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Evolution equations on symmetric spaces

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Survey of collaborations with

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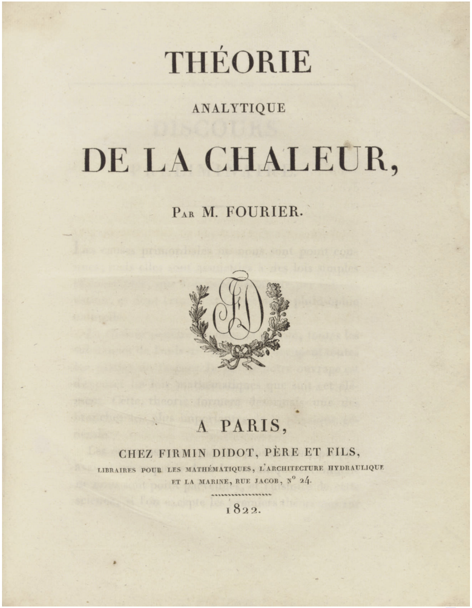
Joseph Fourier (1768 – 1830)



THÉORIE

ANALYTIQUE

DE LA CHALEUR,

PAR M. FOURIER.

A PARIS,

CHEZ FIRMIN DIDOT, PÈRE ET FILS,

LIBRAIRES POUR LES MATHÉMATIQUES, L'ARCHITECTURE HYDRAULIQUE ET LA MARINE, RUE JACOB, N° 24.

1822.

Equations

Heat

$$\begin{cases} \partial_t u(t,x) - \Delta_x u(t,x) = F(t,x) \\ u(0,x) = f(x) \end{cases} \tag{1}$$

Wave

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) = F(t,x) \\ u(0,x) = f_0(x),\ \partial_t|_{t=0}\, u(t,x) = f_1(x) \end{cases} \tag{2}$$

Schrödinger

$$\begin{cases} i\,\partial_t u(t,x) + \Delta_x u(t,x) = F(t,x) \\ u(0,x) = f(x) \end{cases} \tag{3}$$

Equations (continued)

Heat

$$\begin{cases} \partial_t u(t,x) - \Delta_x u(t,x) = F(t,x) \\ u(0,x) = f(x) \end{cases} \qquad (1)$$

Half–wave

$$\begin{cases} i\,\partial_t u(t,x) + (-\Delta_x)^{1/2} u(t,x) = F(t,x) \\ u(0,x) = f(x) \end{cases} \qquad (4)$$

Schrödinger for the fractional Laplacian $(0 < \alpha \leq 2)$

$$\begin{cases} i\,\partial_t u(t,x) + (-\Delta_x)^{\alpha/2} u(t,x) = F(t,x) \\ u(0,x) = f(x) \end{cases} \qquad (5)$$

Spaces

- $X = G/K$ Riemannian symmetric space of noncompact type
  - G noncompact semisimple Lie group (connected, finite center)
  - K maximal compact subgroup
- Cartan decomposition:

$$G = K(\exp \overline{\mathfrak{a}^+})K \quad \rightsquigarrow \quad X = K(\exp \overline{\mathfrak{a}^+})\not{K}/\not{K}$$

- The rank of G/K is the dimension ℓ of $\mathfrak{a}$
- Classification in rank $\ell = 1$

X	$\mathsf{H}^n(\mathbb{R})$	$\mathsf{H}^n(\mathbb{C})$	$\mathsf{H}^n(\mathbb{H})$	$\mathsf{H}^2(\mathbb{O})$
G	$\mathsf{SO}_0(n,1)$	$\mathsf{SU}(n,1)$	$\mathsf{Sp}(n,1)$	$\mathsf{F}_{4(-20)}$
K	$\mathsf{SO}(n)$	$\mathsf{S}[\mathsf{U}(n)\times\mathsf{U}(1)]$	$\mathsf{Sp}(n)$	$\mathsf{SO}(9)$
d	n	$2n$	$4n$	16
ρ	$\frac{n-1}{2}$	n	$2n+1$	11

Plan

1. Heat kernel
   - rank 1
   - higher rank
2. Schrödinger equation
   - rank 1
   - (higher rank)
3. Schrödinger equation for the fractional Laplacian in rank 1
4. Wave equations in rank 1
5. Homogeneous trees

Heat kernel on G/K

- Heat semigroup:
$$e^{t\Delta_{xK}} f(xK) = f \underset{G}{*} h_t(x)$$
- Inverse spherical Fourier transform:
$$h_t(KxK) = \text{const.} \int_{\mathfrak{a}} e^{-t(|\lambda|^2+|\rho|^2)} \varphi_\lambda(x)\, |\mathbf{c}(\lambda)|^{-2} \, d\lambda$$

Estimate in rank 1 [Davies–Mandouvalos]

$$h_t(r) \asymp t^{-\frac{d}{2}} (1+t+r)^{\frac{d-3}{2}} e^{-\rho^2 t - \rho r - \frac{r^2}{4t}}$$

for every $t>0$ and $r\geq 0$.

Estimate in higher rank [A–Ji, A–Ostellari]

$$h_t(\exp x) \asymp t^{-\frac{d}{2}} \Big\{ \prod\nolimits_{\alpha\in R^+_{\text{red}}} (1+t+|\langle\alpha,x\rangle|)^{\frac{m_\alpha+m_{2\alpha}}{2}-1} \Big\} \times$$
$$\times\ e^{-|\rho|^2 t - \langle\rho,x\rangle - \frac{|x|^2}{4t}}$$

for every $t>0$ and $x\in\overline{\mathfrak{a}^+}$.

Applications

- Martin compactification of symmetric spaces $X = G/K$
- Spectral gap λ_0 of locally symmetric spaces $Y = \Gamma \backslash G/K$
  - Rank 1 [Elstrodt, Patterson, Sullivan, Corlette]

$$\lambda_0 = \begin{cases} \rho^2 & \text{if } 0 \le \delta \le \rho \\ \rho(2\rho - \delta) & \text{if } \rho \le \delta \le 2\rho \end{cases}$$

  - Higher rank [Leuzinger, Weber]

$$\begin{cases} \lambda_0 = |\rho|^2 & \text{if } 0 \le \delta \le \tilde{\rho} \\ |\rho|^2 - (\delta - \tilde{\rho})^2 \le \lambda_0 \le |\rho|^2 & \text{if } \tilde{\rho} \le \delta \le \rho \\ |\rho|^2 - (\delta - \tilde{\rho})^2 \le \lambda_0 \le |\rho|^2 - (\delta - |\rho|)^2 & \text{if } \rho \le \delta \le 2\rho \end{cases}$$

Here δ denotes the critical exponent of convergence for the Poincaré series

$$P_z(xK, yK) = \sum\nolimits_{\gamma \in \Gamma} e^{-z\, d(xK, \gamma y K)}$$

and $\tilde{\rho} = \min_{x \in \overline{\mathfrak{a}^+},\, |x|=1} \langle \rho, x \rangle \le |\rho|$

Schrödinger equation on $X = G/K$

- Schrödinger equation:

$$\begin{cases} i\,\partial_t u(t,x) + \Delta_x u(t,x) = F(t,x) \\ u(0,x) = f(x) \end{cases} \tag{3}$$

- Formal solution (Duhamel's formula):

$$u(t,x) = e^{it\Delta_x} f(x) - i\int_0^t e^{i(t-s)\Delta_x} F(s,x)\,ds$$

- Schrödinger propagator:

$$e^{it\Delta_{xK}} f(xK) = f \underset{G}{*} s_t(x)$$

- Inverse spherical Fourier transform:

$$s_t(KxK) = \text{const.} \int_{\mathfrak{a}} e^{-it(|\lambda|^2+|\rho|^2)}\,\varphi_\lambda(x)\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda$$

Schrödinger equation in rank 1

Kernel estimate [A–Pierfelice, Ionescu–Staffilani]

$$|s_t(r)| \lesssim \begin{cases} |t|^{-\frac{d}{2}} \overbrace{(1+r)^{\frac{d-1}{2}} e^{-\rho r}}^{\asymp\, j(r)^{-\frac12}} & \text{if } |t| \le 1+r \\ |t|^{-\frac{3}{2}} \underbrace{(1+r)\, e^{-\rho r}}_{\asymp\, \varphi_0(r)} & \text{if } |t| \ge 1+r \end{cases}$$

Schrödinger equation in rank 1 (continued)

Dispersive estimate

For every $2<q,\tilde{q}\leq\infty$ and $t\in\mathbb{R}^*$,

$$\|e^{it\Delta}\|_{L^{\tilde{q}'}\to L^q} \lesssim \begin{cases} |t|^{-d\max\{\frac{1}{2}-\frac{1}{q},\frac{1}{2}-\frac{1}{\tilde{q}}\}} & \text{if } |t| \text{ is small} \\ |t|^{-\frac{3}{2}} & \text{if } |t| \text{ is large} \end{cases}$$

Main tool for $|t|$ large $=$ following version of the Kunze–Stein phenomenon

Lemma

Assume that g is a radial (measurable) function on X. Then, for every $2\leq q<\infty$,

$$\|f*g\|_{L^q(X)} \leq \|f\|_{L^{q'}(X)} \left\{\int_X dx\,\varphi_0(x)\,|g(x)|^{\frac{q}{2}}\right\}^{\frac{2}{q}}$$

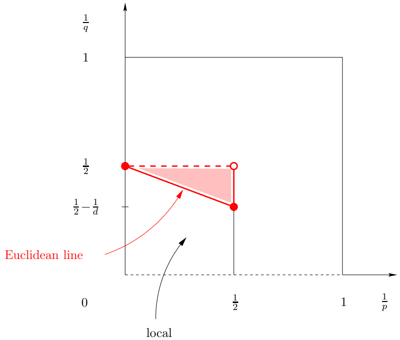
$$\Longrightarrow \quad L^{q'}(G/K)*L^{\tilde{q}}(K\backslash G/K)\subset L^q(G/K) \quad \forall\ \tfrac{q}{2}\leq\tilde{q}<q$$

Schrödinger equation in rank 1 (continued)

Strichartz inequality

$$\|u(t,x)\|_{L^p_{t\in I}L^q_x} \lesssim \|f\|_{L^2} + \|F(t,x)\|_{L^{\tilde{p}'}_{t\in I}L^{\tilde{q}'}_x}$$

for solutions $u(t,x)$ of (3) on $I \times X$ (I interval $\ni 0$).
Here $(\frac{1}{p}, \frac{1}{q})$ and $(\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}})$ are any couple in the following triangle



Application: nonlinear Schrödinger equation in rank 1

Consider power–like nonlinearities $F(t,x) = \widetilde{F}(u(t,x))$

i.e. $\begin{cases} |\widetilde{F}(u)| \lesssim |u|^\gamma \\ |\widetilde{F}(u) - \widetilde{F}(v)| \lesssim \{|u|^{\gamma-1} + |v|^{\gamma-1}\}\,|u-v| \end{cases}$ for some $\gamma > 1$

Theorem

- $\gamma \le 1 + \frac{4}{d}$: **global** well–posedness for **small** L^2 data.
- $\gamma < 1 + \frac{4}{d}$: **local** well–posedness for **arbitrary** L^2 data. Moreover, **global** well–posedness for **arbitrary** L^2 data if $\widetilde{F}(u) = c\,u\,|u|^{\gamma-1}$ with $c \in \mathbb{R}$.

Remarks :

- On $\mathbb{R}^d$, first result holds only in the critical case $\gamma = 1 + \frac{4}{d}$
- Additional smoothness $\rightsquigarrow$ larger powers γ

$$\begin{array}{ccl} L^2(X) & \rightsquigarrow & H^\sigma(X) \text{ Sobolev space} \\ 1+\frac{4}{d} & \rightsquigarrow & 1+\frac{4}{d-2\sigma} \end{array} \qquad (0 < \sigma < \tfrac{d}{2})$$

Schrödinger kernel on general $X = G/K$

Conjectural kernel estimate

For every $t \in \mathbb{R}^*$ and $x \in G$,

$$|s_t(x)| \lesssim |t|^{-\frac{d}{2}}\, e^{-\langle\rho, x^+\rangle} \times$$
$$\times \prod_{\alpha\in R^+_{\text{red}}} (1+\langle\alpha, x^+\rangle)\,(1+|t|+\langle\alpha, x^+\rangle)^{\frac{m_\alpha+m_{2\alpha}}{2}-1}$$

where $x^+ \in \overline{\mathfrak{a}^+}$ denotes the radial component of x in the Cartan decomposition $G = K(\exp\overline{\mathfrak{a}^+})K$

OK in rank $\ell = 1$ or if G is complex

Weaker result [A–Meda–Pierfelice–Vallarino]

$$|s_t(x)| \lesssim \begin{cases} |t|^{-(d-\frac{\ell}{2})}\,(1+|x^+|)^N\, e^{-\langle\rho,x^+\rangle} & \text{if } |t| \text{ is small} \\ |t|^{-\frac{D}{2}}\,(1+|x^+|)^N\, e^{-\langle\rho,x^+\rangle} & \text{if } |t| \text{ is large} \end{cases}$$

where $D = \ell + 2|R^+_{\text{red}}|$ is the *dimension at infinity*

Schrödinger equation for the fractional Laplacian on G/K

Assume that $0<\alpha<1$ or $1<\alpha<2$

- Schrödinger equation for the fractional Laplacian:

$$\begin{cases} i\,\partial_t u(t,x) + (-\Delta_x)^{\alpha/2} u(t,x) = F(t,x) \\ u(0,x) = f(x) \end{cases} \tag{5}$$

- Duhamel's formula:

$$u(t,x) = e^{it(-\Delta_x)^{\alpha/2}} f(x) - i \int_0^t e^{i(t-s)(-\Delta_x)^{\alpha/2}} F(s,x)\, ds$$

Need additional smoothness $\sigma \geq 0$

- Operator:

$$(-\Delta_{xK})^{-\sigma/2}\, e^{it(-\Delta_{xK})^{\alpha/2}} f(xK) = f \underset{G}{*} k_t^\sigma(x)$$

- Inverse spherical Fourier transform:

$$k_t^\sigma(KxK) = \text{const.} \int_{\mathfrak{a}} (|\lambda|^2 + |\rho|^2)^{-\sigma/2}\, e^{it(|\lambda|^2+|\rho|^2)^{\alpha/2}} \times \varphi_\lambda(x)\, \frac{d\lambda}{|\mathbf{c}(\lambda)|^2}\, d\lambda$$

Schrödinger fractional in rank 1

Kernel estimate [A–Sire]

Assume that $\sigma = (2-\alpha)\frac{d}{2}$. Then, for every $t \in \mathbb{R}^*$ and $r \geq 0$,

$$|k_t^\sigma(r)| \lesssim \begin{cases} |t|^{-\frac{n}{2}} (1+r)^{\frac{d-1}{2}} e^{-\rho r} & \text{if } |t| \leq 1+r \\ |t|^{-\frac{3}{2}} (1+r)\, e^{-\rho r} & \text{if } |t| \geq 1+r \end{cases}$$

Dispersive estimate [A–Sire]

Let $2 < q, \tilde{q} \leq \infty$ and $\sigma = (2-\alpha)\, d \max\left\{\frac{1}{2}-\frac{1}{q}, \frac{1}{2}-\frac{1}{\tilde{q}}\right\}$.
Then, for every $t \in \mathbb{R}^*$,

$$\left\| (-\Delta)^{-\sigma/2} e^{i t (-\Delta)^{\alpha/2}} \right\|_{L^{\tilde{q}'} \to L^q} \lesssim \begin{cases} |t|^{-d \max\{\frac{1}{2}-\frac{1}{q}, \frac{1}{2}-\frac{1}{\tilde{q}}\}} & \text{if } t \text{ is small} \\ |t|^{-\frac{3}{2}} & \text{if } t \text{ is large} \end{cases}$$

Schrödinger fractional in rank 1 (continued)

Strichartz inequality [A–Sire]

$$\|(-\Delta)_x^{-\sigma/4} u(t,x)\|_{L^p_{t\in I} L^q_x} \lesssim \|f\|_{L^2} + \|(-\Delta)_x^{\tilde{\sigma}/4} F(t,x)\|_{L^{\tilde{p}'}_{t\in I} L^{\tilde{q}'}_x}$$

for solutions $u(t,x)$ of (3) on $I\times X$, I being any interval $\ni 0$.
Here $(\frac{1}{p},\frac{1}{q})$, $(\frac{1}{\tilde{p}},\frac{1}{\tilde{q}})$ are couples in the same triangle as before and $\sigma=(2-\alpha)\,d\left(\frac{1}{2}-\frac{1}{q}\right)$, $\tilde{\sigma}=(2-\alpha)\,d\left(\frac{1}{2}-\frac{1}{\tilde{q}}\right)$.

Half–wave equation in rank 1

- Different behavior in the limit case $\alpha = 1$
- Restrict to $d \geq 3$
- Analytic family of operators

$$W_t^\sigma = \frac{e^{\sigma^2}}{\Gamma(\frac{n+1}{2} - \sigma)} (I - \Delta)^{-\sigma/2} e^{it(-\Delta)^{\alpha/2}}$$

in the strip $0 \leq \operatorname{Re} \sigma \leq \frac{d+1}{2}$

Kernel estimate [A–Pierfelice]

Assume that $\operatorname{Re} \sigma = \frac{d+1}{2}$. Then, for every $t \in \mathbb{R}^*$ and $r \geq 0$,

$$|w_t^\sigma(r)| \leq C \begin{cases} |t|^{-\frac{d-1}{2}} & \text{if } 0 < |t| \leq 1 \text{ and } 0 \leq r \leq 1 \\ r\, e^{-\rho r} & \text{if } r \geq \max\{1, |t|\} \\ |t|^{-\frac{3}{2}} (1+r)^2 e^{-\rho r} & \text{if } |t| \geq \max\{1, r\} \end{cases}$$

where the constant $C > 0$ doesn't depend on σ, t, r

Half–wave equation in rank 1 (continued)

Dispersive estimate [A–Pierfelice]

Let $2 < q, \tilde{q} < \infty$ and $\sigma = (d+1)\max\{\frac{1}{2}-\frac{1}{q}, \frac{1}{2}-\frac{1}{\tilde{q}}\}$.
Then, for every $t \in \mathbb{R}^*$,

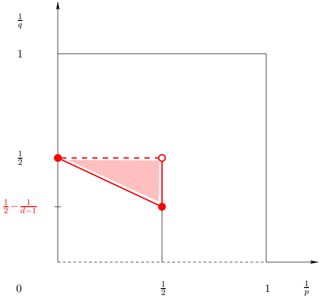
$$\left\| (-\Delta)^{-\sigma/2} e^{it(-\Delta)^{\alpha/2}} \right\|_{L^{\tilde{q}'} \to L^q} \lesssim \begin{cases} |t|^{-(d-1)\max\{\frac{1}{2}-\frac{1}{q}, \frac{1}{2}-\frac{1}{\tilde{q}}\}} & \text{if } t \text{ is small} \\ |t|^{-\frac{3}{2}} & \text{if } t \text{ is large} \end{cases}$$

Half–wave equation in rank 1 (continued)

Strichartz inequality [A–Pierfelice]

$$\|(-\Delta)_x^{-\sigma/4} u(t,x)\|_{L^p_{t\in I} L^q_x} \lesssim \|f\|_{L^2} + \|(-\Delta)_x^{\tilde{\sigma}/4} F(t,x)\|_{L^{\tilde{p}'}_{t\in I} L^{\tilde{q}'}_x}$$

for solutions $u(t,x)$ of (3) on $I \times X$, I being any interval $\ni 0$.
Here $(\frac{1}{p}, \frac{1}{q})$, $(\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}})$ are couples in the following triangle and $\sigma = \frac{d+1}{2}\big(\frac{1}{2} - \frac{1}{q}\big)$, $\tilde{\sigma} = \frac{d+1}{2}\big(\frac{1}{2} - \frac{1}{\tilde{q}}\big)$.

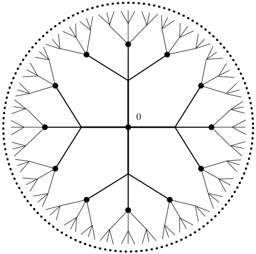


Half–wave equation in rank 1 (continued)

- Similar results for the wave equation
  [Metcalfe–Taylor in 3D, A–Pierfelice in rank 1]
- Local/global wellposedness of the nonlinear wave equation
- No blow–up for small powers γ
  contrarily to the Euclidean setting [John, Strauss, ...]

Homogeneous trees

- $\mathbb{T} = \mathbb{T}_Q$ homogeneous tree with $Q+1 \geq 3$ edges
- Example : $Q = 3$



- Discrete rank 1 symmetric space
- Combinatorial Laplacian on (the vertices of) $\mathbb{T}$:

$$\Delta f(x) = \tfrac{1}{Q+1} \sum_{d(y,x)=1} f(y) - f(x)$$

- Heat kernel
  - Discrete time $\sim$ simple random walk
  - Continuous time [Setti]

Schrödinger equation for the fractional Laplacian on $\mathbb{T}$

Fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ with $0<\alpha\le 2$

Schrödinger equation with continuous time on $\mathbb{T}$

$$\begin{cases} i\,\partial_t u(t,x) + (-\Delta_x)^{\frac{\alpha}{2}} u(t,x) = F(t,x) \\ u(0,x)=f(x) \end{cases} \tag{6}$$

- $\alpha=2$: standard Schrödinger equation [Jamal Eddine]
- $\alpha=1$: half–wave equation [Medolla–Setti]

Schrödinger propagator :

$$e^{it(-\Delta)^{\alpha/2}} f(x) = \underbrace{\sum\nolimits_{y\in\mathbb{T}} f(y)\, k_t^{\alpha}(d(x,y))}_{f * k_t^{\alpha}(x)}$$

Schrödinger fractional on $\mathbb{T}$ (continued)

Inverse spherical Fourier transform

$$k_t^\alpha(r) = \text{const.} \int_0^{\frac{\pi}{\log Q}} d\lambda\, |\mathbf{c}(\lambda)|^{-2}\, \varphi_\lambda(x)\, e^{\,it\,[1-\gamma(\lambda)]^{\alpha/2}}$$

where
$$\gamma(\lambda) = \frac{Q^{i\lambda}+Q^{-i\lambda}}{Q^{1/2}+Q^{-1/2}}$$
$$\mathbf{c}(\lambda) = \frac{1}{Q^{1/2}+Q^{-1/2}}\, \frac{Q^{1/2+i\lambda}-Q^{-1/2-i\lambda}}{Q^{i\lambda}-Q^{-i\lambda}}$$
$$\varphi_\lambda(r) = \mathbf{c}(\lambda)\, Q^{(-1/2+i\lambda)\,r} + \mathbf{c}(-\lambda)\, Q^{(-1/2-i\lambda)\,r}$$

Kernel estimate [A–Sire]

$$|\,k_t^\alpha(r)| \lesssim Q^{-\frac{r}{2}} \qquad \forall\, t \in \mathbb{R}^*,\ \forall\, r \in \mathbb{N}.$$

Moreover there exists a constant $C>0$ such that

$$|\,k_t^\alpha(r)| \lesssim |t|^{-\frac{3}{2}}\,(1+r)\,Q^{-\frac{r}{2}}$$

if $1+r \leq C\,|t|$.

Schrödinger fractional on $\mathbb{T}$ (continued)

Dispersive estimate

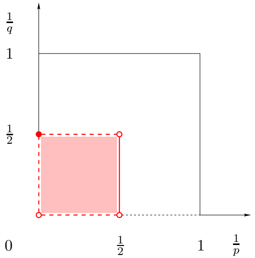
Let $0 < \alpha \leq 2$ and $2 < q, \tilde{q} \leq \infty$. Then

$$\| e^{it(-\Delta)^{\alpha/2}} \|_{\ell^{\tilde{q}'} \to \ell^q} \lesssim (1+|t|)^{-\frac{3}{2}} \qquad \forall\, t \in \mathbb{R}^*$$

Strichartz inequality

$$\|u(t,x)\|_{L^p_t \ell^q_x} \lesssim \|f\|_{\ell^2} + \|F(t,x)\|_{L^{\tilde{p}'}_t \ell^{\tilde{q}'}_x}$$

for all admissible pairs $(\frac{1}{p}, \frac{1}{q})$ and $(\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}})$ in the following square



Thank you for your attention !