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Smooth barycentric decompositions
of Weyl chambers
(a helpful tool to handle
inverse spherical Fourier transforms
on noncompact symmetric spaces of higher rank)

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Riemannian symmetric spaces of noncompact type (RSSN)

- ▶ G semisimple Lie group
(noncompact, connected, finite center)
- ▶ K maximal compact subgroup of G
- ▶ **RSSN**: $X = G/K$ Cartan-Hadamard manifold
- ▶ Cartan decomposition $\sim$ generalized polar decomposition:

$$G = K(\exp \overline{\mathfrak{a}^+})K \ni x = k_1(\exp \mathfrak{x}^+)k_2$$

- ▶ Haar measure: $dx = \text{const. } \delta(\mathfrak{x}^+) dk_1 d\mathfrak{x}^+ dk_2$
 where $\delta(\mathfrak{x}^+) = \prod_{\alpha \in \Sigma^+} (\sinh \langle \alpha, \mathfrak{x}^+ \rangle)^{m_\alpha} \sim e^{\langle 2\rho, \mathfrak{x}^+ \rangle}$
 and $\rho = \sum_{\alpha \in \Sigma^+} \frac{m_\alpha}{2} \alpha \in \mathfrak{a}^+$
- ▶ Dimensions:

$$\ell = \dim \mathfrak{a}$$

rank

$$d = \ell + \sum_{\alpha \in \Sigma^+} m_\alpha$$

dimension

$$D = \ell + 2|\Sigma_r^+|$$

pseudo-dimension

Fourier analysis on RSSN [Harish-Chandra, Helgason]

There is a Fourier transform on RSSN $X = G/K$.

For bi- K -invariant functions $f : G \rightarrow \mathbb{C}$, it reduces to

Spherical Fourier transform

$$\mathcal{H}f(\lambda) = \int_G dx f(x) \varphi_{-\lambda}(x) \quad \forall \lambda \in \mathfrak{a}$$

Inversion formula

$$f(x) = \text{const.} \int_{\mathfrak{a}} d\lambda |\mathbf{c}(\lambda)|^{-2} \varphi_{\lambda}(x) \mathcal{H}f(\lambda)$$

- ▶ The spherical functions $\varphi_{\lambda}(x)$ are analogs of Bessel functions for the Euclidean Fourier transform of radial functions
- ▶ Behavior of $\varphi_{\lambda}(x)$ $\left\{ \begin{array}{l} \text{a lot of information available} \\ \text{still not fully understood} \end{array} \right.$
- ▶ The Plancherel measure $|\mathbf{c}(\lambda)|^{-2}$ is known explicitly [Gindikin-Karpelevich]

Two main dispersive PDE on RSSN

Schrödinger equation

$$(S) \quad \begin{cases} i \partial_t u(t, x) \pm \Delta_x u(t, x) = F(t, x) \\ u(0, x) = f(x) \end{cases}$$

Remark. Heat equation with imaginary time

Wave equation

$$(W) \quad \begin{cases} \partial_t^2 u(t, x) - \Delta_x u(t, x) = F(t, x) \\ u(0, x) = f(x), \partial_t|_{t=0} u(t, x) = g(x) \end{cases}$$

Remarks.

- ▶ Similar analysis and properties
- ▶ Simpler statements for Schrödinger

Schrödinger equation on RSSN

Homogeneous solution

$$u(t, x) = e^{it\Delta} f(x) = (f * s_t)(x)$$

where the Schrödinger kernel

$$s_t(x) = \text{const.} \int_{\mathfrak{a}} d\lambda |\mathbf{c}(\lambda)|^{-2} \varphi_{\lambda}(x) e^{-i(\|\rho\|^2 + \|\lambda\|^2)t}$$

is formally the heat kernel with imaginary time

Inhomogeneous solution (Duhamel's formula)

$$u(t, \cdot) = e^{it\Delta} f - i \int_0^t ds e^{i(t-s)\Delta} F(s, \cdot)$$

Main information = (sharp) kernel estimates

Rank one [A-Pierfelice 2009]

For $t \in \mathbb{R}^*$ and $r \geq 0$,

$$|s_t(r)| \lesssim e^{-\rho r} \times \begin{cases} (1+r)^{\frac{d-1}{2}} |t|^{-\frac{d}{2}} & \text{if } 0 < |t| \leq 1+r \\ (1+r) |t|^{-\frac{3}{2}} & \text{if } |t| \geq 1+r \end{cases}$$

Higher rank [A-Meda-Pierfelice-Vallarino-Zhang 2023]

There exists $n > 0$ such that, for $t \in \mathbb{R}^*$ and $x \in G/K$,

$$|s_t(x)| \lesssim e^{-\langle \rho, x^+ \rangle} (1 + \|x^+\|)^n \times \begin{cases} |t|^{-\frac{d}{2}} & \text{if } 0 < |t| < 1 \\ |t|^{-\frac{D}{2}} & \text{if } |t| \geq 1 \end{cases}$$

Two important inequalities

Dispersive inequality [A-P, A-M-P-V-Z]

Let $2 < q \leq \infty$. Then

$$\|e^{it\Delta}\|_{L^{q'} \rightarrow L^q} \lesssim \begin{cases} |t|^{-(\frac{1}{2} - \frac{1}{q})d} & \text{if } 0 < |t| < 1 \\ |t|^{-\frac{D}{2}} & \text{if } |t| \geq 1 \end{cases}$$

Strichartz mixed norm :

$$\|u(t, x)\|_{L_t^p L_x^q} = \left[\int_{\mathbb{R}} dt \left(\int_{G/K} dx |u(t, x)|^q \right)^{\frac{p}{q}} \right]^{\frac{1}{p}}$$

Strichartz inequality [A-P, A-M-P-V-Z]

Solutions to (S) satisfy

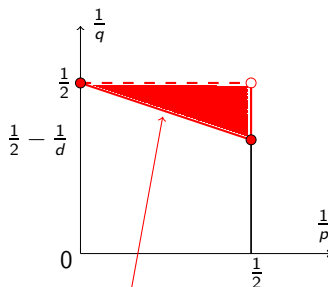
$$\|u(t, x)\|_{L_t^{\tilde{p}} L_x^{\tilde{q}}} \lesssim \|f(x)\|_{L_x^2} + \|F(t, x)\|_{L_t^{p'} L_x^{q'}}$$

for all **admissible** couples $(p, q), (\tilde{p}, \tilde{q})$

Strichartz inequality (continued)

Definition

A couple (p, q) is **admissible** if $(\frac{1}{p}, \frac{1}{q})$ belongs to the triangle

$$\{(\frac{1}{p}, \frac{1}{q}) \in (0, \frac{1}{2}] \times (0, \frac{1}{2}) \mid \frac{1}{p} \geq \frac{d}{2} (\frac{1}{2} - \frac{1}{q})\} \cup \{(0, \frac{1}{2})\}$$


$$\frac{1}{p} = \frac{d}{2} \left(\frac{1}{2} - \frac{1}{q} \right)$$

Main tools

- ▶ Smooth barycentric decomposition of Weyl chambers
 $\rightsquigarrow$ kernel estimates
- ▶ Improved Hadamard parametrix for the wave operator
 $\cos t \sqrt{-\Delta}$ on G/K
- ▶ Kunze-Stein phenomenon
 $\rightsquigarrow$ dispersive inequality for $|t|$ large
- ▶ TT^* argument (Ginibre-Velo, Keel-Tao)
 $\rightsquigarrow$ Strichartz inequality

Suitable version of the Kunze-Stein phenomenon

Let $2 \leq q < \infty$. Then there exists a constant $C > 0$ such that, for every bi- K -invariant (measurable) function $\hat{k}$ on G ,

$$\|\cdot * \hat{k}\|_{L^{q'} \rightarrow L^q} \leq C \left\{ \int_G dx \varphi_0(x) |\hat{k}(x)|^{\frac{q}{2}} \right\}^{\frac{2}{q}}$$

Motivation

The analysis of oscillating integrals such as

$$s_t(x) = \text{const.} \int_{\mathfrak{a}} d\lambda |\mathbf{c}(\lambda)|^{-2} \varphi_\lambda(x) e^{-i(\|\rho\|^2 + \|\lambda\|^2)t}$$

requires **integrations by parts**. The Plancherel density

$$|\mathbf{c}(\lambda)|^{-2} = \prod_{\alpha \in \Sigma_{\text{red}}^+} \left| \mathbf{c}_\alpha \left(\frac{\langle \alpha, \lambda \rangle}{\|\alpha\|^2} \right) \right|^{-2}$$

is a product of one-dimensional differentiable symbols but, in higher rank, it is **not** a differentiable symbol in general.

Thus differentiating arbitrarily $|\mathbf{c}(\lambda)|^{-2}$ produces no additional global decay at infinity.

We overcome this problem

by splitting up each Weyl chamber $w.\mathfrak{a}^+$ into subsectors $w.S_j$, where we differentiate along a suitable direction $w.\lambda_j$.

Rough barycentric decomposition

$$\sum_{w \in W} \sum_{1 \leq j \leq \ell} \mathbb{I}_{w.S_j} = 1 \quad \text{a.e.}$$

- ▶ Simple roots: $\alpha_1, \dots, \alpha_\ell$
- ▶ Dual basis of $\mathfrak{a}$: $\lambda_1, \dots, \lambda_\ell$ fundamental co-weights
 $\rightsquigarrow \overline{\mathfrak{a}^+} = \mathbb{R}_+ \lambda_1 + \dots + \mathbb{R}_+ \lambda_\ell$
- ▶ Subsectors of $\overline{\mathfrak{a}^+}$:
 $S_j = \{H \in \overline{\mathfrak{a}^+} \mid \langle \alpha_j, H \rangle = \max_{1 \leq k \leq \ell} \langle \alpha_k, H \rangle\} \quad \forall 1 \leq j \leq \ell$
 $\rightsquigarrow \overline{\mathfrak{a}^+} = \bigcup_{1 \leq j \leq \ell} S_j$
 $\rightsquigarrow \mathfrak{a} = \bigcup_{w \in W} w.\overline{\mathfrak{a}^+} = \bigcup_{w \in W} \bigcup_{1 \leq j \leq \ell} w.S_j$

Examples of barycentric subdivisions of $\overline{\mathfrak{a}^+}$

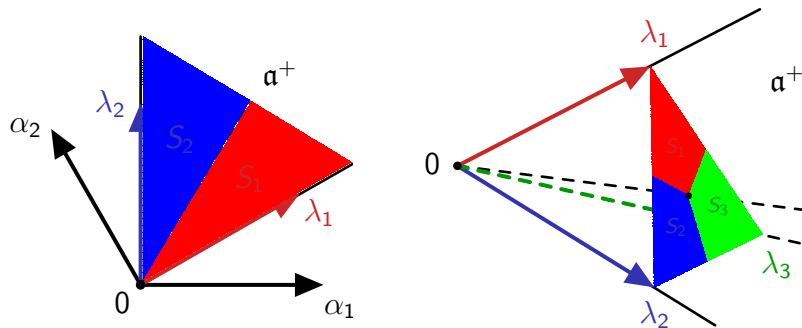


Figure: Root systems A_2 and A_3

Smooth barycentric decomposition

$$\sum_{w \in W} \sum_{1 \leq j \leq \ell} \chi_{w.S_j} = 1 \quad \text{on } \mathfrak{a} \setminus \{0\}$$

- ▶ $\chi_{w.S_j}$ is a smooth homogeneous symbol of order 0 on $\mathfrak{a} \setminus \{0\}$
- ▶ **Dichotomy :** for every $\alpha \in \Sigma$, $w \in W$ and $1 \leq j \leq \ell$,
 - ▶ either $\langle \alpha, w.\lambda_j \rangle = 0$
 - ▶ or $\langle \alpha, \lambda \rangle \asymp \|\lambda\| \quad \forall \lambda \in \text{supp } \chi_{w.S_j}$

Application

Away from the origin, each function

$$\chi_{w.S_j}(\lambda) |\mathbf{c}(\lambda)|^{-2}$$

behaves as a symbol of order $n - \ell$ under differentiation along $w.\lambda_j$,

$$\text{i.e.,} \quad \left| \partial_{w.\lambda_j}^N \{ \chi_{w.S_j}(\lambda) |\mathbf{c}(\lambda)|^{-2} \} \right| \lesssim \|\lambda\|^{d-\ell-N} \quad \forall \|\lambda\| \gtrsim 1$$

Some references

► Rank 1 :

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► Higher rank :

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The End

Thank you for your attention