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Equation des ondes multi-temporelle discrète, avec application à l'unique ergodicité quantique arithmétique

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Joint work with

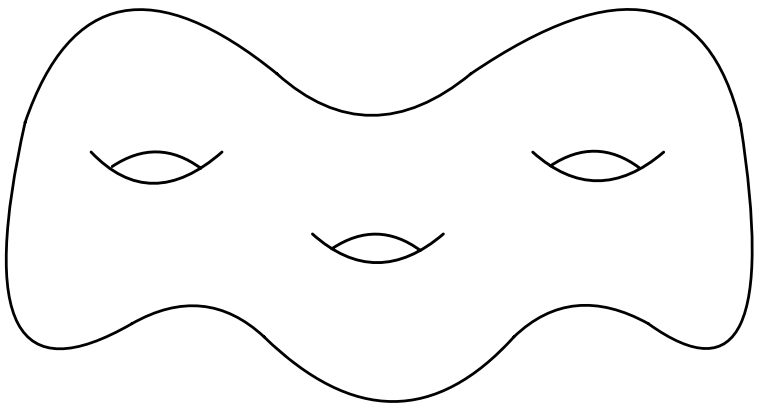
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Content

1. What is quantum unique ergodicity
   for (arithmetic) hyperbolic surfaces ?
2. Multi-temporal waves on non compact symmetric spaces
3. Multi-temporal waves on affine buildings
4. Conclusion

Hyperbolic surfaces

- $\mathbb{H}^2 = \{z = x + iy \in \mathbb{C} \mid y > 0\}$ upper half-space
- $ds^2 = \frac{dx^2+dy^2}{y^2}$ hyperbolic metric
- $\mathrm{PSL}(2,\mathbb{R})$ acts isometrically on $\mathbb{H}^2$ by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right].z = \frac{az+b}{cz+d}$
- Γ discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ $\begin{cases} \text{torsion free} \\ \text{cocompact} \end{cases}$
- $X = \Gamma\backslash\mathbb{H}^2$ compact Riemann surface $\begin{cases} \text{curvature} = -1 \\ \text{genus} \geq 2 \end{cases}$
- All such surfaces are obtained this way



Quantum unique ergodicity

QUE conjecture [Rudnick–Sarnak 1994]
$\sim$ delocalization of high-energy eigenfunctions

Consider any sequence (ϕ_j) of normalized eigenfunctions

$$\begin{cases} -\Delta\phi_j = \lambda_j^2 \phi_j \\ \int_X |\phi_j|^2 \mathrm{dvol} = 1 \end{cases}$$

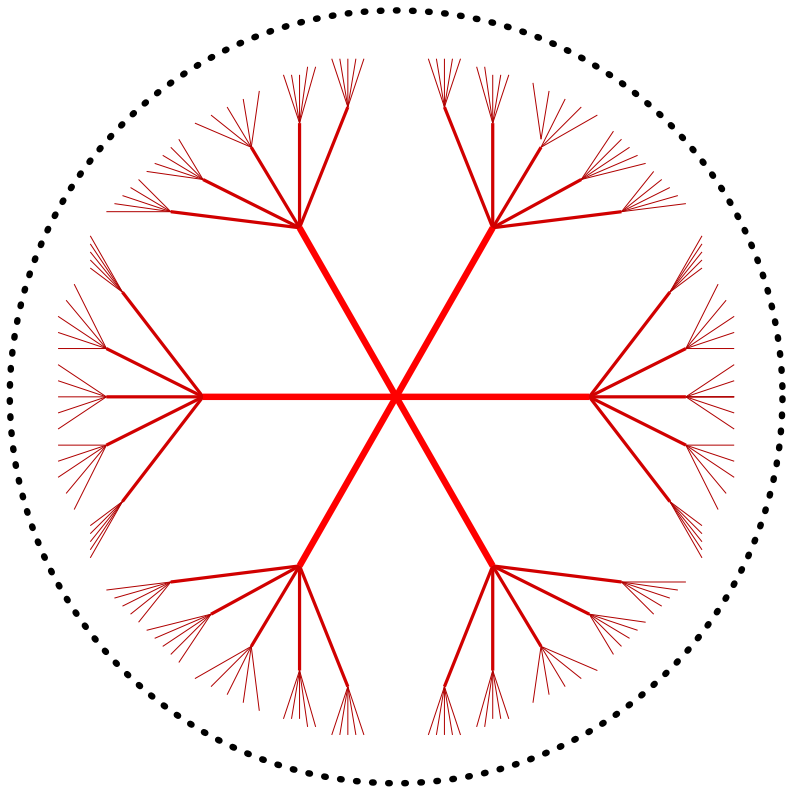
of the laplacian on X with eigenvalues $\lambda_j^2 \nearrow \infty$.
Then the normalized volume $\frac{1}{\mathrm{vol}\, X}\mathrm{dvol}$ is the only possible weak-* cluster value of $|\phi_j|^2 \mathrm{dvol}$.

- Actually, everything is lifted to the unit tangent bundle over X

real versus q-adic

- $\mathrm{PSL}(2,\mathbb{R})/\mathrm{PSO}(2) \approx \mathbb{H}^2$ hyperbolic plane
- q prime number : $\mathrm{PGL}(2,\mathbb{Q}_q)/\mathrm{PGL}(2,\mathbb{Z}_q) \approx \mathbb{T}_q$
  homogeneous tree with $q+1$ edges

$q = 5$:



Arithmetic surfaces and Hecke operators

Example

$$\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}^2 \approx$$
$$\approx \mathrm{PSL}(2,\mathbb{Z}[\tfrac{1}{q}])\backslash \mathrm{PSL}(2,\mathbb{R})\times \mathrm{PGL}(2,\mathbb{Q}_q)/\mathrm{PSO}(2)\times \mathrm{PGL}(2,\mathbb{Z}_q)$$

- $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ has torsion
- $\Gamma\backslash\mathbb{H}^2$ is not compact, but has finite volume
- All arithmetic surfaces are (roughly) constructed this way, by considering finite places and arithmetic type conditions

Commuting operators

- continuous laplacian descending from the hyperbolic plane $\mathrm{PSL}(2,\mathbb{R})/\mathrm{PSO}(2) \approx \mathbb{H}^2$
- discrete laplacians $\sim$ averaging operators (Hecke operators) descending from the homogeneous trees $\mathrm{PGL}(2,\mathbb{Q}_q)/\mathrm{PGL}(2,\mathbb{Z}_q) \approx \mathbb{T}_q$

AQUE for surfaces

Theorem [Lindenstrauss 2006]

QUE holds if

- X is a compact arithmetic surface
- ϕ_j are joint eigenfunctions of the laplacian and all Hecke operators

Theorem [Soundararajan 2010]

Same result for $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}^2$

Theorem [Brooks–Lindenstrauss 2010/2013/2014]

QUE holds if

- X is a compact arithmetic surface
- ϕ_j are joint eigenfunctions of the laplacian and Hecke operators at a single place

Discrete wave equation on homogeneous trees

Brooks & Lindenstrauss use the following discrete wave equation on the homogeneous tree $\mathbb{T}_q$ to construct suitable kernels (Iwaniec amplifiers).

Discrete wave equation on $\mathbb{Z} \times \mathbb{T}_q$

$$\begin{cases} u(t+1,x) + u(t-1,x) = \frac{1}{\sqrt{q}} \sum_{d(y,x)=1} u(t,y) \\ u(0,x) = f(x), \ \frac{u(x,1) - u(x,-1)}{2} = g(x) \end{cases}$$

Properties [Brooks–Lindenstrauss, AMPS]

Assume that the initial data f, g are supported in the ball $B(x_0, r)$. Then

- $u(t, \cdot)$ is supported in the ball $B(x_0, r + |t|)$
- $u(t,x) = O(q^{-|t|/2})$
- The total energy is conserved and asymptotically distributed equally between potential and kinetic energy

Symmetric spaces of noncompact type

- G semisimple Lie group $\begin{cases} \text{connected} \\ \text{non compact} \\ \text{finite center} \end{cases}$
- K maximal compact subgroup
- $X = G/K$ Riemannian manifold $\begin{cases} \text{simply connected } (\approx \mathbb{R}^d) \\ \text{non positive curvature} \end{cases}$
- $G = K(\exp \overline{\mathfrak{a}_+})K$ Cartan decomposition
  (generalized polar decomposition)
  $\mathfrak{a}_+$ cone $\subset \mathfrak{a}$ Euclidean space of dimension ℓ (rank)
- $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ Weyl group

Example

$G = \mathrm{SL}(n, \mathbb{R})$, $K = \mathrm{SO}(n)$, $X = \mathrm{SPos}(n, \mathbb{R})$
$\mathfrak{a} = \{\mathrm{diag}(r_1, \ldots, r_n) \mid r_1 + \cdots + r_n = 0\}$
Cartan $\sim$ diagonalization of positive definite symmetric matrices
$\ell = n - 1$, $W = \mathfrak{S}_n$

Multi-temporal waves

In higher rank $\ell > 1$, the laplacian Δ is replaced by the algebra $\mathbb{D}(X) = \mathbb{C}[\Delta_1 = \Delta, \Delta_2, \ldots, \Delta_\ell]$ of all G-invariant differential operators on $X = G/K$

Multi-temporal wave equation on $\mathfrak{a} \times X$

$$\sigma(D)(\tfrac{\partial}{\partial t})u(t,x) = D_x u(t,x) \qquad \forall\, D \in \mathbb{D}(X)$$

where $\sigma : \mathbb{D}(X) \xrightarrow{\sim} \mathcal{P}(\mathfrak{a})^W$ Harish-Chandra isomorphism (kind of symbol)

- **References**
  - Semenov-Tian-Shansky 1976
  - Phillips-Shahshahani 1993
  - Helgason 1998/2008
- **Aim.** Lax-Phillips scattering theory in higher rank

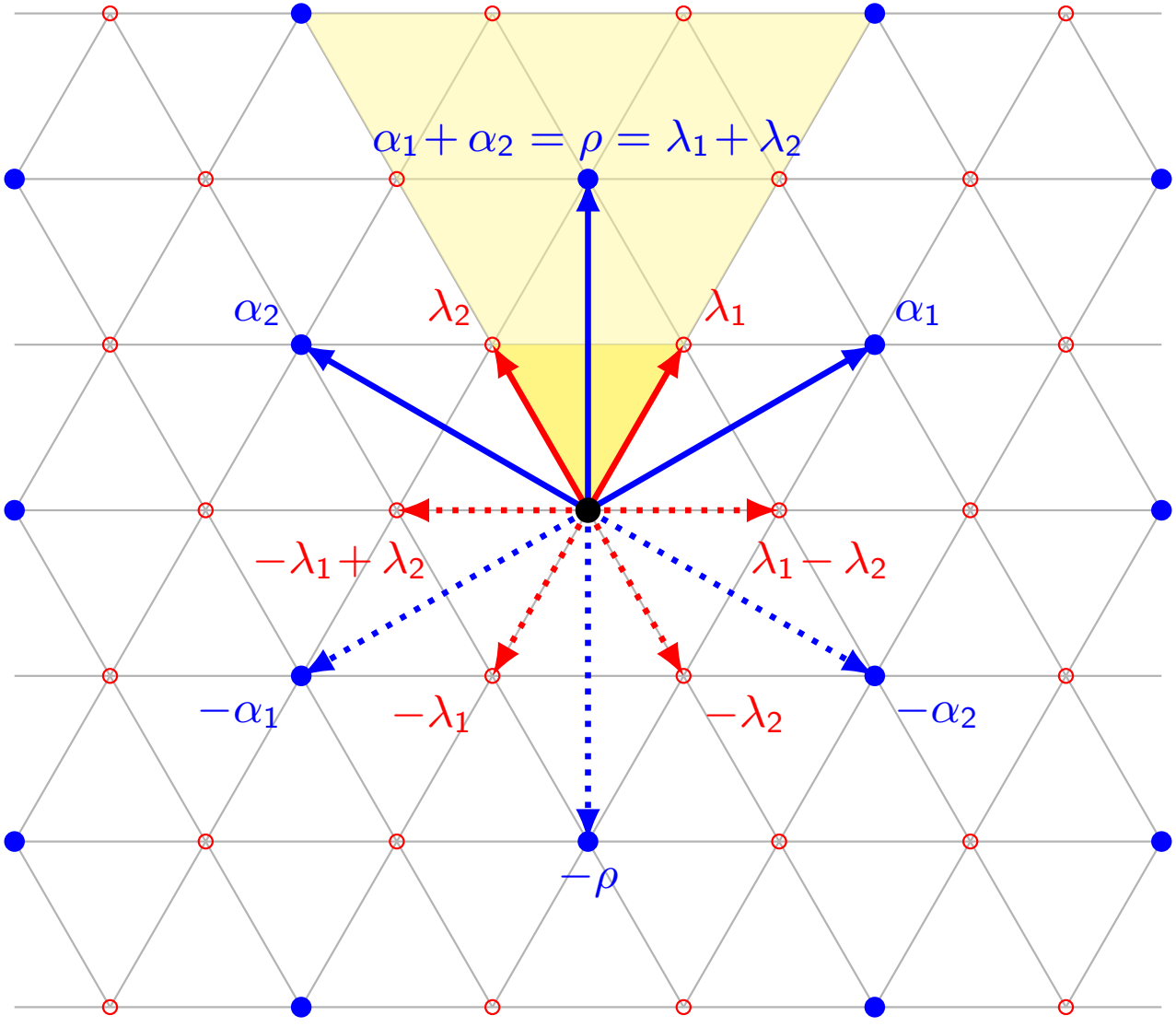
Affine buildings $\sim$ q-adic symmetric spaces

- **Homogeneous trees** are 1-dimensional simplicial complexes, which can be obtained by piecing together, along combinatorial rules, infinitely many exemplars of the tiling of $\mathbb{R}$ by $\mathbb{Z}$.
- Similarly, **affine buildings** are ℓ-dimensional simplicial complexes, which can be obtained by piecing together, along combinatorial rules, infinitely many exemplars (apartments) of a crystallographic tiling of an ℓ-dimensional Euclidean space $\mathfrak{a}$ by a lattice P (co-weights).

Remarks

- Affine buildings in dimension 1 are (semi-)homogenous trees
- Most affine buildings are Bruhat–Tits buildings associated to q-adic Lie groups, e.g. $\mathrm{PGL}(n, \mathbb{Q}_q)/\mathrm{PGL}(n, \mathbb{Z}_q)$ is an affine building of type $\widetilde{A}_{n-1}$

Apartment in buildings of type $\widetilde{A}_2$



Fourier transforms

Fourier transform on the lattice P

$$\mathcal{F}_P f(\vartheta) = \sum_{\lambda \in P} f(\lambda)\, e^{i\langle \lambda, \vartheta \rangle}$$

$$f(\lambda) = (2\pi)^{-\ell} \int_{\mathfrak{a}/2\pi P^*} \mathcal{F}_P f(\vartheta)\, e^{-i\langle \lambda, \vartheta \rangle} \mathrm{d}\vartheta$$

Fourier transform on the set X of vertices of an affine building

$$\mathcal{F}_X f(\vartheta, \omega) = \sum_{x \in X} f(x) \overbrace{\mathcal{P}_\vartheta(x, \omega)}^{\text{Poisson kernel}}$$

$$f(x) = \int_{\mathfrak{a}/2\pi P^*} \int_{\Omega} \mathcal{F}_X f(\vartheta, \omega) \mathcal{P}_{-\vartheta}(x, \omega) \nu(\mathrm{d}\omega) \mu(\mathrm{d}\vartheta)$$

Fourier multipliers: $\forall\, p \in \mathbb{C}[P]$ (trigonometric polynomials)

$$m(p) f = (\mathcal{F}_P)^{-1} \left[\vartheta \mapsto p(i\vartheta)\, \mathcal{F}_P f(\vartheta) \right]$$

$$M(p) f = (\mathcal{F}_X)^{-1} \left[(\vartheta, \omega) \mapsto p(i\vartheta)\, \mathcal{F}_X f(\vartheta, \omega) \right]$$

Multi-temporal waves

Multi-temporal wave equation on $P \times X$

$$\begin{cases} m(p)_t\, u(t,x) = M(p)_x\, u(t,x) & \forall\ p \in \mathbb{C}[P]^W \\ m(p_j)_t\, u(t,x)\big|_{t=0} = f_j(x) & \forall\ 1 \leq j \leq |W| \end{cases}$$

Properties [ART, APRT]

- Existence & uniqueness
- Finite propagation speed
- $|u(t,x)| \leq C\, e^{-c|t|}$
- Conservation of total energy & asymptotic distribution
- Construction of Iwaniec amplifiers
- Basic Lax-Phillips scattering theory

Summary

manifold	$\Gamma\backslash\mathbb{H}^2$ arithmetic hyperbolic surface	$\Gamma\backslash G/K$ arithmetic locally symmetric space
differential operators	$\mathbb{C}[\Delta]$	$\mathbb{D}(G/K)$
Hecke operators	averaging operators on homogeneous tree	averaging operators on affine building
discrete wave equation	one-dimensional discrete time	multi-dimensional discrete time
Iwaniec amplifiers	✓	✓
AQUE	✓	hopefully more cases

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Thank you for your attention