HARMONIC ANALYSIS APPLIED TO PDE : DISPERSIVE INEQUALITIES AND STRICHARTZ ESTIMATES

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1. Restriction theorems

The Fourier transform

$$\widehat{f}(\xi)=\int_{\mathbb{R}^n} dx\, f(x)\, e^{-i\langle x,\xi\rangle}$$

of $f\in L^1(\mathbb{R}^n)$ is a continuous function, which may be restricted to submanifolds S such as spheres. On the other hand, the Fourier transform of $f\in L^2(\mathbb{R}^n)$ is a genuine L^2 function and it makes no sense to restrict it to a sphere or any other subset of measure zero. Stein's restriction problem is concerned with a priori inequalities

$$\|\,\widehat{f}\,\big|_S\|_{L^q(S)}\lesssim \|f\|_{L^{p'}(\mathbb{R}^n)}\,.$$

Such results hold for $q=2$ and p' close to 1 under curvature conditions on S. More information can be found for instance in Stein's book [13].

Theorem 1.1 (Stein–Tomas [16])**.** *Assume that* $1\le p'\le 2\frac{n+1}{n+3}$ *i.e.* $2\frac{n+1}{n-1}\le p\le\infty$. *Then*

$$\|\,\widehat{f}\,\big|_{\mathbb{S}^{n-1}}\|_{L^2(\mathbb{S}^{n-1})}\lesssim \|f\|_{L^{p'}(\mathbb{R}^n)}\qquad \forall\, f\in\mathcal{S}(\mathbb{R}^n).$$

We give three successive proofs (actually four with Remark 1.2), each one improving upon the previous one.

The **first proof** relies on the T^*T method. Consider the operator

$$Tf=\widehat{f}\,\big|_{\mathbb{S}^{n-1}}\,,$$

its formal adjoint

$$T^*g(x)=\int_{\mathbb{S}^{n-1}} d\sigma(\xi)\, g(\xi)\, e^{i\langle x,\xi\rangle}\,,$$

where σ denotes the surface measure on $\mathbb{S}^{n-1}$ induced by the Lebesgue measure on $\mathbb{R}^n$, and their composition

$$\mathcal{T}f(x)=T^*Tf(x)=f*\widehat{\sigma}.$$

Then the following a priori estimates are equivalent :

$$\|Tf\|_{L^2}\lesssim\|f\|_{L^{p'}}\iff \|T^*g\|_{L^p}\lesssim\|g\|_{L^2}\iff \|\mathcal{T}f\|_{L^p}\lesssim\|f\|_{L^{p'}}.$$

In order to prove the third one, we use the estimate

$$|\,\widehat{\sigma}(x)\,|\lesssim\big(1+\|x\|\big)^{-\frac{n-1}{2}}.$$

Such decay at infinity can be obtained by general oscillatory integral methods. In our particular case, it can be also obtained by expressing the Fourier transform

$$\widehat{\sigma}(x)=(2\pi)^{\frac{n}{2}}\,\|x\|^{-\frac{n}{2}+1}J_{\frac{n}{2}-1}(\|x\|)$$

in terms of Bessel functions and by using their behavior at infinity. Hence $\widehat{\sigma}$ belongs to $L^q(\mathbb{R}^n)$ if $q>\frac{2n}{n-1}$ and to the Lorentz space $L^{q,\infty}(\mathbb{R}^n)$ in the limit case $q=\frac{2n}{n-1}$. By using Young's inequality, we deduce the $L^{p'}\!\to L^p$ boundedness of $\mathcal{T}$ when $p\ge 2q=\frac{4n}{n-1}$, which is larger than $2\frac{n+1}{n-1}$. □

The **second proof** relies on a dyadic decomposition. Let $\chi:\mathbb{R}\to[0,1]$ be a smooth cut–off function such that

$$\chi=\begin{cases}1 & \text{on } (-\infty,1]\\ 0 & \text{on } [2,+\infty)\end{cases}$$

and set, for every $j\in\mathbb{N}$ and $x\in\mathbb{R}^n$,

$$\chi_j(x)=\begin{cases}\chi(\|x\|) & \text{if } j=0,\\ \chi(2^{-j}\|x\|)-\chi(2^{1-j}\|x\|) & \text{if } j\in\mathbb{N}^*.\end{cases}$$

Then χ_j is supported in

$$\begin{cases}\text{the ball } \overline{B}(0,2) & \text{if } j=0\\ \text{the dyadic annulus } \Omega_j=\overline{B}(0,2^{j+1})\smallsetminus B(0,2^{j-1}) & \text{if } j\in\mathbb{N}^*\end{cases}$$

and

$$\sum\nolimits_{j\in\mathbb{N}}\chi_j\equiv 1\,.$$

Let us split up $\widehat{\sigma}=\sum_{j\in\mathbb{N}}\overbrace{\chi_j\,\widehat{\sigma}}^{k_j}$ and $\mathcal{T}=\sum_{j\in\mathbb{N}}\mathcal{T}_j$ accordingly. On the one hand,

$$\|\mathcal{T}_j\|_{L^1\to L^\infty}=\|k_j\|_{L^\infty}\lesssim 2^{-\frac{n-1}{2}j}\,. \tag{1}$$

On the other hand,

$$\|\mathcal{T}_j\|_{L^2\to L^2}=\|\widehat{k_j}\|_{L^\infty}\,.$$

We claim that

$$\|\widehat{k_j}\|_{L^\infty}\lesssim 2^j.$$

This estimate is obvious for $\widehat{k_0}=\widehat{\chi_0}*\sigma$. Let us prove it for $\widehat{k_j}=\widehat{\chi_j}*\sigma$ with $j\in\mathbb{N}^*$. As $\widehat{\chi_j}$ is a rescaled Schwartz function, we have

$$|\widehat{\chi_j}(\xi)|\lesssim 2^{nj}(1+2^j\|\xi\|)^{-n}\qquad\forall\,j\in\mathbb{N}^*,\ \forall\,\xi\in\mathbb{R}^n.$$

Hence,

$$\begin{aligned}|\widehat{k_j}(\xi)|&\lesssim\int_{\mathbb{R}^n}d\sigma(\eta)\,2^{nj}(1+2^j\|\xi-\eta\|)^{-n}\\ &\le\int_{B(\xi,2^{-j})}d\sigma(\eta)\,2^{nj}+\sum\nolimits_{k\ge -j}\int_{B(\xi,2^{k+1})\smallsetminus B(\xi,2^k)}d\sigma(\eta)\,2^{-nk}\\ &\le 2^{nj}\sigma(B(\xi,2^{-j}))+\sum\nolimits_{k\ge -j}2^{-nk}\,\sigma(B(\xi,2^k))\\ &\lesssim 2^j+\sum\nolimits_{k\ge -j}2^{-k}\lesssim 2^j.\end{aligned}$$

Here we have used the uniform estimate

$$\sigma(B(\xi,r))\lesssim r^{n-1}\qquad\forall\,\xi\in\mathbb{R}^n,\ \forall\,r>0\,,$$

which matters actually for r small and for $\|\xi\|$ close to 1. Thus

$$\|\mathcal{T}_j\|_{L^2\to L^2}\lesssim 2^j\qquad\forall\,j\in\mathbb{N}. \tag{2}$$

By standard interpolation between (1) and (2), we obtain

$$\|\mathcal{T}_j\|_{L^{p'}\to L^p}\lesssim 2^{-(\frac{n-1}{2}-\frac{n+1}{p})j}\qquad\forall\,j\in\mathbb{N}. \tag{3}$$

By adding up (3) over $j\in\mathbb{N}$, we get the $L^{p'}\to L^p$ boundedness of $\mathcal{T}$ when $p>2\frac{n+1}{n-1}$. □

The **third proof** relies on Stein's interpolation for an analytic family of operators. Let us embed σ into the holomorphic family of tempered distributions

$$\sigma_z=\tfrac{2^{1-z}}{\Gamma(z)}\,(1-\|\xi\|^2)_+^{z-1}.$$

Specifically, σ_z is well–defined in the half–space $\operatorname{Re}z>0$, it extends analytically to $\mathbb{C}$ by means of the functional relation

$$\sigma_z = \Delta\,\sigma_{z+2} + (2z+n)\,\sigma_{z+1}\,,$$

and $\sigma_0 = \sigma$. Moreover,

$$\widehat{\sigma_z}(x) = (2\pi)^{\frac{n}{2}}\,\|x\|^{-z-\frac{n}{2}+1}\,J_{z+\frac{n}{2}-1}(\|x\|).$$

Consider the analytic family of operators $\Sigma_z f = f * \widehat{\sigma_z}$ in the strip $-\frac{n-1}{2} \le \operatorname{Re} z \le 1$. On the one hand, if $\operatorname{Re} z = -\frac{n-1}{2}$, then Σ_z is bounded from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ and

$$\|\Sigma_z\|_{L^1\to L^\infty} = \|\widehat{\sigma_z}\|_{L^\infty}$$

grows at most exponentially in $\operatorname{Im} z$. On the other hand, if $\operatorname{Re} z = 1$, then Σ_z is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and

$$\|\Sigma_z\|_{L^2\to L^2} = (2\pi)^n\,\|\sigma_z\|_{L^\infty}$$

grows at most exponentially in $\operatorname{Im} z$. In conclusion, Stein's interpolation theorem yields the $L^{p'}\!\to L^p$ boundedness of $\Sigma_0 = \mathcal{T}$ for the endpoint $p = 2\,\frac{n+1}{n-1}$. □

Remark 1.2. *As was recently observed in [3], the endpoint result can be also obtained by resuming the second proof above and by using real interpolation. The argument is based on Bourgain's trick (see Lemma 1.3 below) which is closely connected with the method of Keel and Tao (see Step 4 in the proof of Theorem 2.3 below). Let us reprove the endpoint result along these lines. By standard interpolation between* (2) *and*

$$\|\mathcal{T}_j\|_{L^1\to L^{p'}} = \|\mathcal{T}_j\|_{L^p\to L^\infty} = \|k_j\|_{L^{p'}} \lesssim 2^{(\frac{n}{2}-\frac{n}{p}+\frac{1}{2})j}\,, \tag{4}$$

we get

$$\|\mathcal{T}_j\|_{L^{p'}\to L^q} \lesssim 2^{\kappa(p,q)\,j} \qquad \forall\ 2\le p,q\le\infty\,, \tag{5}$$

where $\kappa(p,q) = \frac{n+1}{2}\big(\frac{1}{p}+\frac{1}{q}\big) - \frac{n-1}{2}\big(1-\big|\frac{1}{p}-\frac{1}{q}\big|\big)$. *This result can be improved by real interpolation. Specifically, by moving* (p,q) *and by applying Lemma 1.3 below, we deduce that the operator* $\sum_{j\in\mathbb{N}} 2^{-\kappa(p,q)\,j}\,\mathcal{T}_j$ *maps* $L^{p',r}$ *into* $L^{q,r}$, *for every* $2<p,q<\infty$ *and* $1\le r\le\infty$. *In particular,* $\mathcal{T} = \sum_{j\in\mathbb{N}} \mathcal{T}_j$ *maps* $L^{p'}\subset L^{p',2}$ *into* $L^{q,2}\subset L^q$ *whenever* $\kappa(p,q)=0$, *which is the case if* $p=q=2\,\frac{n+1}{n-1}$.

Lemma 1.3 (variant of Bourgain's trick)**.** *Let* $1\le p_0\ne p_1\le\infty$, $1\le q_0\ne q_1\le\infty$ *and* $-\infty<\kappa_0\ne\kappa_1<+\infty$. *Assume that a sequence* $\{\mathcal{T}_j\}_{j\in\mathbb{N}}$ *of linear operator satisfies*

$$\|\mathcal{T}_j\|_{L^{p_0}\to L^{q_0}} \lesssim 2^{\kappa_0 j} \quad \textit{and} \quad \|\mathcal{T}_j\|_{L^{p_1}\to L^{q_1}} \lesssim 2^{\kappa_1 j} \quad (j\in\mathbb{N}).$$

Then, for every $0<\theta<1$, $\sum_{j\in\mathbb{N}} 2^{-\kappa j}\,\mathcal{T}_j$ *maps* $L^{p,r}$ *into* $L^{q,r}$, *where* $\kappa=(1-\theta)\kappa_0+\theta\kappa_1$, $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$, $\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$ *and* $1\le r\le\infty$.

Remark 1.4 (Knapp)**.** *Theorem 1.1 doesn't hold for* $p'>2\,\frac{n+1}{n+3}$ *i.e.* $p<2\,\frac{n+1}{n-1}$. *For small* $\varepsilon>0$, *consider indeed the function*

$$f_\varepsilon(x) = (4\pi)^n\,\frac{\sin\sqrt{\varepsilon}\,x_1}{x_1}\,\dots\,\frac{\sin\sqrt{\varepsilon}\,x_{n-1}}{x_{n-1}}\,\frac{\sin\frac{\varepsilon}{2}x_n}{x_n}\,e^{\,i\,x_n}\,,$$

whose Fourier transform is the characteristic function of the set

$$[-\sqrt{\varepsilon},+\sqrt{\varepsilon}\,]^{n-1}\times[1-\tfrac{\varepsilon}{2},1+\tfrac{\varepsilon}{2}]\,.$$

Then $\|f_\varepsilon\|_{L^{p'}} \asymp \varepsilon^{\frac{n+1}{2p}}$ *while* $\|\,Tf_\varepsilon\|_{L^2} \asymp \varepsilon^{\frac{n-1}{4}}$. *By letting* $\varepsilon\to0$, *we see that* $p\ge2\,\frac{n+1}{n-1}$ *is a necessary condition for Theorem 1.1 to hold.*

Theorem 1.5 (Strichartz [14])**.** *Consider the paraboloid*

$$S = \{(\tau,\xi)\in\mathbb{R}\times\mathbb{R}^n \mid \tau = -\|\xi\|^2\}$$

in $\mathbb{R}\times\mathbb{R}^n$ *and assume that* $p'=2\,\frac{n+2}{n+4}$ *i.e.* $p=2\,\frac{n+2}{n}$. *Then*

$$\big\|\,\widehat{u}\,\big|_S\big\|_{L^2(S)} \lesssim \|u\|_{L^{p'}(\mathbb{R}\times\mathbb{R}^n)} \qquad \forall\,u\in\mathcal{S}(\mathbb{R}\times\mathbb{R}^n).$$

Proof. Let us prove Theorem 1.5 by adapting the third proof above. We have now

$$Tu=\widehat{u}\,\big|_S\,,\quad T^*v(t,x)=\int_{\mathbb{R}^n}d\xi\,v(-\|\xi\|^2,\xi)\,e^{\,i(-t\|\xi\|^2+\langle x,\xi\rangle)}$$

and

$$\mathcal{T}u=T^*Tu=u*\widehat{\sigma}\,,$$

where

$$\int_{\mathbb{R}\times\mathbb{R}^n}d\sigma(\tau,\xi)\;v(\tau,\xi)\;=\int_{\mathbb{R}^n}d\xi\,v(-\|\xi\|^2,\xi)$$

and

$$\widehat{\sigma}(t,x)\;=\int_{\mathbb{R}\times\mathbb{R}^n}d\xi\,e^{\,i(t\|\xi\|^2-\langle x,\xi\rangle)}\;=\;\underbrace{\big(i\,\tfrac{\pi}{t}\big)^{\frac{n}{2}}}_{\pi^{\frac{n}{2}}\,e^{\,i\,\mathrm{sign}(t)\,n\frac{\pi}{4}}\,|t|^{-\frac{n}{2}}}\;e^{-i\,\frac{|x|^2}{4\,t}}$$

Consider the analytic family of tempered distributions

$$\sigma_z(\tau,\xi)=\tfrac{1}{\Gamma(z)}\,(\tau+\|\xi\|^2)_+^{\,z-1}$$

and the associated operators $\Sigma_z u=u*\widehat{\sigma_z}$. The $(n{+}1)$–dimensional Fourier transform

$$\widehat{\sigma_z}(t,x)=\pi^{\frac{n}{2}}\,e^{-i\,\frac{\pi}{2}\,z}\,e^{\,i\,\mathrm{sign}(t)\,n\frac{\pi}{4}}\,(t-i\,0)^{-z}\,|t|^{-\frac{n}{2}}\,e^{\,i\,\frac{|x|^2}{4\,t}}\,,$$

in the sense of distributions, is computed by combining the 1–dimensional Fourier transform

$$\tfrac{1}{\Gamma(z)}\int_{\mathbb{R}}d\tau\,\tau_+^{\,z-1}\,e^{-i\,t\,\tau}\;=\;e^{-i\,\frac{\pi}{2}\,z}\,(t-i\,0)^{-z}\,,$$

which yields

$$\tfrac{1}{\Gamma(z)}\int_{\mathbb{R}}d\tau\,(\tau+\|\xi\|^2)_+^{\,z-1}\,e^{-i\,t\,\tau}\;=\;e^{\,i\,\frac{\pi}{2}\,z}\,(t-i\,0)^{-z}\,e^{\,i\,t\,\|\xi\|^2}\,,$$

with the n–dimensional Fourier transform

$$\int_{\mathbb{R}^n}d\xi\;e^{\,i\,t\,\|\xi\|^2}\,e^{-i\langle x,\xi\rangle}\;=\;\pi^{\frac{n}{2}}\,e^{\,i\,\mathrm{sign}(t)\,n\frac{\pi}{4}}\,|t|^{-\frac{n}{2}}\,e^{\,i\,\frac{|x|^2}{4\,t}}\,.$$

On the one hand, if $\operatorname{Re}z=-\frac{n}{2}$, then Σ_z is bounded from $L^1(\mathbb{R}\times\mathbb{R}^n)$ to $L^\infty(\mathbb{R}\times\mathbb{R}^n)$ and

$$\|\Sigma_z\|_{L^1\to L^\infty}=\|\widehat{\sigma_z}\|_{L^\infty}$$

grows at most exponentially in $\operatorname{Im}z$. On the other hand, if $\operatorname{Re}z=1$, then Σ_z is bounded from $L^2(\mathbb{R}\times\mathbb{R}^n)$ to $L^2(\mathbb{R}\times\mathbb{R}^n)$ and

$$\|\Sigma_z\|_{L^2\to L^2}=(2\pi)^{n+1}\,\|\sigma_z\|_{L^\infty}$$

grows at most exponentially in $\operatorname{Im}z$. In conclusion, Stein's interpolation theorem yields the $L^{p'}\!\to L^p$ boundedness of $\Sigma_0=\mathcal{T}$ for $p'=2\,\frac{n+2}{n+4}$ i.e. $p=2\,\frac{n+2}{n}$. □

Remark 1.6. *Theorem 1.5 doesn't hold for* $p'\neq2\,\frac{n+2}{n+4}$ *i.e.* $p\neq2\,\frac{n+2}{n}$. *This is easily proved by rescaling. Specifically, let*

$$(\delta_s u)(t,x)=u(s^2t,sx)\,,\quad\forall\,s>0,\,\forall\,(t,x)\in\mathbb{R}\times\mathbb{R}^n.$$

Then $\|\delta_s u\|_{L^{p'}}=s^{-\frac{n+2}{p'}}\,\|u\|_{L^{p'}}$ *while* $\|T(\delta_s u)\|_{L^2}=s^{-\frac{n}{2}-2}\,\|Tu\|_{L^2}$, *as*

$$\widehat{\delta_s u}(\tau,\xi)=s^{-n-2}\,\widehat{u}(s^{-2}\tau,s^{-1}\xi)\,.$$

By letting $s\to0$ *or* $s\to+\infty$, *we see that* $p=2\,\frac{n+2}{n+4}$ *is a necessary condition for Theorem 1.5 to hold.*

Let us next apply Theorem 1.5 to the Schrödinger equation

$$\begin{cases}i\,\partial_t u(t,x)=-\,\Delta_x u(t,x),\\ u(t,x)=f(x),\end{cases}\tag{6}$$

Via the Fourier transform, (6) becomes

$$\begin{cases}i\,\partial_t\widehat{u}(t,\xi)=\|\xi\|^2\,\widehat{u}(t,\xi),\\ \widehat{u}(t,\xi)=\widehat{f}(\xi),\end{cases}$$

whose solution is given by

$$\widehat{u}(t,\xi) = e^{-it\|\xi\|^2}\,\widehat{f}(\xi)\,.$$

By applying the inverse Fourier transform, we get

$$u(t,x) = (2\pi)^{-n}\int_{\mathbb{R}^n} d\xi\,\widehat{f}(\xi)\,e^{\,i(-t\|\xi\|^2+\langle x,\xi\rangle)}\,,$$

which boils down to the expression of the operator T^*.

Corollary 1.7. *Let $p = 2\frac{n+2}{n+4}$. Then the following a priori inequality holds for solutions to* (6)*:*

$$\|u\|_{L^p(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}\,.$$

Proof. This result is a restatement of the $L^2 \to L^p$ boundedness of the operator T^*. Specifically,

$$\|u\|_{L^p(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|\widehat{f}\|_{L^2(\mathbb{R}^n)} = (2\pi)^{\frac{n}{2}}\,\|f\|_{L^2(\mathbb{R}^n)}\,. \qquad \square$$

Remark 1.8. *In his seminal work [14], Strichartz studies restriction theorems for general quadratic hypersurfaces and applies some of them to related PDE. Another important example is the cone*

$$S = \{(\tau,\xi)\in\mathbb{R}\times\mathbb{R}^n \mid \tau^2 = \|\xi\|^2\}$$

and the wave equation

$$\begin{cases} \partial_t^2 u(t,x) = \Delta_x u(t,x), \\ u(t,x) = f(x),\ \partial_t|_{t=0}\,u(t,x) = g(x). \end{cases} \tag{7}$$

In this case, $p' = 2\frac{n+1}{n+3}$ i.e. $p = 2\frac{n+1}{n-1}$, the restriction theorem reads

$$\big\|\,\widehat{u}\,\big|_S\big\|_{L^2(S)} \lesssim \|u\|_{L^{p'}(\mathbb{R}\times\mathbb{R}^n)} \qquad \forall\, u\in\mathcal{S}(\mathbb{R}\times\mathbb{R}^n),$$

and the following a priori inequality holds for solutions to (7)*:*

$$\|u\|_{L^p(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^{1/2}(\mathbb{R}^n)} + \|g\|_{\dot{H}^{-1/2}(\mathbb{R}^n)}\,,$$

where the initial data belong to homogeneous Sobolev spaces $\dot{H}^{\pm\frac{1}{2}}(\mathbb{R}^n) = (-\Delta)^{\mp\frac{1}{2}}L^2(\mathbb{R}^n)$.

2. Schrödinger equation on $\mathbb{R}^n$ (linear case)

In this section we consider the linear Schrödinger equation

$$\begin{cases} i\,\partial_t u(t,x) + \Delta_x u(t,x) = F(t,x) \\ u(t,x) = f(x) \end{cases} \tag{8}$$

on $\mathbb{R}\times\mathbb{R}^n$ and we prove two fundamental inequalities in Theorem 2.1 and Theorem 2.3, the latter improving upon Corollary 1.7. Our main references are [8] and [10].

Consider first the homogeneous Schrödinger equation

$$\begin{cases} i\,\partial_t u(t,x) + \Delta_x u(t,x) = 0, \\ u(t,x) = f(x), \end{cases} \tag{6}$$

whose solution is given by

$$u(t,x) = e^{it\Delta} f(x) = f * s_t(x)\,, \tag{9}$$

where

$$s_t(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} d\xi\, e^{-it\|\xi\|^2}\,e^{\,i\langle x,\xi\rangle} = \overbrace{(4\pi i t)^{-\frac{n}{2}}}^{e^{-i\,\mathrm{sign}(t)\,n\,\frac{\pi}{4}}\,(4\,\pi\,|t|)^{-\frac{n}{2}}}\,e^{\,i\frac{|x|^2}{4\,t}}\,, \tag{10}$$

is the heat kernel with imaginary time.

Theorem 2.1. *Let $1\le q'\le 2$ i.e. $2\le q\le\infty$. Then the following dispersive estimate holds, for every $t\in\mathbb{R}^*$:*

$$\|\,e^{it\Delta}\|_{L^{q'}\to L^q} \lesssim |t|^{-n(\frac{1}{2}-\frac{1}{q})}$$

Proof. This result is obtained by standard interpolation between the elementary estimate

$$\| e^{it\Delta} \|_{L^1 \to L^\infty} = \| s_t \|_{L^\infty} \lesssim |t|^{-\frac{n}{2}}$$

and the L^2 conservation

$$\| e^{it\Delta} f \|_{L^2} = \| f \|_{L^2} \,.$$

□

Let us turn to the inhomogeneous equation (8), whose solution is given by Duhamel's formula :

$$u(t,x) = \overbrace{e^{it\Delta_x} f(x)}^{\text{homogeneous}} - i \overbrace{\int_0^t ds\, e^{i(t-s)\Delta_x} F(s,x)}^{\text{inhomogeneous}} \,. \tag{11}$$

Definition 2.2. *A couple (p,q) of indices is called admissible if*

$$2 \le p \le \infty,\ 2 \le q < \infty \quad \textit{and} \quad \tfrac{1}{p} = \tfrac{n}{2}\left(\tfrac{1}{2} - \tfrac{1}{q}\right).$$

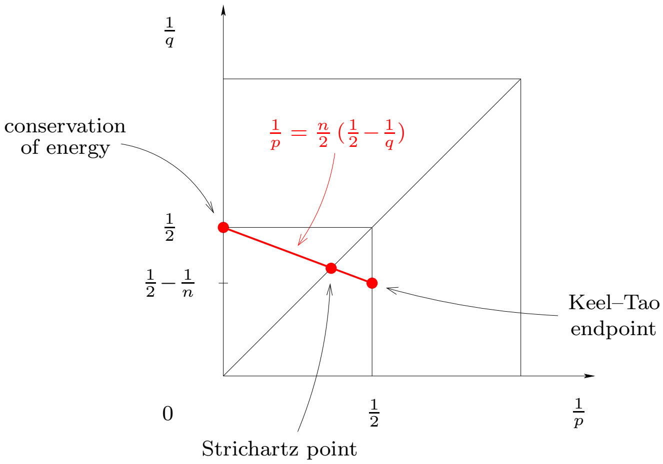

Figure 1. Admissibility for $\mathbb{R}^n$

Theorem 2.3. *Let (p,q) and $(\tilde{p},\tilde{q})$ be two admissible couples of indices. Then the following Strichartz estimate holds for solutions (11) to (8) :*

$$\|u\|_{L^{\tilde{p}'}(\mathbb{R},L^{\tilde{q}'}(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))}$$

We shall prove Theorem 2.3 in several steps.

Step 1 : TT^* method revisited.
Consider the operator

$$Tf(t,x) = e^{it\Delta_x} f(x)\,,$$

its formal adjoint

$$T^*F(x) = \int_{-\infty}^{+\infty} ds\, e^{-is\Delta_x} F(s,x)\,,$$

their composition

$$\mathcal{T}F(t,x) = TT^*F(t,x) = \int_{-\infty}^{+\infty} ds\, e^{i(t-s)\Delta_x} F(s,x)\,,$$

and its truncated version

$$\widetilde{\mathcal{T}}F(t,x) = \int_{-\infty}^{t} ds\, e^{i(t-s)\Delta_x} F(s,x)\,. \tag{12}$$

Then the following a priori estimates are equivalent :

$$\|Tf\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)}\,, \tag{13}$$

$$\|T^*F\|_{L^2(\mathbb{R}^n)} \lesssim \|F\|_{L^{p'}(\mathbb{R},L^{q'}(\mathbb{R}^n))}\,, \tag{14}$$

$$\|\mathcal{T}F\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))} \lesssim \|F\|_{L^{p'}(\mathbb{R},L^{q'}(\mathbb{R}^n))}\,. \tag{15}$$

Step 2 : Proof of (13), (14), (15) when (p,q) is admissible and $p>2$.
On the one hand, according to Theorem 2.1,

$$\|\,\mathcal{T}F(t,x)\|_{L^q_x} \lesssim \int_{-\infty}^{+\infty} ds\; |t-s|^{-\alpha}\, \|F(s,x)\|_{L^{q'}_x}$$

where $\alpha = n\big(\frac12-\frac1q\big)$. On the other hand, according to the Hardy–Littlewood–Sobolev inequality, the convolution kernel $|t-s|^{-\alpha}$ defines a bounded operator from $L^{p'}(\mathbb{R})$ to $L^p(\mathbb{R})$, provided that $0\le\alpha<1$ and $\frac1p=\frac{\alpha}{2}$. These two results yield (15), hence (13) and (14), under the assumptions above.

Step 3 : Decoupling indices.
Let (p,q) and $(\tilde p,\tilde q)$ be admissible couples with $p>2$ and $\tilde p>2$. By combining the $L^{\tilde p'}(\mathbb{R},L^{\tilde q'}(\mathbb{R}^n))\to L^2(\mathbb{R}^n)$ boundedness of T^* with the $L^2(\mathbb{R}^n)\to L^p(\mathbb{R},L^q(\mathbb{R}^n))$ boundedness of T, we obtain the $L^{\tilde p'}(\mathbb{R},L^{\tilde q'}(\mathbb{R}^n))\to L^p(\mathbb{R},L^q(\mathbb{R}^n))$ boundedness of $\mathcal{T}$. Moreover, as $\tilde p'<p$, the same result holds true for the truncated operator $\widetilde{\mathcal{T}}$, according to the Christ–Kiselev lemma [6].

Step 4 : Endpoint estimates when $n\ge3$ and $(p,q)=(\tilde p,\tilde q)=(2,2\frac{n}{n-2})$.
The following arguments are due to Keel and Tao [10]. Instead of $\mathcal{T}$ and $\widetilde{\mathcal{T}}$, consider the bilinear (actually Hermitian) form

$$\mathcal{B}(F,G) = \iint_{\mathbb{R}^2} ds\,dt \int_{\mathbb{R}^n} dx\; e^{-is\Delta_x}F(s,x)\;\overline{e^{-it\Delta_x}G(t,x)}$$

and its truncated version

$$\widetilde{\mathcal{B}}(F,G) = \iint_{s<t} ds\,dt \int_{\mathbb{R}^n} dx\; e^{-is\Delta_x}F(s,x)\;\overline{e^{-it\Delta_x}G(t,x)}\,. \tag{16}$$

In order to estimate (16), let us split up dyadically $\widetilde{\mathcal{B}}=\sum_{j\in\mathbb{Z}}\widetilde{\mathcal{B}}_j$, where

$$\widetilde{\mathcal{B}}_j(F,G) = \iint_{2^j\le t-s<2^{j+1}} ds\,dt \int_{\mathbb{R}^n} dx\; e^{-is\Delta_x}F(s,x)\;\overline{e^{-it\Delta_x}G(t,x)}\,,$$

and let us further split up

$$F(s,x) = \sum_{k=-\infty}^{+\infty} \underbrace{\mathbb{1}_{[k2^j,(k+1)2^j)}(s)\,F(s,x)}_{F_k^{(j)}(s,x)} \quad\text{and}\quad G(t,x) = \sum_{\ell=-\infty}^{+\infty} \underbrace{\mathbb{1}_{[\ell2^j,(\ell+1)2^j)}(t)\,G(t,x)}_{G_\ell^{(j)}(t,x)}\,.$$

Notice the orthogonality

$$\|F\|_{L^{\tilde p'}L^{\tilde q'}} = \Big\{\sum\nolimits_{k=-\infty}^{+\infty}\|F_k^{(j)}\|_{L^{\tilde p'}L^{\tilde q'}}^{\tilde p'}\Big\}^{1/\tilde p'}\,,\;\; \|G\|_{L^2L^{q'}} = \Big\{\sum\nolimits_{\ell=-\infty}^{+\infty}\|G_\ell^{(j)}\|_{L^{p'}L^{q'}}^{p'}\Big\}^{1/p'} \tag{17}$$

and the almost orthogonality

$$\widetilde{\mathcal{B}}_j(F,G) = \sum\nolimits_{\substack{k,\ell\in\mathbb{Z}\\1\le\ell-k\le2}} \widetilde{\mathcal{B}}_j(F_k^{(j)},G_\ell^{(j)})\,. \tag{18}$$

We claim that, for all indices corresponding to the region depicted in Figure 2, we have

$$|\,\widetilde{\mathcal{B}}_j(F_k^{(j)},G_\ell^{(j)})| \lesssim 2^{\kappa(\tilde q,q)j}\,\|F_k^{(j)}\|_{L^2L^{\tilde q'}}\|G_\ell^{(j)}\|_{L^2L^{q'}} \quad \forall\, j,k,\ell\in\mathbb{Z}, \tag{19}$$

where $\kappa(\tilde q,q)=\frac n2\big(\frac1{\tilde q}+\frac1q\big)-\frac{n-2}{2}$.

This estimate is obtained by standard interpolation between the following three cases:

$$\begin{cases} \text{(a)}\ \ 2\le\tilde{q}<2\frac{n}{n-2}\ \text{ and }\ q=2,\\ \text{(b)}\ \ \tilde{q}=2\ \text{ and }\ 2\le q<2\frac{n}{n-2},\\ \text{(c)}\ \ 2<\tilde{q}=q<\infty. \end{cases}$$

∘ *Case* (a): Let us estimate

$$|\,\widetilde{\mathcal{B}}_j(F_k^{(j)},G_\ell^{(j)})|\lesssim\sup_{t\in\mathbb{R}}\Big\|\int_{t-2^{j+1}}^{t-2^j}ds\;e^{-is\Delta_x}F_k^{(j)}(s,x)\Big\|_{L^2_x}\int_{-\infty}^{+\infty}dt\,\big\|e^{-it\Delta_x}G_\ell^{(j)}(t,x)\big\|_{L^2_x}\,.$$

On the one hand, as T^* is bounded from $L^{\tilde{p}'}(\mathbb{R},L^{\tilde{q}'}(\mathbb{R}^n))$ into $L^2(\mathbb{R}^n)$, with $\frac{1}{\tilde{p}}=\frac{n}{2}(\frac{1}{2}-\frac{1}{\tilde{q}})$,

$$\Big\|\int_{\mathbb{R}}ds\;\mathbb{1}_{(t-2^{j+1},t-2^j]}(s)\;e^{-is\Delta_x}F_k^{(j)}(s,x)\Big\|_{L^2_x}\lesssim\big\|\mathbb{1}_{(t-2^{j+1},t-2^j]}(s)\,F_k^{(j)}(s,x)\big\|_{L_s^{\tilde{p}'}L_x^{\tilde{q}'}}$$

On the other hand,

$$\big\|e^{-it\Delta_x}G_\ell^{(j)}(t,x)\big\|_{L^2_x}\lesssim\big\|G_\ell^{(j)}(t,x)\big\|_{L^2_x}\,.$$

By combining these estimates and by using Hölder's inequality in time, we conclude that

$$|\,\widetilde{\mathcal{B}}_j(F_k^{(j)},G_\ell^{(j)})|\lesssim 2^{\frac{1}{\tilde{p}'}j}\,\big\|F_k^{(j)}\big\|_{L^2L^{\tilde{q}'}}\big\|G_\ell^{(j)}\big\|_{L^2L^2}$$

with $\frac{1}{\tilde{p}'}=\kappa(\tilde{q},2)$.

∘ *Case* (b) is handled similarly.

∘ *Case* (c): Let us estimate this time

$$|\,\widetilde{\mathcal{B}}_j(F_k^{(j)},G_\ell^{(j)})|\lesssim\iint_{2^j\le t-s<2^{j+1}}ds\,dt\,\big\|e^{\,i(t-s)\Delta_x}F_k^{(j)}(s,x)\big\|_{L^q_x}\big\|\,G_\ell^{(j)}(t,x)\big\|_{L^{q'}_x}.$$

According to the dispersive estimate in Theorem 2.1,

$$\big\|e^{\,i(t-s)\Delta_x}F_k^{(j)}(s,x)\big\|_{L^q_x}\lesssim|t-s|^{-n(\frac{1}{2}-\frac{1}{q})}\,\big\|\,F_k^{(j)}(s,x)\big\|_{L^{q'}_x}.$$

By using again Hölder's inequality in time, we conclude that

$$\begin{aligned}|\,\widetilde{\mathcal{B}}_j(F_k^{(j)},G_\ell^{(j)})|&\lesssim 2^{\,n(\frac{1}{2}-\frac{1}{q})j}\,\big\|F_k^{(j)}\big\|_{L^1L^{q'}}\big\|G_\ell^{(j)}\big\|_{L^1L^{q'}}\\&\lesssim 2^{\,\kappa(q,q)\,j}\,\big\|F_k^{(j)}\big\|_{L^2L^{q'}}\big\|G_\ell^{(j)}\big\|_{L^2L^{q'}}\,.\end{aligned}$$

By adding up (19) over $k,\ell\in\mathbb{Z}$ and by using the orthogonality properties (17) and (18), we deduce that

$$\sup\nolimits_{j\in\mathbb{Z}}\;\overbrace{2^{-\kappa(\tilde{q},q)\,j}}^{w_{\tilde{q},q}(j)}\,|\widetilde{\mathcal{B}}_j(F,G)|\lesssim\|F\|_{L^2L^{\tilde{q}'}}\,\|G\|_{L^2L^{q'}}\,.$$

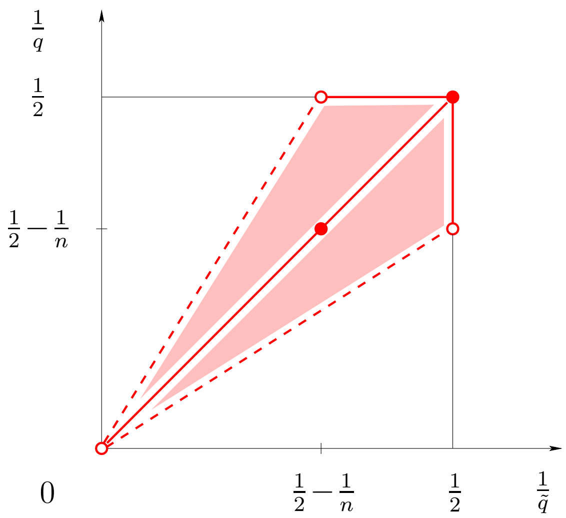

Figure 2. Interpolation region

In other words, the sequence $\{\widetilde{\mathcal{B}}_j\}_{j\in\mathbb{Z}}$ defines a bounded bilinear (actually Hermitian) operator from the product space $L^2(\mathbb{R},L^{\tilde q'}(\mathbb{R}^n))\times L^2(\mathbb{R},L^{q'}(\mathbb{R}^n))$ into the weighted space $\ell^\infty(\mathbb{Z},w_{\tilde q,q})$. This result can be improved by real interpolation. Specifically, by moving $(\tilde q,q)$ and varying $w_{\tilde q,q}$, one gets a bounded map into $\ell^1(\mathbb{Z},w_{\tilde q,q})$ instead of $\ell^\infty(\mathbb{Z},w_{\tilde q,q})$, according to Lemma 2.4 below. When $\tilde q=q=2\frac{n}{n-2}$, we have in particular $\kappa(\tilde q,q)=0$ i.e. $w_{\tilde q,q}=1$. Hence

$$|\widetilde{\mathcal{B}}(F,G)|\leq\sum\nolimits_{j\in\mathbb{Z}}|\widetilde{\mathcal{B}}_j(F,G)|\lesssim\|F\|_{L^2L^{\tilde q'}}\|G\|_{L^2L^{q'}}.$$

We conclude with two elementary observations. On the one hand, the truncated form

$$\mathcal{B}(F,G)-\widetilde{\mathcal{B}}(F,G)=\iint_{s>t}ds\,dt\int_{\mathbb{R}^n}dx\;e^{-is\Delta_x}F(s,x)\,\overline{e^{-it\Delta_x}G(t,x)}\,.$$

and hence $\mathcal{B}(F,G)$ are estimated in the same way. On the other hand, the truncated operator defined by

$$\int_0^t ds\;e^{i(t-s)\Delta_x}F(s,x)$$

is deduced from (12) by multiplying $F(s,x)$ by $1\!\!1_{\mathbb{R}_+}(s)$.

Step 5 : Endpoint estimates when $n\geq3$ and either $(p,q)=(\tilde p,\tilde q)=(2,2\frac{n}{n-2})$ or $(\tilde p,\tilde q)=(2,2\frac{n}{n-2})$.

This case is simpler and relies on the same arguments. □

Lemma 2.4 (O'Neil, see [2, § 3.13, Exercise 5.(b)]). *Within the standard real interpolation setting, assume that a bilinear operator* $\mathfrak{B}$ *maps*

$$\widetilde{E}_0\times E_0\longrightarrow F_0\,,\quad \widetilde{E}_0\times E_1\longrightarrow F_1\quad\textit{and}\quad \widetilde{E}_1\times E_0\longrightarrow F_1\,.$$

Then $\mathfrak{B}$ *maps*

$$(\widetilde{E}_0,\widetilde{E}_1)_{\widetilde\theta,\widetilde r}\times(E_0,E_1)_{\theta,r}\longrightarrow(F_0,F_1)_{\eta,s}\,,$$

where $0<\widetilde\theta,\theta,\eta<1$ *and* $1\leq\widetilde r,r,s\leq\infty$ *satisfy* $\widetilde\theta+\theta=\eta$ *and* $\frac{1}{\widetilde r}+\frac{1}{r}\geq\frac{1}{s}$.

3. Schrödinger equation on $\mathbb{R}^n$ (nonlinear case)

In this section, we consider the semilinear Schrödinger equation

$$\begin{cases} i\,\partial_t u(t,x)+\Delta_x u(t,x)=F(u(t,x)),\\ u(t,x)=f(x),\end{cases}\tag{20}$$

with power–like nonlinearities F. By this we mean that there exist constants $C>0$ and $\gamma>1$ such that

$$\begin{cases} |F(u)|\leq C\,|u|^\gamma\,,\\ |F(u)-F(\tilde u)|\leq C\,\{|u|^{\gamma-1}+|\tilde u|^{\gamma-1}\}\,|u-\tilde u|\,.\end{cases}\tag{21}$$

Typical examples are

$$F(u)=\text{const.}\times\begin{cases}|u|^\gamma,\\ u\,|u|^{\gamma-1}.\end{cases}$$

Following the strategy developed by Kato, Ginibre–Velo [8] and Keel–Tao [10], we apply the Strichartz estimates in Theorem 2.3 to the well–posedness of (20), which means roughly existence and uniqueness of solutions in some suitable function space.

Theorem 3.1. (a) Critical case : *Assume that* $\gamma=1+\frac{4}{n}$.
Then (20) *is globally well–posed in* $L^2(\mathbb{R}^n)$ *for small initial data* f.
(b) Subcritical case : *Assume that* $1<\gamma<1+\frac{4}{n}$.
Then (20) *is locally well–posed in* $L^2(\mathbb{R}^n)$ *for arbitrary initial data* f.
(c) Subcritical case : *Assume again that* $1<\gamma<1+\frac{4}{n}$ *and assume that* $F(u)$ *is a real multiple of* $u\,|u|^{\gamma-1}$.
Then (20) *is globally well–posed in* $L^2(\mathbb{R}^n)$ *for arbitrary initial data* f.

Proof of (a). Define $u=\Phi(v)$ as the solution to the Cauchy problem

$$\begin{cases} i\,\partial_t u(t,x)+\Delta_x u(t,x)=F(v(t,x))\,, \\ u(t,x)=f(x), \end{cases} \tag{22}$$

which is given by Duhamel's formula (11):

$$u(t,x)=e^{\,it\Delta_x}f(x)-i\int_0^t ds\;e^{\,i(t-s)\Delta_x}F(s,x).$$

According to Theorem 2.3, the following Strichartz estimate holds

$$\|u\|_{L^\infty(\mathbb{R},L^2(\mathbb{R}^n))}+\|u\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))}\le C\,\|f\|_{L^2(\mathbb{R}^n)}+C\,\|F(v)\|_{L^{\tilde p'}(\mathbb{R},L^{\tilde q'}(\mathbb{R}^n))} \tag{23}$$

for all couples (p,q) and $(\tilde p,\tilde q)$ satisfying the admissibility conditions

$$\begin{cases} 2\le p\le\infty\,,\;2\le q<\infty\,,\;\frac{1}{p}=\frac{n}{2}\,(\frac12-\frac1q)\,, \\ 2\le \tilde p\le\infty\,,\;2\le \tilde q<\infty\,,\;\frac{1}{\tilde p}=\frac{n}{2}\,(\frac12-\frac1{\tilde q})\,. \end{cases} \tag{24}$$

Moreover

$$\|F(v)\|_{L^{\tilde p'}(\mathbb{R},L^{\tilde q'}(\mathbb{R}^n))}\le C\,\|\,|v|^\gamma\|_{L^{\tilde p'}(\mathbb{R},L^{\tilde q'}(\mathbb{R}^n))}\le C\,\|v\|^\gamma_{L^{\gamma\tilde p'}(\mathbb{R},L^{\gamma\tilde q'}(\mathbb{R}^n))}\,.$$

by our nonlinear assumption (21). Thus

$$\|u\|_{L^\infty(\mathbb{R},L^2(\mathbb{R}^n))}+\|u\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))}\le C\,\|f\|_{L^2(\mathbb{R}^n)}+C\,\|v\|^\gamma_{L^{\gamma\tilde p'}(\mathbb{R},L^{\gamma\tilde q'}(\mathbb{R}^n))}\,. \tag{25}$$

In order to remain within the same function space, we require that

$$\begin{cases} p=\gamma\,\tilde p' & \Longleftrightarrow\quad \frac{\gamma}{p}+\frac{1}{\tilde p}=1\,, \\ q=\gamma\,\tilde q' & \Longleftrightarrow\quad \frac{\gamma}{q}+\frac{1}{\tilde q}=1\,. \end{cases} \tag{26}$$

All conditions (24) and (26) can be fulfilled provided that $\gamma=1+\frac4n$. In this case, one may consider for instance the Strichartz point, which is given by

$$p=q=\tilde p=\tilde q=\gamma+1=2+\tfrac4n.$$

For such a choice, Φ maps $L^\infty(\mathbb{R},L^2(\mathbb{R}^n))\cap L^p(\mathbb{R},L^q(\mathbb{R}^n))$ into itself, and actually $X=C(\mathbb{R},L^2(\mathbb{R}^n))\cap L^p(\mathbb{R},L^q(\mathbb{R}^n))$ into itself. As X is a Banach space for the norm

$$\|u\|_X=\|u\|_{L^\infty(\mathbb{R},L^2(\mathbb{R}^n))}+\|u\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))}\,,$$

it remains for us to show that Φ is a contraction in the ball

$$X_\varepsilon=\{\,u\in X\mid\|u\|_X\le\varepsilon\,\}\,,$$

provided that $\varepsilon>0$ and $\|f\|_{L^2}$ are sufficiently small. Let $v,\tilde v\in X$ and $u=\Phi(v)$, $\tilde u=\Phi(\tilde v)$. Arguying as above and using in addition Hölder's inequality, we estimate

$$\begin{aligned}\|u-\tilde u\|_X&\le C\,\|F(v)-F(\tilde v)\|_{L^{\tilde p'}(\mathbb{R},L^{\tilde q'}(\mathbb{R}^n))}\\ &\le C\,\|\{|v|^{\gamma-1}+|\tilde v|^{\gamma-1}\}\,|v-\tilde v|\,\|_{L^{\tilde p'}(\mathbb{R},L^{\tilde q'}(\mathbb{R}^n))}\\ &\le C\,\{\|v\|^{\gamma-1}_{L^p(\mathbb{R},L^q(\mathbb{R}^n))}+\|\tilde v|^{\gamma-1}_{L^p(\mathbb{R},L^q(\mathbb{R}^n))}\}\,\|v-\tilde v\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))}\,,\end{aligned}$$

hence

$$\|u-\tilde u\|_X\le C\,\{\|v\|_X^{\gamma-1}+\|\tilde v\|_X^{\gamma-1}\}\,\|v-\tilde v\|_X\,. \tag{27}$$

If we assume $\|v\|_X\le\varepsilon$, $\|\tilde v\|_X\le\varepsilon$ and $\|f\|_{L^2}\le\delta$, then (25) and (27) yield

$$\|u\|_X\le C\,\delta+C\,\varepsilon^\gamma\,,\quad \|\tilde u\|_X\le C\,\delta+C\,\varepsilon^\gamma\quad\text{and}\quad \|u-\tilde u\|_X\le 2\,C\,\varepsilon^{\gamma-1}\,\|v-\tilde v\|_X\,.$$

Thus

$$\|u\|_X\le\varepsilon\,,\quad \|\tilde u\|_X\le\varepsilon\quad\text{and}\quad \|u-\tilde u\|_X\le\tfrac12\,\|v-\tilde v\|_X\,,$$

if $C\,\varepsilon^{\gamma-1}\le\frac14$ and $C\,\delta\le\frac34\,\varepsilon$. We conclude by applying the fixed point theorem in the complete metric space X_ε. □

Proof of (b). In the subcritical case $\gamma<1+\frac4n$, the same arguments yield the local well–posedness of (20) in $L^2(\mathbb{R}^n)$ for arbitrary initial data f. Specifically, we restrict to a small time interval $I=[-T,+T]$ and we proceed as above, except that we apply in addition Hölder's inequality in time. This way, we get the Strichartz estimate

$$\|u\|_X \leq C\,\|f\|_{L^2} + C\,T^{\lambda}\,\|v\|_X^{\gamma}\,, \tag{28}$$

where $X=C(I,L^2(\mathbb{R}^n))\cap L^p(I,L^q(\mathbb{R}^n))$ and $\lambda=1-\frac{\gamma}{p}-\frac{1}{\tilde{p}}>0$, and the related estimate

$$\|u-\tilde{u}\|_X \leq C\,T^{\lambda}\left\{\|v\|_X^{\gamma-1}+\|v\|_X^{\gamma-1}\right\}\|v-\tilde{v}\|_X\,. \tag{29}$$

As a consequence, we deduce that Φ is a contraction in the ball

$$X_M = \{\, u\in X \mid \|u\|_X \leq M\,\}\,,$$

provided $M>0$ is large enough and $T>0$ small enough, more precisely $\frac{3}{4}\,M\geq C\,\|f\|_{L^2}$ and $C\,T^{\lambda}M^{\gamma-1}\leq\frac{1}{4}$. We conclude as before. Notice that the size of T depends only on the L^2 norm of the initial data f.

Proof of (c). Assume moreover that $F(u)=c\,u\,|u|^{\gamma-1}$ with $c\in\mathbb{R}$. Then the expression

$$i\int_{\mathbb{R}^n} dx\;\partial_t u(t,x)\,\overline{u(t,x)} = -\int_{\mathbb{R}^n} dx\;\Delta_x u(t,x)\,\overline{u(t,x)} + c\int_{\mathbb{R}^n} dx\;|u(t,x)|^{\gamma+1}$$

is real, hence

$$\partial_t\int_{\mathbb{R}^n} dx\,|u(t,x)|^2 = 2\,\mathrm{Re}\int_{\mathbb{R}^n} dx\,\partial_t u(t,x)\overline{u(t,x)}$$

vanishes and we have L^2 conservation:

$$\int_{\mathbb{R}^n} dx\,|u(t,x)|^2 = \int_{\mathbb{R}^n} dx\,|f(x)|^2\,. \tag{30}$$

As the time interval in part (b) depends only on (30), we may iterate and deduce global existence from local existence, for arbitrary initial data $f\in L^2$. We refer to [5] for more information about conservation properties of the Schödinger equation. □

Remark 3.2. *Similar results hold if* $1+\frac{4}{n}$ *is replaced by* $1+\frac{4}{n-2\sigma}$ *and* $L^2(\mathbb{R}^n)$ *by* $H^{\sigma}(\mathbb{R}^n)$*, with* $0<\sigma<\frac{n}{2}$.

Remark 3.3. *The semilinear wave equation*

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) = F(u(t,x)),\\ u(t,x) = f(x),\;\partial_t|_{t=0}\,u(t,x) = g(x), \end{cases} \tag{31}$$

with power–like nonlinearities F, can be studied in a similar way but its analysis is more involved. We refer to [7, 8, 10, 11, 15] for more information.

4. Schrödinger equation on hyperbolic spaces $\mathbb{H}^n$

This section consists in an introduction to my joint work [1] with Vittoria Pierfelice, where we have investigated the semilinear Schrödinger equation (20) on real hyperbolic spaces $\mathbb{H}^n$. Among other references, let us mention on the one hand the earlier works [4, 12] and on the other hand [9], which is mainly devoted to scattering theory in $H^1(\mathbb{H}^n)$. As might be expected, dispersion properties are better in negative curvature. Consequently, Strichartz estimates hold for a wider range and one obtains stronger well–posedness results.

Fourier analysis on $\mathbb{H}^n$ yields the following explicit expression for the Schrödinger kernel (i.e. the heat kernel with imaginary time).

Lemma 4.1. *For every* $t\in\mathbb{R}^*$ *and* $r\geq 0$, *we have*

$$s_t(r) = \text{const.}\,(it)^{-\frac{1}{2}}\,e^{-i(\frac{n-1}{2})^2 t}\,\big(-\tfrac{1}{\sinh r}\tfrac{\partial}{\partial r}\big)^{\frac{n-1}{2}}\,e^{\frac{i}{4}\frac{r^2}{t}}\,.$$

Here $(it)^{-\frac{1}{2}} = e^{-i\,\mathrm{sign}(t)\frac{\pi}{4}}\,|t|^{-\frac{1}{2}}$ *and, in the even dimensional case, the fractional derivative reads*

$$\big(-\tfrac{1}{\sinh r}\tfrac{\partial}{\partial r}\big)^{\frac{n-1}{2}}\,e^{\frac{i}{4}\frac{r^2}{t}} = \tfrac{1}{\sqrt{\pi}}\int_{|r|}^{+\infty}\tfrac{\sinh s\,ds}{\sqrt{\cosh s-\cosh r}}\,\big(-\tfrac{1}{\sinh s}\tfrac{\partial}{\partial s}\big)^{\frac{n}{2}}e^{\frac{i}{4}\frac{s^2}{t}}\,.$$

We deduce first the following sharp kernel estimates.

Corollary 4.2. *For every* $t\in\mathbb{R}^*$ *and* $r\geq 0$, *we have*

$$|s_t(r)| \lesssim \begin{cases} |t|^{-\frac{n}{2}}\,(1+r)^{\frac{n-1}{2}}\,e^{-\frac{n-1}{2}r} & \text{if } |t|\leq 1+r,\\ |t|^{-\frac{3}{2}}\,(1+r)\,e^{-\frac{n-1}{2}r} & \text{if } |t|\geq 1+r.\end{cases}$$

Remark 4.3. *Notice that*

$$(1+r)^{\frac{n-1}{2}}\,e^{-\frac{n-1}{2}r} \asymp j(r)^{-\frac{1}{2}},$$

where j *is the jacobian of the exponential map, and*

$$(1+r)\,e^{-\frac{n-1}{2}r} \asymp \varphi_0(r)$$

where φ_0 *is the radial ground state.*

We deduce next the following dispersive estimates.

Theorem 4.4. *For every* $2<q\leq\infty$ *and* $t\in\mathbb{R}^*$, *we have*

$$\|e^{it\Delta}\|_{L^{q'}(\mathbb{H}^n)\to L^q(\mathbb{H}^n)} \lesssim \begin{cases} |t|^{-n(\frac{1}{2}-\frac{1}{q})} & \text{if } 0<|t|<1,\\ |t|^{-\frac{3}{2}} & \text{if } |t|\geq 1.\end{cases}$$

Hint of proof. The first estimate is obtained by standard interpolation. The second one is obtained by applying the following version of the Kunze–Stein phenomenon:

$$L^{q'}(\mathbb{H}^n) * L^{\tilde{q}}_{\text{rad}}(\mathbb{H}^n) \subset L^q(\mathbb{H}^n)$$

if $2<q,\tilde{q}<\infty$ satisfy $\frac{q}{2}<\tilde{q}<q$. □

Consider the linear Schrödinger equation

$$\begin{cases} i\,\partial_t u(t,x) + \Delta_x u(t,x) = F(t,x)\\ u(t,x) = f(x)\end{cases} \tag{8}$$

on $\mathbb{R}\times\mathbb{H}^n$.

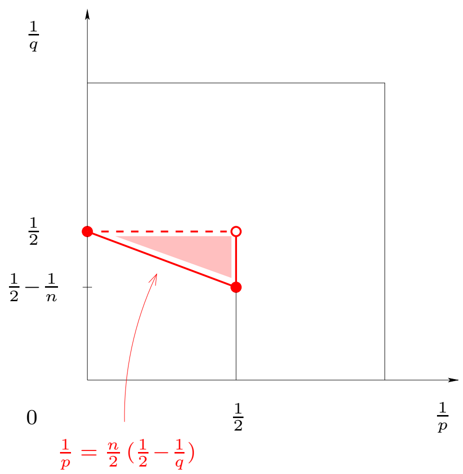

FIGURE 3. Admissibility for $\mathbb{H}^n$

Theorem 4.5. *The Strichartz estimate*

$$\|u\|_{L^{\tilde{p}'}(\mathbb{R},L^{\tilde{q}'}(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))}$$

holds for (8) *and for all admissible couples* (p,q), $(\tilde{p},\tilde{q})$ *corresponding to the triangle depicted in Figure 3.*

Consider eventually the semilinear Schrödinger equation

$$\begin{cases} i\,\partial_t u(t,x) + \Delta_x u(t,x) = F(u(t,x)), \\ u(t,x) = f(x), \end{cases} \tag{20}$$

on $\mathbb{R}\times\mathbb{H}^n$, with power–like nonlinearities F as in (21).

Theorem 4.6. *Assume that* $1<\gamma\le 1+\frac{4}{n}$.
Then (20) *is globally well–posed in* $L^2(\mathbb{R}^n)$ *for small initial data* f.

Remark 4.7. *Parts* (b) *and* (c) *in Theorem 3.1 hold also on* $\mathbb{H}^n$.

Remark 4.8. *All these results extend straightforwardly to the so–called rank one case, which include all hyperbolic spaces, as well as Damek–Ricci spaces.*

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