HASS (Harmonic Analysis and Symmetric Spaces)
October 27–29, 2021
online conference organized by
Simon Marshall (University of Wisconsin – Madison)

# Dispersive PDE on noncompact symmetric spaces

Jean–Philippe Anker (Orléans)

in collaboration with

Stefano Meda (Milano)
Vittoria Pierfelice (Orléans)
Maria Vallarino (Torino)
Hong–Wei Zhang (Orléans/Ghent)

UNIVERSITE D'ORLEANS
INSTITUT DENIS POISSON ORLÉANS · TOURS Mathématiques & Physique Théorique
cnrs

# Setting

**Riemannian symmetric spaces of non compact type**

$X = G/K$ where

- $G$ noncompact semisimple Lie group (connected, finite center)
- $K$ maximal compact subgroup of $G$

for instance

- $\mathbb{H}^n(\mathbb{R}) = \mathrm{SO}(n,1)^\circ/\mathrm{SO}(n)$ in rank 1
- $\mathrm{SL}(n,\mathbb{C})/\mathrm{SU}(n)$ or $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ in higher rank

**Decompositions**

- Cartan : $G = K(\exp \overline{\mathfrak{a}^+})K \ni g = k_1(\exp g^+)\, k_2$
- Iwasawa : $G = N(\exp \mathfrak{a})K \ni g = n(\exp A(g))\, k$

$d = \dim X$, $D = \ell + 2|\Sigma_0^+|$ dimension at infinity
$\ell = \dim \mathfrak{a}$ rank, $\Sigma_0^+$ set of positive indivible roots
$\rho = \sum_{\alpha\in\Sigma^+} \frac{m_\alpha}{2}\alpha \in \mathfrak{a}^+$, $[|\rho|^2, +\infty)$ $L^2$ spectrum of $-\Delta$

# Three evolution equations

## Heat equation

$$\begin{cases} \partial_t u(t,x) - \Delta_x u(t,x) = F(t,x) \\ u(0,x) = f(x) \end{cases}$$

## Schrödinger equation

$$\begin{cases} i\,\partial_t u(t,x) \pm \Delta_x u(t,x) = F(t,x) \\ u(0,x) = f(x) \end{cases}$$

## Wave equation

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) = F(t,x) \\ u(0,x) = f(x), \ \partial_t|_{t=0}\, u(t,x) = g(x) \end{cases}$$

# Main problem : sharp estimates for fundamental solutions

Convolution operators :

$$e^{t\Delta} f = f * h_t$$

$$e^{\pm it\Delta} f = f * s_t$$

$$(-\Delta)^{-\frac{\sigma}{2}} e^{it\sqrt{-\Delta}} f = f * w_t^{\sigma} \quad (\sigma \geq 0)$$

Expression of the corresponding bi−$K$−invariant kernels by using the inverse spherical Fourier transform :

$$h_t(g) = \int_{\mathfrak{a}} \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} \varphi_\lambda(g)\, e^{-t(|\lambda|^2+|\rho|^2)}$$

$$s_t(g) = \int_{\mathfrak{a}} \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} \varphi_\lambda(g)\, e^{\mp it(|\lambda|^2+|\rho|^2)}$$

$$w_t^{\sigma}(g) = \int_{\mathfrak{a}} \frac{d\lambda}{|\mathbf{c}(\lambda)|^2} \varphi_\lambda(g)\, (|\lambda|^2+|\rho|^2)^{-\frac{\sigma}{2}} e^{it\sqrt{|\lambda|^2+|\rho|^2}}$$

# Heat kernel estimates

**A–Ji 1999, A–Ostellari 2003**

$$h_t(g) \asymp t^{-\frac{d}{2}} \prod_{\alpha\in\Sigma_0^+} (1+\langle\alpha, g^+\rangle)(1+t+\langle\alpha, g^+\rangle)^{\frac{m_\alpha+m_{2\alpha}}{2}}$$
$$\times\ e^{-|\rho|^2 t - \langle\rho, g^+\rangle - \frac{|g^+|^2}{4t}}$$

for every $t>0$ and $g\in G$

# Schrödinger kernel

**A–Meda–Pierfelice–Vallarino–Zhang 2021**

There exist $C>0$ and $M>0$ such that the following estimate holds, for every $t\in\mathbb{R}^*$ and $g\in G$,

$$|s_t(g)| \leq C\,(1+|g^+|)^M e^{-\langle\rho,g^+\rangle} \times \begin{cases} |t|^{-\frac{d}{2}} & \text{if } 0<|t|<1 \\ |t|^{-\frac{D}{2}} & \text{if } |t|\geq 1 \end{cases}$$

## Remarks

- $M$ not optimal but harmless for applications
- Explicit expressions in some particular cases :
  - rank 1
    (Banica 2007, Ionescu–Staffilani 2009, A–Pierfelice 2009)
  - $G$ complex
    (Gangolli 1968, Chanillo 2007)

# Proof for large $|t|$ (suprisingly simple)

- Substitute the Harish–Chandra integral formula
$$\varphi_\lambda(g) = \int_K dk\, e^{\langle \rho + i\lambda, A(kg)\rangle}$$
in
$$s_t(g) = \int_{\mathfrak{a}} d\lambda\, |\mathbf{c}(\lambda)|^{-2}\, \varphi_\lambda(g)\, e^{-it(|\lambda|^2+|\rho|^2)}$$
and split up
$$\int_{\mathfrak{a}} d\lambda = \int_{|\lambda|\lesssim t^{-1/2}} d\lambda\, \chi(\sqrt{|t|}\lambda) + \int_{|\lambda|\gtrsim t^{-1/2}} d\lambda\, (1-\chi(\sqrt{|t|}\lambda)) \quad (1)$$
- Elementary estimates of the Plancherel density :
$$\begin{cases} |\mathbf{c}(\lambda)|^{-2} \lesssim |\lambda|^{D-\ell} & \text{if } |\lambda| \le 1 \\ |\nabla_\lambda^j |\mathbf{c}(\lambda)|^{-2}| \lesssim |\lambda|^{d-\ell} & \text{if } |\lambda| \le 1 \end{cases}$$
- The contribution of the first integral in (1) is estimated by
$$|t|^{-\frac{D}{2}}\, \varphi_0(g) \lesssim |t|^{-\frac{D}{2}}\, (1+|g^+|)^{\frac{D-\ell}{2}}\, e^{-\langle \rho, g^+\rangle}$$

# Proof for large $|t|$ (sequel)

- Substitute the Harish–Chandra integral formula
$$\varphi_\lambda(g) = \int_K dk\, e^{\langle \rho + i\lambda, A(kg)\rangle}$$
in
$$s_t(g) = \int_{\mathfrak{a}} d\lambda\, |\mathbf{c}(\lambda)|^{-2}\, \varphi_\lambda(g)\, e^{-it(|\lambda|^2+|\rho|^2)}$$
and split up
$$\int_{\mathfrak{a}} d\lambda = \int_{|\lambda|\lesssim t^{-1/2}} d\lambda\, \chi(\sqrt{|t|}\lambda) + \int_{|\lambda|\gtrsim t^{-1/2}} d\lambda\, (1-\chi(\sqrt{|t|}\lambda)) \quad (1)$$
- Elementary estimates of the Plancherel density :
$$\begin{cases} |\mathbf{c}(\lambda)|^{-2} \lesssim |\lambda|^{D-\ell} & \text{if } |\lambda| \leq 1 \\ |\nabla_\lambda^k |\mathbf{c}(\lambda)|^{-2}| \lesssim |\lambda|^{d-\ell} & \text{if } |\lambda| \leq 1 \end{cases}$$
- After performing several integrations by parts based on
$$e^{-it|\lambda|^2} = \frac{i}{2t} \sum_{j=1}^{\ell} \frac{\lambda_j}{|\lambda|^2} \frac{\partial}{\partial \lambda_j}\, e^{-it|\lambda|^2}$$
the contribution of the second integral in (1) is estimated by
$$|t|^{-\frac{D}{2}}\, (1+|g^+|)^M\, e^{-\langle \rho, g^+\rangle}$$

# Proof for small $|t|$ (suprisingly difficult)

The previous arguments yield

$$|s_t(g)| \lesssim |t|^{-d} (1+|g^+|)^M e^{-\langle \rho, g^+ \rangle}$$

Our aim is to reduce $d$ to $\frac{d}{2}$

**Main problem (well–known)**

In some cases (for instance in rank 1 or if $G$ is complex), the Plancherel density $|\mathbf{c}(\lambda)|^{-2}$ is a symbol. But not in general, where it is just a product of 1D symbols. Thus arbitrary integrations by parts don't improve its behavior at infinity.

**Explicit expression of the Plancherel density**

$$|\mathbf{c}(\lambda)|^{-2} = \mathbf{c}(\lambda)^{-1} \mathbf{c}(-\lambda)^{-1}$$

where

$$\mathbf{c}(\lambda)^{-1} = \text{const.} \prod_{\alpha \in \Sigma_0^+} \mathbf{c}_\alpha \Big( \frac{\langle \alpha, \lambda \rangle}{|\alpha|^2} \Big)^{-1}$$

and

$$\mathbf{c}_\alpha(z)^{-1} = \frac{\Gamma(iz + \frac{m_\alpha}{2})}{\Gamma(iz)} \frac{\Gamma(\frac{iz}{2} + \frac{m_\alpha}{4} + \frac{m_{2\alpha}}{2})}{\Gamma(\frac{iz}{2} + \frac{m_\alpha}{4})}$$

# New tool : barycentric decomposition (A–Zhang 2020)

## Decomposition in $\mathfrak{a}\smallsetminus\{0\}$

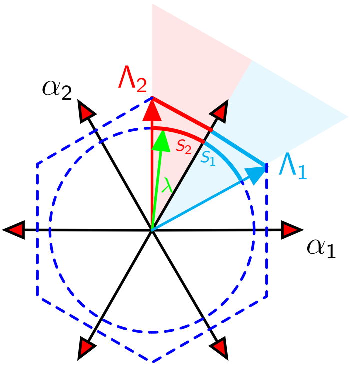
In $\mathfrak{a}\smallsetminus\{0\}$, there exist

- a smooth partition of unity
$$\sum_{w\in W}\sum_{1\le j\le\ell}\chi_{w.S_j}=1$$
consisting of homogeneous symbols of order 0
- nonzero vectors $w.\Lambda_j\in\operatorname{supp}\chi_{w.S_j}$

such that the following dichotomy holds,
for every $\alpha\in\Sigma$, $w\in W$ and $1\le j\le\ell$ :

- either $\langle\alpha,w.\Lambda_j\rangle=0$
- or $|\langle\alpha,\lambda\rangle|\asymp|\lambda|\ \ \forall\,\lambda\in\operatorname{supp}\chi_{w.S_j}$

# Root system $A_2$

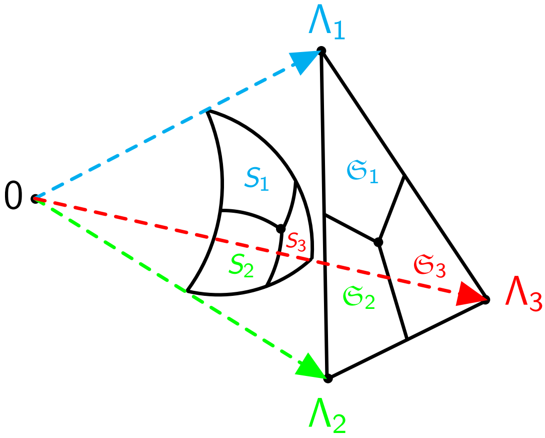



Simple roots : $\alpha_1$, $\alpha_2$
Dual basis : $\Lambda_1$, $\Lambda_2$

Dichotomy for $w = \mathrm{id}$ and $j = 2$ :

- $\langle \alpha, \Lambda_2 \rangle = 0 \iff \alpha = \pm\alpha_1$
- $|\langle \alpha, \lambda \rangle| \asymp |\lambda| \quad \begin{cases} \forall\, \alpha \neq \pm\alpha_1 \\ \forall\, \lambda \in \operatorname{supp} \chi_{S_2} \end{cases}$

# Root system $A_3$



## Application to the Plancherel density

The Plancherel density $|\mathbf{c}(\lambda)|^{-2}$ behaves as it were a symbol, if we differentiate it along $w.\Lambda_j$ in $\operatorname{supp}\chi_{w.S_j}$ :

$$\left|\partial_{w.\Lambda_j}^{k}|\mathbf{c}(\lambda)|^{-2}\right| \lesssim (1+|\lambda|)^{d-\ell-k} \quad \forall\,\lambda\in\operatorname{supp}\chi_{w.S_j}$$

Indeed,

$$|\mathbf{c}(\lambda)|^{-2} = \text{const.}\prod_{\alpha\in\Sigma_0^+}\left|\mathbf{c}_\alpha\left(\tfrac{\langle\alpha,\lambda\rangle}{|\alpha|^2}\right)\right|^{-2}$$

and either

$$\partial_{w.\Lambda_j}\left|\mathbf{c}_\alpha\left(\tfrac{\langle\alpha,\lambda\rangle}{|\alpha|^2}\right)\right|^{-2} = 0$$

or

$$\begin{aligned}\left|\partial_{w.\Lambda_j}^{k}\left|\mathbf{c}_\alpha\left(\tfrac{\langle\alpha,\lambda\rangle}{|\alpha|^2}\right)\right|^{-2}\right| &\lesssim (1+|\langle\alpha,\lambda\rangle|)^{m_\alpha+m_{2\alpha}-k}\\ &\asymp (1+|\lambda|)^{m_\alpha+m_{2\alpha}-k}\end{aligned}$$

# Other classical tools

- stationary phase analysis
- subordination :

$$e^{it\Delta}f = \pi^{-\frac{1}{2}}\, e^{-i\frac{\pi}{4}\,\mathrm{sign}(t)}\,|t|^{-\frac{1}{2}} \int_0^{+\infty} ds\, e^{i\frac{s^2}{4t}}\, \overbrace{(\cos s\sqrt{-\Delta})f}^{f*c_s}$$

- Hadamard parametrix (improved for $X = G/K$) :

$$c_s(g) = s\, J(g)^{-\frac{1}{2}} \sum\nolimits_{0\le k\le\frac{d}{2}} u_k(g)\, \frac{(s^2-|g^+|^2)_+^{k-\frac{d+1}{2}}}{\Gamma(k-\frac{d-1}{2})} + E_s(g)$$

where

  - $J(g) = \prod\nolimits_{\alpha\in\Sigma^+} \big(\frac{\sinh\langle\alpha,g^+\rangle}{\langle\alpha,g^+\rangle}\big)^{m_\alpha}$ jacobian exponential map
  - the coefficients $u_k$ are smooth bi–$K$–invariant functions on $G$, which are bounded together with their derivatives,
  - remainder estimate :

$$|E_s(g)| \lesssim (1+s)^{3(\frac{d}{2}+1)}\, e^{-\langle\rho,g^+\rangle}$$

# Dispersive estimates for Schrödinger on $X = G/K$

**AMPVZ 2021**

Let $2 < p < \infty$. Then

$$\|e^{it\Delta}\|_{L^{q'} \to L^q} \lesssim \begin{cases} |t|^{-\left(\frac{1}{2}-\frac{1}{q}\right)d} & \text{if } 0 < |t| < 1 \\ |t|^{-\frac{D}{2}} & \text{if } |t| \geq 1 \end{cases}$$

**Kunze–Stein phenomenon (special version)**

Let $2 \leq q < \infty$ and let $\kappa$ be a (reasonable) bi–$K$–invariant function on $G$. Then

$$\|\cdot * \kappa\|_{L^{q'}(X) \to L^q(X)} \lesssim \left( \int_G dg\, \varphi_0(g)\, |\kappa(g)|^{\frac{q}{2}} \right)^{\frac{2}{q}}$$

# Strichartz inequality for Schrödinger on $X = G/K$

## Linear Cauchy problem

$$\begin{cases} i\,\partial_t u(t,x) + \Delta_x u(t,x) = F(t,x) \\ u(0,x) = f(x) \end{cases}$$

Duhamel's formula :

$$u(t,x) = e^{it\Delta_x} f(x) - i \int_0^t ds\, e^{i(t-s)\Delta_x} F(s,x)$$

Space–time Strichartz norm :

$$\|u\|_{L^p(\mathbb{R};L^q(X))} = \Big[ \int_{\mathbb{R}} dt \Big( \int_X dx\, |u(t,x)|^q \Big)^{\frac{p}{q}} \Big]^{\frac{1}{p}}$$

## (Global–in–time) Strichartz inequality

$$\|u\|_{L^p(\mathbb{R};L^q(X))} \lesssim \|f\|_{L^2(X)} + \|F\|_{L^{\tilde{p}'}(\mathbb{R};L^{\tilde{q}'}(X))}$$

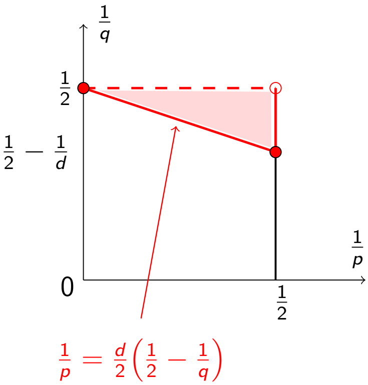
for all admissible pairs $(p,q)$ and $(\tilde{p},\tilde{q})$

# Admissible pairs for $X = G/K$

**Definition**

$(p, q)$ is admissible if $(\frac{1}{p}, \frac{1}{q})$ belongs to the triangle

$$\{(\tfrac{1}{p}, \tfrac{1}{q}) \in (0, \tfrac{1}{2}] \times (0, \tfrac{1}{2}) \mid \tfrac{1}{p} \geq \tfrac{d}{2}(\tfrac{1}{2} - \tfrac{1}{q})\} \cup \{(0, \tfrac{1}{2})\}$$



# Semi–linear Schrödinger equation on $X = G/K$

## Semi–linear Cauchy problem

(NLS) $$\begin{cases} i\partial_t u(t,x) + \Delta_x u(t,x) = F(t,x) \\ u(0,x) = f(x) \end{cases}$$

with power–like nonlinearities of order $\gamma > 1$:

$$\begin{cases} |F(u)| \lesssim |u|^\gamma \\ |F(u) - F(v)| \lesssim (|u|^{\gamma-1} + |v|^{\gamma-1})\,|u-v| \end{cases}$$

## Global well–posedness

- For every $1 < \gamma \le 1 + \frac{4}{d}$, (NLS) is globally well posed for small $L^2$ initial data
- For every $1 < \gamma \le 1 + \frac{4}{d-2}$, (NLS) is globally well posed for small $H^1$ initial data

# Semi–linear Schrödinger equation on $X = G/K$

## Scattering for small initial data

- If $1 < \gamma \le 1 + \frac{4}{d}$, there exists $u_\pm \in L^2(X)$ such that
$$\| u(t,\cdot) - e^{it\Delta} u_\pm \|_{L^2(X)} \longrightarrow 0 \qquad \text{as } t \to \pm\infty$$
- If $1 < \gamma \le 1 + \frac{4}{d-2}$, there exists $u_\pm \in H^1(X)$ such that
$$\| u(t,\cdot) - e^{it\Delta} u_\pm \|_{H^1(X)} \longrightarrow 0 \qquad \text{as } t \to \pm\infty$$

## On $\mathbb{R}^d$

scattering fails for powers $\gamma > 1$ small enough

# Wave equation on $X = G/K$

Similar results with some differences

- additional smoothness conditions $\rightsquigarrow$ statements more involved
- finite propagation speed $\rightsquigarrow$ helpful
- kernel estimates are somewhat $\begin{cases} \text{harder for large time} \\ \text{easier for small time} \end{cases}$
- global well–posedness holds for small initial data
  $\rightsquigarrow$ no blow–up for small powers $\gamma > 1$
  in contrast with $\mathbb{R}^d$ (Strauss conjecture)
  (also Sire–Sogge–Wang 2019, Sire–Sogge–Wang–Zhang 2020)

Thank you for your attention