NON-AUTONOMOUS PARABOLIC IMPLOSION

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Abstract. We study parabolic implosion in a general non-autonomous setting. Let $f(w) = w + w^2 + O(w^3)$ be a holomorphic germ tangent to the identity. We consider the iteration of non-autonomous perturbations of the form

$$w_{j+1} = f(w_j) + \varepsilon_{j,n}^2.$$

We show that, when the $\varepsilon_{j,n}^2$'s satisfy a Lavaurs-type condition, the element w_n can be described by means of a suitable Lavaurs map L_{u_n}, whose phase u_n is an explicit function of the perturbation parameters. In particular, whenever $u_n \to u \in \mathbb{C}$, the non-autonomous dynamics converges locally uniformly on compact subsets of the parabolic basin to the corresponding Lavaurs map L_u.

Our study provides a general description of additive non-autonomous parabolic implosion and yields several deterministic and random convergence results as corollaries, as well as a unified proof of several previous results. As an application, we also obtain strong discontinuity results for the Julia sets of fibered holomorphic endomorphisms of $\mathbb{P}^2(\mathbb{C})$.

1. Introduction

1.1. **Parabolic points and implosion.** Parabolic fixed points are among the main sources of bifurcations in one–dimensional holomorphic dynamics; we refer to [Mil11; CG13] for background. Let

$$f(w) = w + w^2 + O(w^3)$$

be a holomorphic germ tangent to the identity at the origin. The local dynamics of f is described by attracting and repelling Fatou coordinates and the associated horn maps, which also encode the analytic classification of simple parabolic germs (Écalle–Voronin theory [Éca85; Vor81]), see e.g. [DS14; DS15].

In the classical *autonomous* setting, parabolic implosion concerns the limiting dynamics of suitably rescaled perturbations of f. Following the seminal work of Douady–Hubbard and Lavaurs [DH85; DH84; Lav89], one considers additive perturbations

$$f_\varepsilon(w) = f(w) + \varepsilon^2,$$

with $\varepsilon \to 0$ along sequences of the form

$$\varepsilon_n = \frac{\pi}{n} + \frac{\pi\sigma}{n^2} + O\Big(\frac{1}{n^{2+\alpha}}\Big), \qquad \alpha > 0.$$

Then the renormalized dynamics after n iterates converges (locally uniformly on compact subsets of the parabolic basin $\mathcal{B}_f$) to a *Lavaurs map* L_σ associated with f. Parabolic implosion has deep consequences in complex dynamics; for instance it explains the discontinuity of Julia sets at parabolic parameters [Dou94] and it plays a key role in Shishikura's proof that the boundary of the Mandelbrot set has Hausdorff dimension two [Shi98]. We refer to [PV20] for an overview of the theory and to [BC12; CS15; IS06] for further consequences and applications.

Parabolic implosion techniques have also been developed in higher dimension, where these phenomena appear naturally in the study of semi-parabolic or tangent-to-the-identity dynamics. In particular, they have been used to construct genuinely higher–dimensional phenomena such as wandering Fatou components for polynomial skew products [Ast+16; ABP23; AT26], and to obtain discontinuity statements for two–dimensional families [BSU17; DL15; Bia19; ALR26] and structural properties for the chaotic sets [AB25].

The purpose of this paper is to study an analogue of parabolic implosion in a general *non-autonomous* setting, where the perturbation is allowed to vary at each iterate. This type of problem is motivated both by higher–dimensional skew-product dynamics and by random models. In one complex dimension, a first result in this direction is due to Vivas [Viv20], see also the recent work [HSV26]. Here we develop a general additive non-autonomous theory for arbitrary parabolic germs and obtain an explicit description of the limiting Lavaurs phase.

1.2. **Non-autonomous perturbations and main result.** Let

$$f(w) = w + w^2 + O(w^3) \tag{1}$$

be a holomorphic germ at the origin. For each integer n, we consider sequences of perturbations $\varepsilon_{k,n}$ and define the non-autonomous iteration

$$w_{k+1}^{(n)} = f(w_k^{(n)}) + \varepsilon_{k,n}^2, \qquad k = 0, \ldots, n-1. \tag{2}$$

We are interested in the asymptotic behavior of $w_n^{(n)}$ as $n \to \infty$, under perturbations of Lavaurs type

$$\varepsilon_{k,n} = \frac{\pi}{n} + \frac{\pi}{n^2}\sigma_{k,n} + O\Big(\frac{1}{n^{2+\alpha}}\Big), \tag{3}$$

where $\alpha > 0$ can be taken arbitrarily small (in particular, for simplicity, we will always assume that $\alpha < 1$). In contrast with the autonomous case, now the perturbation not only depends on n but is also allowed to vary with the time index k. Our main result shows that the non-autonomous system still converges to a Lavaurs map, whose phase is obtained as an explicit weighted average of the perturbation parameters.

Let L_u denote the classical Lavaurs map associated with f and phase $u \in \mathbb{C}$. This map depends only on f and u and has an explicit expression, see Section 2.2.

Theorem 1.1. *Let f, $\{w_k^{(n)}\}$, and $\varepsilon_{k,n}$ be as in* (1)*,* (2)*, and* (3)*. Assume that the $(\sigma_{k,n})$ as in* (3) *are uniformly bounded. Then*

$$w_n^{(n)} = L_{u_n}(w_0) + o(1), \qquad n \to \infty,$$

locally uniformly on the parabolic basin $\mathcal{B}_f$, where the phase u_n is given by

$$u_n = \frac{1}{n}\sum_{k=0}^{n-1} \sigma_{k,n}\, G\left(\frac{k+1}{n}\right), \qquad G(x) = 2\sin^2(\pi x).$$

In particular, whenever $u_n \to u$, the iterates converge to the Lavaurs map L_u. We observe here that the function G is universal: it does not depend on the higher-order terms of f.

1.3. **Consequences.** We list here a few consequences of Theorem 1.1. As a first application we recover the following result, which is the core technical part of [Ast+16].

Corollary 1.2 (Proposition A, [Ast+16]). *Take $F(z,w) = (p(z), q(w) + \frac{\pi^2}{4}z)$, where $p(z) = z - z^2 + O(z^3)$ and $q(w) = w + w^2 + O(w^3)$. Then*

$$F^{2n+1}(p^{n^2}(z), w) = (0, L_0(w) + o(1))$$

with local uniform convergence on $\mathcal{B}_p \times \mathcal{B}_q$.

We also recover a more general version of the main result of [Viv20], which was proved in the particular case where $f(w) = \frac{w}{1-w}$, see also [HSV26]. Observe that, for this choice of f, the maps ϕ^ι and ϕ^o as in Section 2.1 are equal to $-1/w$ (see (6) for the general expression), hence the map L_0 is equal to the identity.

Corollary 1.3. *Let f, $\{w_k^{(n)}\}$, and $\varepsilon_{k,n}$ be as in (1), (2), and (3), and assume that*

$$\sigma_{k,n} + \sigma_{n-2-k,n} = O\left(\frac{1}{n}\right) \qquad \text{for every } n \in \mathbb{N} \text{ and } 0 \le k \le n-2. \tag{4}$$

Then

$$w_n^{(n)} = L_0(w_0) + o(1).$$

We can also see the $\sigma_{k,n}$'s as random variables. For instance, we have the following probabilistic version of Theorem 1.1.

Corollary 1.4. *Let f, $(w_k^{(n)})$, and $\varepsilon_{k,n}$ be as in (1), (2), and (3), where $(\sigma_{k,n}) = (\sigma_k)$ is a sequence of uniformly bounded random variables such that*

$$\frac{1}{n}\sum_{k=0}^{n-1} \sigma_k \to u \in \mathbb{C} \qquad \text{almost surely.}$$

Then, for every $w_0 \in \mathcal{B}_f$, the sequence of random variables $(w_n^{(n)})$ converges to $L_u(w_0)$ almost surely.

Finally, one can also consider the case where the σ_k's arise (deterministically) from the action of a second dynamical system.

Corollary 1.5. *Let $(\Omega, \mathcal{T}, \mu)$ be a probability space, $T : \Omega \to \Omega$ be an ergodic transformation, and take $\sigma \in L^\infty(\Omega, \mathcal{T}, \mu)$. Consider a sequence of measurable skew-products of the form*

$$f_n(z,w) = (T(z), q(w) + \varepsilon_n(z)^2), \qquad \text{where } \varepsilon_n(z) := \frac{\pi}{n} + \frac{\pi}{n^2}\sigma(z) + O\Big(\frac{1}{n^{2+\alpha}}\Big).$$

Then for μ-almost every z,

$$\lim_{n\to\infty} \pi_2 \circ f_n^n(z,w) = L_u(w), \qquad \text{where} \quad u := \int \sigma(z) d\mu(z)$$

and $\pi_2 : \Omega \times \mathbb{C}$ denotes the projection on the second coordinate.

Finally, in the spirit of the discontinuity of the Julia sets in one-dimensional dynamics, we have the following result about the limits of Julia sets (i.e., the supports of the unique measure of maximal entropy) for endomorphisms of $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$.

Corollary 1.6. *Let $F : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic endomorphism which, in some affine chart, can be written in the form $F(z, w) = (p(z), q(w))$, where p is a rational map and q is a polynomial map of the form $q(w) = w + w^2 + O(w^3)$. Let F_n be a sequence of holomorphic endomorphisms of $\mathbb{P}^2$ which, in the same affine chart, have the form*

$$F_n(z, w) = F(z, w) + \left(0, \left(\frac{\pi}{n} + \frac{a(z)}{n^2}\right)^2\right).$$

Assume that there exist three periodic points $z^{(1)}, z^{(2)}, z^{(3)} \in J(p)$ with periods m_1, m_2, m_3 such that the three numbers

$$A_j := \frac{1}{m_j} \sum_{\ell=0}^{m_j-1} a\left(p^\ell(z^{(j)})\right), \qquad j = 1, 2, 3,$$

are not collinear in $\mathbb{C}$. Then there exists a nonempty open subset $U \subset \mathbb{C}$ such that

$$J(p) \times U \subset \liminf_{n\to\infty} J(F_n),$$

where $J(F_n)$ denotes the Julia set of F_n.

Corollary 1.6 will follow from a more general result about the limit *Julia-Lavaurs sets* of perturbations as in the statement, see Proposition 6.1, as well as from some thermodynamics formalism arguments, see Lemma 6.2. Observe that no hyperbolicity assumption is made on the rational map p. As a special case of Corollary 1.6, we can consider a rational map p whose Julia set is equal to the Riemann sphere (e.g., a Lattés map). In this case, when $\deg p \geq 4$, we can always choose the function $a(\cdot)$ so that the non-collinearity condition is satisfied, and we obtain that $\liminf_{n\to\infty} J(F_n)$ contains a nonempty open subset of $\mathbb{P}^2$.

1.4. **Structure of the paper.** In Section 2 we recall the necessary background on Fatou coordinates and classical parabolic implosion, as well as set some notation for the rest of the paper. In Section 3 we construct approximate Fatou coordinates adapted to the non-autonomous perturbations and give the main error terms with respect to actual (non-autonomous) Fatou coordinates. In Section 4 we prove Theorem 1.1. Sections 5 and 6 are devoted to the proofs of the corollaries of the main theorem.

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2. Preliminaries and notations

In this section we recall the necessary background on parabolic germs, Fatou coordinates, and classical (autonomous) parabolic implosion. We also fix notation that will be used throughout the paper.

2.1. **Parabolic germs and Fatou coordinates.** Let

$$f(w) = w + w^2 + aw^3 + O(w^4)$$

be a holomorphic germ at the origin. The fixed point at $w = 0$ is parabolic with multiplier 1 and one attracting and one repelling direction. For a sufficiently small $r > 0$, define the *attracting* and *repelling petals* as

$$P^\iota = \{|w + r| < r\} \quad \text{and} \quad P^o = \{|w - r| < r\}$$

The attracting petal P^ι is forward invariant and we have $f^n(w) \to 0$ for every $w \in P^\iota$. The *parabolic basin* $\mathcal{B}_f$ consists of all points w with $f^n(w) \to 0$ and is equal to $\cup_n f^{-n}(P^\iota)$.

It is classical that there exist attracting and repelling Fatou coordinates on the petals, that is, holomorphic maps $\phi^\iota \colon \mathcal{P}^\iota \to \mathbb{C}$ and $\phi^o \colon P^o \to \mathbb{C}$ such that

$$\phi^\iota(f(w)) = \phi^\iota(w) + 1, \qquad \phi^o(f(w)) = \phi^o(w) + 1. \tag{5}$$

The image of P^ι by ϕ^ι (resp. the image of P^o by ϕ^o) contains a right (resp. left) half plane. Moreover, ϕ^ι and ϕ^o can be chosen[1] to satisfy

$$\begin{cases} \phi^\iota(w) = -\frac{1}{w} + (1-a)\log(-w) + o(1) \\ \phi^o(w) = -\frac{1}{w} + (1-a)\log w + o(1), \end{cases} \qquad w \to 0. \tag{6}$$

Thanks to (5), ϕ^ι can also be extended to $\mathcal{B}_f$. Similarly, the inverse $(\phi^o)^{-1}$ can be extended to the complex plane.

2.2. **Lavaurs maps.** The now classical Douady–Lavaurs construction [Lav89; Dou94] associates to f a one-parameter family of holomorphic maps $(L_u)_{u\in\mathbb{C}}$ on $\mathcal{B}_f$ defined by

$$L_u = (\phi^o)^{-1} \circ T_u \circ \phi^\iota,$$

where $T_u(\zeta) = \zeta + u$ denotes translation by u. These maps arise as limits of suitable perturbations of f. More precisely, if one considers autonomous perturbations

$$f_\varepsilon(w) = f(w) + \varepsilon^2$$

and a sequence $\{\varepsilon_n\}$ with $\varepsilon_n = \pi/n + \pi u/n^2 + o(1/n^2)$, then

$$f_{\varepsilon_n}^{\circ n} \longrightarrow L_u$$

locally uniformly on $\mathcal{B}_f$.

[1]The maps ϕ^ι and ϕ^o are unique up to an additive constant, which we fix by requiring the asymptotics (6), where $\log(\cdot)$ denotes the principal branch of the logarithm.

3. Approximate Fatou coordinates for non-autonomous perturbations

In this section we construct approximate Fatou coordinates adapted to the non-autonomous iteration (2), where the $\varepsilon_{k,n}$ are as in (3). The goal is to conjugate the dynamics of $f_{k,n} := f + \varepsilon_{k,n}^2$ to an approximate translation, with an error that can be explicitly controlled with respect to k and n. In order to simplify some notations, we will drop the index n whenever possible and denote $f_k = f_{k,n}$ and $w_k = w_k^{(n)}$. Throughout this section, we fix a sequence of integers $(k_n)_{n\geq 1}$ such that $1 \leq k_n \leq n/2$, $k_n \to \infty$, and $k_n = o(n)$. The precise choice of k_n needed in the proof of Theorem 1.1 will be made in Section 4.

3.1. The coordinate ψ.

We begin with the following lemma, which gives an approximation of the fixed points of the maps f_k. Recall that we assume $0 < \alpha < 1$ in (3).

Lemma 3.1. *Set* $\zeta^{\pm} := \pm\frac{i\pi}{n} + \frac{a\pi^2}{2n^2}$. *Then*

$$f_k(\zeta^{\pm}) = \zeta^{\pm} + \frac{\delta_k}{n^3} + O\Big(\frac{1}{n^{3+\alpha}}\Big), \qquad \textit{where} \qquad \delta_k := 2\pi^2\sigma_k.$$

Proof. Let us set $\zeta_k^{\pm} := \pm\frac{i\pi}{n} + \frac{\alpha_2^{\pm}(k)}{n^2}$. A direct computation gives

$$\begin{aligned} f_k(\zeta_k^{\pm}) - \zeta_{k+1}^{\pm} =& \zeta_k^{\pm} + (\zeta_k^{\pm})^2 + a(\zeta_k^{\pm})^3 + O((\zeta_k^{\pm})^4) + \varepsilon_{k,n}^2 - \zeta_{k+1}^{\pm} \\ =& \pm\frac{i\pi}{n} + \frac{\alpha_2^{\pm}(k)}{n^2} - \frac{\pi^2}{n^2} \pm \frac{2i\pi\alpha_2^{\pm}(k)}{n^3} \mp a\frac{i\pi^3}{n^3} \\ &+ \frac{\pi^2}{n^2} + \frac{2\pi^2\sigma_k}{n^3} \mp \frac{i\pi}{n} - \frac{\alpha_2^{\pm}(k+1)}{n^2} + O\Big(\frac{1}{n^{3+\alpha}}\Big) \\ =& \frac{\alpha_2^{\pm}(k) - \alpha_2^{\pm}(k+1)}{n^2} + \frac{2\pi^2\sigma_k \pm 2i\pi\alpha_2^{\pm}(k) \mp ai\pi^3}{n^3} + O\Big(\frac{1}{n^{3+\alpha}}\Big). \end{aligned}$$

Choosing $\alpha_2^{\pm}(k) = \frac{a\pi^2}{2}$ for all $k \geq 0$, we obtain $\zeta_k^{\pm} = \zeta^{\pm}$ and

$$f_k(\zeta_k^{\pm}) - \zeta_{k+1}^{\pm} = \frac{2\pi^2\sigma_k}{n^3} + O\Big(\frac{1}{n^{3+\alpha}}\Big),$$

which is the desired expression. □

Observe that, in particular, $\zeta^{\pm}$ as above do not depend on k (we allowed for a dependence $\alpha_2(k)$ in the proof to show that adding such a dependence would not have improved the estimates).

Definition 3.2. *We denote*

$$\psi(w) := \frac{1}{2i\pi}\log\Big(\frac{w-\zeta^+}{w-\zeta^-}\Big) \qquad \textit{and} \qquad A_k := A_k(w) := \psi(w_{k+1}) - \psi(w_k).$$

Here and in the rest of the paper, we denote by $\log(\cdot)$ the principal branch of the logarithm. Note that the map ψ sends the real axis to the segment $(0, 1)$. Circles through ζ^+ and ζ^- are mapped to vertical lines in the strip above.

The map ψ is a first approximation of the non-autonomous Fatou coordinates that we will use. We will now give an estimate for the error term A_k, for w in a suitable domain. We begin by giving the following expression for the inverse of ψ.

Lemma 3.3. *We have*

$$\psi^{-1}(W) = -\frac{\pi}{n}\cot(\pi W) + O\Big(\frac{1}{n^2}\Big).$$

Proof. Recall that $\cot z = i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}$. Moreover, by the expression of ψ we see that

$$\psi^{-1}(W) = \frac{e^{i\pi W}\zeta^- - e^{-i\pi W}\zeta^+}{e^{i\pi W} - e^{-i\pi W}}.$$

It follows from the definition of $\zeta^\pm$ that

$$\psi^{-1}(W) = -\frac{\pi}{n}\cot(\pi W) + \frac{a\pi^2}{2n^2},$$

which gives the desired estimate. □

Definition 3.4. *We define the rectangle*

$$R_n := \{W \in \mathbb{C} : \mathrm{Re}(W) \in \Big(\frac{k_n}{10n}, 1 - \frac{k_n}{10n}\Big), \quad \mathrm{Im}(W) \in (-1,1)\}.$$

Observe that R_n is contained in the image of ψ for every n. For a fixed n we will mainly consider orbits $(w_k^{(n)})$ such that $\psi(w_k^{(n)}) \in R_n$ for the relevant times k. The following lemma gives a simple estimate that will be able to apply to such orbits.

Lemma 3.5. *There exists a constant $c > 0$ such that $|\psi^{-1}(W) - \zeta^\pm| \geq \frac{c}{n}$ for every n and $W \in R_n$.*

Proof. By Lemma 3.1, it is enough to prove that

$$|\psi^{-1}(W) \pm \frac{i\pi}{n}| \geq \frac{c}{n} \text{ for all } W \text{ with } |\mathrm{Im}(W)| \leq 1,$$

for some suitable constant c as in the statement. By Lemma 3.3, this reduces to

$$|\cot(\pi W) \pm i| \geq c \text{ for some positive } c.$$

The above holds since $\pm i$ are not in the image of the map $W \mapsto \cot(\pi W)$, hence $n\zeta^\pm$ are uniformly far from the image of $\overline{R_n}$ under $n\psi^{-1}$. □

Proposition 3.6. *For every $n \in \mathbb{N}$, $0 \leq k < n$, and $w \in \mathcal{B}_f$ with $\psi(w_k), \psi(w_{k+1}) \in R_n$ we have*

$$A_k(w) = \frac{1}{n} - (1-a)\frac{w_k}{n} + H_{k,n}(w_{k+1}) + O\Big(\frac{w_k^2}{n}, \frac{1}{n^{2+\alpha}}\Big)$$

where

$$H_{k,n}(w) := \frac{\delta_k}{2i\pi n^3}\left(\frac{1}{w-\zeta_+} - \frac{1}{w-\zeta^-}\right) \tag{7}$$

and the estimate is locally uniform in $w \in \mathcal{B}_f$.

Proof. By definition, we have

$$A_k = \frac{1}{2i\pi}\log\frac{w_{k+1}-\zeta^+}{w_{k+1}-\zeta^-} - \frac{1}{2i\pi}\log\frac{w_k-\zeta^+}{w_k-\zeta^-}.$$

Define

$$A_k' := \frac{1}{2i\pi}\log\frac{w_{k+1}-f_k(\zeta^+)}{w_{k+1}-f_k(\zeta^-)} - \frac{1}{2i\pi}\log\frac{w_k-\zeta^+}{w_k-\zeta^-}.$$

We split the proof into two main steps, where we estimate A'_k (the main term) and $A_k - A'_k$ (the correction term), respectively.

Step 1. Estimate of the main term A'_k. We claim that we have

$$A'_k = \frac{1}{n} - (1-a)\frac{w_k}{n} + O\Big(\frac{w_k^2}{n}, \frac{1}{n^3}\Big). \tag{8}$$

for every n sufficiently large and every $0 \le k < n$.

Proof of Step 1. Since $w_{k+1} = f_k(w_k)$, we can rewrite A'_k as

$$A'_k = \frac{1}{2i\pi}\log\frac{f_k(w_k) - f_k(\zeta^+)}{f_k(w_k) - f_k(\zeta^-)} - \frac{1}{2i\pi}\log\frac{w_k - \zeta^+}{w_k - \zeta^-}.$$

As f_k differs from f by an additive constant, we deduce

$$A'_k = \frac{1}{2i\pi}\log\frac{f(w_k) - f(\zeta^+)}{f(w_k) - f(\zeta^-)} - \frac{1}{2i\pi}\log\frac{w_k - \zeta^+}{w_k - \zeta^-}.$$

Equivalently, we have

$$A'_k = \frac{1}{2i\pi}\log\frac{f(w_k) - f(\zeta^+)}{w_k - \zeta^+} - \frac{1}{2i\pi}\log\frac{f(w_k) - f(\zeta^-)}{w_k - \zeta^-}.$$

Set

$$\xi^\pm(t) := \pm i\pi t + \frac{a\pi^2}{2}t^2, \qquad t := \frac{1}{n},$$

so that $\zeta^\pm = \xi^\pm(1/n)$. Consider the function

$$Q(x,\xi) := \frac{f(x) - f(\xi)}{x - \xi}.$$

Since f is holomorphic, Q extends holomorphically across the diagonal $x = \xi$, and

$$Q(0,0) = f'(0) = 1.$$

Therefore, on a sufficiently small neighborhood of $(0,0)$, we may choose a holomorphic branch of $\log Q(x,\xi)$. We then define

$$\mathcal{A}'(x,t) := \frac{1}{2i\pi}\Big(\log Q(x,\xi^+(t)) - \log Q(x,\xi^-(t))\Big).$$

This is a holomorphic function of (x,t) in a neighborhood of $(0,0)$, independent of k and n, and by construction we have

$$A'_k = \mathcal{A}'(w_k, 1/n).$$

We now expand $\mathcal{A}'(x,t)$ at $t = 0$. First observe that $\xi^\pm(-t) = \xi^\mp(t)$, hence $\mathcal{A}'(x,-t) = -\mathcal{A}'(x,t)$. So $\mathcal{A}'(x,t)$ is odd in t, and in particular all even Taylor coefficients in t vanish. Next, differentiating with respect to t, we get

$$\partial_t \log Q(x,\xi^\pm(t)) = -(\xi^\pm)'(t)\left(\frac{f'(\xi^\pm(t))}{f(x) - f(\xi^\pm(t))} - \frac{1}{x - \xi^\pm(t)}\right).$$

Since $\xi^\pm(0) = 0$, $(\xi^\pm)'(0) = \pm i\pi$, and $f'(0) = 1$, we obtain

$$\partial_t \log Q(x,\xi^\pm(t))\Big|_{t=0} = \pm i\pi\left(\frac{1}{x} - \frac{1}{f(x)}\right).$$

Therefore, we obtain

$$\partial_t \mathcal{A}'(x,0) = \frac{1}{x} - \frac{1}{f(x)}.$$

We conclude that the Taylor expansion of $\mathcal{A}'$ at $t=0$ has the form

$$\mathcal{A}'(x,t) = \left(\frac{1}{x} - \frac{1}{f(x)}\right) t + O(t^3), \tag{9}$$

where in particular the error is uniform for x in a sufficiently small fixed neighbourhood of 0.

We now verify that $x = w_k$ belongs to such a neighbourhood, for every n sufficiently large. Indeed, by the assumption $\psi(w_k) \in R_n$ and the formula for ψ^{-1}, we have

$$w_k = \psi^{-1}(\psi(w_k)) = -\frac{\pi}{n}\cot(\pi\psi(w_k)) + O\Big(\frac{1}{n^2}\Big).$$

Since $\psi(w_k) \in R_n$, the distance of $\psi(w_k)$ to $\mathbb{Z}$ is bounded below by $k_n/(10n)$, hence

$$|\cot(\pi\psi(w_k))| \leq C\frac{n}{k_n}$$

for some constant $C > 0$ independent of k and n. Thus

$$w_k = O\Big(\frac{1}{k_n}\Big) + O\Big(\frac{1}{n^2}\Big).$$

Since $k_n \to \infty$, it follows that for n large enough all such w_k belong to a fixed small neighbourhood of 0.

Thanks to the above, we can evaluate (9) at $(x,t) = (w_k, 1/n)$ and obtain

$$A'_k = \left(\frac{1}{w_k} - \frac{1}{f(w_k)}\right)\frac{1}{n} + O\Big(\frac{1}{n^3}\Big). \tag{10}$$

Using the expansion $f(w) = w + w^2 + aw^3 + O(w^4)$, we get

$$\frac{1}{f(w)} = \frac{1}{w}\,\frac{1}{1 + w + aw^2 + O(w^3)} = \frac{1}{w}\Big(1 - w + (1-a)w^2 + O(w^3)\Big),$$

hence

$$\frac{1}{w} - \frac{1}{f(w)} = 1 - (1-a)w + O(w^2).$$

Applying this at $w = w_k$ and inserting it into (10), we obtain (8). □

Step 2. Estimate of the correction term $A_k - A'_k$. We claim that we have

$$A_k - A'_k = H_{k,n}(w_{k+1}) + O\Big(\frac{1}{n^{2+\alpha}}\Big) \tag{11}$$

for every n sufficiently large and every $0 \leq k < n$.

Proof of Step 2. By definition, we have

$$A_k - A'_k = \frac{1}{2i\pi}\log\frac{w_{k+1} - \zeta^+}{w_{k+1} - \zeta^-} - \frac{1}{2i\pi}\log\frac{w_{k+1} - f_k(\zeta^+)}{w_{k+1} - f_k(\zeta^-)},$$

which we can rewrite as

$$A_k - A'_k = \frac{1}{2i\pi}\left(\log \frac{w_{k+1} - \zeta^+}{w_{k+1} - f_k(\zeta^+)} - \log \frac{w_{k+1} - \zeta^-}{w_{k+1} - f_k(\zeta^-)}\right). \tag{12}$$

Set

$$\eta^{\pm}_{k,n} := f_k(\zeta^{\pm}) - \zeta^{\pm}.$$

By Lemma 3.1, we have

$$\eta^{\pm}_{k,n} = \frac{\delta_k}{n^3} + O\Big(\frac{1}{n^{3+\alpha}}\Big),$$

where the error term is uniform in k. Since the sequence (δ_k) is uniformly bounded, this implies

$$\eta^{\pm}_{k,n} = O\Big(\frac{1}{n^3}\Big), \tag{13}$$

where again the error term is uniform in k. On the other hand, since $\psi(w_{k+1}) \in R_n$, Lemma 3.5 gives

$$|w_{k+1} - \zeta^{\pm}| \geq \frac{c}{n} \tag{14}$$

for some $c > 0$ independent of k and n. Hence

$$\left|\frac{\eta^{\pm}_{k,n}}{w_{k+1} - \zeta^{\pm}}\right| \leq C\frac{1/n^3}{1/n} = O\Big(\frac{1}{n^2}\Big)$$

uniformly in k and n. In particular, for n large enough we obtain

$$\log \frac{w_{k+1} - \zeta^{\pm}}{w_{k+1} - f_k(\zeta^{\pm})} = -\log\left(1 - \frac{\eta^{\pm}_{k,n}}{w_{k+1} - \zeta^{\pm}}\right) = \frac{\eta^{\pm}_{k,n}}{w_{k+1} - \zeta^{\pm}} + O\left(\frac{|\eta^{\pm}_{k,n}|^2}{|w_{k+1} - \zeta^{\pm}|^2}\right).$$

Using again (13) and (14), the error term above satisfies

$$O\left(\frac{|\eta^{\pm}_{k,n}|^2}{|w_{k+1} - \zeta^{\pm}|^2}\right) = O\Big(\frac{1/n^6}{1/n^2}\Big) = O\Big(\frac{1}{n^4}\Big).$$

Moreover, we also have

$$\frac{\eta^{\pm}_{k,n}}{w_{k+1} - \zeta^{\pm}} = \frac{\delta_k}{n^3}\frac{1}{w_{k+1} - \zeta^{\pm}} + O\Big(\frac{1}{n^{3+\alpha}} \cdot \frac{1}{|w_{k+1} - \zeta^{\pm}|}\Big) = \frac{\delta_k}{n^3}\frac{1}{w_{k+1} - \zeta^{\pm}} + O\left(\frac{1}{n^{2+\alpha}}\right).$$

Since $O(n^{-4}) = O(n^{-2-\alpha})$ (recall that we always assume $0 < \alpha < 1$), we conclude that

$$\log \frac{w_{k+1} - \zeta^{\pm}}{w_{k+1} - f_k(\zeta^{\pm})} = \frac{\delta_k}{n^3}\frac{1}{w_{k+1} - \zeta^{\pm}} + O\left(\frac{1}{n^{2+\alpha}}\right), \tag{15}$$

uniformly in k and n.

Combining (12) and (15), we obtain

$$A_k - A'_k = \frac{\delta_k}{2i\pi n^3}\left(\frac{1}{w_{k+1} - \zeta^+} - \frac{1}{w_{k+1} - \zeta^-}\right) + O\Big(\frac{1}{n^{2+\alpha}}\Big).$$

By the definition of $H_{k,n}$, this gives (11) and concludes the proof. □

Combining the estimates in Steps 1 and 2, we obtain

$$A_k = \frac{1}{n} - (1-a)\frac{w_k}{n} + H_{k,n}(w_{k+1}) + O\left(\frac{w_k^2}{n}, \frac{1}{n^3}, \frac{1}{n^{2+\alpha}}\right).$$

Since $1/n^3 = O(1/n^{2+\alpha})$, this simplifies to

$$A_k = \frac{1}{n} - (1-a)\frac{w_k}{n} + H_{k,n}(w_{k+1}) + O\left(\frac{w_k^2}{n}, \frac{1}{n^{2+\alpha}}\right),$$

which is the desired estimate. Observe that all the implicit constants above are independent of k and n, and are uniform for w in a compact subset of $\mathcal{B}_f$. This proves the local uniformity in w. □

3.2. **The coordinate ϕ.** We will now post-compose ψ with a further suitable change of coordinate to get rid of the error term in $(1-a)w_k/n$ appearing in Proposition 3.6. Observe that this step is not necessary when $a = 1$, as $\phi = \psi$ in this case.

Definition 3.7. *We set*

$$\chi(W) := W - \frac{1}{n}(1-a)\log\sin(\pi W), \quad \phi := \chi \circ \psi, \quad \textit{and} \quad \tilde{A}_k := \phi(w_{k+1}) - \phi(w_k).$$

For convenience, we will denote $U_k := \psi(w_k)$ and $W_k := \phi(w_k)$.

Lemma 3.8. *The following properties hold.*

(1) For every n and every $W \in R_n$, the inverse branch of ϕ defined by $\phi^{-1}(W) := \psi^{-1}(\chi^{-1}(W))$ satisfies

$$\phi^{-1}(W) = -\frac{\pi}{n}\cot(\pi W) + O\left(\frac{1}{n^2}\right) + O\left(\frac{\log(n/k_n)}{k_n^2}\right).$$

(2) There exists a constant $c > 0$ such that

$$|\phi^{-1}(W) - \zeta_\pm| \geq \frac{c}{n} \qquad \textit{for every } n \textit{ and every } W \in R_n.$$

Proof. (1) Fix $W \in R_n$ and set $U := \chi^{-1}(W)$, so that $\phi^{-1}(W) = \psi^{-1}(U)$. Since $\mathrm{Re}(W) \in \left(\frac{k_n}{10n}, 1 - \frac{k_n}{10n}\right)$ and $|\mathrm{Im}(W)| < 1$, we have $|\sin(\pi W)| \asymp \mathrm{dist}(W, \mathbb{Z}) \gtrsim \frac{k_n}{n}$, hence

$$|\log\sin(\pi W)| \lesssim 1 + \log(n/k_n),$$

where the implicit constants are independent of n and W. Therefore, from $W = U - \frac{1-a}{n}\log\sin(\pi U)$ we obtain

$$U = W + O\left(\frac{1 + \log(n/k_n)}{n}\right).$$

By Lemma 3.3, we have

$$\phi^{-1}(W) = \psi^{-1}(U) = -\frac{\pi}{n}\cot(\pi W) + O\left(\frac{1}{n^2}\right) + O\left(\frac{1 + \log(n/k_n)}{k_n^2}\right),$$

where we used that $|\cot'(\pi Z)| = O\big((n/k_n)^2\big)$ on R_n and the previous bound on $|U - W|$.

(2) The bound follows using the development $\zeta_\pm = \pm i\pi/n + O(1/n^2)$ (see Lemma 3.1) and the lower bound for $|\cot(\pi W) \pm i|$ given by Lemma 3.5. □

Proposition 3.9. *For every $n \in \mathbb{N}$, $0 \le k < n$, and $w \in \mathcal{B}_f$ with $W_k, W_{k+1} \in R_n$ we have*

$$\tilde{A}_k = \frac{1}{n} + H_{k,n}(-\frac{\pi}{n}\cot(\pi W_{k+1})) + O\Big(\frac{w_k^2}{n}, \frac{1}{n^{2+\alpha}}, \frac{\log(n/k_n)}{n\,k_n^2}\Big).$$

Proof. By the definitions of ϕ, $\tilde{A}_k$, and A_k, we see that

$$\begin{aligned}\tilde{A}_k &= W_{k+1} - W_k = A_k - \frac{1}{n}(1-a)\big(\log\sin(\pi U_{k+1}) - \log\sin(\pi U_k)\big)\\ &= \frac{1}{n} - (1-a)\frac{w_k}{n} + H_{k,n}(w_{k+1}) - \frac{1}{n}(1-a)\log\frac{\sin(\pi U_{k+1})}{\sin(\pi U_k)} + O\Big(\frac{w_k^2}{n}, \frac{1}{n^{2+\alpha}}\Big).\end{aligned} \tag{16}$$

Hence, we need to compare $H_{k,n}(w_{k+1})$ with $H_{k,n}(-\frac{\pi}{n}\cot(\pi W_{k+1}))$ and estimate the term $\log\frac{\sin(\pi U_{k+1})}{\sin(\pi U_k)}$. For convenience, we will set $\widehat{w}_k := -\frac{\pi}{n}\cot(\pi W_k)$.

Claim 1. We have

$$H_{k,n}(w_{k+1}) = H_{k,n}(\widehat{w}_{k+1}) + O\left(\frac{1}{n^3}, \frac{\log(n/k_n)}{n\,k_n^2}\right).$$

Proof. By the definition (7) of $H_{k,n}(w)$, we have

$$H'_{k,n}(w) = \frac{\delta_k}{2i\pi n^3}\left(-\frac{1}{(w-\zeta_+)^2} + \frac{1}{(w-\zeta_-)^2}\right).$$

Since $W_{k+1} \in R_n$, by Lemma 3.8(2) we have $|w_{k+1} - \zeta_\pm| \ge c/n$. Moreover, $\widehat{w}_{k+1} = \phi^{-1}(W_{k+1}) + O(\frac{1}{n^2}, \frac{\log(n/k_n)}{k_n^2})$ by Lemma
3.8(1), hence also $|\widehat{w}_{k+1} - \zeta_\pm| \ge c'/n$ for some $c' > 0$ (using Lemmas 3.5 and 3.8(2)). Therefore, for all w on the segment joining w_{k+1} to $\widehat{w}_{k+1}$ we have $|w - \zeta_\pm| \ge c''/n$ for some $c'' > 0$, and thus

$$|H'_{k,n}(w)| \le \frac{C}{n^3}\left(\frac{n^2}{(c'')^2} + \frac{n^2}{(c'')^2}\right) \le \frac{C_1}{n},$$

for some positive constants C, C_1, where we used the fact that the sequence (δ_k) is uniformly bounded. By the mean value theorem, we have

$$|H_{k,n}(w_{k+1}) - H_{k,n}(\widehat{w}_{k+1})| \le \sup_{[w_{k+1}, \widehat{w}_{k+1}]} |H'_{k,n}| \cdot |w_{k+1} - \widehat{w}_{k+1}| \le \frac{C_1}{n} \cdot O\left(\frac{1}{n^2}, \frac{\log(n/k_n)}{k_n^2}\right),$$

which gives the assertion. □

Claim 2. We have $\log\sin(\pi U_{k+1}) - \log\sin(\pi U_k) = -w_k + O(w_k^2, \frac{1}{n^2})$.

Proof. We have

$$\begin{aligned}\log\sin(\pi U_{k+1}) - \log\sin(\pi U_k) &= \log\frac{\sin(\pi(U_k + A_k))}{\sin(\pi U_k)}\\ &= \log\frac{\cos(\pi A_k)\sin(\pi U_k) + \sin(\pi A_k)\cos(\pi U_k)}{\sin(\pi U_k)}\\ &= \log(\cos(\pi A_k) + \sin(\pi A_k)\cot(\pi U_k)).\end{aligned} \tag{17}$$

Since $A_k = O(1/n)$ (see Proposition 3.6), we can expand

$$\cos(\pi A_k) = 1 + O(A_k^2) \quad \text{and} \quad \sin(\pi A_k) = \pi A_k + O(A_k^3),$$

which give

$$\cos(\pi A_k) + \sin(\pi A_k)\cot(\pi U_k) = 1 + \pi A_k \cot(\pi U_k) + O\big(A_k^2(1+\cot^2(\pi U_k))\big). \tag{18}$$

Recalling that $U_k = \psi(w_k)$, we deduce from Lemma 3.3 that

$$w_k = \psi^{-1}(U_k) = -\frac{\pi}{n}\cot(\pi U_k) + O\left(\frac{1}{n^2}\right), \quad \text{hence} \quad \cot(\pi U_k) = -\frac{n}{\pi} w_k + O\left(\frac{1}{n}\right).$$

Combining this with expansion of A_k given in Proposition 3.6 gives

$$\begin{aligned}\pi A_k \cot(\pi U_k) &= \pi\Big(\frac{1}{n} - \frac{(1-a)w_k}{n} + O\Big(\frac{w_k^2}{n}\frac{1}{n^3}\Big)\Big)\Big(-\frac{n}{\pi}w_k + O\Big(\frac{1}{n}\Big)\Big)\\ &= -w_k + (1-a)w_k^2 + O(w_k^2) + O\left(\frac{1}{n^2}\right).\end{aligned} \tag{19}$$

Moreover, we also have

$$A_k^2(1+\cot^2(\pi U_k)) = O\left(\frac{1}{n^2}\right) + O\left(\frac{\cot^2(\pi U_k)}{n^2}\right) = O\left(\frac{1}{n^2}\right) + O(w_k^2), \tag{20}$$

using again the expression $\cot(\pi U_k) = -(n/\pi)w_k + O(1/n)$. Therefore, we deduce from (18), (19), and (20) that

$$\cos(\pi A_k) + \sin(\pi A_k)\cot(\pi U_k) = 1 - w_k + (1-a)w_k^2 + O(w_k^2) + O\left(\frac{1}{n^2}\right).$$

We conclude from (17) and the above expression that

$$\log\sin(\pi U_{k+1}) - \log\sin(\pi U_k) = -w_k + O(w_k^2) + O\left(\frac{1}{n^2}\right),$$

as claimed □

The assertion follows from (16) and the above claims. □

4. Proof of Theorem 1.1

In this section we prove our main Theorem 1.1. The proof follows the orbits and is divided into three main steps. In Section 4.1 we consider the first $\sim k_n$ points of the orbits, which still look close to the autonomous system f. In Section 4.2 we study the central part of the orbit, from $k \sim k_n$ to $k \sim n - k_n$, where the perturbations $\varepsilon_{k,n}$ dominate. This is where the estimates of the previous sections will be mainly used and the accumulated errors leading to the formula for the phase in the statement will appear. Finally, in Section 4.3 we consider the last $\sim k_n$ points of the orbit, for which again the main contribution is given by the dynamics of f. In Section 4.4 we will put all the estimates together, completing the proof of Theorem 1.1.

We fix below a point $w_0 \in \mathcal{B}_f$. All the estimates will be uniform for w in a given compact neighbourhood of w_0 in $\mathcal{B}_f$. From now on, we will assume that the sequence k_n satisfies $n = o(k_n^2/\log(n/k_n))$ and $k_n = o(n^{2/3})$, i.e., that

$$\frac{k_n^3}{n^2}, \frac{n\log(n/k_n)}{k_n^2} \to 0, \qquad n \to \infty. \tag{21}$$

For instance, we can fix $1/2 < \beta < 2/3$ and take $k_n := \lfloor n^\beta \rfloor$. Observe that, with these assumptions, the error term in Proposition 3.9 becomes $o(1/n^2)$.

4.1. **Entering the eggbeater.** In this section we compare the readings of the first approximately k_n points of an orbit in the approximate Fatou coordinate ϕ introduced above and the actual Fatou coordinate ϕ^ι (see Section 2). We start with the following lemma about ϕ^ι.

Lemma 4.1. *We have $w_{k_n} = -\frac{1+o(1)}{k_n}$ and $\phi^\iota(w_{k_n}) = \phi^\iota(w_0) + k_n + o(1)$.*

Proof. Recall that $w_{k+1} = f(w_k) + \varepsilon_{k,n}^2$, and that $|\varepsilon_{k,n}^2| = O(\frac{1}{n^2})$. Moreover, we have $\phi^\iota(f(w)) = \phi^\iota(w) + 1$ and $(\phi^\iota)'(w_k) = O(\frac{1}{w_k^2})$. These identities and estimates give

$$\begin{aligned}\phi^\iota(w_{k+1}) &= \phi^\iota(f_k(w_k)) = \phi^\iota(f(w_k) + \varepsilon_{k,n}^2)\\ &= \phi^\iota(f(w_k)) + O(|\varepsilon_{k,n}^2||(\phi^\iota)'(f(w_k))|)\\ &= \phi^\iota(w_k) + 1 + O\Big(\frac{1}{|w_{k+1}|^2 \cdot n^2}\Big).\end{aligned}$$

By induction and the asymptotic (6) of ϕ^ι, the above shows that $w_k = -\frac{1+o(1)}{k}$ for every $0 \le k \le k_n$. In particular, this holds for $k = k_n$ and we have

$$\phi^\iota(w_{k_n}) = \phi^\iota(w_0) + k_n + O\Big(\frac{k_n^3}{n^2}\Big).$$

The assertion follows from the assumption (21) on k_n. □

Proposition 4.2. *We have*

$$n\phi(w_{k_n}) = \phi^\iota(w_{k_n}) - (1-a)\log(\pi/n) + o(1). \tag{22}$$

Moreover, we have $W_{k_n}, W_{k_n+1} \in R_n$ for every n sufficiently large.

Proof. Let us start proving that

$$n\psi(w_{k_n}) = -\frac{1}{w_{k_n}} + o(1), \qquad n \to \infty. \tag{23}$$

Indeed, for all k we have

$$n\psi(w_k) = n\frac{1}{2i\pi}\log\frac{w_k - \zeta^+}{w_k - \zeta^-} = n\frac{1}{2i\pi}\log\frac{1-\frac{\zeta^+}{w_k}}{1-\frac{\zeta^-}{w_k}} = n\frac{1}{2i\pi}\Big(\log(1-\frac{\zeta^+}{w_k}) - \log(1-\frac{\zeta^-}{w_k})\Big).$$

By Lemma 3.1, we have $\zeta^+ - \zeta^- = \frac{2i\pi}{n}$. Moreover, by Lemma 4.1 we also have $|1/w_{k_n}| \sim k_n = o(n)$. Hence, $\frac{\zeta^\pm}{w_{k_n}} = o(1)$ and

$$n\psi(w_{k_n}) = n\frac{1}{2i\pi}\frac{\zeta^+ - \zeta^-}{w_{k_n}} + nO(\frac{1}{n^3 \cdot |w_{k_n}|^2}) = -\frac{1}{w_{k_n}} + O\Big(\frac{k_n^2}{n^2}\Big) = -\frac{1}{w_{k_n}} + o(1).$$

Recall that $\phi = \chi \circ \psi$ and that ϕ^ι satisfies (6). In order to establish (22), we need to prove that

$$\log \sin(\pi\psi(w_{k_n})) = -\log(-w_{k_n}) + \log(\pi/n) + o(1)$$

(observe that we do not need to do this estimate if $a = 1$, as $\phi = \psi$ in that case). By the estimate (23) for $\psi(w_{k_n})$ and the facts that $w_{k_n} \sim 1/k_n$ and $\frac{k_n^3}{n^3} = o(\frac{1}{n})$ by the assumption (21), we have

$$\begin{aligned}\log\sin(\pi\psi(w_{k_n})) &= \log\sin(\pi(-\frac{1}{nw_{k_n}} + o(\frac{1}{n}))) = \log(-\frac{\pi}{nw_{k_n}} + o(\frac{1}{n}) + O(\frac{k_n^3}{n^3}))\\ &= \log(-\frac{\pi}{nw_{k_n}} + o(\frac{1}{n})) = \log(-\frac{\pi}{nw_{k_n}}) + \log(1 - w_{k_n}o(1)).\\ &= \log(-\frac{\pi}{nw_{k_n}}) + o(1) = -\log(-w_{k_n}) + \log(\pi/n) + o(1).\end{aligned}$$

This concludes the proof of (22).

Let us now show that $W_{k_n} \in R_n$ for every n sufficiently large. By Lemma 4.1 we have $w_{k_n} = -(1 + o(1))/k_n$, hence $|\zeta_\pm|/|w_{k_n}| = O(k_n/n) = o(1)$. Using the definition of ψ we get

$$\psi(w_{k_n}) = \frac{1}{2i\pi}\log\frac{w_{k_n} - \zeta_+}{w_{k_n} - \zeta_-} = \frac{k_n}{n} + o\left(\frac{k_n}{n}\right).$$

Since $\chi(W) = W - \frac{1-a}{n}\log\sin(\pi W)$ satisfies $\chi(W) = W + O(\log(n/k_n)/n)$ on R_n, it follows that $W_{k_n} := \phi(w_{k_n}) = \chi(\psi(w_{k_n})) \in R_n$ for n large.

Finally, from $w_{k_n+1} = f(w_{k_n}) + \varepsilon^2_{k_n,n}$ and $f(w) = w + w^2 + O(w^3)$ we have $\frac{1}{w_{k_n+1}} = \frac{1}{w_{k_n}} - 1 + O(w_{k_n})$, hence

$$\psi(w_{k_n+1}) - \psi(w_{k_n}) = \frac{1}{n} + O\left(\frac{1}{nk_n}\right) = \frac{1}{n} + o\left(\frac{1}{n}\right).$$

Therefore $W_{k_n+1} - W_{k_n} = \chi(\psi(w_{k_n+1})) - \chi(\psi(w_{k_n})) = \frac{1}{n} + o(1/n)$. Since $W_{k_n} \sim k_n/n$ lies at distance at least $k_n/2n$ from the boundary of R_n, we also get $W_{k_n+1} \in R_n$. This concludes the proof. □

4.2. **Passing through the eggbeater.** As a result of the estimates of the previous section and the definitions of ϕ and R_n, the element w_{k_n} satisfies $\phi(w_{k_n}) \in R_n$ as soon as n is sufficiently large. The goal of this section is to show that $\phi(w_k)$ stays in R_n for every $k_n \le k \le n - k_n$, and to estimate the error made at every step with respect to the autonomous system. This is the point where we use the control on the error terms in Section 3. We start with the following preliminary estimate.

Lemma 4.3. *For every $k_n \le j \le n - k_n$, we have $W_j \in R_n$ and*

$$W_j = W_{k_n} + \frac{j - k_n}{n} + O\Big(\frac{j}{n^2}\Big).$$

Observe that, by the assumptions (21) on k_n, we have $\frac{\log(n/k_n)}{k_n^2} = o(1/n)$, hence errors of this magnitude can be considered negligible in the proof below.

Proof. We argue by induction on j in the range $k_n \le j \le n - k_n$.

Base step. By Proposition 4.2 we have $W_{k_n}, W_{k_n+1} \in R_n$. The estimate is trivial for $j = k_n$, and the proof of Proposition 4.2 yields

$$W_{k_n+1} = W_{k_n} + \frac{1}{n} + O\left(\frac{1}{n^2}\right),$$

so it holds for $j = k_n + 1$ as well.

Induction step. Fix J with $k_n \le J \le n - k_n - 2$ and assume that

- $W_j \in R_n$ for all $k_n \le j \le J + 1$, and
- for all $k_n \le j \le J + 1$,

$$W_j = W_{k_n} + \frac{j - k_n}{n} + O\left(\frac{j}{n^2}\right).$$

We prove that (for every n sufficiently large) we have $W_{J+2} \in R_n$ and the estimate holds for $j = J + 2$. Observe that we first need to prove $W_{J+2} \in R_n$ by a rough one-step estimate, since Proposition 3.9 is stated under the a priori assumption $W_{J+2} \in R_n$.

Step 1 (one–step invariance): $W_{J+2} \in R_n$. From the induction hypothesis we have

$$\mathrm{Re}(W_{J+1}) = \mathrm{Re}(W_{k_n}) + \frac{J + 1 - k_n}{n} + O\left(\frac{J+1}{n^2}\right),$$

hence for n large we have

$$\mathrm{dist}(W_{J+1}, \partial R_n) \ge \frac{k_n}{20n}.$$

Write $w_{J+2} = f(w_{J+1}) + \varepsilon^2_{J+1,n}$. Then

$$w_{J+2} - w_{J+1} = w^2_{J+1} + O(w^3_{J+1}) + O(1/n^2).$$

Since $W_{J+1} \in R_n$, Lemma 3.8(2) gives $|w_{J+1} - \zeta_\pm| \ge c/n$ and Lemma 3.8(1) gives $|w_{J+1}| \lesssim 1/k_n$ on R_n (we use here that $|\cot(\pi W_{J+1})| = O(n/k_n)$). Therefore

$$|w_{J+2} - w_{J+1}| \lesssim \frac{1}{k_n^2} + \frac{1}{n^2}.$$

Denote as above $U_{J+1} := \psi(w_{J+1})$. Differentiating the definition of ψ yields

$$\psi'(w) = \frac{1}{n(w - \zeta_+)(w - \zeta_-)}.$$

Using $|w_{J+1} - \zeta_\pm| \ge c/n$, we get $|\psi'(w_{J+1})| \lesssim \frac{1}{n \max(|w_{J+1}|, 1/n)^2}$, hence

$$|\psi'(w_{J+1})| \cdot |w_{J+2} - w_{J+1}| \lesssim \frac{1}{n}.$$

A Taylor expansion gives $A_{J+1} = \psi(w_{J+2}) - \psi(w_{J+1}) = O(1/n)$.

Recall that we denote $W = \chi(U)$. On R_n we have $|\cot(\pi U)| \lesssim n/k_n$, hence

$$\chi'(U) = 1 - \frac{1-a}{n} \pi \cot(\pi U) = 1 + O(1/k_n),$$

so χ is $1 + o(1)$–Lipschitz on R_n. Therefore

$$|W_{J+2} - W_{J+1}| = |\chi(U_{J+2}) - \chi(U_{J+1})| \le (1 + o(1))|A_{J+1}| = O(1/n).$$

Since $\mathrm{dist}(W_{J+1}, \partial R_n) \geq k_n/(20n)$, we conclude that $W_{J+2} \in R_n$ for every n sufficiently large, as desired.

Step 2: estimate for W_{J+2}. Since $W_{J+1}, W_{J+2} \in R_n$, we may apply Proposition 3.9 at time $J+1$ to get (as observed above, the term $\frac{\log(n/k_n)}{k_n^2}$ is negligible with respect to $1/n$)

$$W_{J+2} - W_{J+1} = \frac{1}{n} + O\left(\frac{1}{n^2}\right),$$

since $H_{J+1,n}(\cdot) = O(1/n^2)$ on R_n (by Lemma 3.8(2). Combining this with the induction hypothesis for W_{J+1} yields

$$W_{J+2} = W_{k_n} + \frac{J+2-k_n}{n} + O\left(\frac{J+2}{n^2}\right).$$

This concludes the induction step and the proof. □

Proposition 4.4. *For every $k_n \leq j \leq n - k_n$ we have $W_j \in R_n$ and*

$$W_j = \left(W_{k_n} + \frac{j-k_n}{n} + \sum_{k=k_n}^{j-1} H_{k,n}\Big(-\frac{\pi}{n}\cot\Big(\frac{(k+1)\pi}{n}\Big)\Big)\right) + o\Big(\frac{1}{n}\Big).$$

A more precise error term in the expression above would be $O\Big(\frac{1}{n^{1+\alpha}}, \frac{\log(n/k_n)}{k_n^2}\Big)$. This is indeed $o(1/n)$ because of the assumptions (21) on k_n. An error $o(1/n)$ will be enough for our final estimate in Section 4.4.

Proof. We refine the estimate of Lemma 4.3. We only need to prove the formula. We again work by induction and prove that

$$W_j = \left(W_{k_n} + \frac{j-k_n}{n} + \sum_{k=k_n}^{j-1} H_{k,n}(-\frac{\pi}{n}\cot(\frac{(k+1)\pi}{n}))\right) + o(\frac{j}{n^2})$$

for every j as in the statement. For $j = k_n$ the estimate is trivial (the sum is empty in this case). For $k_n < J < n - k_n$, by Lemma 4.3 we can apply Proposition 3.9, which gives

$$\begin{aligned} W_{J+1} = W_J + \tilde{A}_J = &\left(W_{k_n} + \frac{J-k_n}{n} + \sum_{k=k_n}^{J-1} H_{k,n}(-\frac{\pi}{n}\cot(\frac{(k+1)\pi}{n}))\right) + o(\frac{J}{n^2}) \\ &+ \left(\frac{1}{n} + H_{J,n}(-\frac{\pi}{n}\cot(\pi W_{J+1})) + o(\frac{1}{n^2})\right), \end{aligned}$$

where we used the assumption (21) on k_n to bound with $o(1/n^2)$ the error term in Proposition 3.9 (here, we may get a more precise error term $O\Big(\frac{1}{n^{2+\alpha}}, \frac{\log(n/k_n)}{n\,k_n^2}\Big)$ leading to the more precise error term mentioned above). It follows that

$$W_{J+1} = W_{k_n} + \frac{J+1-k_n}{n} + \left(\sum_{k=k_n}^{J-1} H_{k,n}(-\frac{\pi}{n}\cot(\frac{(k+1)\pi}{n})) + H_{J,n}(-\frac{\pi}{n}\cot(\pi W_{J+1})) \right) + o\Big(\frac{J}{n^2}\Big).$$

To conclude, we need to prove that

$$H_{J,n}(-\frac{\pi}{n}\cot(\pi W_{J+1})) - H_{J,n}(-\frac{\pi}{n}\cot(\frac{(J+1)\pi}{n})) = o(\frac{1}{n^2})$$

(observe that each term $H_{k,n}$ is $O(1/n^2)$). By the definition of $H_{k,n}$, it is enough to prove that

$$\cot(\pi W_{J+1}) - \cot(\frac{(J+1)\pi}{n}) = o(1) \tag{24}$$

By Lemma 4.3, we have $W_{J+1} = \frac{J+1}{n} + O(\frac{1}{n})$. Hence, we have

$$|\cot(\pi W_{J+1}) - \cot(\frac{(J+1)\pi}{n})| \lesssim \cot'(\frac{(J+1)\pi}{n}) \cdot O(\frac{1}{n})$$

Since $\cot(\cdot)$ has a simple pole at 0, its derivative has a double pole at the same point. Since $J \geq k_n$, we deduce that $|\cot'(\frac{(J+1)\pi}{n})| = O(\frac{n^2}{k_n^2})$. Hence,

$$|\cot(\pi W_{J+1}) - \cot(\frac{(J+1)\pi}{n})| = O(\frac{n}{k_n^2}) = o(1),$$

where in the last step we used the assumption (21) on k_n. □

We conclude this part with the following estimate for the point w_{n-k_n}, which we will need to initialize the next part (we will actually need only the bound $w_{n-k_n} = O(1/k_n)$, but the proof for the precise expression is essentially the same).

Lemma 4.5. *We have*

$$w_{n-k_n} = \frac{1}{k_n} + O\Big(\frac{1}{n}\Big).$$

In particular, $w_{n-k_n} = O(1/k_n)$.

Proof. By Lemma 4.3 we have

$$W_{n-k_n} = \frac{n-k_n}{n} + O\Big(\frac{1}{n}\Big) = 1 - \frac{k_n}{n} + O\Big(\frac{1}{n}\Big).$$

Set $\eta_n := 1 - W_{n-k_n}$, so that $\eta_n = \frac{k_n}{n} + O(1/n)$ and $n\eta_n = k_n + O(1)$. By Lemma 3.8(1), we have

$$w_{n-k_n} = -\frac{\pi}{n}\cot(\pi W_{n-k_n}) + O\Big(\frac{1}{n^2}\Big) + O\Big(\frac{\log(n/k_n)}{k_n^2}\Big).$$

Since $\cot(\pi(1-\eta)) = -\cot(\pi\eta) = -(\pi\eta)^{-1} + O(\eta)$ as $\eta \to 0$, we obtain

$$-\frac{\pi}{n}\cot(\pi W_{n-k_n}) = \frac{1}{n\eta_n} + O\Big(\frac{\eta_n}{n}\Big) = \frac{1}{k_n} + O\Big(\frac{1}{n}\Big).$$

This gives the assertion (recall that, by (21), $\frac{\log(n/k_n)}{k_n^2}$ is negligible with respect to $\frac{1}{n}$). □

4.3. **Exiting the eggbeater.** The next two estimates mirror those of Proposition 4.2 and Lemma 4.1.

Proposition 4.6. *We have*

$$n\big(\phi(w_{n-k_n}) - 1\big) = \phi^o(w_{n-k_n}) - (1-a)\log(\pi/n) + o(1).$$

Proof. Set $\eta_n := \psi(w_{n-k_n}) - 1$. Since $\psi(w_{n-k_n}) \in (0,1) + i(-1,1)$ and $w_{n-k_n} = O(1/k_n)$ (by Lemma 4.5) a Taylor expansion of the logarithm in the definition of ψ shows that

$$n\eta_n = -\frac{1}{w_{n-k_n}} + o(1).$$

Moreover, we have

$$\sin(\pi\psi(w_{n-k_n})) = \sin(\pi(1+\eta_n)) = -\sin(\pi\eta_n) = -\pi\eta_n + o(1/n),$$

hence

$$\log\sin(\pi\psi(w_{n-k_n})) = \log\Big(\frac{\pi}{n\,w_{n-k_n}}\Big) + o(1) = -\log w_{n-k_n} + \log(\pi/n) + o(1).$$

Recalling that $\phi = \chi \circ \psi$, we get

$$\begin{aligned} n(\phi(w_{n-k_n}) - 1) &= n(\psi(w_{n-k_n}) - 1) - (1-a)\log\sin(\pi\psi(w_{n-k_n})) \\ &= -\frac{1}{w_{n-k_n}} + (1-a)\log w_{n-k_n} - (1-a)\log(\pi/n) + o(1). \end{aligned}$$

Using the asymptotic (6) for ϕ^o gives the assertion. □

Lemma 4.7. *We have*

$$\phi^o(w_n) = \phi^o(w_{n-k_n}) + k_n + o(1).$$

Proof. Arguing exactly as in Lemma 4.1, but starting at time $n-k_n$, and using the estimate for w_{n-k_n} given by Lemma 4.5, we obtain

$$w_{n-k_n+\ell} = \frac{1+o(1)}{k_n - \ell} \qquad \text{for } 0 \le \ell \le k_n - 1. \tag{25}$$

Since $\phi^o \circ f = \phi^o + 1$, $(\phi^o)'(w) = O(1/w^2)$ near 0, and $|\varepsilon_{j,n}|^2 = O(1/n^2)$, for every $n - k_n \le j \le n-1$ we get

$$\begin{aligned} \phi^o(w_{j+1}) = \phi^o(f(w_j) + \varepsilon_{j,n}^2) &= \phi^o(f(w_j)) + O\bigg(\frac{1}{n^2}\left|(\phi^o)'(f(w_j))\right|\bigg) \\ &= \phi^o(w_j) + 1 + O\bigg(\frac{1}{n^2|w_{j+1}|^2}\bigg). \end{aligned}$$

Therefore,

$$\phi^o(w_n) - \phi^o(w_{n-k_n}) = k_n + O\Bigg(\frac{1}{n^2}\sum_{\ell=0}^{k_n-1}\frac{1}{|w_{n-k_n+\ell+1}|^2}\Bigg).$$

Using (25), we obtain

$$\phi^o(w_n) - \phi^o(w_{n-k_n}) = k_n + O\Bigg(\frac{1}{n^2}\sum_{m=1}^{k_n} m^2\Bigg) = k_n + O\bigg(\frac{k_n^3}{n^2}\bigg) = k_n + o(1),$$

where in the last step we use the assumption (21) on k_n. □

4.4. **End of the proof of Theorem 1.1.** We can now conclude the proof of Theorem 1.1. Applying Proposition 4.4 with $j = n - k_n$ and multiplying by n, we obtain

$$nW_{n-k_n} = nW_{k_n} + n - 2k_n + n \sum_{m=k_n}^{n-k_n-1} H_{m,n}\Big(-\frac{\pi}{n}\cot\Big(\frac{(m+1)\pi}{n}\Big)\Big) + o(1).$$

By Proposition 4.2 and Lemma 4.1, we have

$$nW_{k_n} = \phi^\iota(w_{k_n}) - (1-a)\log(\pi/n) + o(1) = \phi^\iota(w_0) + k_n - (1-a)\log(\pi/n) + o(1).$$

On the outgoing side, Proposition 4.6 gives

$$\phi^o(w_{n-k_n}) = nW_{n-k_n} - n + (1-a)\log(\pi/n) + o(1).$$

Combining the previous identities, we get

$$\phi^o(w_{n-k_n}) = \phi^\iota(w_0) - k_n + n \sum_{m=k_n}^{n-k_n-1} H_{m,n}\Big(-\frac{\pi}{n}\cot\Big(\frac{(m+1)\pi}{n}\Big)\Big) + o(1).$$

Finally, applying Lemma 4.7 we obtain

$$\phi^o(w_n) = \phi^\iota(w_0) + n \sum_{m=k_n}^{n-k_n-1} H_{m,n}\Big(-\frac{\pi}{n}\cot\Big(\frac{(m+1)\pi}{n}\Big)\Big) + o(1).$$

We can also extend the sum above to all indices $0 \le m \le n-1$, since the omitted boundary terms contribute $o(1)$ after multiplication by n. Indeed, for the boundary indices one has

$$\begin{cases} G\big(\frac{m+1}{n}\big) = O\Big(\frac{(m+1)^2}{n^2}\Big) & \text{for } m < k_n, \\ G\big(\frac{m+1}{n}\big) = O\Big(\frac{(n-m)^2}{n^2}\Big) & \text{for } n-k_n \le m \le n-1. \end{cases}$$

Hence the omitted contribution is

$$O\left(\frac{1}{n}\sum_{m<k_n}\frac{(m+1)^2}{n^2}\right) + O\left(\frac{1}{n}\sum_{n-k_n\le m\le n-1}\frac{(n-m)^2}{n^2}\right) = O\Big(\frac{k_n^3}{n^3}\Big) = o(1),$$

where in the last step we used the assumption (21) on k_n. As a result, we obtain

$$\phi^o(w_n) = \phi^\iota(w_0) + n \sum_{m=0}^{n-1} H_{m,n}\Big(-\frac{\pi}{n}\cot\Big(\frac{(m+1)\pi}{n}\Big)\Big) + o(1).$$

To conclude, we need to compute the sum on the right hand side. It follows from the definition (7) of $H_{k,n}$ (see Lemma 3.1 for the definition of δ_k) that

$$\begin{aligned} H_{k,n}(w) &= \frac{2\pi^2\sigma_k}{2i\pi n^3}\frac{\zeta^+ - \zeta^-}{w^2 - (\zeta^+ + \zeta^-)w + \zeta^+\zeta^-} = \frac{2\pi^2\sigma_k}{2i\pi n^3}\frac{\frac{2i\pi}{n}}{w^2 - \frac{a\pi^2}{n^2}w + \frac{\pi^2}{n^2} + O(\frac{1}{n^3})} \\ &= \frac{2\pi^2\sigma_k}{n^4}\frac{1}{w^2 - \frac{a\pi^2}{n^2}w + \frac{\pi^2}{n^2} + O(\frac{1}{n^3})}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} H_{k,n}\Big(-\frac{\pi}{n}\cot\Big(\frac{(k+1)\pi}{n}\Big)\Big) \\ &= \frac{2\pi^2\sigma_k}{n^4}\left(\frac{\pi^2}{n^2}\cot^2\Big(\frac{(k+1)\pi}{n}\Big)+\frac{a\pi^3}{n^3}\cot\Big(\frac{(k+1)\pi}{n}\Big)+\frac{\pi^2}{n^2}+O(\frac{1}{n^3})\right)^{-1} \\ &= \frac{2\sigma_k}{n^2}\left(\cot^2\Big(\frac{(k+1)\pi}{n}\Big)+\frac{a\pi}{n}\cot\Big(\frac{(k+1)\pi}{n}\Big)+1+O(\frac{1}{n})\right)^{-1}. \end{aligned}$$

It follows that

$$\begin{aligned} n\sum_{k=0}^{n-1}H_{k,n}\Big(-\frac{\pi}{n}\cot\Big(\frac{(k+1)\pi}{n}\Big)\Big) \\ &= \frac{2}{n}\sum_{k=0}^{n-1}\sigma_k\left(\cot^2\Big(\frac{(k+1)\pi}{n}\Big)+\frac{a\pi}{n}\cot\Big(\frac{(k+1)\pi}{n}\Big)+1+O(\frac{1}{n})\right)^{-1} \\ &= \left(\frac{1}{n}\sum_{k=0}^{n-1}G(\frac{(k+1)}{n})\sigma_k\right)+o(1). \end{aligned}$$

The proof is complete.

Remark 4.8. *We observe that* $\int_0^1 G(x)dx = 1$*, so that if* $\sigma_k \equiv \sigma$ *independently of* k*, we do find again* $\frac{\sigma}{n}\sum_{k=0}^{n-1} G(\frac{k+1}{n}) = \sigma + o(1)$ *(as a Riemann sum), from which we recover the usual statement of the autonomous parabolic implosion.*

5. Proof of Corollaries 1.2–1.5

Proof of Corollary 1.2. Fix $n \in \mathbb{N}$. For $j \geq 0$, it will be convenient to set $\varepsilon_j := \varepsilon_{j,n} = \frac{\pi}{2}\sqrt{p^{n^2+j}(z)}$, where $\sqrt{\cdot}$ denotes the square root with positive real part. For $z \in \mathcal{B}_p$, we have $p^k(z) = \frac{1}{k+O(\ln k)}$ as $k \to \infty$, so that

$$\varepsilon_j = \frac{\pi}{2n} - \frac{\pi}{2n}\frac{j}{2n^2} + O\left(\frac{\ln n}{n^3}\right).$$

Denote $N := 2n$ and $\sigma_j := -\frac{j}{n} = -\frac{2j}{N}$, so that

$$\varepsilon_j = \frac{\pi}{N} + \frac{\pi}{N^2}\sigma_j + O\left(\frac{\ln N}{N^3}\right).$$

Let $(z_j, w_j) = F^j(p^{n^2}(z), w)$. Then, by Theorem 1.1 we have $w_N = L_u(w) + o(1)$, where $u := \frac{1}{N}\sum_{j=0}^{N-1}\sigma_j G(\frac{j+1}{N})$. Hence,

$$u = -\frac{1}{N}\sum_{j=0}^{N-1}\frac{2j}{N}G(\frac{j+1}{N}) = -\int_0^1 4x\sin^2(\pi x)dx + o(1) = -1 + o(1).$$

Therefore $w_N = L_{-1}(w) + o(1)$, and so $w_{2n+1} = L_0(w) + o(1)$, as required. □

Proof of Corollary 1.3. By Theorem 1.1, it suffices to prove that

$$u_n := \frac{1}{n}\sum_{k=0}^{n-1}\sigma_{k,n}\, G\Big(\frac{k+1}{n}\Big) \longrightarrow 0, \qquad n\to\infty.$$

Observe that, since $G(1)=2\sin^2(\pi)=0$, the term corresponding to $k=n-1$ in the sum above vanishes. Moreover, since $G(x)=G(1-x)$ for every $x\in[0,1]$, for every $0\le k\le n-2$ we have

$$G\Big(\frac{k+1}{n}\Big) = G\Big(1-\frac{k+1}{n}\Big) = G\Big(\frac{n-1-k}{n}\Big) = G\Big(\frac{(n-2-k)+1}{n}\Big).$$

Therefore, we obtain

$$u_n = \frac{1}{2n}\sum_{k=0}^{n-2}(\sigma_{k,n}+\sigma_{n-2-k,n})\, G\Big(\frac{k+1}{n}\Big).$$

Since G is bounded on $[0,1]$, using (4) we obtain

$$|u_n| \le \frac{\|G\|_\infty}{2n}\sum_{k=0}^{n-2}|\sigma_{k,n}+\sigma_{n-2-k,n}| = O\Big(\frac{1}{n}\Big).$$

Hence $u_n\to 0$, and the conclusion follows from Theorem 1.1. □

Proof of Corollary 1.4. By Theorem 1.1, it suffices to prove that the phases

$$u_n := \frac{1}{n}\sum_{k=0}^{n-1}\sigma_{k,n}\, G\Big(\frac{k+1}{n}\Big)$$

converge almost surely to u, where $G(x)=2\sin^2(\pi x)$ is as in Theorem 1.1.

Set $b_j := \sigma_{j-1}$ for $j\ge 1$. Then we have

$$u_n = \frac{1}{n}\sum_{j=1}^{n} b_j\, G\Big(\frac{j}{n}\Big) \qquad\text{and}\qquad \frac{1}{n}\sum_{j=1}^{n} b_j = \frac{1}{n}\sum_{j=0}^{n-1}\sigma_j \longrightarrow u \qquad \text{almost surely.}$$

Therefore, Lemma 5.1 applied to the sequence (b_j) and the function $g=G$ gives

$$u_n \longrightarrow u\int_0^1 G(x)\,dx \qquad \text{almost surely.}$$

Since $\int_0^1 2\sin^2(\pi x)\,dx = 1$, we obtain $u_n\to u$ almost surely. Hence we have $w_n^{(n)}\to L_u(w_0)$ almost surely, for every $w_0\in B_f$. This concludes the proof. □

The following lemma is an elementary consequence of the discrete integration by parts. We give a proof for the reader's convenience. It will also be used in the proof of Corollary 1.6, see Proposition 6.1. The assumption on the derivative of g is not essential, but the proof is easier assuming it.

Lemma 5.1. *Let $(b_k)_{k\ge 1}$ be a bounded sequence such that*

$$\frac{1}{n}\sum_{k=1}^{n} b_k \longrightarrow L\in\mathbb{C} \qquad (n\to\infty).$$

Then for every $g \in C^1([0,1])$ one has

$$\frac{1}{n}\sum_{k=1}^{n} b_k\, g\Big(\frac{k}{n}\Big) \longrightarrow L\int_0^1 g(x)\,dx \qquad (n\to\infty).$$

Proof. Set $S_0 := 0$ and $S_j := \sum_{k=1}^{j} b_k$ for $j \geq 1$. The assumption implies $S_j/j \to L$, hence $S_j = Lj + o(j)$. As $b_k = S_k - S_{k-1}$, a summation by parts gives

$$\sum_{k=1}^{n} b_k\, g\Big(\frac{k}{n}\Big) = S_n\, g(1) - \sum_{k=1}^{n-1} S_k \cdot \Big(g\Big(\frac{k+1}{n}\Big) - g\Big(\frac{k}{n}\Big)\Big),$$

which gives

$$\frac{1}{n}\sum_{k=1}^{n} b_k\, g\Big(\frac{k}{n}\Big) = L\, g(1) + o(1) - \frac{1}{n}\sum_{k=1}^{n-1} S_k\Big(g\Big(\frac{k+1}{n}\Big) - g\Big(\frac{k}{n}\Big)\Big),$$

using that $S_n/n \to L$. By the mean value theorem, we have

$$g\Big(\frac{k+1}{n}\Big) - g\Big(\frac{k}{n}\Big) = \frac{1}{n}\, g'(\xi_{k,n}) \quad \text{for some } \xi_{k,n} \in (k/n, (k+1)/n),$$

hence

$$\frac{1}{n}\sum_{k=1}^{n-1} S_k\Big(g\Big(\frac{k+1}{n}\Big) - g\Big(\frac{k}{n}\Big)\Big) = \frac{1}{n^2}\sum_{k=1}^{n-1} S_k\, g'(\xi_{k,n}).$$

Using that $S_k = Lk + o(k)$ and the boundedness of g', we get

$$\frac{1}{n^2}\sum_{k=1}^{n-1} S_k\, g'(\xi_{k,n}) = \frac{L}{n^2}\sum_{k=1}^{n-1} k\, g'(\xi_{k,n}) + o(1).$$

Moreover,

$$\frac{1}{n^2}\sum_{k=1}^{n-1} k\, g'(\xi_{k,n}) = \frac{1}{n}\sum_{k=1}^{n-1} \frac{k}{n}\, g'(\xi_{k,n}) \longrightarrow \int_0^1 x\, g'(x)\,dx,$$

since $\xi_{k,n} \to k/n$ uniformly in k. Therefore

$$\frac{1}{n}\sum_{k=1}^{n} b_k\, g\Big(\frac{k}{n}\Big) \longrightarrow L\, g(1) - L\int_0^1 x\, g'(x)\,dx. \tag{26}$$

Finally, an integration by parts gives

$$g(1) - \int_0^1 x\, g'(x)\,dx = \int_0^1 g(x)\,dx.$$

Hence, the limit in (26) is equal to $L\int_0^1 g(x)\,dx$. The assertion follows. □

Proof of Corollary 1.5. For a μ-generic z, we apply Corollary 1.4 to the bounded sequence $\sigma_k := \sigma(T^k z)$. Then, by Birkhoff's ergodic theorem, we have

$$\frac{1}{n}\sum_{k=0}^{n-1} \sigma(T^k z) \longrightarrow \int_\Omega \sigma\, d\mu.$$

Hence, the corresponding fiber dynamics converges to L_u, with $u = \int_\Omega \sigma \, d\mu$ as desired. □

6. Proof of Corollary 1.6

Let F be an endomorphism of $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$ of algebraic degree $d \geq 2$ which is fibered over a rational map p of degree d on $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$, i.e., such that there exists a dominant rational map $\pi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ such that

$$\pi \circ F = p \circ \pi.$$

More explicitly, after choosing homogeneous coordinates such that $\pi([z_1, z_2, w]) = [z_1, z_2]$, any such endomorphism can be written in the form

$$F([z_1, z_2, w]) = [p_1(z_1, z_2), p_2(z_1, z_2), Q(z_1, z_2, w)]$$

where $p = [p_1, p_2]$ is the expression of the rational map p in homogeneous coordinates. Geometrically, the map F sends each fiber of π to another fiber, and the induced dynamics on the base $\mathbb{P}^1$, which parametrizes the fibers, is given by p. Recall from [Jon99; DT21] that the Julia set $J(F)$ and the equilibrium measure μ_F of F admit a decomposition and a disintegration of the form

$$J(F) = \bigcup_{z \in J(p)} J_z \text{ and } \mu_F = \int \mu_z d\mu_p(z)$$

where $J_z \subset \pi^{-1}(z)$ and μ_z is a probability measure on $\pi^{-1}(z)$.

Proposition 6.1. *Let $F : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic endomorphism which, in some affine chart, can be written in the form $F(z, w) = (p(z), q(w))$, where p is a rational map and q is a polynomial map of the form $q(w) = w + w^2 + O(w^3)$. Let F_n be a sequence of holomorphic endomorphisms of $\mathbb{P}^2$ which, in the same affine chart, have the form*

$$F_n(z, w) = F(z, w) + \left(0, \left(\frac{\pi}{n} + \frac{a(z)}{n^2}\right)^2\right).$$

Then, for every ergodic p-invariant probability measure ν and for ν-a.e. $z \in J(p)$ we have

$$\liminf_{n \to \infty} J_z(F_n) \supset J_{\text{Lav}}\left(q, \frac{1}{\pi} \int a \, d\nu\right),$$

As in [Lav89], given $u \in \mathbb{C}$, the *Julia-Lavaurs set* $J_{\text{Lav}}(q, u)$ is defined as

$$J_{\text{Lav}}(q, u) := \overline{\left\{ w \ : \ \exists m \geq 0 \text{ such that } L_u^{\circ m}(w) \text{ is defined and } L_u^{\circ m}(w) \in J(q) \right\}}.$$

Observe that we have $J(q) \subset J_{\text{Lav}}(q, u)$ for every $u \in \mathbb{C}$ and $\cup_{u \in \mathbb{C}} J_{\text{Lav}}(q, u) = J \cup \mathcal{B}_q$, see for instance [Lav89; Dou94].

Proof of Proposition 6.1. For $z \in J_p$ and $n \geq 1$, the fiber dynamics of F_n over z is described by the non-autonomous iteration

$$w_{k+1} = q(w_k) + \varepsilon_{k,n}(z)^2, \qquad \varepsilon_{k,n}(z) := \frac{\pi}{n} + \frac{a(p^k(z))}{n^2} \qquad 0 \leq k \leq n-1.$$

Thus, it fits the framework of Theorem 1.1 with

$$\sigma_{k,n}(z) = \frac{1}{\pi}\, a(p^k(z)).$$

Fix an ergodic p-invariant measure ν on J_p and let z be ν-generic. Then, by Birkhoff's ergodic theorem, we have

$$\frac{1}{n}\sum_{k=0}^{n-1} a(p^k(z)) \;\longrightarrow\; \int a\, d\nu.$$

Since $\int_0^1 G(t)\, dt = 1$, Lemma 5.1 gives

$$\frac{1}{n}\sum_{k=0}^{n-1} \sigma_{k,n}(z)\, G\Big(\frac{k+1}{n}\Big) = \frac{1}{\pi n}\sum_{k=0}^{n-1} a(p^k(z))\, G\Big(\frac{k+1}{n}\Big) \;\longrightarrow\; \frac{1}{\pi}\int a\, d\nu.$$

Hence, by Theorem 1.1, the (non-autonomous) fiber dynamics of F_n over such z converges to the Lavaurs map of phase $u = \frac{1}{\pi}\int a\, d\nu$. In particular, we have

$$\liminf_{n\to\infty} J_z(F_n) \;\supset\; J_{\mathrm{Lav}}(q, u).$$

This inclusion is proved as in the one-dimensional case [Lav89] and is a consequence of the lower semi continuity of the fibered Julia sets, which in turn is a consequence of the continuity (with respect to the map) of the corresponding measures μ_z [Jon99; DT21]. □

Corollary 1.6 will follow combining the above proposition and the following lemma.

Lemma 6.2. *Let $p\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geq 2$ and let $a\colon J(p) \to \mathbb{C}$ be Hölder continuous. Assume that, for every $v \in \mathbb{C}\setminus\{0\}$, the function $z \mapsto v \cdot a(z)$ is not cohomologous to a constant on J_p, where $\cdot$ denotes the standard inner product on $\mathbb{R}^2 \simeq \mathbb{C}$. Then the set*

$$W := \left\{ \int_{J_p} a\, d\nu : \; \nu \textit{ ergodic } p\textit{-invariant and } \mathrm{supp}(\nu) = J_p \right\}$$

has nonempty interior in $\mathbb{C}$.

Proof. For $\tau = (\tau_1, \tau_2) \in \mathbb{R}^2$ consider the real Hölder continuous potential

$$\varphi_\tau := \tau_1 \operatorname{Re}(a) + \tau_2 \operatorname{Im}(a).$$

For every τ sufficiently small, the potential φ_τ admits a unique equilibrium state ν_τ for φ_τ, see for instance [BD23; DPU96]. Moreover, ν_τ is ergodic and has full support. Let $P(\tau) := P(\varphi_\tau)$ be the topological pressure. Again for τ sufficiently small, by [BD24] the function $\tau \mapsto P(\tau)$ is real-analytic and satisfies

$$\nabla P(\tau) = \left(\int \operatorname{Re}(a)\, d\nu_\tau,\; \int \operatorname{Im}(a)\, d\nu_\tau \right).$$

We observe here that the assumption **(A)** in [BD23; BD24] is not needed in our case, since in dimension 1 all periodic critical points are outside of the Julia set. The assumption on a implies that $\det \operatorname{Hess} P(0) \neq 0$. To see this, for $v \in \mathbb{C} \sim \mathbb{R}^2$, define the function $g_v(z) := v \cdot a(z)$. Then g_v is Hölder-continuous and we have

$$v^\top \operatorname{Hess} P(0)\, v = \sigma^2(g_v) \geq 0, \qquad \text{where} \quad \sigma^2(g_v) := \int g_v^2 \, d\nu_0 + 2 \sum_{j=1}^{\infty} \int g_\nu (g_v \circ p^j) \, d\nu_0$$

denotes the asymptotic variance of g_v with respect to ν_0, see for instance [PP90, Section 4] and [BD24, Lemma 5.7]. If $\operatorname{Hess} P(0)$ were not positive definite, there would exist $v \neq 0$ with $v^\top \operatorname{Hess} P(0) v = 0$, hence $\sigma^2(g_v) = 0$. But, again by [PP90; BD24], this implies that g_v is cohomologous to a constant. Therefore $\operatorname{Hess} P(0)$ is positive definite, hence invertible.

In particular, by the inverse function theorem, ∇P is a local diffeomorphism near $0 \in \mathbb{R}^2$. Therefore, its image contains an open subset, as desired. □

We can now prove Corollary 1.6.

Proof of Corollary 1.6. We first observe that the assumption on a in Corollary 1.6 implies that of Lemma 6.2.

Indeed, fix $v \neq 0$ and suppose by contradiction that $v \cdot a$ is cohomologous to a constant, i.e., that the function $z \mapsto v \cdot a(z) = c + h - h \circ p$ for some continuous function $h \colon J_p \to \mathbb{C}$. This implies that, for $i = 1, 2, 3$ we have

$$v \cdot A_i = \frac{1}{m_i} \sum_{j=0}^{m_i - 1} v \cdot a(p^j(z^{(i)})) = c.$$

Thus the three points A_1, A_2, A_3 lie on the affine real line

$$\{z \in \mathbb{C} : \ v \cdot z = c\},$$

contradicting the assumption that they are not collinear.

By the above, we can apply Lemma 6.2. Hence, the set of phases

$$W = \left\{ \frac{1}{\pi} \int a \, d\nu : \ \nu \text{ ergodic, } \operatorname{supp}(\nu) = J_p \right\}$$

contains a nonempty open set $\Omega \subset \mathbb{C}$. The Julia–Lavaurs set $J_{\text{Lav}}(q, u)$ depends non-trivially on u as the dependence is a translation in Fatou coordinates. Hence, $\bigcup_{u \in \Omega} J_{\text{Lav}}(q, u)$ contains a nonempty open subset $U \subset \mathbb{C}$.

The above, Proposition 6.1, and the fact that the measures in the definition of W above have full support in $J(p)$ imply that every $z \in J_p$ can be approximated by a sequence $z_m \in J_p$ with the property that, for every m, we have

$$U \subset \liminf_{n \to \infty} J_{z_m}(F^n).$$

As $J_{z_m}(F^n) \subset J(F^n)$ for every n and m, the assertion follows from a diagonal argument. □

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