmetric case by Heckman and Opdam in the 1980s, and in the non–symmetric case by Opdam and Cherednik in the 1990s. This theory of special functions in several variables encompasses

- Euclidean Fourier analysis (which corresponds to the multiplicity $k=0$),
- Jacobi functions in dimension 1,
- spherical functions associated with Riemannian symmetric spaces of noncompact type (which correspond to a discrete set of multiplicities k).

In this subsection, we use [64] as our primary reference and quote only later works. We resume the notation of Section 3, with some modifications:

- the root system R is now assumed to be crystallographic but not necessarily reduced,
- $\widetilde{R}$ denotes the subsystem of non–multipliable roots,
- the reference density in the case $k\geqslant 0$ is now $\delta(x)=\prod_{\alpha\in R^+}\bigl|\,2\sinh\frac{\langle\alpha,x\rangle}{2}\,\bigr|^{2k_\alpha}$,

and some addenda:

- Q denotes the root lattice, $Q^\vee$ the coroot lattice and P the weight lattice,
- $\rho=\sum_{\alpha\in\mathbb{R}^+}\frac{k_\alpha}{2}\,\alpha$.

4.1. Cherednik operators.

Definition 4.1. *The trigonometric Dunkl operators, which are often called Cherednik operators, are the differential–difference operators defined by*

$$D_\xi f(x)=\partial_\xi f(x)+\sum\nolimits_{\alpha\in R^+}k_\alpha\,\frac{\langle\alpha,\xi\rangle}{1-e^{-\langle\alpha,x\rangle}}\,\bigl\{f(x)-f(r_\alpha x)\bigr\}-\langle\rho,\xi\rangle\,f(x) \tag{42}$$

for every $\xi\in\mathfrak{a}$.

Notice that the counterpart of Remark 3.2 holds in the present setting. In next theorem, we collect properties of Cherednik operators. The main one is again commutativity, which leads to Cherednik operators D_p, for every polynomial $p\in\mathcal{P}(\mathfrak{a})$, and to their symmetric parts $\widetilde{D}_p$ on W–invariant functions.

Theorem 4.2.

- *For any fixed multiplicity k, the Cherednik operators* (42) *commute pairwise.*
- *The Cherednik operators map the following function spaces into themselves:*

  $$\mathbb{C}\bigl[e^P\bigr],\ \mathcal{P}(\mathfrak{a}),\ \mathcal{C}^\infty(\mathfrak{a}),\ \mathcal{C}_c^\infty(\mathfrak{a}),\ \mathcal{S}^2(\mathfrak{a})=\bigl(\mathrm{Cosh}_\rho\bigr)^{-1}\mathcal{S}(\mathfrak{a}),\ \dots$$

  where $\mathbb{C}\bigl[e^P\bigr]$ denotes the algebra of polynomials in e^λ ($\lambda\in P$) and Cosh_ρ is defined in (23).
- W–equivariance: *For every $w\in W$ and $\xi\in\mathfrak{a}$, we have*

  $$\bigl(w\circ D_\xi\circ w^{-1}\bigr)f(x)=D_{w\xi}f(x)+\sum\nolimits_{\alpha\in R^+\cap wR^-}k_\alpha\,\langle\alpha,w\xi\rangle\,f(r_\alpha x)\,.$$

  Hence $\widetilde{D}_{pq}=\widetilde{D}_p\circ\widetilde{D}_q$ for all symmetric polynomials $p,q\in\mathcal{P}(\mathfrak{a})^W$.
- Adjointness: *Assume that $k\geqslant 0$. Then, for every $\xi\in\mathfrak{a}$,*

  $$\int_{\mathfrak{a}}dx\;\delta(x)\,D_\xi f(x)\,g(-x)=\int_{\mathfrak{a}}dx\;\delta(x)\,f(x)\,D_\xi g(-x)\,.$$

Example 4.3.

- *The Heckman–Opdam Laplacian is given by*

$$\begin{aligned}\Delta f(x) = \sum\nolimits_{j=1}^{n} D_j^2 f(x) = &\overbrace{\sum\nolimits_{j=1}^{n} \partial_j^2 f(x) + \sum\nolimits_{\alpha\in R^+} k_\alpha \coth\tfrac{\langle\alpha,x\rangle}{2}\,\partial_\alpha f(x) + |\rho|^2 f(x)}^{\text{differential part }\widetilde{\Delta} f(x)}\\ &\underbrace{-\sum\nolimits_{\alpha\in R^+} k_\alpha\,\tfrac{|\alpha|^2}{4\sinh^2\frac{\langle\alpha,x\rangle}{2}}\,\big\{f(x)-f(r_\alpha x)\big\}}_{\text{difference part}}\,,\end{aligned}$$

where D_j, respectively ∂_j denote the Cherednik operators, respectively the partial derivatives with respect to an orthonormal basis of $\mathfrak{a}$.

- *In dimension 1, the Cherednik operator is given by*

$$\begin{aligned}Df(x) &= \big(\tfrac{\partial}{\partial x}\big)f(x) + \big\{\tfrac{k_1}{1-e^{-x}} + \tfrac{2\,k_2}{1-e^{-2x}}\big\}\,\big\{f(x)-f(-x)\big\} - \rho\,f(x)\\ &= \big(\tfrac{\partial}{\partial x}\big)f(x) + \big\{\tfrac{k_1}{2}\coth\tfrac{x}{2} + k_2\coth x\big\}\,\big\{f(x)-f(-x)\big\} - \rho\,f(-x)\end{aligned}$$

and the Heckman–Opdam Laplacian by

$$\begin{aligned}\Delta f(x) = &\big(\tfrac{\partial}{\partial x}\big)^2 f(x) + \big\{k_1\coth\tfrac{x}{2} + 2\,k_2\coth x\big\}\,\big(\tfrac{\partial}{\partial x}\big)f(x) + \rho^2 f(x)\\ &-\Big\{\tfrac{k_1}{4\sinh^2\frac{x}{2}} + \tfrac{k_2}{\sinh^2 x}\Big\}\,\big\{f(x)-f(-x)\big\}\,,\end{aligned}$$

where $\rho = \frac{k_1}{2} + k_2$.

4.2. **Hypergeometric functions.**

Theorem 4.4. *Assume that $k \geqslant 0$. Then, for every $\lambda\in\mathfrak{a}_{\mathbb{C}}$, the system*

$$\begin{cases} D_\xi G_\lambda = \langle\lambda,\xi\rangle\,G_\lambda & \forall\,\xi\in\mathfrak{a},\\ G_\lambda(0) = 1. \end{cases}$$

has a unique smooth solution on $\mathfrak{a}$, which is called the Opdam hypergeometric function.

Definition 4.5. *The Heckman–Opdam hypergeometric function is the average*

$$F_\lambda(x) = \tfrac{1}{|W|}\sum\nolimits_{w\in W} G_\lambda(wx)\,. \tag{43}$$

Remark 4.6.

- *Conversely, $G_\lambda(x)$ can be recovered by applying to $F_\lambda(x)$ a linear differential operator in x whose coefficients are rational functions of λ.*
- *The expression $G_\lambda(x)$ extends to a holomorphic function of $\lambda\in\mathfrak{a}_{\mathbb{C}}$, $x\in\mathfrak{a}+i\,U$ and $k\in V$, where U is a W–invariant open neighborhood of 0 in $\mathfrak{a}$ and V is a W–invariant open neighborhood of $\{k\in K \mid k\geqslant 0\}$.*
- *The Heckman–Opdam hypergeometric function (43) is characterized by the system*

$$\begin{cases} \widetilde{D}_p\,F_\lambda = p(\lambda)\,F_\lambda & \forall\,p\in\mathcal{P}(\mathfrak{a})^W,\\ F_\lambda(0) = 1. \end{cases}$$

- *In dimension 1, the Heckman–Opdam hypergeometric function reduces to the Gauss hypergeometric function ${}_2\mathrm{F}_1$ or, equivalently, to the Jacobi functions $\varphi_\lambda^{\alpha,\beta}$:*

$$F_\lambda(x) = {}_2\mathrm{F}_1\big(\rho+\lambda,\rho-\lambda;k_1+k_2+\tfrac{1}{2};-\sinh^2\tfrac{x}{2}\big) = \varphi_{i\,2\lambda}^{k_1+k_2-\frac{1}{2},\,k_2-\frac{1}{2}}\big(\tfrac{x}{2}\big)\,,$$

and the Opdam hypergeometric function to a combination of two such functions:

$$G_\lambda(x) = \varphi_{i\,2\lambda}^{k_1+k_2-\frac{1}{2},\,k_2-\frac{1}{2}}\big(\tfrac{x}{2}\big) + \tfrac{\rho+\lambda}{2k_1+2k_2+1}\,(\sinh x)\,\varphi_{i\,2\lambda}^{k_1+k_2+\frac{1}{2},\,k_2+\frac{1}{2}}\big(\tfrac{x}{2}\big)\,.$$

- *When* $k=0$, $G_\lambda(x)$ *reduces to the exponential* $e^{\langle\lambda,\,x\rangle}$ *and* $F_\lambda(x)$ *to* $\operatorname{Cosh}_\lambda(x)$.
- *The functions* $G_{-\rho}$ *and* $F_{-\rho}$ *are equal to* 1.
- *Spherical functions* φ_λ^G *on Riemannian symmetric space* G/K *of noncompact type are Heckman–Opdam hypergeometric functions. Specifically, if*
$$\begin{cases}\mathcal{R} \text{ is the root system of } (\mathfrak{g},\mathfrak{a}),\\ m_\alpha=\dim\mathfrak{g}_\alpha,\end{cases}$$
set
$$\begin{cases}R=2\,\mathcal{R},\\ k_{2\alpha}=\tfrac12 m_\alpha.\end{cases}$$
Then
$$\varphi_\lambda^G(\exp x)\,=\,F_{i\frac{\lambda}{2}}(2x)\,.$$

We collect in the next two propositions asymptotics and estimates of the hypergeometric functions.

Proposition 4.7. *The following Harish–Chandra type expansions hold*:
$$F_\lambda(x)\,=\,\sum\nolimits_{w\in W}\mathbf{c}(w\lambda)\,\Phi_{w\lambda}(x)\,,$$
$$G_\lambda(x)\,=\,\frac{1}{\prod_{\alpha\in\widetilde{R}^+}\bigl(\langle\lambda,\alpha^\vee\rangle)-\frac12 k_{\alpha/2}-k_\alpha\bigr)}\,\sum\nolimits_{w\in W}\mathbf{c}(w\lambda)\,\Psi_{w,\lambda}(x)\,.$$
Here
$$\mathbf{c}(\lambda)\,=\,c_0\prod\nolimits_{\alpha\in R^+}\frac{\Gamma(\langle\lambda,\alpha^\vee\rangle+\frac12 k_{\alpha/2})}{\Gamma(\langle\lambda,\alpha^\vee\rangle+\frac12 k_{\alpha/2}+k_\alpha)}\,,$$
where c_0 *is a positive constant such that* $\mathbf{c}(\rho)=1$, *and*
$$\Phi_\lambda(x)=\sum\nolimits_{\ell\in Q^+}\Gamma_\ell(\lambda)\,e^{\langle\lambda-\rho-\ell,\,x\rangle},\ \ \Psi_{w,\lambda}(x)\,=\sum\nolimits_{\ell\in Q^+}\Gamma_\ell(w,\lambda)\,e^{\langle w\lambda-\rho-\ell,\,x\rangle}$$
are converging series, for generic $\lambda\in\mathfrak{a}_{\mathbb{C}}$ *and for every* $x\in\mathfrak{a}^+$.

Proposition 4.8. *Assume that* $k\geqslant0$.
- *All functions* G_λ *with* $\lambda\in\mathfrak{a}$ *are strictly positive.*
- *The ground function* G_0 *has the following behavior*:
$$G_0(x)\,\asymp\,\Bigl\{\prod\nolimits_{\substack{\alpha\in\widetilde{R}^+\\ \langle\alpha,x\rangle\geqslant0}}\bigl(1+\langle\alpha,x\rangle\bigr)\Bigr\}\,e^{-\langle\rho,x^+\rangle}\qquad\forall\;x\in\mathfrak{a}.$$
In particular,
$$G_0(x)\,\asymp\,\Bigl\{\prod\nolimits_{\alpha\in\widetilde{R}^+}\bigl(1+\langle\alpha,x\rangle\bigr)\Bigr\}\,e^{-\langle\rho,x^+\rangle}$$
if $x\in\overline{\mathfrak{a}^+}$, *while*
$$G_0(x)\asymp e^{-\langle\rho,x^+\rangle}$$
if $x\in-\overline{\mathfrak{a}^+}$.
- *For every* $\lambda\in\mathfrak{a}_{\mathbb{C}}$, $\mu\in\mathfrak{a}$ *and* $x\in\mathfrak{a}$, *we have*
$$|G_{\lambda+\mu}(x)|\leqslant e^{\langle(\operatorname{Re}\lambda)^+,\,x^+\rangle}\,G_\mu(x)\,.$$
In particular, the following estimates hold, for every $\lambda\in\mathfrak{a}_{\mathbb{C}}$ *and* $x\in\mathfrak{a}$,
$$|G_\lambda(x)|\leqslant G_{\operatorname{Re}\lambda}(x)\leqslant G_0(x)\,e^{\langle(\operatorname{Re}\lambda)^+,\,x^+\rangle}\,.$$

4.3. **Cherednik transform.** From now on, we assume that $k \geqslant 0$.

Definition 4.9. *The Cherednik transform is defined by*

$$\mathcal{H}f(\lambda) = \int_{\mathfrak{a}} dx\, \delta(x)\, f(x)\, G_{i\lambda}(-x)\,. \tag{44}$$

In next theorem, we collect the main properties of the Cherednik transform.

Theorem 4.10.

- *The Cherednik transform is an isomorphism between the L^2 Schwartz space*
$$\mathcal{S}^2(\mathfrak{a}) = \big(\operatorname{Cosh}_\rho\big)^{-1}\,\mathcal{S}(\mathfrak{a})$$
and the Euclidean Schwartz space $\mathcal{S}(\mathfrak{a})$.
- Paley–Wiener Theorem: *The Cherednik transform is an isomorphism between $\mathcal{C}_c^\infty(\mathfrak{a})$ and the Paley–Wiener space $\mathcal{PW}(\mathfrak{a}_{\mathbb{C}})$. More precisely, let C be a W–invariant convex compact neighborhood of the origin in $\mathfrak{a}$ and let $\chi(\lambda) = \max_{x\in C}\langle\lambda, x\rangle$ be the associated gauge. Then the support of $f \in \mathcal{C}_c^\infty(\mathfrak{a})$ is contained in C if and only if $h = \mathcal{H}f$ satisfies* (28).
- Inversion formula:
$$f(x) = c_{\text{trig}}^{-2}\int_{\mathfrak{a}} d\lambda\, \widetilde{\delta}(\lambda)\, \mathcal{H}f(\lambda)\, G_{i\lambda}(x)\,, \tag{45}$$
where
$$\begin{aligned}\widetilde{\delta}(\lambda) &= \tfrac{c_0^2}{|\mathbf{c}(i\lambda)|^2}\,\prod\nolimits_{\alpha\in\widetilde{R}^+} \tfrac{-i\langle\lambda,\alpha^\vee\rangle+\frac12 k_{\alpha/2}+k_\alpha}{-i\langle\lambda,\alpha^\vee\rangle}\\ &= \prod\nolimits_{\alpha\in R^+} \tfrac{\Gamma(i\langle\lambda,\alpha^\vee\rangle+\frac12 k_{\alpha/2}+k_\alpha)}{\Gamma(i\langle\lambda,\alpha^\vee\rangle+\frac12 k_{\alpha/2})}\,\tfrac{\Gamma(-i\langle\lambda,\alpha^\vee\rangle+\frac12 k_{\alpha/2}+k_\alpha+1)}{\Gamma(-i\langle\lambda,\alpha^\vee\rangle+\frac12 k_{\alpha/2}+1)}\end{aligned} \tag{46}$$
and c_{trig} is a positive constant.

Remark 4.11.

- *In the W–invariant case, the Cherednik transform* (44) *reduces to*
$$\mathcal{H}f(\lambda) = \int_{\mathfrak{a}} dx\, \delta(x)\, f(x)\, F_{i\lambda}(-x) \tag{47}$$
and its inverse (45) *to*
$$f(x) = c_{\text{trig}}^{-2}\int_{\mathfrak{a}} d\lambda\, \tfrac{c_0^2}{|\mathbf{c}(i\lambda)|^2}\, \mathcal{H}f(\lambda)\, F_{i\lambda}(x)\,. \tag{48}$$
It is an isomorphism between $\mathcal{S}^2(\mathfrak{a})^W = \big(\operatorname{Cosh}_\rho\big)^{-1}\mathcal{S}(\mathfrak{a})^W$ and $\mathcal{S}(\mathfrak{a})^W$, which extends to an isometric isomorphism, up to a positive constant, between $L^2(\mathfrak{a}, \delta(x)\,dx)^W$ and $L^2(\mathfrak{a}, |\mathbf{c}(i\lambda)|^{-2}d\lambda)^W$.
- *Formulae* (47) *and* (48) *are not symmetric, as the spherical Fourier transform* (11) *and its inverse* (12) *on hyperbolic spaces $\mathbb{H}^n$, or their counterparts* (15) *and* (16) *on homogeneous trees $\mathbb{T}_q$. The asymmetry is even greater between* (44) *and* (45), *where the density* (46) *is complex–valued.*
- *There is no straightforward Plancherel identity for the full Cherednik transform* (44). *Opdam has defined in* [63] *a vector–valued transform leading to a Plancherel identity in the non–W–invariant case.*

4.4. **Rational limit.** Rational Dunk theory (in the crystallographic case) is a suitable limit of trigonometric Dunk theory, as Hankel analysis on $\mathbb{R}^n$ is a limit of spherical Fourier analysis on $\mathbb{H}^n$. More precisely, assume that the root system R is both crystallographic and reduced. Then,

- the Dunkl kernel is the following limit of Opdam hypergeometric functions:
$$E_\lambda(x) = \lim\nolimits_{\varepsilon\to 0} G_{\varepsilon^{-1}\lambda}(\varepsilon x)\,,$$

- the Dunkl transform $\mathcal{H}_{\text{rat}}$ is a limit case of the Cherednik transform $\mathcal{H}_{\text{trig}}$:
$$(\mathcal{H}_{\text{rat}} f)(\lambda) = \lim_{\varepsilon \to 0} \varepsilon^{-n-2\gamma} \big\{\mathcal{H}_{\text{trig}}[f(\varepsilon^{-1}\,.\,)]\big\}(\varepsilon^{-1}\lambda)\,,$$
- likewise for the inversion formulae (25) and (45):
$$(\mathcal{H}_{\text{rat}}^{-1} f)(x) = \text{const.} \lim_{\varepsilon \to 0} \varepsilon^{n+2\gamma} \big\{\mathcal{H}_{\text{trig}}^{-1}[f(\varepsilon\,.\,)]\big\}(\varepsilon x)\,.$$

4.5. **Intertwining operator and (dual) Abel transform.** In the trigonometric setting, consider again the Abel transform
$$\mathcal{A} = \mathcal{F}^{-1} \circ \mathcal{H}\,,$$
which is obtained by composing the Cherednik transform $\mathcal{H}$ with the inverse Euclidean Fourier transform $\mathcal{F}^{-1}$ on $\mathfrak{a}$, and the dual Abel transform $\mathcal{A}^*$, which satisfies
$$\int_{\mathfrak{a}} dx\; \delta(x)\, f(x)\, \mathcal{A}^* g(x) \;=\; \int_{\mathfrak{a}} dy\; \mathcal{A}f(y)\, g(y)\,.$$

Remark 4.12. *In* [85], *$\mathcal{A}^* = \mathcal{V}$ is called the trigonometric Dunkl intertwining operator and $\mathcal{A} = \mathcal{V}^*$ the dual operator.*

Proposition 4.13.
- *For every $\xi \in \mathfrak{a}$,*
$$\mathcal{A} \circ D_\xi = \partial_\xi \circ \mathcal{A} \qquad \textit{and} \qquad \mathcal{V} \circ \partial_\xi = D_\xi \circ \mathcal{V}\,.$$
- *For every $x \in \mathfrak{a}$, there is a unique tempered distribution μ_x on $\mathfrak{a}$ such that*
$$\mathcal{V} g(x) = \langle\, \mu_x, g\,\rangle\,.$$
Moreover, the support of μ_x is contained in the convex hull of Wx.

Corollary 4.14. *For every $\lambda \in \mathfrak{a}_{\mathbb{C}}$, we have*
$$G_\lambda(x) = \langle\, \mu_x, e^\lambda\,\rangle \quad \textit{and} \quad F_\lambda(x) = \langle\, \mu_x^W, \operatorname{Cosh}_\lambda\,\rangle\,, \tag{49}$$
where $\operatorname{Cosh}_\lambda$ *is defined in* (23) *and* $\mu_x^W = \frac{1}{|W|} \sum_{w \in W} \mu_{wx}\,.$

Remark 4.15.

- *The distribution μ_x is most likely a probability measure, as in the rational setting.*
- *This is true in dimension* 1 (*hence in the product case*), *where*
$$d\mu_x(y) = \begin{cases} d\delta_x(y) & \textit{if } x=0 \textit{ or if } k_1=k_2=0.\\ \mu(x,y)\,dy & \textit{otherwise.}\end{cases}$$
As far as it is concerned, the density $\mu(x,y)$ vanishes unless $|y|<|x|$. In the generic case, where $k_1>0$ and $k_2>0$, it is given explicitly by
$$\begin{aligned}\mu(x,y) \;=\; & 2^{k_1+k_2-2}\, \tfrac{\Gamma(k_1+k_2+\frac12)}{\sqrt{\pi}\,\Gamma(k_1)\,\Gamma(k_2)}\, |\sinh\tfrac{x}{2}|^{-2k_1}\, |\sinh x|^{-2k_2}\\ & \times \int_{|y|}^{|x|} dz\, (\sinh\tfrac{z}{2})\, (\cosh\tfrac{z}{2} - \cosh\tfrac{y}{2})^{k_1-1}\, (\cosh x - \cosh z)^{k_2-1}\\ & \qquad \times (\operatorname{sign} x) \big\{ e^{\frac{x}{2}}\, (2\cosh\tfrac{x}{2}) - e^{-\frac{y}{2}}\, (2\cosh\tfrac{z}{2}) \big\}\,.\end{aligned} \tag{50}$$
In the limit case, where $k_1=0$ and $k_2>0$,
$$\mu(x,y) = 2^{k_2-1}\, \tfrac{\Gamma(k_2+\frac12)}{\sqrt{\pi}\,\Gamma(k_2)}\, |\sinh x|^{-2k_2}\, (\cosh x - \cosh y)^{k_2-1}\, (\operatorname{sign} x)\, (e^x - e^{-y})\,. \tag{51}$$
In the other limit case, where $k_1>0$ and $k_2=0$, the density is half of (51), *with k_2, x, y replaced respectively by k_1, $\frac{x}{2}$, $\frac{y}{2}$.*

4.6. **Generalized translations, convolution and product formula.**

Definition 4.16.

- *The generalized convolution corresponds, via the Cherednik transform, to pointwise multiplication*:

$$(f*g)(x) = c_{\mathrm{trig}}^{-2} \int_{\mathfrak{a}} d\lambda\, \widetilde{\delta}(\lambda)\, \mathcal{H}f(\lambda)\, \mathcal{H}g(\lambda)\, G_{i\lambda}(x)\,.$$

- *The generalized translations are defined by*

$$(\tau_y f)(x) = c_{\mathrm{trig}}^{-2} \int_{\mathfrak{a}} d\lambda\, \widetilde{\delta}(\lambda)\, \mathcal{H}f(\lambda)\, G_{i\lambda}(x)\, G_{i\lambda}(y)\,. \tag{52}$$

The key objects are again the tempered distributions

$$f \longmapsto \langle\, \nu_{x,y}\,, f\,\rangle$$

defined by (52) and their averages

$$\nu_{x,y}^{W} = \tfrac{1}{|W|} \sum\nolimits_{w\in W} \nu_{wx,wy}\,,$$

which enter the product formulae

$$G_\lambda(x)\, G_\lambda(y) = \langle\, \nu_{x,y}, G_\lambda\,\rangle$$

and

$$F_\lambda(x)\, F_\lambda(y) = \langle\, \nu_{x,y}^{W}, F_\lambda\,\rangle\,. \tag{53}$$

Example 4.17. *In dimension* 1, *the distributions* $\nu_{x,y}$ *are signed measures, which are uniformly bounded in* x *and* y. *Explicitly* [9],

$$d\nu_{x,y}(z) = \begin{cases} \nu(x,y,z)\, dz & \text{if } x,y\in\mathbb{R}^*,\\ d\delta_y(z) & \text{if } x=0,\\ d\delta_x(z) & \text{if } y=0, \end{cases}$$

where the density $\nu(x,y,z)$ *is given by the following formulae, when* $x,y,z\in\mathbb{R}^*$ *satisfy the triangular inequality*

$$\big|\,|x|-|y|\,\big| < |z| < |x|+|y|\,,$$

and vanishes otherwise.

- *Assume that* $k_1>0$ *and* $k_2>0$. *Then*

$$\begin{aligned} \nu(x,y,z) = {} & 2^{k_1-2}\, \tfrac{\Gamma(k_1+k_2+\frac12)}{\sqrt{\pi}\,\Gamma(k_1)\,\Gamma(k_2)}\, \operatorname{sign}(xyz)\, \big|\sinh\tfrac{x}{2}\sinh\tfrac{y}{2}\big|^{-2k_1-2k_2} (\cosh\tfrac{z}{2})^{2k_2}\\ & \times \int_0^\pi d\chi\, (\sin\chi)^{2k_2-1}\\ & \quad \times \Big[\cosh\tfrac{x}{2}\cosh\tfrac{y}{2}\cosh\tfrac{z}{2}\cos\chi - \tfrac{1+\cosh x+\cosh y+\cosh z}{4}\Big]_+^{k_1-1}\\ & \quad \times \Big[\sinh\tfrac{x+y+z}{2} - 2\cosh\tfrac{x}{2}\cosh\tfrac{y}{2}\sinh\tfrac{z}{2}\\ & \qquad + \tfrac{k_1+2k_2}{k_2}\cosh\tfrac{x}{2}\cosh\tfrac{y}{2}\cosh\tfrac{z}{2}\,(\sin\chi)^2\\ & \qquad + \tfrac{\sinh z-\sinh x-\sinh y}{2}\cos\chi\,\Big]\,. \end{aligned}$$

- *Assume that* $k_1 = 0$ *and* $k_2 > 0$. *Then*
$$\nu(x,y,z) = 2^{2k_2-1}\,\tfrac{\Gamma(k_2+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(k_2)}\;\operatorname{sign}(xyz)\,|(\sinh x)(\sinh y)|^{-2k_2}$$
$$\times\left[\sinh\tfrac{x+y+z}{2}\,\sinh\tfrac{-x+y+z}{2}\,\sinh\tfrac{x-y+z}{2}\,\sinh\tfrac{x+y-z}{2}\right]^{k_2}$$
$$\times\left[\sinh\tfrac{x+y-z}{2}\right]^{-1} e^{\frac{x+y-z}{2}}\,.$$
- *In the other limit case, where* $k_1 > 0$ *and* $k_2 = 0$, *the density is again half of the previous one, with* k_2, x, y *replaced respectively by* k_1, $\frac{x}{2}$, $\frac{y}{2}$.

In higher dimension, we have the trigonometric counterparts of Problems 3.25 & 3.26 but fewer results than in the rational case. In particular, there is no formula like (40) for radial functions. A new property is the following Kunze–Stein phenomenon, which is typical of the semisimple setting and which was proved in [9] (see also [15]) and [86].

Proposition 4.18. *Let* $1 \leqslant p < 2$. *Then there exists a constant* $C > 0$ *such that*
$$\|\,f * g\,\|_{L^2} \leqslant C\,\|\,f\,\|_{L^p}\,\|g\|_{L^2}\,,$$
for every $f \in L^p(\mathfrak{a},\delta(x)dx)$ *and* $g \in L^2(\mathfrak{a},\delta(x)dx)$.

4.7. **Comments, references and further results.**

- The joint action of the Cherednik operators D_p, with $p \in \mathcal{P}(\mathfrak{a})$, and of the Weyl group W may look intricate. It corresponds actually to a faithful representation of a graded affine Hecke algebra [63].
- Heckman [43] considered initially the following trigonometric version
$$'D_\xi f(x) = \partial_\xi f(x) + \sum\nolimits_{\alpha\in R^+}\tfrac{k_\alpha}{2}\,\langle\alpha,\xi\rangle\,\coth\tfrac{\langle\alpha,x\rangle}{2}\,\big\{f(x)-f(r_\alpha x)\big\}$$
of rational Dunkl operators, which are closely connected to (42):
$$D_\xi f(x) = {}'D_\xi f(x) - \sum\nolimits_{\alpha\in R^+}\tfrac{k_\alpha}{2}\,\langle\alpha,\xi\rangle\,f(r_\alpha x)\,.$$
These operators are W–equivariant:
$$w\circ{}'D_\xi\circ w^{-1} = {}'D_{w\xi}\,,$$
and skew–invariant:
$$\int_{\mathfrak{a}} dx\;\delta(x)\,({}'D_\xi f)(x)\,g(x) = -\int_{\mathfrak{a}} dx\;\delta(x)\,f(x)\,({}'D_\xi g)(x)\,,$$
but they don't commute:
$$[{}'D_\xi,{}'D_\eta]\,f(x) = \sum\nolimits_{\alpha,\beta\in R^+}\tfrac{k_\alpha k_\beta}{4}\,\big\{\langle\alpha,\xi\rangle\langle\beta,\eta\rangle-\langle\beta,\xi\rangle\langle\alpha,\eta\rangle\big\}\,f(r_\alpha r_\beta x)\,.$$
- The hypergeometric functions $x \longmapsto G_\lambda(x)$ and $x \longmapsto F_\lambda(x)$ extend holomophically to a tube $\mathfrak{a}+i\,U$ in $\mathfrak{a}_{\mathbb{C}}$. The optimal width for F_λ was investigated in [52].
- Proposition 4.7 was obtained in [63]. The asymptotic behavior of F_λ was fully determined in [59]. This paper contains in particular a proof of the estimate
$$F_\lambda(x) \asymp \Big\{\prod\nolimits_{\substack{\alpha\in\widetilde{R}^+\\ \langle\alpha,\lambda\rangle\neq 0}}\big(1+\langle\alpha,x\rangle\big)\Big\}\,e^{\langle\lambda-\rho,x\rangle}\qquad\forall\;\lambda,x\in\overline{\mathfrak{a}^+}\,,$$
which was stated in [80] (see also [78]), and the following generalization of a celebrated result of Helgason & Johnson in the symmetric space case:

  F_λ is bounded if and only if λ belongs to the convex hull of $W\rho$.
- The sharp estimates in Proposition 4.8 were obtained in [80] (see also [78]) and [71].

- As in the rational case, we have not discussed the shift operators, which move the multiplicity k by integers and which have proven useful in the W–invariant setting (see [44, Part I, Ch. 3], [64, Section 5]).
- Rational limits in Subsection 4.4 have a long prehistory. In the Dunkl setting, they have been used for instance in [72], [16], [48], [4], [2], [77], ... (seemingly first and independently in preprint versions of [16] and [48]).
- There are other interesting limits between special functions occurring in Dunkl theory. For instance, in [71] and [74], Heckman–Opdam hypergeometric functions associated with the root system A_{n-1} are obtained as limits of Heckman–Opdam hypergeometric functions associated with the root system BC_n, when some multiplicities tend to infinity. See [73] for a similar result about generalized Bessel functions.
- The expressions (49) are substitutes for the integral representations (10) and (19). A different integral representation of F_λ is established in [83].
- Formula (50) was obtained in [8] and used there to prove the positivity of μ_x when $k_1>0$ and $k_2>0$. A more complicated expression was obtained previously in [38] and in [15]. It was used in [38] to disprove mistakenly the positivity of μ_x. Another approach, which consists in proving the positivity of a heat type kernel, was followed in [87], [88], [89], [90]. But, as pointed out in [8, Remark 3.4], the same flaw occurs in [88], [89], [90].
- It is natural to look for recurrence formulae over n for the five families of classical crystallographic root systems A_n, B_n, C_n, BC_n, D_n (see the appendix). In the case of A_n, an integral recurrence formula for F_λ (or for Jack polynomials) was discovered independently by several authors (see for instance [76], [60], [42]). An explicit expression of μ_x^W is deduced in [76] and [77], first for $x\in\mathfrak{a}^+$ and next for any $x\in\overline{\mathfrak{a}^+}$. In particular, if $k>0$, then μ_x^W is a probability measure, whose support is equal to the convex hull of Wx and which is absolutely continuous with respect to the Lebesgue measure, except for $x=0$ where $\mu_x^W=\delta_0$.
- As in the rational case (see the eighth item in Subsection 3.7), an explicit product formula was obtained in [70] and [92] for Heckman–Opdam hypergeometric functions associated with root systems of type BC and for certain continuous families of multiplicities.
- Probabilistic aspects of trigonometric Dunkl theory were studied in [80] and [79] (see also [78]). Regarding the heat kernel $h_t(x,y)$, the estimate (20) was shown to hold for $h_t(x,0)$ and some asymptotics were obtained for $h_t(x,y)$. But there is no trigonometric counterpart of the expression (29), neither precise information like (30) about the full behavior of $h_t(x,y)$.
- The bounded harmonic functions for the Heckman–Opdam Laplacian were determined in [81].

Appendix A. Root systems

In this appendix, we collect some information about root systems and reflection groups. More details can be found in classical textbooks such as [46] or [50].

Definition A.1. *Let $\mathfrak{a}\approx\mathbb{R}^n$ be a Euclidean space.*

- *A (crystallographic) root system in $\mathfrak{a}$ is a finite set R of nonzero vectors satisfying the following conditions:*
  (a) *for every $\alpha\in R$, the reflection $r_\alpha(x)=x-2\,\frac{\langle\alpha,x\rangle}{|\alpha|^2}\,\alpha$ maps R onto itself,*
  (b) *$2\,\frac{\langle\alpha,\beta\rangle}{|\alpha|^2}\in\mathbb{Z}$ for all $\alpha,\beta\in R$.*

- *A root system R is reducible if it can be splitted into two orthogonal root systems, and irreducible otherwise.*

Remark A.2.

- *Unless specified, we shall assume that R spans $\mathfrak{a}$.*
- *$\alpha^\vee = \frac{2}{|\alpha|^2}\,\alpha$ denotes the coroot corresponding to a root α. If R is a root system, then $R^\vee$ is again a root system.*
- *Most root systems are reduced, which means that*
  (c) *the roots proportional to any root α are reduced to $\pm\alpha$.*
  Otherwise the only possible alignment of roots is
  $$-2\alpha, -\alpha, +\alpha, +2\alpha\,.$$
  A root α is called
  ○ *indivisible if $\frac{\alpha}{2}$ is not a root,*
  ○ *non–multipliable if 2α is not a root.*
- *We shall also consider non–crytallographic reduced root systems R, which satisfy* (a) *and* (c)*, but not necessarily* (b).

Definition A.3.

- *The connected components of*
  $$\{\, x\in\mathfrak{a} \mid \langle\alpha,x\rangle\neq 0 \;\;\forall\;\alpha\in R\,\}$$
  are called Weyl chambers. We choose any of them, which is called positive and denoted by $\mathfrak{a}^+$. R^+ denotes the set of roots which are positive on $\mathfrak{a}^+$.
- *The Weyl or Coxeter group W associated with R is the finite subgroup of the orthogonal group $\mathrm{O}(\mathfrak{a})$ generated by the root reflections $\{\, r_\alpha \mid \alpha\in R\,\}$.*

Remark A.4.

- *The group W acts simply transitively on the set of Weyl chambers.*
- *The longest element w_0 in W interchanges $\mathfrak{a}^+$ and $-\mathfrak{a}^+$.*
- *Every $x\in\mathfrak{a}$ belongs to the W–orbit of a single $x^+\in\overline{\mathfrak{a}^+}$.*

There are six classical families of irreducible root systems:

- A_n $(n\geqslant 1)$: $\mathfrak{a}=\{\, x\in\mathbb{R}^{n+1} \mid x_0+x_1+\ldots+x_n=0\,\}$
  $R=\{\, e_i-e_j \mid 0\leqslant i\neq j\leqslant n\,\}$
  $\mathfrak{a}^+=\{\, x\in\mathfrak{a} \mid x_0>x_1>\ldots>x_n\,\}$
  $W=\mathrm{S}_{n+1}$
- B_n $(n\geqslant 2)$: $\mathfrak{a}=\mathbb{R}^n$
  $R=\{\,\pm e_i \mid 1\leqslant i\leqslant n\,\}\cup\{\,\pm e_i\pm e_j \mid 1\leqslant i<j\leqslant n\,\}$
  $\mathfrak{a}^+=\{\, x\in\mathbb{R}^n \mid x_1>\ldots>x_n>0\,\}$
  $W=\{\pm1\}^n\rtimes\mathrm{S}_n$
- C_n $(n\geqslant 2)$: $\mathfrak{a}=\mathbb{R}^n$
  $R=\{\,\pm 2\,e_i \mid 1\leqslant i\leqslant n\,\}\cup\{\,\pm e_i\pm e_j \mid 1\leqslant i<j\leqslant n\,\}$
  $\mathfrak{a}^+=\{\, x\in\mathbb{R}^n \mid x_1>\ldots>x_n>0\,\}$
  $W=\{\pm1\}^n\rtimes\mathrm{S}_n$
- BC_n $(n\geqslant 1)$: $\mathfrak{a}=\mathbb{R}^n$
  $R=\{\,\pm e_i,\pm 2\,e_i \mid 1\leqslant i\leqslant n\,\}\cup\{\,\pm e_i\pm e_j \mid 1\leqslant i<j\leqslant n\,\}$
  $\mathfrak{a}^+=\{\, x\in\mathbb{R}^n \mid x_1>\ldots>x_n>0\,\}$
  $W=\{\pm1\}^n\rtimes\mathrm{S}_n$
- D_n $(n\geqslant 3)$: $\mathfrak{a}=\mathbb{R}^n$
  $R=\{\,\pm e_i\pm e_j \mid 1\leqslant i<j\leqslant n\,\}$
  $\mathfrak{a}^+=\{\, x\in\mathbb{R}^n \mid x_1>\ldots>|x_n|\,\}$
  $W=\{\,\varepsilon\in\{\pm1\}^n \mid \varepsilon_1\ldots\varepsilon_n=1\,\}\rtimes\mathrm{S}_n$

- $\mathrm{I}_2(m)$ $(m\geqslant 3)$: $\mathfrak{a}=\mathbb{C}$
$$R=\{\,e^{i\pi\frac{j}{m}}\,|\,0\leqslant j<2m\,\}$$
$$\mathfrak{a}^+=\{\,z\in\mathbb{C}^*\,|\,\big(\tfrac{1}{2}-\tfrac{1}{m}\big)\,\pi<\arg z<\tfrac{\pi}{2}\,\}$$
$$W=(\mathbb{Z}/m\mathbb{Z})\rtimes(\mathbb{Z}/2\mathbb{Z})\ \text{(dihedral group)}$$

The full list of irreducible root systems (crystallographic or reduced) includes in addition a finite number of exceptional cases:
$$\mathrm{E}_6,\,\mathrm{E}_7,\,\mathrm{E}_8,\,\mathrm{F}_4,\,\mathrm{G}_2,\,\mathrm{H}_3,\,\mathrm{H}_4.$$

Remark A.5. *In the list above,*
- *the non crystallographic root systems are*
$$\mathrm{H}_3,\,\mathrm{H}_4\ \textit{and}\ \mathrm{I}_2(m)\ \textit{with}\ \begin{cases} m=5,\\ m\geqslant 7,\end{cases}$$
- *all root systems are reduced, with the exception of* BC_n,
- *there are some redundancies in low dimension*:
$$\begin{cases} \mathrm{A}_1\times\mathrm{A}_1\approx\mathrm{D}_2\approx\mathrm{I}_2(2) & \\ \mathrm{B}_2\approx\mathrm{C}_2\approx\mathrm{I}_2(4) & (\textit{up to the root length})\\ \mathrm{A}_2\approx\mathrm{I}_2(3) & \\ \mathrm{G}_2\approx\mathrm{I}_2(6) & (\textit{up to the root length})\end{cases}$$

The 2–dimensional root systems (crystallographic or reduced) are depicted in Figure 9.

Definition A.6. *A multiplicity is a W–invariant function* $k:R\longrightarrow\mathbb{C}$.

Remark A.7.
- *In Dunkl theory, one assumes most of the time that* $k\geqslant 0$.
- *Assume that R is crystallographic and irreducible. Then two roots belong to the same W–orbit if and only if they have the same length. Thus k takes at most three values. In the non crystallographic case, there are one or two W–orbits in R. Specifically, by resuming the classification of root systems, k takes*
  - 1 *value in the following cases*:
  $$\mathrm{A}_n,\,\mathrm{D}_n,\,\mathrm{E}_6,\,\mathrm{E}_7,\,\mathrm{E}_8,\,\mathrm{H}_3,\,\mathrm{H}_4,\,\mathrm{I}_2(m)\ \textit{with } m \textit{ odd},$$
  - 2 *values in the following cases*:
  $$\mathrm{B}_n,\,\mathrm{C}_n,\,\mathrm{F}_4,\,\mathrm{G}_2,\,\mathrm{I}_2(m)\ \textit{with } m \textit{ even},$$
  - 3 *values in the case of* BC_n.

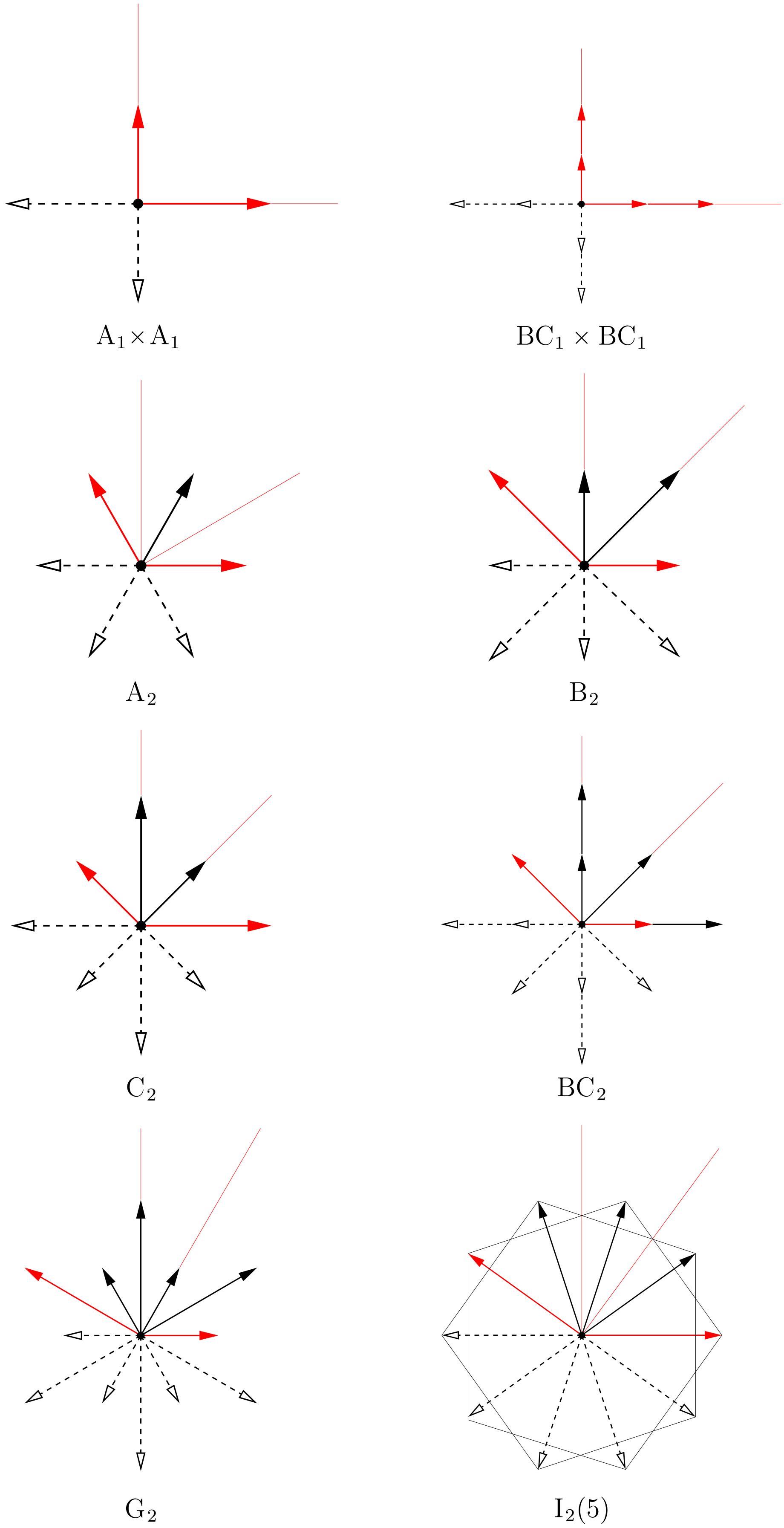

FIGURE 9. 2–dimensional root systems

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