PARABOLIC IMPLOSION IN DIMENSION 2

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Abstract. In this paper, we extend the theory of parabolic implosion in complex dimension 2 to the case of holomorphic maps tangent to the identity at order 2. We investigate the bifurcation phenomena that occur when a fully parabolic fixed point is perturbed. Under the assumption of a non-degenerate characteristic direction with a formal invariant curve and director α satisfying $\operatorname{Re}\alpha > 2$, we establish the existence of Lavaurs maps as limits of iterates $f_{\epsilon_n}^n$ for specific sequences of the perturbation parameter ϵ_n. Finally, we apply these results to prove the discontinuity of the Julia sets J_1 and J_2 for holomorphic endomorphisms of $\mathbb{P}^2$, generalizing classical one-dimensional results to this higher-dimensional setting.

1. Introduction

Parabolic implosion is the study of the bifurcation phenomena which occur when a multiple (i.e., parabolic) fixed point is perturbed and splits into several fixed points or periodic cycles. It was first developed by Lavaurs in his PhD thesis ([Lav89]). A first consequence of this theory is a precise description of the discontinuity (*enrichment*) of Julia sets with respect to the parameter, in the presence of a non-persistent parabolic cycle. This was used by Shishikura ([Shi98]) to prove that the boundary of the Mandelbrot set has Hausdorff dimension 2. A refinement of parabolic implosion (*near-parabolic renormalization*), developed by Inou and Shishikura ([IS06]), has led to remarkable results, such as the construction of quadratic Julia sets with positive area by Buff and Chéritat ([BC12]) or progress towards the hyperbolicity conjecture ([CS15]).

More recently, the theory of parabolic implosion has started to develop and to find successful applications in higher dimension. In [BSU17], Bedford, Smillie and Ueda develop a parabolic implosion theory in the setting of semi-parabolic diffeomorphisms in dimension 2, i.e., in the case of a fixed point with one attracting direction and the other one with multiplier equal to 1. In particular, in the important case of a dissipative Hénon map, they were able to deduce the discontinuity of several dynamically defined sets (including the forward Julia set J^+ and the closure J^* of the saddle periodic points) with respect to the parameter. Building on their result, Bianchi and the first named author proved in [AB25] the existence of perturbations of such Hénon maps whose forward Julia set J^+ has large Hausdorff dimension.

In [DL15], Dujardin and Lyubich adapted the results of [BSU17] to construct homoclinic tangencies for perturbations of dissipative Hénon maps with a semi-parabolic periodic cycle, with applications to bifurcation theory. In [ABD+16], the authors used parabolic implosion techniques to construct the first examples of polynomial maps (in dimension 2) with a wandering Fatou component (see also [ABTP23], [ABT26]). The dynamical systems under consideration are polynomial skew-products, hence the techniques employed can be seen as a non-autonomous version of one-dimensional parabolic implosion. Finally, in [Bia19b], Bianchi obtained results analogous to those of [BSU17] but for the more difficult case of maps with a fully parabolic fixed point, i.e., in the case where the differential at the fixed point is the identity. The purpose of this article is to extend the results from [Bia19b].

Let us now provide a quick overview of classical parabolic implosion in dimension one, in the simplest case of a parabolic fixed point with just one attracting and one repelling petal. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic map of the form $f(z) = z + z^2 + \mathcal{O}(z^3)$. By the classical Leau-Fatou theorem, there exists $r > 0$ and univalent maps $\phi^\iota : \mathbb{D}(-r, r) \to \mathbb{C}$ and $\phi^o : \mathbb{D}(r, r) \to \mathbb{C}$ such that

(1) $P^\iota := \mathbb{D}(-r, r)$ is forward invariant under f, and $P^o := \mathbb{D}(r, r)$ is invariant under the branch of f^{-1} fixing the origin;
(2) $\phi^\iota \circ f = \phi^\iota + 1$, and $\phi^o \circ f = \phi^o + 1$.

The domains P^ι and P^o are respectively called incoming and outgoing petals and the maps ϕ^ι and ϕ^o are called incoming and outgoing Fatou coordinates. Let

$$\mathcal{B} := \{z \in \mathbb{C} : f^n(z) \to 0 \text{ and } f^n(z) \neq 0 \quad \forall n \geq 0\}$$

be the *parabolic basin*. Then $P^\iota \subset \mathcal{B}$, and moreover $\mathcal{B} = \bigcup_{n \geq 0} f^{-n}(P^\iota)$. The incoming Fatou coordinate extends to a holomorphic map $\phi^\iota : \mathcal{B} \to \mathbb{C}$ and the inverse $(\phi^o)^{-1}$ of the outgoing Fatou coordinate extends to a holomorphic map $\psi^o : \mathbb{C} \to \mathbb{C}$, called the outgoing Fatou parametrization. We refer the reader to [Mil11] for details. In particular, for any $\sigma \in \mathbb{C}$, the change of coordinate $\mathcal{L}_\sigma := \psi^o \circ T_\sigma \circ \phi^\iota$ is well-defined on $\mathcal{B}$, where $T_\sigma(z) := z + \sigma$ is the translation of vector σ. It is called the *Lavaurs map* of phase σ.[1]

Consider now the family of perturbations $f_\varepsilon(z) = f(z) + \varepsilon^2$, $\varepsilon \in \mathbb{C}$. For ε small but non-zero, the double fixed point at the origin for f splits into 2 simple fixed points for f_ε, of the form $z^\pm(\varepsilon) = \pm i\varepsilon + \mathcal{O}(\varepsilon^2)$. If we take say $\varepsilon > 0$, then Lavaurs proved that orbits under f_ε starting from a point in P^ι will approach the origin, then cross the "gate" given by the vertical segment $[z^-(\varepsilon), z^+(\varepsilon)]$ between $z^-(\varepsilon)$ and $z^+(\varepsilon)$ and then move away from the origin inside P^o. Since f_ε is close to the identity near 0, as $\varepsilon \to 0$ it takes more and more iterations to do this. At the limit, we obtain in this way a "transit map" from P^ι to P^o, which is useful for studying the dynamics of f_ε. It turns out that this transit map is exactly the Lavaurs map defined above. More precisely:

Theorem (Lavaurs, [Lav89])**.** *Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers, and $\sigma \in \mathbb{C}$. Assume that* $\lim_{n\to\infty} (n - \pi/\varepsilon_n) = \sigma$. *Then $f^n_{\varepsilon_n} \to \mathcal{L}_\sigma$ locally uniformly on $\mathcal{B}$.*

We now move on to the setting of complex dimension 2. Let $f : U \subset \mathbb{C}^2 \to \mathbb{C}^2$ be a holomorphic map on a neighborhood U of the origin, with a power series expansion of the form

$$f = \mathrm{Id} + P_2 + P_3 + \dots$$

where the P_j are homogeneous degree j polynomial maps from $\mathbb{C}^2$ to $\mathbb{C}^2$, and $P_2 \not\equiv 0$. Such a map is called *tangent to the identity at order 2.* Following Hakim [Hak98] and Écalle [É85], we say that $v \in \mathbb{C}^2 \setminus \{(0, 0)\}$ is a *characteristic direction* for f if there exists $\lambda \in \mathbb{C}$ so that $P_k(v) = \lambda v$. If $\lambda \neq 0$ then v is said to be *non-degenerate.* We shall denote by $v \mapsto [v]$ the canonical projection of $\mathbb{C}^2\setminus\{(0, 0)\}$ onto $\mathbb{P}^1$. The *director* of a non-degenerate characteristic direction v is the eigenvalue of the linear operator

$$d(P_2)_{[v]} - \mathrm{Id} : T_{[v]}\mathbb{P}^1 \to T_{[v]}\mathbb{P}^1.$$

If the real part of the director of a non-degenerate characteristic direction v is strictly positive, Hakim proved in [Hak97] that for any $C > 0$ there exist incoming and outgoing petals P^ι_C and P^o_C and incoming/outgoing Fatou coordinates $\Phi^{\iota/o} : P^{\iota/o}_C \to \mathbb{C}^2$

[1] Fatou coordinates are in fact only unique up to addition by a constant; thus $\mathcal{L}_\sigma$ may be thought of as $(\phi^o)^{-1} \circ \phi^\iota$, for a different choice of ϕ^ι. In practice, we will work with some unique normalizations of Fatou coordinates based on their formal expansion at 0.

conjugating f to a translation of vector $(1,0)$ (see Propositions 3.2 and 3.3 for details). Let

$$\mathcal{B}_{U,v} := \{(x,y) \in U : f^n(x,y) \to (0,0) \text{ tangentially to } v\}$$

denote the *parabolic basin associated to* v. Similarly to the one-dimensional case, we have that $\mathcal{B}_{U,v} = \bigcup_{C>0} \bigcup_{n\geq 0} f^{-n}(P_C^\iota)$ (see [LHR25]).

We say that a formal non-singular curve $\mathcal{C}$ is invariant for f if given a parametrization $\gamma(t)$ of $\mathcal{C}$ (i.e., $\gamma(t) \in \mathbb{C}[[t]]^2$ with $\gamma(0) = (0,0)$ and $\gamma'(0) \neq (0,0)$) there exists $h \in t\mathbb{C}[[t]]$ with $h'(0) \neq 0$ such that

$$f \circ \gamma = \gamma \circ h.$$

The tangent of $\mathcal{C}$ is, by definition, $\mathbb{C} \cdot \gamma'(0)$.

From now on, we will assume that $f : U \subset \mathbb{C}^2 \to \mathbb{C}^2$ is a holomorphic map defined on a neighborhood U of the origin and which satisfies the following assumption:

(H_1) The map f is tangent to the identity at order 2 and has a non-degenerate characteristic direction v, with a formal non-singular invariant curve $\mathcal{C}$ tangent to v, and a director α such that $\operatorname{Re}\alpha > 2$.

It is worth mentioning that the existence of a non-singular formal invariant curve tangent to a non-degenerate characteristic direction v is a generic hypothesis. It is equivalent to the existence of an analytic curve C tangent to v which is preserved up to order $k = \lfloor \operatorname{Re}\alpha \rfloor + 2$ in the following sense: if $\gamma(t)$ is a parametrization of C then there exists $h \in t\mathbb{C}\{t\}$ such that

$$f \circ \gamma - \gamma \circ h \in \langle t^{k+1} \rangle$$

and this condition is always satisfied when $\alpha \notin \mathbb{N}$ (see [[Hak98], Section 3]).

We now state a first, non-technical version of our main result:

Theorem 1 (Non-technical version)**.** *Let $f : U \subset \mathbb{C}^2 \to \mathbb{C}^2$ be a holomorphic map satisfying (H_1). Then for any $q \in \mathbb{C}$ there exists a holomorphic family of holomorphic maps $(f_\varepsilon : U \to \mathbb{C}^2)_{\varepsilon\in\mathbb{D}}$ with $f_0 = f$ such that for any $\sigma \in \mathbb{C}$ and for any compact set $K \subset \mathcal{B}_{U,v}$ there exist $N \in \mathbb{N}$ and a sequence $\varepsilon_n \to 0$ such that $f_{\varepsilon_n}^{n-N} \to (\Phi^o)^{-1} \circ A_{\sigma-N,q} \circ \Phi^\iota$ uniformly on K, where $A_{\sigma-N,q}(X,Y) := (X + \sigma - N, e^{\pi q} Y)$.*

Since the maps f and f_ε are only defined on U, in general the iterates $f_{\varepsilon_n}^n$ may not be well-defined; however Theorem 1 implies that there is a constant integer N such that $f_{\varepsilon_n}^{n-N}$ is well-defined on K for all n large enough. Moreover, since the map

$$\mathcal{L}_{\sigma-N,q} = (\Phi^o)^{-1} \circ A_{\sigma-N,q} \circ \Phi^\iota$$

satisfies

$$f \circ \mathcal{L}_{\sigma-N,q} = \mathcal{L}_{\sigma-N,q} \circ f = \mathcal{L}_{\sigma-N+1,q},$$

it is not difficult to see that we also have $f_{\varepsilon_n}^{n-N'} \to \mathcal{L}_{\sigma-N',q}$ for any $N' \in \mathbb{Z}$ such that $\mathcal{L}_{\sigma-N',q}(K) \subset U$. In particular, in the case where the maps f_ε are global endomorphisms of a complex manifold we may simply take $N = 0$.

We can be more precise on the requirements for the family of perturbations $(f_\varepsilon)_{\varepsilon\in\mathbb{D}}$, and interpret the constants σ and q in terms of f_ε. We will make the following assumptions:

(H_2) The family $(f_\varepsilon)_{\varepsilon\in\mathbb{D}}$ preserves the formal curve $\mathcal{C}$ to order $m+1$, where $m := \lfloor \operatorname{Re}\alpha \rfloor + 1$, in the following sense: if $\gamma(t)$ is a parametrization of $\mathcal{C}$, there exists $h_\varepsilon \in \mathcal{O}(\mathbb{D})[[t]]$ such that

$$f_\varepsilon \circ \gamma - \gamma \circ h_\varepsilon \in \langle t^{m+2}, \varepsilon t^{m+1}, \ldots, \varepsilon^{m+2} \rangle.$$

(H_3) For all ε in a neighborhood of 0, f_ε has exactly 4 fixed points $z_i(\varepsilon)$ near the origin ($1 \leq i \leq 4$) and counted with multiplicity, which depend holomorphically on ε and such that $z_i'(0) := \frac{d}{d\varepsilon}_{|\varepsilon=0} z_i(\varepsilon)$ are non-zero and pairwise distinct.

Note that the existence of 4 fixed points depending holomorphically on ε is always satisfied up to passing to a branched cover in parameter space (although doing so affects the derivatives $z_i'(0)$).

With a slight abuse of notation we will say that $(f_\varepsilon)_{\varepsilon\in\mathbb{D}}$ *satisfies* $(H_1)-(H_3)$ when f_0 satisfies (H_1) and $(f_\varepsilon)_{\varepsilon\in\mathbb{D}^*}$ satisfies (H_2) and (H_3).

We can interpret the constants σ and q in terms of the multipliers of the fixed points of f_ε. To make this precise, we first need the following Proposition, whose proof is deferred until the next section:

Proposition 1. *Assume that $(f_\varepsilon : U \to \mathbb{C}^2)_{\varepsilon\in\mathbb{D}}$ satisfies $(H_1)-(H_3)$. Then there are exactly two fixed points of f_ε, say $z_1(\varepsilon), z_2(\varepsilon)$, which are asymptotically tangent to v for ε small, i.e.,*

$$z_i'(0) \in \mathbb{C}^* v, \quad 1 \leq i \leq 2.$$

Moreover, if the eigenvalues $\lambda_i(\varepsilon), \mu_i(\varepsilon)$ of $z_i(\varepsilon)$ satisfy that $\lambda_i'(0) \neq \mu_i'(0)$ then one of the eigenspaces of $z_i(\varepsilon)$ tends to $\mathbb{C}v$ as $\varepsilon \to 0$, and the condition $\lambda_i'(0) \neq \mu_i'(0)$ holds for at least one of the fixed points.

Remark 1. *If (H_1) and (H_2) are satisfied, then the proof of Proposition 1 will show that (H_3) can be replaced by the following slightly weaker condition:*

(H_3') The jet of order $m+1$ of h_ε has two fixed points $w_\pm(\varepsilon)$, and $w_+'(0) \neq w_-'(0)$.

Note that even though the fixed points $w_\pm(\varepsilon)$ depend on the higher order terms of h_ε, their derivatives at 0 do not.

We can now state a second version of our results, which is more explicit on the requirements on the family of perturbations $(f_\varepsilon)_{\varepsilon\in\mathbb{D}}$. To do so, we will use the following convention: let $z_1(\varepsilon)$ and $z_2(\varepsilon)$ be the fixed points from Proposition 1 and let $\lambda_1(\varepsilon), \mu_1(\varepsilon)$ and $\lambda_2(\varepsilon), \mu_2(\varepsilon)$ be their eigenvalues. If $\lambda_i'(0) \neq \mu_i'(0)$, we denote by $\rho_T^i(\varepsilon)$ the eigenvalue whose eigenspace tends to $\mathbb{C}v$ as $\varepsilon \to 0$ and by $\rho_N^i(\varepsilon)$ the other one; if $\lambda_i'(0) = \mu_i'(0)$, we assign the names $\rho_T^i(\varepsilon)$ and $\rho_N^i(\varepsilon)$ indifferently to the two eigenvalues $\lambda_i(\varepsilon)$ and $\mu_i(\varepsilon)$.

Theorem 2 (Coordinate-free version)**.** *Let $(f_\varepsilon : U \to \mathbb{C}^2)_{\varepsilon\in\mathbb{D}}$ be a family of holomorphic maps satisfying $(H_1)-(H_3)$. Let $z_1(\varepsilon), z_2(\varepsilon)$ be the fixed points from Proposition 1 and denote by $\rho_T^i(\varepsilon)$ and $\rho_N^i(\varepsilon)$ ($1 \leq i \leq 2$) their eigenvalues, using the convention above. Let*

$$q := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \frac{\rho_N^1(\varepsilon) + \rho_N^2(\varepsilon) - 2}{\rho_T^1(\varepsilon) + \rho_T^2(\varepsilon)}.$$

There exists a constant $\sigma_0 \in \mathbb{C}$ such that for any $1 \leq i \leq 2$, for any $\sigma \in \mathbb{C}$, for any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that

$$\frac{2i\pi}{\rho_T^i(\varepsilon_n) - 1} = n - \sigma + o(1),$$

and for any compact set $K \subset \mathcal{B}_{U,v}$, there exists $N \in \mathbb{N}$ such that

$$f_{\varepsilon_n}^{n-N} \to (\Phi^o)^{-1} \circ A_{\sigma+\sigma_0-N,q} \circ \Phi^\iota$$

uniformly on K, where $A_{\sigma+\sigma_0-N,q}(X,Y) := (X + \sigma + \sigma_0 - N, e^{\pi q} Y)$.

We finally state a third version of our main result, explicitly expressed in coordinates:

Theorem 3 (Coordinate version)**.** *Let*

$$g_\varepsilon(x,y) = (x + (x^2+\varepsilon^2)a_\varepsilon(x) + y b_\varepsilon(x,y), y + y c_\varepsilon(x,y) + d_\varepsilon(x))$$

be a family of holomorphic maps defined on a neighborhood U of $(0,0)$, where $a_\varepsilon, b_\varepsilon, c_\varepsilon$ and d_ε depend holomorphically on ε, and assume that

(1) $a_0(0) = 1$, $b_0(0,0) = 0$

(2) $c_\varepsilon(x,y) = \eta x + q\varepsilon + cy + \mathcal{O}_2(x,y,\varepsilon)$, *with* $\operatorname{Re}\eta > 3$, $q, c \in \mathbb{C}$ *and* $d_\varepsilon(x) = \mathcal{O}(x^{m+3}) + \varepsilon\mathcal{O}_{m+1}(x,\varepsilon)$ *where* $m = \lfloor \operatorname{Re}\eta \rfloor$.

Let (ε_n) *be a sequence such that* $n - \pi/\varepsilon_n = \sigma + o(1)$. *Then for any compact set* $K \subset \mathcal{B}_{U,(1,0)}$ *there exists* $N \in \mathbb{N}$ *such that*

$$g_{\varepsilon_n}^{n-N} \to (\Phi^o)^{-1} \circ A_{\sigma-N,q} \circ \Phi^\iota$$

uniformly on K, *where* $A_{\sigma-N,q}(X,Y) := (X + \sigma - N, e^{\pi q}Y)$.

Theorem 3 can be interpreted as a generalization of Bianchi's main result in [Bia19b], Theorem 1.4. Let us comment here on the differences between Theorem 3 and [Bia19b, Theorem 1.4]. First, [Bia19b, Theorem 1.4] applies only to maps satisfying a strong assumption, namely that they leave invariant the 3 complex lines $x = \pm i\varepsilon$ and $y = 0$ (which amounts to taking $b_\varepsilon(x,y) = d_\varepsilon(x) = 0$ with our notations). If that is the case, then the formal curve corresponding to our assumptions (H_1) and (H_2) is the curve $y = 0$, and it is invariant for the whole family of perturbations. Secondly, [Bia19b, Theorem 1.4] only proves convergence near the line $y = 0$ (instead of the whole parabolic basin associated to $v = (1,0)$), and only up to extraction. In particular, it does not rule out the possibility of the sequence $(g_{\varepsilon_n}^n)_{n\in\mathbb{N}}$ having more than one limit value. Finally, in [Bia19b, Theorem 1.4] the possible limits of $(g_{\varepsilon_n}^n)_{n\in\mathbb{N}}$ are not described explicitly as maps of the form $(\Phi^o)^{-1} \circ A_{\sigma,q} \circ \Phi^\iota$, and only depend on the parameter σ since in his case $q = 0$. On the other hand, we must note that [Bia19b, Theorem 1.4] only assumes $\operatorname{Re}\eta > 1$, compared to our assumption that $\operatorname{Re}\eta > 3$, so our results do not strictly imply his.

Let us now give an application case of our main results. A holomorphic endomorphism of $\mathbb{P}^2$ is a map which may be written in homogeneous coordinates as

$$f([z_0 : z_1 : z_2]) = [P_0(z) : P_1(z) : P_2(z)],$$

where $P_0, P_1, P_2 : \mathbb{C}^3 \to \mathbb{C}$ are homogeneous polynomials of degree d with no common factors. The integer d is called the algebraic degree of f. Given such an endomorphism f, one can define two distinct notions of Julia sets: the set $J_1(f)$, which may be defined as the non-normality locus; and the set $J_2(f)$, also sometimes called the small Julia set, which may be defined as the support of the unique measure of maximal entropy. Alternatively, $J_m(f)$ $(1 \leq m \leq 2)$ may be defined as the support of $T_f^{\wedge m}$, where T_f is the so-called Green current of f (this construction is not specific to the case of dimension 2). In general, $J_2(f) \subsetneq J_1(f)$, and even for very simple maps (such as product maps) there is no equality. We refer the reader to the survey [DS10] for more details.

In the following, for any $(\sigma, q) \in \mathbb{C}^2$ we will let $\mathcal{L}_{\sigma,q} := \Psi^o \circ A_{\sigma,q} \circ \Phi^\iota$, where Ψ^o is the extension of $(\Phi^o)^{-1}$ to $\mathbb{C}^2$. For any endomorphism f of $\mathbb{P}^2$ satisfying (H_1), we will set

$$J^1(f, \mathcal{L}_{\sigma,q}) := \overline{\{z \in \mathbb{P}^2 : \exists p \in J_1(f)\ \exists n \in \mathbb{N}\ \mathcal{L}_{\sigma,q}^n(p) = z\}}$$

Corollary 1 (Compare to [Bia19b, Theorem 1.6]). *Let* $f : \mathbb{P}^2 \to \mathbb{P}^2$ *be an endomorphism satisfying* (H_1) *and of algebraic degree* $d > \operatorname{Re}\alpha + 1$. *Then for any* $q \in \mathbb{C}$ *there exists a family* $(f_\varepsilon)_{\varepsilon\in\mathbb{D}}$ *of endomorphisms of* $\mathbb{P}^2$ *of algebraic degree* d *satisfying* $(H_1)-(H_3)$, *with* q *as Theorem 2. Moreover, for any* $\sigma \in \mathbb{C}$,

$$\liminf_{n\to+\infty} J_1(f_{\varepsilon_n}) \supset J_1(f, \mathcal{L}_{\sigma,q})$$

where $(\varepsilon_n)_{n\in\mathbb{N}}$ *is as in Theorem 2.*

Corollary 2. *Let* $f : \mathbb{P}^2 \to \mathbb{P}^2$ *be an endomorphism satisfying* (H_1) *and of algebraic degree* $d > \operatorname{Re}\alpha + 1$. *Assume moreover that* $\Psi^o(\mathbb{C}^2) \cap J_2(f) \neq \emptyset$. *Then the map* $\mathcal{H}_d(\mathbb{P}^2) \ni g \mapsto J_2(g)$ *is discontinuous at* f, *where* $\mathcal{H}_d(\mathbb{P}^2)$ *denotes the space of degree* d *endomorphisms of* $\mathbb{P}^2$.

Even if we drop the assumption that $\Psi^o(\mathbb{C}^2)\cap J_2(f) \neq \emptyset$, our arguments still prove the discontinuity of the closure of the set of repelling periodic points; however, as mentioned above, the Julia set J_2 may be smaller than this closure. This hypothesis is not easy to check in practice; let us however give a concrete example. Let $\eta \in \mathbb{C}$ with $\operatorname{Re}\eta > 3$, and let $d > \operatorname{Re}\eta$. Let

$$f(x,y) = (x + x^2 + axy + by^2 + x^d, y + \eta xy + cy^2 + y^d).$$

The polynomial map $f : \mathbb{C}^2 \to \mathbb{C}^2$ extends to an endomorphism of $\mathbb{P}^2$. If $(a,b) = (0,0)$, then the map f is a polynomial skew-product. In that case, Ψ^o is of the form $\Psi^o(x,y) = (\Psi^o_p(x), \Psi^o_2(x,y))$, where Ψ^o_p is an outgoing Fatou parametrization of the base polynomial map $p(z) := z + z^2 + z^d$. By [Jon99], $J_2(f) := \overline{\bigcup_{z\in J(p)} J_z}$, where $J(p)$ denotes the Julia set of p and J_z is the non-normality locus of $\{f^n : n \in \mathbb{N}\}$ restricted to the vertical line $x = z$. Since Ψ^o_p is non-constant and entire, it omits at most one value, so there exists $x_0 \in J(p)$ and $X_0 \in \mathbb{C}$ such that $\Psi^o_p(X_0) = x_0$. Similarly, the map $Y \mapsto \Psi^o_2(X_0, Y)$ is entire and non-constant and J_{x_0} is uncountable; so $\Psi^o(\mathbb{C}^2) \cap (J_2(f)) \neq \emptyset$. Now, the set $J_2(f)$ varies lower semi-continuously with respect to the parameters (a,b,c); and the map Ψ^o depends holomorphically (hence continuously) on (a,b,c). (For a proof of this fact in dimension one, see the Appendix in [ABD+16]; the argument remains valid in higher dimension). Therefore, there exists some open set $W \subset \mathbb{C}^3$ such that for all $(a,b,c) \in \mathbb{C}^3$, the map f satisfies $\Psi^o_f(\mathbb{C}^2) \cap J_2(f) \neq \emptyset$.

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Outline of the paper. In Section 2, we give a proof of Theorem 1 and Theorem 2 assuming Theorem 3. Sections 3 to 5 are devoted to the proof of Theorem 3. In Section 3, we introduce the incoming and outgoing petals for g_0 and recall the construction of Fatou coordinates and compute their asymptotics. Section 4 is devoted to the construction of so-called *approximate Fatou coordinates* for g_ε, which are in a sense close to the actual Fatou coordinates of g_0 and which nearly conjugate the dynamics of g_{ε_n} to a translation. In Section 5 we provide precise estimates of the orbit under g_{ε_n} in the parabolic basin and complete the proof of Theorem 3. Finally, Corollaries 1 and 2 are proved in Section 6.

2. Proof of Theorems 1 and 2 from Theorem 3

Let us first show how Theorem 3 implies Theorem 1. Let $f : U \subset \mathbb{C}^2 \to \mathbb{C}^2$ be a holomorphic map satisfying (H_1). If we choose coordinates (x,y) such that $v = (1,0)$ we have that

$$f(x,y) = \big(x + \lambda x^2 + \mathcal{O}(x^3, xy, y^2), y + \eta xy + \mathcal{O}(x^2y, y^2, x^3)\big)$$

with $\lambda \neq 0$ and $\eta/\lambda = \alpha + 1$, where α is the director of v. Up to conjugating by the linear map $(x,y) \mapsto (\lambda x, y)$, we can assume that $\lambda = 1$, so $\eta = \alpha + 1$. In those coordinates, the formal invariant curve $\mathcal{C}$ has a parametrization $\gamma(t) = (t, \zeta(t))$, with $\zeta(t) \in t^2\mathbb{C}[[t]]$. If we take $m := \lfloor \operatorname{Re}\eta \rfloor$ and $\Psi(x,y) := (x, y - J_{m+2}\zeta(x))$, where $J_{m+2}\zeta$ is the jet of order $m+2$ of ζ, we have that $g_0(x,y) = \Psi \circ f \circ \Psi^{-1}$ has the form

$$g_0(x,y) = \big(x + x^2 a_0(x) + y b_0(x,y), y + y c_0(x,y) + \mathcal{O}(x^{m+3})\big),$$

with $a_0(x) = 1 + \mathcal{O}(x)$, $b_0(0,0) = 0$ and $c_0(x,y) = \eta x + \mathcal{O}(x^2, y)$. Then Theorem 1 follows immediately considering a family (g_ε) and a sequence (ε_n) as in Theorem 3 and taking $f_\varepsilon := \Psi^{-1} \circ g_\varepsilon \circ \Psi$.

Let us now obtain Theorem 2 from Theorem 3. Consider a family $(f_\varepsilon : U \subset \mathbb{C}^2 \to \mathbb{C}^2)_{\varepsilon \in \mathbb{D}}$ of holomorphic maps satisfying $(H_1) - (H_3)$. We will start making several successive changes of coordinates until we obtain the form of Theorem 3.

As above, we first choose coordinates in which $v = (1,0)$ and the curve $\mathcal{C}$ has a parametrization $\gamma(t) := (t, \zeta(t))$, with $\zeta(t) \in t^2\mathbb{C}[[t]]$. Let $m := \lfloor \operatorname{Re} \alpha \rfloor + 1$ and $\Psi(x,y) := (x, y - J_{m+1}\zeta(x))$, where $J_{m+1}\zeta$ is the jet of order $m+1$ of ζ, and set $\widetilde{g}_\varepsilon := \Psi \circ f_\varepsilon \circ \Psi^{-1}$. Since the family (f_ε) preserves $\mathcal{C}$ up to order $m+1$ by hypothesis (H_2), we have

$$\widetilde{g}_\varepsilon(x, 0) = (p_\varepsilon(x), \mathcal{O}_{m+2}(x, \varepsilon)),$$

where p_ε depends holomorphically on ε, and $p_0(x) = x + \lambda x^2 + \mathcal{O}(x^3)$ for some constant $\lambda \neq 0$ (as above, the fact that $\lambda \neq 0$ follows from the assumptions of f_0 having order 2 and v being non-degenerate). Up to conjugating by the linear map $(x,y) \mapsto (\lambda x, y)$, we may assume without loss of generality that $\lambda = 1$.

In particular, there exist holomorphic functions $\widetilde{b}_\varepsilon$ and $\widetilde{c}_\varepsilon$ such that

$$\widetilde{g}_\varepsilon(x,y) = (p_\varepsilon(x) + y\widetilde{b}_\varepsilon(x,y), y(1 + \widetilde{c}_\varepsilon(x,y)) + \mathcal{O}_{m+2}(x, \varepsilon)). \tag{1}$$

Since $\widetilde{g}_0$ is tangent to the identity, we also have $\widetilde{b}_0(0,0) = \widetilde{c}_0(0,0) = 0$. Moreover, by hypotheses (H_3), $\widetilde{g}_\varepsilon$ has 4 fixed points $\widetilde{z}_i(\varepsilon) = \Psi(z_i(\varepsilon))$ $(1 \leq i \leq 4)$ near the origin, which depend holomorphically on ε and satisfy that $\widetilde{z}_i'(0)$ are non-zero and pairwise different.

Lemma 2.1. *There are exactly two fixed points $\widetilde{z}_1(\varepsilon), \widetilde{z}_2(\varepsilon)$ of $\widetilde{g}_\varepsilon$ (counting multiplicity) which are asymptotically tangent to v for ε small, i.e.,*

$$\widetilde{z}_i'(0) \in \mathbb{C}^* v, \quad 1 \leq i \leq 2.$$

Moreover, if $w_\pm(\varepsilon)$ denote the two fixed points of p_ε, then $w_\pm(\varepsilon)$ depend holomorphically on ε and $\widetilde{z}_i(\varepsilon) = (w_\pm(\varepsilon), 0) + \mathcal{O}(\varepsilon^2)$. In particular, $w_+'(0) \neq w_-'(0)$.

Proof. Let us write $\widetilde{g}_\varepsilon(x,y) = \sum_{n \geq 0} P_n(x,y,\varepsilon)$, where $P_n : \mathbb{C}^3 \to \mathbb{C}^2$ is homogeneous polynomial map of degree n. Since $\widetilde{g}_0$ is tangent to the identity, we have $P_0 = 0$ and $P_1(x,y,\varepsilon) = (x,y) + (\gamma\varepsilon, \delta\varepsilon)$ for some $\gamma, \delta \in \mathbb{C}$. We claim that under our assumptions, we must have $\gamma = \delta = 0$. Indeed, since $\widetilde{g}_\varepsilon(\widetilde{z}_i(\varepsilon)) = \sum_{n \geq 0} P_n(\widetilde{z}_i(\varepsilon), \varepsilon)$ and $\widetilde{z}_i(0) = 0$, by differentiating $\widetilde{g}_\varepsilon(\widetilde{z}_i(\varepsilon)$, we obtain

$$(\gamma, \delta) = \partial_\varepsilon \widetilde{g}_\varepsilon(\widetilde{z}_i(\varepsilon)|_{\varepsilon=0} = (0,0).$$

In particular, this means that

$$p_\varepsilon(x) = x + x^2 + a_{1,1}\varepsilon x + a_{0,2}\varepsilon^2 + \mathcal{O}_3(x, \varepsilon).$$

By Weierstrass' Preparation Theorem applied to $p_\varepsilon - \mathrm{Id}$, there exists holomorphic maps $\varepsilon \mapsto \alpha(\varepsilon), \varepsilon \mapsto \beta(\varepsilon)$ and $(\varepsilon, x) \mapsto u(\varepsilon, x)$ with $\alpha(0) = \beta(0) = 0$ and $u(0,0) = 1$ such that

$$p_\varepsilon(x) = x + (x^2 + \alpha(\varepsilon)x + \beta(\varepsilon))u(\varepsilon, x).$$

Moreover, $\alpha(\varepsilon) = a_{1,1}\varepsilon + \mathcal{O}(\varepsilon^2)$ and $\beta(\varepsilon) = a_{0,2}\varepsilon^2 + \mathcal{O}(\varepsilon^3)$. The fixed points $w_\pm(\varepsilon)$ of p_ε are the zeros of $x \mapsto x^2 + \alpha(\varepsilon)x + \beta(\varepsilon)$. It follows that $w_\pm(\varepsilon) = \frac{-\alpha(\varepsilon) \pm \sqrt{\alpha(\varepsilon)^2 - 4\beta(\varepsilon)}}{2}$. In particular,

$$w_\pm(\varepsilon) = \frac{-a_{1,1} \pm \sqrt{a_{1,1}^2 - 4a_{0,2}}}{2}\varepsilon + \mathcal{O}(\varepsilon^{3/2}),$$

and $w_\pm$ are complex differentiable at $\varepsilon = 0$, with $w_\pm'(0) = \frac{-a_{1,1} \pm \sqrt{a_{1,1}^2 - 4a_{0,2}}}{2}$.

It is not yet clear that w_+ and w_- are holomorphic near 0; note however that if

$$w'_+(0) - w'_-(0) = \sqrt{a_{1,1}^2 - 4a_{0,2}} \neq 0$$

then $\varepsilon \mapsto \alpha(\varepsilon)^2 - 4\beta(\varepsilon)$ vanishes exactly at order 2 and in this case the two fixed points $w_\pm(\varepsilon)$ depend holomorphically on ε.

Next, we write $p_\varepsilon(x) = x + (x - w_-(\varepsilon))(x - w_+(\varepsilon))(1 + \mathcal{O}(x, \varepsilon))$, $\widetilde{b}_\varepsilon(x, y) = b_1\varepsilon + b_2 x + b_3 y + \mathcal{O}_2(x, y, \varepsilon)$ and $\widetilde{c}_\varepsilon(x, y) = c_1\varepsilon + c_2 x + c_3 y + \mathcal{O}_2(x, y, \varepsilon)$. Note that $c_2 = \alpha + 1$, where α is the director of v, so $c_2 \neq 0$ by our assumptions. Let

$$H_\varepsilon(X, Y) := \frac{\widetilde{g}_\varepsilon(\varepsilon X, \varepsilon Y) - (\varepsilon X, \varepsilon Y)}{\varepsilon^2}$$

so that $H_\varepsilon(X, Y) = (0, 0)$ if and only if $(\varepsilon X, \varepsilon Y)$ is a fixed point of $\widetilde{g}_\varepsilon$. Then

$$H_\varepsilon(X, Y) = \big((X - w'_-(0))(X - w'_+(0)) + Y(b_1 + b_2 X + b_3 Y), Y(c_1 + c_2 X + c_3 Y)\big) + \mathcal{O}(\varepsilon),$$

so as $\varepsilon \to 0$, the map H_ε converges locally uniformly to

$$H(X, Y) = \big((X - w'_-(0))(X - w'_+(0)) + Y(b_1 + b_2 X + b_3 Y), Y(c_1 + c_2 X + c_3 Y)\big).$$

Since $c_2 \neq 0$, it is then straightforward to check that the set $H^{-1}(0, 0)$ is finite and contains 4 elements counted with multiplicity. Moreover, $H(X, 0) = (0, 0)$ if and only if $X = w'_\pm(0)$.

Since proper intersections of analytic sets persist under perturbations, for all ε small enough the set $H_\varepsilon^{-1}(0, 0)$ is has the same cardinality as $H^{-1}(0, 0)$ and its elements are close to those of $H^{-1}(0, 0)$. Therefore, for small ε the fixed points of the map $\widetilde{g}_\varepsilon$ are of the form $\varepsilon v_i + \mathcal{O}(\varepsilon^2)$, where $H(v_i) = (0, 0)$.

In particular, $\widetilde{g}_\varepsilon$ has exactly two fixed points $\widetilde{z}_i(\varepsilon)$ (with $1 \leq i \leq 2$) which are asymptotically tangent to $v = (1, 0)$ and moreover, $\widetilde{z}_i(\varepsilon) = (w_\pm(\varepsilon), 0) + \mathcal{O}(\varepsilon^2)$. Finally, the assertion that $w'_+(0) \neq w'_-(0)$ follows from the fact that $\widetilde{z}'_1(0) \neq \widetilde{z}'_2(0)$. In particular, $\varepsilon \mapsto w_\pm(\varepsilon)$ are indeed holomorphic. □

Remark 2.2. *As we mentioned in Remark 1, in our results hypothesis* (H_3) *can be replaced by the weaker assumption* (H'_3)*. To show this, it suffices to note that if we impose hypothesis* (H'_3) *then the jet or order* $m+1$ *of* $p_\varepsilon(x)$ *has two fixed points* $\hat{w}_\pm(\varepsilon)$ *with* $\hat{w}'_+(0) \neq \hat{w}'_-(0)$*, so* $p_\varepsilon(x)$ *has two fixed points* $w_\pm(\varepsilon)$ *with* $w'_+(0) \neq w'_-(0)$ *and then Lemma 2.1 also holds with the same proof.*

Lemma 2.3. *Up to replacing* ε *by* $\widetilde{\varepsilon} := \mu\varepsilon$ *for some* $\mu \neq 0$*, there is a family of polynomial automorphisms* $\Theta_\varepsilon : \mathbb{C}^2 \to \mathbb{C}^2$*, depending holomorphically on* ε *near* 0 *and such that* $\Theta_0 = \mathrm{Id}$*, such that the maps* $g_\varepsilon := \Theta_\varepsilon \circ \widetilde{g}_\varepsilon \circ \Theta_\varepsilon^{-1}$ *have the form*

$$g_\varepsilon(x, y) = (x + (x^2 + \varepsilon^2)a_\varepsilon(x) + y b_\varepsilon(x, y), y + y c_\varepsilon(x, y) + d_\varepsilon(x))$$

with $a_\varepsilon, b_\varepsilon, c_\varepsilon$ *and* d_ε *depending holomorphically on* ε *and*

(1) $a_0(0) = 1$*,* $b_0(0, 0) = 0$

(2) $c_\varepsilon(x, y) = \eta x + q\varepsilon + cy + \mathcal{O}_2(x, y, \varepsilon)$*, with* $\mathrm{Re}\,\eta > 3$*,* $q, c \in \mathbb{C}$ *and* $d_\varepsilon(x) = \mathcal{O}(x^{m+3}) + \varepsilon\mathcal{O}_{m+1}(x, \varepsilon)$ *where* $m = \lfloor \mathrm{Re}\,\eta \rfloor$*.*

Proof. By Lemma 2.1, the fixed points $w_\pm(\varepsilon)$ of p_ε depend holomorphically on ε and satisfy $w'_+(0) \neq w'_-(0)$. Therefore, there exists $\mu \in \mathbb{C}^*$ such that if we replace ε by $\widetilde{\varepsilon} := \mu\varepsilon$ then $\frac{d}{d\widetilde{\varepsilon}}_{|\widetilde{\varepsilon}=0}(w_+(\widetilde{\varepsilon}) - w_-(\widetilde{\varepsilon})) = 2i$. By an abuse of notation, we will still denote $\widetilde{\varepsilon}$ by ε: in other words, we assume from now on that $w'_+(\varepsilon) - w'_-(\varepsilon)) = 2i$.

For $\varepsilon \neq 0$, let N_ε be the unique affine automorphism of $\mathbb{C}$ mapping $w_\pm(\varepsilon)$ to $\pm i\varepsilon$ (note that N_ε is well-defined for ε small since $w_+(\varepsilon) \neq w_-(\varepsilon)$). More explicitly,

$$N_\varepsilon(x) = \frac{2i\varepsilon}{w_+(\varepsilon) - w_-(\varepsilon)}(x - w_+(\varepsilon)) + i\varepsilon$$

and $\varepsilon \mapsto N_\varepsilon$ extends holomorphically to a neighborhood of 0 with

$$N_0(x) := \frac{2i}{w'_+(0) - w'_-(0)} x = x.$$

Let $M_\varepsilon(x,y) := (N_\varepsilon(x), y)$ and $\widehat{g}_\varepsilon := M_\varepsilon \circ \widetilde{g}_\varepsilon \circ M_\varepsilon^{-1}$. A direct computation using expression (1) shows that $\widehat{g}_\varepsilon$ is of the form

$$\widehat{g}_\varepsilon(x,y) = \left(N_\varepsilon \circ p_\varepsilon \circ N_\varepsilon^{-1}(x) + y\widehat{b}_\varepsilon(x,y), y(1 + \widehat{c}_\varepsilon(x,y)) + \widehat{d}_\varepsilon(x) \right)$$

with $\widehat{b}_\varepsilon, \widehat{c}_\varepsilon, \widehat{d}_\varepsilon$ depending holomorphically on ε and $\widehat{d}_\varepsilon(x) = \mathcal{O}(x^{m+2})$. Moreover, by construction, $N_\varepsilon \circ p_\varepsilon \circ N_\varepsilon^{-1}$ has fixed points at $\pm i\varepsilon$ so we can write $N_\varepsilon \circ p_\varepsilon \circ N_\varepsilon^{-1}(x) = x + (x^2 + \varepsilon^2)\widehat{a}_\varepsilon(x)$, with $\widehat{a}_\varepsilon$ depending holomorphically on ε. Therefore

$$\widehat{g}_\varepsilon(x,y) = \left(x + (x^2+\varepsilon^2)\widehat{a}_\varepsilon(x) + y\widehat{b}_\varepsilon(x,y), y + y\widehat{c}_\varepsilon(x,y) + \widehat{d}_\varepsilon(x) \right).$$

Since $N_0 = \mathrm{Id}$ and $p_0(x) = x + x^2 + \mathcal{O}(x^3)$, we have $\widehat{a}_0(0) = 1$. And since $M_0 = \mathrm{Id}$, we also have $\widehat{b}_0(0,0) = \widehat{c}_0(0,0) = 1$. Write

$$\widehat{c}_\varepsilon(x,y) = \eta x + q\varepsilon + \mathcal{O}(x^2, y, x\varepsilon, \varepsilon^2); \quad \widehat{d}_\varepsilon(x) = dx^{m+2} + \mathcal{O}(x^{m+3}) + \varepsilon\mathcal{O}_{m+1}(x,\varepsilon)$$

for some $q, d \in \mathbb{C}$, where $\eta = \alpha + 1$, so $\operatorname{Re}\eta > 3$ and $m = \lfloor \operatorname{Re}\eta \rfloor$. Now, we consider the polynomial change of coordinates Ψ_ε given by

$$\Psi_\varepsilon(x,y) = \left(x, y - \frac{d}{m+1-\eta} x^{m-1}(x^2+\varepsilon^2) \right).$$

Let $g_\varepsilon := \Psi_\varepsilon \circ \widehat{g}_\varepsilon \circ \Psi_\varepsilon^{-1}$. If we denote $(x_1, v_1) := g_\varepsilon(x, v)$, we have that

$$x_1 = x + (x^2+\varepsilon^2)a_\varepsilon(x) + vb_\varepsilon(x,v)$$

for some a_ε and b_ε depending holomorphically on ε and such that $a_0(0) = \widehat{a}_0(0) = 1$ and $b_0(0,0) = \widehat{b}_0(0,0) = 0$. Now, denote $\ell_\varepsilon(x,y) = (x^2+\varepsilon^2)\widehat{a}_\varepsilon(x) + y\widehat{b}_\varepsilon(x,y)$, so that $x_1 = x + \ell_\varepsilon(x,y)$. Given $k \in \mathbb{N}$, we have that

$$x_1^k = x^k + kx^{k-1}\ell_\varepsilon(x,y) + \sum_{j=2}^{k} \binom{k}{j} x^{k-j}\ell_\varepsilon(x,y)^j.$$

Since $\ell_\varepsilon(x,y) = x^2 + \varepsilon^2 + \mathcal{O}_3(x,\varepsilon) + y\mathcal{O}(x,y,\varepsilon)$, we have

$$x^{k-1}\ell_\varepsilon(x,y) = x^{k+1} + x^{k-1}\varepsilon^2 + \mathcal{O}_{k+2}(x,\varepsilon) + y\mathcal{O}_k(x,y,\varepsilon)$$

and since $\ell_\varepsilon(x,y) = \mathcal{O}_2(x,y,\varepsilon)$ we have

$$x^{k-j}\ell_\varepsilon(x,y)^j = \mathcal{O}_{k+2}(x,y,\varepsilon)$$

for all $2 \leq j \leq k$. Therefore

$$x_1^k = x^k + kx^{k+1} + kx^{k-1}\varepsilon^2 + \mathcal{O}_{k+2}(x,\varepsilon) + y\mathcal{O}_k(x,y,\varepsilon) + \mathcal{O}_{k+2}(x,y,\varepsilon),$$

so

$$x_1^{m+1} + x_1^{m-1}\varepsilon^2 = x^{m+1} + (m+1)x^{m+2} + x^{m-1}\varepsilon^2 + \mathcal{O}(x^{m+3}) + \varepsilon\mathcal{O}_{m+1}(x,\varepsilon) + y\mathcal{O}_{m+1}(x,y,\varepsilon).$$

Then we have,

$$\begin{aligned}
v_1 &= y\left(1+\eta x+q\varepsilon+\mathcal{O}\left(x^2,y,x\varepsilon,\varepsilon^2\right)\right)+dx^{m+2}+\mathcal{O}(x^{m+3})+\varepsilon\mathcal{O}_{m+1}(x,\varepsilon)+y\mathcal{O}_{m+1}(x,y,\varepsilon)\\
&\quad-\frac{d}{m+1-\eta}\left(x^{m+1}+(m+1)x^{m+2}+x^{m-1}\varepsilon^2\right)\\
&=\left(v+\frac{d}{m+1-\eta}\left(x^{m+1}+x^{m-1}\varepsilon^2\right)\right)\left(1+\eta x+q\varepsilon+\mathcal{O}\left(x^2,v,x\varepsilon,\varepsilon^2\right)\right)+dx^{m+2}\\
&\quad-\frac{d}{m+1-\eta}\left(x^{m+1}+(m+1)x^{m+2}+x^{m-1}\varepsilon^2\right)+\mathcal{O}(x^{m+3})+\varepsilon\mathcal{O}_{m+1}\left(x,\varepsilon\right)\\
&=v\left(1+\eta x+q\varepsilon+\mathcal{O}\left(x^2,v,x\varepsilon,\varepsilon^2\right)\right)+\frac{d}{m+1-\eta}\left(x^{m+1}+\eta x^{m+2}+x^{m-1}\varepsilon^2\right)+dx^{m+2}\\
&\quad-\frac{d}{m+1-\eta}\left(x^{m+1}+(m+1)x^{m+2}+x^{m-1}\varepsilon^2\right)+\mathcal{O}(x^{m+3})+\varepsilon\mathcal{O}_{m+1}\left(x,\varepsilon\right)\\
&=v\left(1+\eta x+q\varepsilon+\mathcal{O}\left(x^2,v,x\varepsilon,\varepsilon^2\right)\right)+\mathcal{O}(x^{m+3})+\varepsilon\mathcal{O}_{m+1}\left(x,\varepsilon\right)
\end{aligned}$$

and the Lemma is proved taking $\Theta_\varepsilon := \Psi_\varepsilon \circ M_\varepsilon$. □

We will also need the following result.

Lemma 2.4. *Let $\varepsilon \mapsto A(\varepsilon)$ be a holomorphic map from $\mathbb{D}$ to $\mathcal{M}_2(\mathbb{C})$. Assume that $A(0) = \mathrm{Id}$, and $A'(0) = \begin{pmatrix} u & w \\ 0 & v \end{pmatrix}$. Then, for $\varepsilon \neq 0$ small enough, $A(\varepsilon)$ has two eigenvalues $\lambda_1(\varepsilon) = 1 + u\varepsilon + \mathcal{O}(\varepsilon^2)$, $\lambda_2(\varepsilon) = 1 + v\varepsilon + \mathcal{O}(\varepsilon^2)$. Moreover, if $u \neq v$ then the eigenspace associated to $\lambda_1(\varepsilon)$ tends to $\mathbb{C}(1,0)$ as $\varepsilon \to 0$.*

Observe that the condition $u \neq v$ is necessary: the matrices $A(\varepsilon) = \begin{pmatrix} 1+u\varepsilon & \varepsilon^2 \\ \varepsilon^2 & 1+u\varepsilon \end{pmatrix}$ have eigenvectors $(1,1)$ and $(1,-1)$ for all $\varepsilon \neq 0$.

Proof. Write $A(\varepsilon) = (a_{ij}(\varepsilon))_{1\le i,j\le 2}$, and let

$$p(\varepsilon,\mu) := \frac{1}{\varepsilon^2}\begin{vmatrix} a_{11}(\varepsilon)-1-\mu\varepsilon & a_{12}(\varepsilon) \\ a_{21}(\varepsilon) & a_{22}(\varepsilon)-1-\mu\varepsilon \end{vmatrix} = \frac{\chi_{A(\varepsilon)}(1+\mu\varepsilon)}{\varepsilon^2}.$$

Then $p(\varepsilon,\mu) = \begin{vmatrix} u-\mu & w \\ 0 & v-\mu \end{vmatrix} + \mathcal{O}(\varepsilon)$; this proves that for $\varepsilon \neq 0$ small enough the eigenvalues of $A(\varepsilon)$ satisfy $\lambda_1(\varepsilon) = 1+u\varepsilon+\mathcal{O}(\varepsilon^2)$, $\lambda_2(\varepsilon) = 1+v\varepsilon+\mathcal{O}(\varepsilon^2)$. In particular, if $u \neq v$ then for $\varepsilon \neq 0$ small enough they are simple.

Assume now that $u \neq v$ and let us prove the assertion about the eigenspace associated to $\lambda_1(\varepsilon)$. Let $X(\varepsilon)$ be a family of eigenvectors associated to $\lambda_1(\varepsilon) = 1 + u\varepsilon + \mathcal{O}(\varepsilon^2)$ for $\varepsilon \neq 0$, and let $\varepsilon_n \to 0$ be any sequence. Assume without loss of generality that $\|X(\varepsilon_n)\| = 1$ for all $n \in \mathbb{N}$. For every n, we have

$$A(\varepsilon_n)X(\varepsilon_n) = \lambda_1(\varepsilon_n)X(\varepsilon_n)$$

and then

$$\frac{A(\varepsilon_n)-\mathrm{Id}}{\varepsilon_n}X(\varepsilon_n) = uX(\varepsilon_n) + \mathcal{O}(\varepsilon_n);$$

therefore any adherence value $X \in \mathbb{C}^2$ of the sequence $(X(\varepsilon_n))_{n\ge 0}$ satisfies $A'(0)X = uX$. In particular, X is in $\mathbb{C}(1,0)$. Therefore, $\lim_{n\to\infty}[X(\varepsilon_n)] = [1:0]$. □

We can now conclude the proof of Proposition 1 and Theorem 2. The first statement of Proposition 1 follows from Lemma 2.1. Let us compute the differential of g_ε at the fixed points $\widehat{z}_1(\varepsilon) = \Theta_\varepsilon(\widetilde{z}_1(\varepsilon)) = (i\varepsilon,0)+\mathcal{O}(\varepsilon^2)$ and $\widehat{z}_2(\varepsilon) = \Theta_\varepsilon(\widetilde{z}_2(\varepsilon)) = (-i\varepsilon,0)+\mathcal{O}(\varepsilon^2)$.

We have

$$\operatorname{Jac} g_\varepsilon(\widehat{z}_1(\varepsilon)) = \begin{pmatrix} 1+2i\varepsilon+\mathcal{O}(\varepsilon^2) & b_\varepsilon(i\varepsilon,0)+\mathcal{O}(\varepsilon^2) \\ \mathcal{O}(\varepsilon^2) & 1+q\varepsilon+i\eta\varepsilon+\mathcal{O}(\varepsilon^2) \end{pmatrix}.$$

and

$$\operatorname{Jac} g_\varepsilon(\widehat{z}_2(\varepsilon)) = \begin{pmatrix} 1-2i\varepsilon+\mathcal{O}(\varepsilon^2) & b_\varepsilon(-i\varepsilon,0)+\mathcal{O}(\varepsilon^2) \\ \mathcal{O}(\varepsilon^2) & 1+q\varepsilon-i\eta\varepsilon+\mathcal{O}(\varepsilon^2) \end{pmatrix}.$$

Then, by Lemma 2.4, $\operatorname{Jac} g_\varepsilon(\widehat{z}_1(\varepsilon))$ has eigenvalues

$$\rho_T^1(\varepsilon) = 1+2i\varepsilon+\mathcal{O}(\varepsilon^2), \quad \rho_N^1(\varepsilon) = 1+(q+i\eta)\varepsilon+\mathcal{O}(\varepsilon^2)$$

and $\operatorname{Jac} g_\varepsilon(\widehat{z}_2(\varepsilon))$ has eigenvalues

$$\rho_T^2(\varepsilon) = 1-2i\varepsilon+\mathcal{O}(\varepsilon^2), \quad \rho_N^2(\varepsilon) = 1+(q-i\eta)\varepsilon+\mathcal{O}(\varepsilon^2).$$

If $(\rho_T^i)'(0) \neq (\rho_N^i)'(0)$ then again by Lemma 2.4 the eigenspace associated to $\rho_T^i(\varepsilon)$ tends to $\mathbb{C}(1,0)$ as $\varepsilon \to 0$. Moreover, this condition happens for at least one of the fixed points, since either $q+i\eta \neq 2i$ or $q-i\eta \neq -2i$ (or both). This finishes the proof of Proposition 1.

Let us now prove Theorem 2. Observe first that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \frac{\rho_N^1(\varepsilon)+\rho_N^2(\varepsilon)-2}{\rho_T^1(\varepsilon)+\rho_T^2(\varepsilon)} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \frac{2q\varepsilon+\mathcal{O}(\varepsilon^2)}{2+\mathcal{O}(\varepsilon^2)} = q.$$

Now, consider a sequence $\varepsilon_n \to 0$ be such that $\frac{2i\pi}{\rho_T^i(\varepsilon_n)-1} = n-\sigma+o(1)$ for some $1 \leq i \leq 2$ and for some constant $\sigma \in \mathbb{C}$. Up to replacing ε by $-\varepsilon$ in the family (f_ε), we can assume without loss of generality that $\rho_T^i = \rho_T^1$. Then, writing $\rho_T^1(\varepsilon) = 1+2i\varepsilon_n+\beta\varepsilon_n^2+\mathcal{O}(\varepsilon_n^3)$ (where β depends only on the family (f_{ε_n})) we have

$$\frac{2i\pi}{2i\varepsilon_n+\beta\varepsilon_n^2+\mathcal{O}(\varepsilon_n^3)} = n-\sigma+o(1),$$

and therefore $n-\pi/\varepsilon_n = \sigma+\sigma_0+o(1)$, with $\sigma_0 = i\pi\beta/2$. By Theorem 3 we have that for any compact set K contained in the parabolic basin of g_0 associated to $(1,0)$ there exists $N \in \mathbb{N}$ such that

$$g_{\varepsilon_n}^{n-N} \to \mathcal{L}_{\sigma+\sigma_0-N,q}^{(g_0)}$$

uniformly on K, where $\mathcal{L}_{\sigma+\sigma_0-N,q}^{(g_0)} := (\Phi_{(g_0)}^o)^{-1} \circ A_{\sigma+\sigma_0-N,q} \circ \Phi_{(g_0)}^\iota$ and $\Phi_{(g_0)}^\iota$, $\Phi_{(g_0)}^o$ denote respectively the incoming and outgoing Fatou coordinates for g_0. Since $f_\varepsilon = \Psi^{-1} \circ \Theta_\varepsilon^{-1} \circ g_\varepsilon \circ \Theta_\varepsilon \circ \Psi$ and $\Theta_0 = \mathrm{Id}$, we deduce that

$$\lim_{n\to\infty} f_{\varepsilon_n}^{n-N} = \Psi^{-1} \circ \mathcal{L}_{\sigma+\sigma_0-N,q}^{(g_0)} \circ \Psi.$$

It is straightforward to see that $\Phi_{(f_0)}^{\iota,o} := \Phi_{(g_0)}^{\iota,o} \circ \Psi$ are respectively the incoming and outgoing Fatou coordinates for f_0. It follows that for any compact K contained in the parabolic basin of f_0 associated to v there exists N such that $\lim_{n\to\infty} f_{\varepsilon_n}^{n-N} = (\Phi_{(f_0)}^o)^{-1} \circ A_{\sigma+\sigma_0-N,q} \circ \Phi_{(f_0)}^\iota$ and Theorem 2 is proved.

3. Asymptotics of Fatou coordinates

Consider a family (g_ε) as in Theorem 3, so $g := g_0$ has the form

$$g(x,y) = (x+x^2a_0(x)+yb_0(x,y), y+yc_0(x,y)+d_0(x))$$

with $a_0(0)=1$, $b_0(0,0)=0$, $c_0(x,y)=\eta x+cy+\mathcal{O}_2(x,y,\varepsilon)$ and $d_0(x)=\mathcal{O}(x^{m+3})$ with $\rho = \operatorname{Re}\eta > 3$ and $m = \lfloor \rho \rfloor$. Then, we can write

$$g(x,y) = \left(x+x^2+ax^3+\mathcal{O}\left(x^4,xy,y^2\right), y+\eta xy+\mathcal{O}\left(x^2y,y^2,x^{m+3}\right)\right).$$

In this section we recall the construction and asymptotics of Fatou coordinates for g. Although these results are essentially contained in [Hak97] (see also [LHR25]) we provide all the proofs for the sake of completeness.

In the following, log refers to the principal branch of the complex logarithm, defined on $\mathbb{C} \setminus (-\infty, 0]$. The expression $\left(-\frac{1}{x}\right)^\eta$ (defined in particular when $\operatorname{Re} x < 0$) means by definition $\exp\left(\eta \log\left(-\frac{1}{x}\right)\right)$.

Denote, for any $C > 0$ and any $r > 0$

$$P^\iota(r, C) = \left\{(x, y) \in \mathbb{C}^2 : |x + r| < r, \left|\frac{y}{(-x)^\eta}\right| < C\right\}.$$

Although we will not mention it explicitly, in the following computations we always assume that r is small enough such that $P^\iota(r, C) \subset U$, where U is the domain of definition of (g_ε).

Lemma 3.1. *For any $C > 0$ there exists $r_0(C) > 0$ such that if $0 < r \leq r_0(C)$ and $(x, y) \in P^\iota(r, C)$ then*

$$g^n(x, y) \in P^\iota(r, C + 1)$$

for every $n \geq 0$. Moreover, if $(x, y) \in P^\iota(r, C)$ and we denote $(x_n, y_n) := g^n(x, y)$ we have that

$$-\frac{1}{x_1} = -\frac{1}{x} + 1 + (a-1)x + \mathcal{O}\left(x^2\right), \quad \frac{y_1}{(-x_1)^\eta} = \frac{y}{(-x)^\eta} + \mathcal{O}\left(x^2\right) \tag{2}$$

as $(x, y) \to 0$ (the constants in $\mathcal{O}$ being allowed to depend on C),

$$\lim_{n\to\infty} \frac{-1}{n x_n} = 1 \quad \text{and} \quad |x_n| \leq \left(\operatorname{Re}\left(-\frac{1}{x}\right) + \frac{n}{2}\right)^{-1} \text{ for any } n \geq 1.$$

Proof. Fix $C > 0$ and set $(x_1, y_1) = g(x, y)$. Let $0 < r_0 < r_1 \leq 1$ and consider $(x, y) \in P^\iota(r, C)$. By the expression of g, if $|y(-x)^{-\eta}| < C+1$ (so in particular $y = \mathcal{O}(x^3)$ since $\rho > 3$) we have that $x_1 = x + x^2 + ax^3 + \mathcal{O}(x^4)$ and $y_1 = y(1 + \eta x + \mathcal{O}(x^2)) + \mathcal{O}(x^{m+3})$ (where the constants in $\mathcal{O}(x^4)$ and $\mathcal{O}(x^2)$ depend on C), so

$$-\frac{1}{x_1} = -\frac{1}{x\left(1 + x + ax^2 + \mathcal{O}\left(x^3\right)\right)} = -\frac{1}{x} + 1 + (a-1)x + \mathcal{O}\left(x^2\right)$$

and, as long as $\operatorname{Re} x < 0$ and $|x|$ is small enough so that $(-x_1)^\eta$ is defined,

$$\begin{aligned} \frac{y_1}{(-x_1)^\eta} &= \frac{y\left(1 + \eta x + \mathcal{O}(x^2)\right) + \mathcal{O}(x^{m+3})}{(-x)^\eta\left(1 + x + \mathcal{O}(x^2)\right)^\eta} \\ &= \frac{y}{(-x)^\eta}\left(1 + \mathcal{O}(x^2)\right) + \mathcal{O}(x^{m+3-\rho}) = \frac{y}{(-x)^\eta} + \mathcal{O}\left(x^2\right), \end{aligned}$$

(using in the last equality that $m+3-\rho > 2$ by definition of m) so there exists a constant $\widetilde{C} > 0$, depending on C, such that

$$\left|\frac{y_1}{(-x_1)^\eta}\right| \leq \left|\frac{y}{(-x)^\eta}\right| + \widetilde{C}|x|^2.$$

By the expression of $-1/x_1$, we can choose $r_0 = r_0(C) > 0$ small enough such that if $0 < r \leq r_0$ and $|x + r| < r$ (so $\operatorname{Re}(-1/x) > (2r)^{-1}$) then

$$\operatorname{Re}\left(-\frac{1}{x_1}\right) \geq \operatorname{Re}\left(-\frac{1}{x}\right) + \frac{1}{2},$$

so in particular $|x_1 + r| < r$ and moreover

$$|x_1| \leq \left(\operatorname{Re}\left(-\frac{1}{x}\right) + \frac{1}{2}\right)^{-1}.$$

Up to decreasing r_0 if necessary, we can also assume that

$$\widetilde{C}\sum_{n\geq 0}\left(\frac{1}{2r_0}+\frac{n}{2}\right)^{-2}<1.$$

Take $(x,y)\in P^\iota(r,C)$ with $0<r\leq r_0$. Since $P^\iota(r,C)\subset P^\iota(r,C+1)$, we have by the previous computation that $|x_1+r|<r$ (so in particular $(-x_1)^\eta$ is defined) and

$$\left|y_1(-x_1)^{-\eta}\right|<C+\widetilde{C}|x|^2<C+\widetilde{C}(2r)^2<C+1,$$

so $(x_1,y_1)\in P^\iota(r,C+1)$. Arguing recursively, we obtain that $|x_n+r|<r$,

$$|x_n|\leq\left(\mathrm{Re}\left(-\frac{1}{x}\right)+\frac{n}{2}\right)^{-1}$$

and

$$\begin{aligned}\left|\frac{y_n}{(-x_n)^\eta}\right| &\leq \left|\frac{y}{(-x)^\eta}\right|+\widetilde{C}\sum_{j=0}^{n-1}|x_j|^2<C+\widetilde{C}\sum_{j=0}^{n-1}\left(\mathrm{Re}\left(-\frac{1}{x}\right)+\frac{j}{2}\right)^{-2}\\ &\leq C+\widetilde{C}\sum_{j=0}^{n-1}\left(\frac{1}{2r}+\frac{j}{2}\right)^{-2}<C+1,\end{aligned}$$

for every $n\geq 1$, where $(x_n,y_n)=g^n(x,y)$, so $(x_n,y_n)\in P^\iota(r,C+1)$ for every $n\geq 1$. Moreover, since

$$-\frac{1}{x_1}=-\frac{1}{x}+1+h(x,y)$$

where $h(x,y)=\mathcal{O}(x)$, we have that

$$-\frac{1}{x_n}=-\frac{1}{x}+n+\sum_{j=0}^{n-1}h(x_j,y_j)$$

and therefore $\frac{-1}{nx_n}\to 1$ as $n\to\infty$. □

Our next goal is to prove the following Propositions:

Proposition 3.2. *For any $C>0$ there exists $0<r^\iota(C)\leq r_0(C)$ such that for any $0<r\leq r^\iota(C)$ there exists a holomorphic univalent map $\Phi^\iota:P^\iota(r,C)\to\mathbb{C}^2$ which is an incoming Fatou coordinate for g (i.e., $\Phi^\iota\circ g=\Phi^\iota+(1,0)$) and satisfies*

$$\Phi^\iota(x,y)=\left(-\frac{1}{x}+(1-a)\log(-x)+o(1),\frac{y}{(-x)^\eta}+o(1)\right),$$

as $\mathrm{Re}(-1/x)\to+\infty$ inside $P^\iota(r,C)$. Moreover,

$$\{X\in\mathbb{H}_{r^{-1}}:|\mathrm{Im}X|<2|\mathrm{Re}\,X|\}\times\mathbb{D}(0,C-1)\subset\Phi^\iota(P^\iota(r,C)),$$

where $\mathbb{H}_t:=\{X\in\mathbb{C}:\mathrm{Re}\,X>t\}$.

Proposition 3.3. *For any $C>0$ there exists $r^o(C)>0$ such that for any $0<r\leq r^o(C)$ there exists a holomorphic univalent map $\Phi^o:P^o(r,C)\to\mathbb{C}^2$, where*

$$P^o(r,C)=\left\{(x,y)\in\mathbb{C}^2:|x-r|<r,\left|\frac{y}{x^\eta}\right|<C\right\},$$

which is an outgoing Fatou coordinate for g (i.e., $\Phi^o\circ g=\Phi^o+(1,0)$) and satisfies

$$\Phi^o(x,y)=\left(-\frac{1}{x}+(1-a)\log x+o(1),\frac{y}{x^\eta}+o(1)\right),$$

as $\mathrm{Re}(-1/x)\to-\infty$ inside $P^o(r,C)$. Moreover,

$$-\{X\in\mathbb{H}_{r^{-1}}:|\mathrm{Im}X|<2|\mathrm{Re}\,X|\}\times\mathbb{D}(0,C-1)\subset\Phi^o(P^o(r,C)).$$

Set

$$\Phi_0^\iota(x,y) := \left(-\frac{1}{x}, \frac{y}{(-x)^\eta}\right).$$

It is straightforward to check that Φ_0^ι is well-defined and univalent on $\{(x,y)\in\mathbb{C}^2 : \operatorname{Re} x < 0\}$ and that

$$\left(\Phi_0^\iota\right)^{-1}(X,Y) = \left(-\frac{1}{X}, \frac{Y}{X^\eta}\right).$$

By Lemma 3.1, for any $C>0$ there exists $r_0(C)>0$ such that for any $0<r\le r_0(C)$ the map

$$G := \Phi_0^\iota \circ g \circ \left(\Phi_0^\iota\right)^{-1}$$

is well-defined on

$$\Phi_0^\iota\left(P^\iota(r,C)\right) = \mathbb{H}_{(2r)^{-1}} \times \mathbb{D}(0,C),$$

where $\mathbb{H}_{(2r)^{-1}} = \left\{X\in\mathbb{C} : \operatorname{Re} X > (2r)^{-1}\right\}$, and $G^n(X,Y) \in \Phi_0^\iota\left(P^\iota(r,C+1)\right)$ for any $n\ge 1$ and for any $(X,Y)\in \mathbb{H}_{(2r)^{-1}}\times\mathbb{D}(0,C)$. Moreover, using (2) we have that

$$G(X,Y) = \left(X + 1 + \frac{1-a}{X} + \mathcal{O}\left(\frac{1}{X^2}\right), Y + \mathcal{O}\left(\frac{1}{X^2}\right)\right) \tag{3}$$

in $\mathbb{H}_{(2r)^{-1}}\times\mathbb{D}(0,C)$ as $\operatorname{Re} X\to+\infty$ (the constants in $\mathcal{O}$ being allowed to depend on C).

Lemma 3.4. *For any $C>0$ and any $0<r\le r_0(C)$, where $r_0(C)$ is given by Lemma 3.1, the map*

$$\psi(X,Y) := \lim_{n\to\infty} Y_n,$$

where $(X_n,Y_n) = G^n(X,Y)$, is well-defined in $\mathbb{H}_{(2r)^{-1}}\times\mathbb{D}(0,C)$, satisfies $\psi\circ G = \psi$ and has the form $\psi(X,Y) = Y + o(1)$ as $\operatorname{Re} X\to+\infty$. Moreover, the map $\Phi_1^\iota : \mathbb{H}_{(2r)^{-1}}\times\mathbb{D}(0,C)\to\mathbb{C}^2$ defined by

$$\Phi_1^\iota(X,Y) := (X, \psi(X,Y))$$

is injective.

Proof. By (3), we have that $Y_1 = Y + k(X,Y)$ for some holomorphic map $k(X,Y) = \mathcal{O}(X^{-2})$ and for every $(X,Y)\in\mathbb{H}_{(2r)^{-1}}\times\mathbb{D}(0,C)$. Therefore, for every n we have that

$$Y_n = Y + \sum_{j=0}^{n-1} k(X_j,Y_j).$$

Using the bound $|X_j|^{-1} \le (\operatorname{Re} X + j/2)^{-1} \le 2/j$ from Lemma 3.1 we have that the series $\sum_{j=0}^{\infty} k(X_j,Y_j)$ is uniformly convergent in $\mathbb{H}_{(2r)^{-1}}\times\mathbb{D}(0,C)$, so Y_n converges uniformly to a holomorphic function

$$\psi(X,Y) = Y + v(X,Y)$$

where $v(X,Y) = \sum_{j=0}^{\infty} k(X_j,Y_j)$. The invariance of ψ is immediate, by construction. Moreover, since

$$\sum_{j=0}^{\infty} \frac{1}{|X_j|^2} \le \frac{1}{(\operatorname{Re} X)^2} + \int_0^\infty \frac{1}{(\operatorname{Re} X + t/2)^2} dt = \frac{1}{(\operatorname{Re} X)^2} + \frac{2}{\operatorname{Re} X}$$

we have that $v(X,Y) = o(1)$ as $\operatorname{Re} X\to+\infty$. Finally, since Y_n is injective when X is fixed and ψ is not constant, we obtain that Φ_1^ι is injective. $\square$

By Lemma 3.4, for any $C > 0$ and any $0 < r \le r_0(C)$ the map

$$\widetilde{G} := \Phi_1^\iota \circ G \circ (\Phi_1^\iota)^{-1}$$

is well-defined on $\Phi_1^\iota\left(\mathbb{H}_{(2r)^{-1}} \times \mathbb{D}(0,C)\right)$ and $\widetilde{G}^n(X,Y) \in \Phi_1^\iota\left(\mathbb{H}_{(2r)^{-1}} \times \mathbb{D}(0,C+1)\right)$ for any $n \ge 1$ and for any $(X,Y) \in \Phi_1^\iota\left(\mathbb{H}_{(2r)^{-1}} \times \mathbb{D}(0,C)\right)$. Moreover, using (3) and Lemma 3.4 we have that

$$\widetilde{G}(X,Y) = \left(X + 1 + \frac{1-a}{X} + \mathcal{O}\left(\frac{1}{X^2}\right), Y\right) \tag{4}$$

in $\Phi_1^\iota\left(\mathbb{H}_{(2r)^{-1}} \times \mathbb{D}(0,C)\right)$ as $\operatorname{Re} X \to +\infty$. Observe also that since $\psi(X,Y) = Y + o(1)$ as $\operatorname{Re} X \to +\infty$, there exists $0 < r_1(C) \le r_0(C)$ such that for every $0 < r \le r_1(C)$

$$\Phi_1^\iota\left(\mathbb{H}_{(2r)^{-1}} \times \mathbb{D}(0,C)\right) \subset \Phi_0^\iota\left(\mathbb{H}_{(2r)^{-1}} \times \mathbb{D}(0,C+1)\right).$$

Lemma 3.5. *For any $C > 0$ there exists $r_2(C) > 0$ such that for any $0 < r \le r_2(C)$, the map*

$$\varphi(X,Y) := \lim_{n\to\infty} \left[X_n - n - (1-a)\log n\right],$$

where $(X_n, Y) := \widetilde{G}^n(X,Y)$, is well-defined in $\Phi_1^\iota\left(\mathbb{H}_{(2r)^{-1}} \times \mathbb{D}(0,C)\right)$, satisfies $\varphi \circ \widetilde{G} = \varphi + 1$ and has the form $\varphi(X,Y) = X - (1-a)\log X + o(1)$ as $\operatorname{Re} X \to +\infty$. Moreover, the map $\Phi_2^\iota : \Phi_1^\iota\left(\mathbb{H}_{(2r)^{-1}} \times \mathbb{D}(0,C)\right) \to \mathbb{C}^2$ defined by

$$\Phi_2^\iota(X,Y) := (\varphi(X,Y), Y)$$

is injective.

Proof. Fix $C > 0$, set $r_2(C) = \min\{r_0(C+1), r_1(C)\}$ and take $0 < r \le r_2(C)$. Thanks to (4), for any $(X,Y) \in \Phi_1^\iota\left(\mathbb{H}_{(2r)^{-1}} \times \mathbb{D}(0,C)\right)$ we have that

$$X_1 - (1-a)\log X_1 = X + 1 - (1-a)\log X + h(X,Y)$$

with $h(X,Y) = \mathcal{O}\left(1/X^2\right)$, so

$$X_n - (1-a)\log X_n = X + n - (1-a)\log X + \sum_{j=0}^{n-1} h(X_j, Y).$$

Since $\Phi_1^\iota\left(\mathbb{H}_{(2r)^{-1}} \times \mathbb{D}(0,C)\right) \subset \Phi_0^\iota\left(\mathbb{H}_{(2r)^{-1}} \times \mathbb{D}(0,C+1)\right)$ and $0 < r \le r_0(C+1)$, by Lemma 3.1 we have that $|X_j|^{-1} \le (\operatorname{Re} X + j/2)^{-1} \le 2/j$ for every j, so the series $\sum_{j=0}^\infty h(X_j, Y)$ is normally convergent and hence $X_n - (1-a)\log X_n - n$ converges uniformly to a holomorphic map of the form

$$X - (1-a)\log X + H(X,Y).$$

Moreover, arguing as in the proof of Lemma 3.4, $H(X,Y) = o(1)$ as $\operatorname{Re} X \to +\infty$. Now define

$$\varphi_n(X,Y) := X_n - n - (1-a)\log n.$$

Then, rewriting

$$\varphi_n(X,Y) = X_n - (1-a)\log X_n - n + (1-a)\log\left(\frac{X_n}{n}\right)$$

and using the fact that $X_n/n \to 1$ as $n \to \infty$ by Lemma 3.1, we have that the sequence φ_n converges uniformly in $\Phi_1^\iota\left(\mathbb{H}_{(2r)^{-1}} \times \mathbb{D}(0,C)\right)$ to a map φ of the form

$$\varphi(X,Y) = X - (1-a)\log X + o(1)$$

as $\operatorname{Re} X \to +\infty$. Since $\varphi_n \circ \widetilde{G} = \varphi_{n+1} + 1 + (1-a)\log(1 + 1/n)$, we obtain that $\varphi \circ \widetilde{G} = \varphi + 1$. Injectivity of Φ_2^ι follows from the injectivity of φ_n when Y is fixed and the fact that φ is not constant. $\square$

We are now ready to prove Proposition 3.2.

Proof of Proposition 3.2. By Lemmas 3.4 and 3.5, for any $C > 0$ and any $0 < r \leq r_2(C)$ the map $\Phi_G := \Phi_2^\iota \circ \Phi_1^\iota$ is well-defined on $\mathbb{H}_{(2r)^{-1}} \times \mathbb{D}(0, C)$ and

$$\Phi_G(X, Y) = (X - (1 - a) \log X, Y) + o(1),$$

where the convergence in the $o(1)$ term is uniform as $\operatorname{Re} X \to +\infty$ in $\mathbb{H}_{(2r_2(C))^{-1}} \times \mathbb{D}(0, C)$. Moreover, $\Phi_G \circ G = \Phi_G + (1, 0)$.

We will prove that there exists $0 < r^\iota(C) \leq r_2(C)$ such that for all $0 < r \leq r^\iota(C)$

$$\{X \in \mathbb{H}_{r^{-1}} : |\operatorname{Im} X| < 2\operatorname{Re} X\} \times \mathbb{D}(0, C - 1) \subset \Phi_G\left(\mathbb{H}_{(2r)^{-1}} \times \mathbb{D}(0, C)\right) \tag{5}$$

Let us set $u := \Phi_G - \operatorname{Id}$ and write $u = (u_1, u_2)$. There exists $0 < r^\iota(C) \leq r_2(C)$ such that for all $(X, Y) \in \{X \in \mathbb{H}_{(2r^\iota(C))^{-1}} : |\operatorname{Im} X| < 5\operatorname{Re} X\} \times \mathbb{D}(0, C)$

$$|u_1(X, Y)| = |(1 - a) \log X + o(1)| \leq \frac{1}{3}\operatorname{Re} X \quad \text{and} \quad |u_2(X, Y)| \leq 1.$$

Now, consider $(X_0, Y_0) \in \{X \in \mathbb{H}_{r^{-1}} : |\operatorname{Im} X| < 2\operatorname{Re} X\} \times \mathbb{D}(0, C-1)$ with $0 < r \leq r^\iota(C)$ and let us show that there exists $(X, Y) \in \{X \in \mathbb{H}_{(2r)^{-1}} : |\operatorname{Im} X| < 5\operatorname{Re} X\} \times \mathbb{D}(0, C)$ with $\Phi_G(X, Y) = (X_0, Y_0)$. Let $h_0(X, Y) := (X_0, Y_0) - u(X, Y)$, so $\Phi_G(X, Y) = (X_0, Y_0)$ if and only if (X, Y) is a fixed point of h_0, and let $K := \overline{\mathbb{D}}(X_0, \operatorname{Re} X_0/2) \times \overline{\mathbb{D}}(Y_0, 1)$. Observe that if $(X, Y) \in K$ we have that $|Y| \leq C$, $\operatorname{Re} X \geq \operatorname{Re} X_0/2 > (2r)^{-1}$ and

$$|\operatorname{Im} X| \leq |\operatorname{Im} X_0| + \frac{\operatorname{Re} X_0}{2} < \frac{5}{2}\operatorname{Re} X_0 \leq 5\operatorname{Re} X$$

so $K \subset \{X \in \mathbb{H}_{(2r)^{-1}} : |\operatorname{Im} X| < 5\operatorname{Re} X\} \times \mathbb{D}(0, C)$. Then if $(X_1, Y_1) = h_0(X, Y)$ we have

$$|X_1 - X_0| = |u_1(X, Y)| \leq \frac{1}{3}\operatorname{Re} X \leq \frac{1}{2}\operatorname{Re} X_0 \quad \text{and} \quad |Y_1 - Y_0| = |u_2(X, Y)| \leq 1$$

so $(X_1, Y_1) \in K$. Moreover, since $u(X, Y) = ((a - 1) \log X, 0) + o(1)$ as $\operatorname{Re} X \to +\infty$ in $\mathbb{H}_{(2r_2(C))^{-1}} \times \mathbb{D}(0, C)$, using Cauchy's estimate and the fact that $\log'(X) = 1/X$ and reducing $r^\iota(C)$ if necessary we have that for all $(X, Y) \in \mathbb{H}_{(2r^\iota(C))^{-1}} \times \mathbb{D}(0, C)$,

$$\|\partial_X u(X, Y)\| \leq \frac{1}{2} \quad \text{and} \quad \|\partial_Y u(X, Y)\| \leq \frac{1}{2}.$$

Then, the map $h_0 : K \to K$ is contracting on K, so by Banach fixed point theorem h_0 has a fixed point in K and (5) is proved.

Finally, Proposition 3.2 follows by defining $\Phi^\iota := \Phi_G \circ \Phi_0^\iota$. □

Proof of Proposition 3.3. This follows immediately from Proposition 3.2. Since g^{-1} is of the form

$$g^{-1}(x, y) = \left(x - x^2 + (2 - a)x^3 + \mathcal{O}(x^4, xy, y^2), y - \eta xy + \mathcal{O}(x^2 y, y^2, x^{m+3})\right),$$

if we take $\Psi(x, y) := (1/x, -y/x^\eta)$ we have that

$$\Psi \circ g^{-1} \circ \Psi^{-1}(X, Y) = \left(X + 1 - \frac{1 - a}{X} + \mathcal{O}\left(\frac{1}{X^2}\right), Y + \mathcal{O}\left(\frac{1}{X^2}\right)\right)$$

so repeating the construction above we find an incoming Fatou coordinate for g^{-1} in $P^o(r, C)$ of the form

$$\Phi^\iota_{g^{-1}}(x, y) = \left(\frac{1}{x} - (1 - a) \log(x) + o(1), -\frac{y}{x^\eta} + o(1)\right),$$

as $(x, y) \to (0, 0)$ inside $P^o(r, C)$. Then, to obtain Proposition 3.3 it suffices to define $\Phi^o := -\Phi^\iota_{g^{-1}}$. □

Finally, we will also need the following asymptotic expansion for $(\Phi^\iota)^{-1}$:

Lemma 3.6. *Let $C > 0$, $0 < r \le r^\iota(C)$. If $(X, Y) \in \Phi^\iota\left(P^\iota(r, C)\right)$, then*

$$(\Phi^\iota)^{-1}(X, Y) = \left(-\frac{1}{X} + \mathcal{O}\left(\frac{\log X}{X^2}\right), \frac{Y}{X^\eta} + o\left(\frac{1}{X^\eta}\right)\right),$$

as $\operatorname{Re} X \to +\infty$ inside $\Phi^\iota\left(P^\iota(r, C)\right)$, where the convergence in the term $o(X^{-\eta})$ and the implicit constant in the term $\mathcal{O}(X^{-2}\log X)$ are uniform on $\Phi^\iota\left(P^\iota(r, C)\right)$.

Proof. Take $(x, y) \in P^\iota(r, C)$ and set $(X, Y) := \Phi^\iota(x, y)$. By Proposition 3.2 we have

$$X = -\frac{1}{x} + (1-a)\log(-x) + o(1) = -\frac{1}{x} + o\left(\frac{1}{x}\right)$$

as $\operatorname{Re} X \to +\infty$, so $x = -1/X + o\left(1/X\right)$. Hence

$$X = -\frac{1}{x} + (1-a)\log\left(\frac{1}{X} + o\left(\frac{1}{X}\right)\right) + o(1) = -\frac{1}{x} + \mathcal{O}(\log X),$$

and so

$$x = \frac{-1}{X + \mathcal{O}\left(\log X\right)} = -\frac{1}{X} + \mathcal{O}\left(\frac{\log X}{X^2}\right).$$

Similarly:

$$Y = \frac{y}{(-x)^\eta} + o(1),$$

so

$$y = Y(-x)^\eta + o((-x)^\eta) = \frac{Y}{X^\eta}\left(1 + \mathcal{O}\left(\frac{\log X}{X}\right)\right)^\eta + o\left(\frac{1}{X^\eta}\right) = \frac{Y}{X^\eta} + o\left(\frac{1}{X^\eta}\right).$$

□

Similarly, we can compute the asymptotics of $(\Phi^o)^{-1}$:

Lemma 3.7. *Let $C > 0$, $0 < r \le r^0(C)$. If $(X, Y) \in \Phi^o\left(P^o(r, C)\right)$, then*

$$(\Phi^o)^{-1}(X, Y) = \left(-\frac{1}{X} + \mathcal{O}\left(\frac{\log X}{X^2}\right), \frac{Y}{(-X)^\eta} + o\left(\frac{1}{X^\eta}\right)\right),$$

as $\operatorname{Re} X \to -\infty$ inside $\Phi^o\left(P^o(r, C)\right)$, where the convergence in the term $o(X^{-\eta})$ and the implicit constant in the term $\mathcal{O}(X^{-2}\log X)$ are uniform on $\Phi^o\left(P^o(r, C)\right)$.

4. Approximate Fatou coordinates and estimation of the error terms

Consider a family (g_ε) as in Theorem 3, so

(6) $$g_\varepsilon(x, y) = (x + (x^2 + \varepsilon^2)a_\varepsilon(x) + y b_\varepsilon(x, y), y + y c_\varepsilon(x, y) + d_\varepsilon(x)),$$

where $a_\varepsilon, b_\varepsilon, c_\varepsilon$ and d_ε depend holomorphically on ε, $a_0(0) = 1$, $b_0(0, 0) = 0$,

$$c_\varepsilon(x, y) = \eta x + q\varepsilon + cy + \mathcal{O}_2(x, y, \varepsilon) \quad \text{and} \quad d_\varepsilon(x) = \mathcal{O}(x^{m+3}) + \varepsilon\mathcal{O}_{m+1}(x, \varepsilon),$$

with $\operatorname{Re}\eta > 3$ and $m = \lfloor \operatorname{Re}\eta \rfloor$. We consider a sequence (ε_n) such that $n - \pi/\varepsilon_n = \sigma + o(1)$ (so $\varepsilon_n \sim \pi/n$) and write

$$a_{\varepsilon_n}(x) = 1 + ax + p\varepsilon + \mathcal{O}(x^2, x\varepsilon_n, \varepsilon_n^2); \quad b_{\varepsilon_n}(x, y) = bx + \mathcal{O}(x^2, y, \varepsilon_n);$$
$$c_{\varepsilon_n}(x, y) = \eta x + q\varepsilon_n + \mathcal{O}(x^2, y, x\varepsilon_n, \varepsilon_n^2); \quad d_{\varepsilon_n}(x) = \mathcal{O}(x^{m+3}) + \varepsilon_n\mathcal{O}_{m+1}(x, \varepsilon_n)$$

for some $a, b, p \in \mathbb{C}$. As in [BSU17], up to considering the change of variables $(x, y, \varepsilon) \mapsto (\widetilde{x}, y, \widetilde{\varepsilon})$ with

$$x = \widetilde{x}(1 - p\widetilde{\varepsilon}); \quad \varepsilon = \widetilde{\varepsilon}(1 - p\widetilde{\varepsilon}),$$

we can assume that $p = 0$.

Define

$$\begin{aligned} w_{\varepsilon_n}(x) :&= \frac{1}{\varepsilon_n}\arctan\left(\frac{x}{\varepsilon_n}\right) + \frac{\pi}{2\varepsilon_n} + \frac{1-a}{2}\log(x^2+\varepsilon_n^2) \\ &= \frac{1}{2i\varepsilon_n}\log\left(\frac{i\varepsilon_n - x}{i\varepsilon_n + x}\right) + \frac{\pi}{2\varepsilon_n} + \frac{1-a}{2}\log(x^2+\varepsilon_n^2) \end{aligned}$$

and

$$t_{\varepsilon_n}(x,y) = \frac{y}{(x^2+\varepsilon_n^2)^{\eta/2}}.$$

We set

$$\Phi^\iota_{\varepsilon_n}(x,y) := (w_{\varepsilon_n}(x), t_{\varepsilon_n}(x,y)), \qquad \Phi^o_{\varepsilon_n}(x,y) := \Phi^\iota_{\varepsilon_n}(x,y) - \left(\frac{\pi}{\varepsilon_n}, 0\right)$$

defined on $(\mathbb{C}\setminus L_{\varepsilon_n})\times\mathbb{C}$, where $L_{\varepsilon_n} := \{it\varepsilon_n, t \in (-\infty,-1]\cup[1,+\infty)\}$, and we fix a constant $\gamma \in (1/2, 2/3)$ chosen so that

$$\gamma\rho > 2.$$

Definition 4.1. *Set $k_n := \lfloor n^\gamma \rfloor$. We define, for any $C > 1$, the set $\mathcal{R}_n(C) \subset (\mathbb{C}\setminus L_{\varepsilon_n})\times\mathbb{C}$ as*

$$\mathcal{R}_n(C) := \left\{(x,y) : \mathrm{Re}\ (\varepsilon_n w_{\varepsilon_n}(x)) \in \left[\frac{\pi k_n}{10n}, \pi - \frac{\pi k_n}{10n}\right], |\mathrm{Im}(\varepsilon_n w_{\varepsilon_n}(x))| \le C\frac{\pi}{n} \text{ and } \frac{1}{C} < |t_{\varepsilon_n}(x,y)| < C\right\}.$$

Remark 4.2. *Although we will not explicitly mention it, all the terms o and O appearing in the results in this Section are uniform on $\mathcal{R}_n(C)$.*

Proposition 4.3. *For all $C > 0$ and for all n large enough, $(\Phi^\iota_{\varepsilon_n})^{-1}$ is well-defined on $\Phi^\iota_{\varepsilon_n}(\mathcal{R}_n(C))$ and*

$$(\Phi^\iota_{\varepsilon_n})^{-1}(X,Y) = \left(-\varepsilon_n\cot\left(\varepsilon_n X + \mathcal{O}\left(n^{-1}\log n\right)\right), Y\frac{\varepsilon_n^\eta}{\sin^\eta\left(\varepsilon_n X + \mathcal{O}\left(n^{-1}\log n\right)\right)}\right)$$

Proof. Let

$$\widetilde{\Phi}_{\varepsilon_n}(x,y) := \left(\frac{1}{\varepsilon_n}\arctan\left(\frac{x}{\varepsilon_n}\right) + \frac{\pi}{2\varepsilon_n}, \frac{y}{(x^2+\varepsilon_n^2)^{\eta/2}}\right).$$

We claim that the map $\widetilde{\Phi}_{\varepsilon_n} : (\mathbb{C}\setminus L_{\varepsilon_n})\times\mathbb{C} \to \{(X,Y)\in\mathbb{C}^2 : \mathrm{Re}\,(\varepsilon_n X) \in (0,\pi)\}$ is well-defined and bijective, and its inverse is given by

$$(\widetilde{\Phi}_{\varepsilon_n})^{-1}(X,Y) = \left(-\varepsilon_n\cot(\varepsilon_n X), Y\frac{\varepsilon_n^\eta}{\sin^\eta(\varepsilon_n X)}\right).$$

Indeed, set $(X,Y) := \widetilde{\Phi}_{\varepsilon_n}(x,y)$. We have $\varepsilon_n X = \arctan\,(x/\varepsilon_n) + \pi/2$ and therefore

$$x = \varepsilon_n\tan\left(\varepsilon_n X - \frac{\pi}{2}\right) = -\varepsilon_n\cot(\varepsilon_n X).$$

On the other hand, since $Y = y\left(x^2+\varepsilon_n^2\right)^{-\eta/2}$ we get

$$y = Y(x^2+\varepsilon_n^2)^{\eta/2} = Y\varepsilon_n^\eta(1+\cot^2(\varepsilon_n X))^{\eta/2} = Y\frac{\varepsilon_n^\eta}{\sin^\eta(\varepsilon_n X)}.$$

This proves the claim.

Now denote $\mathcal{G}_{\varepsilon_n} := \Phi^\iota_{\varepsilon_n}\circ(\widetilde{\Phi}_{\varepsilon_n})^{-1}$, defined in $\{(X,Y)\in\mathbb{C}^2 : \mathrm{Re}\,(\varepsilon_n X)\in(0,\pi)\}$. Since

$$\Phi^\iota_{\varepsilon_n}(x,y) = \widetilde{\Phi}_{\varepsilon_n}(x,y) + \left(\frac{1-a}{2}\log(x^2+\varepsilon_n^2), 0\right)$$

we have by the claim above that

$$\begin{aligned}\mathcal{G}_{\varepsilon_n}(X,Y) &= (X,Y) + \left(\frac{1-a}{2}\log(\varepsilon_n^2 + \varepsilon_n^2\cot^2(\varepsilon_n X)), 0\right)\\ &= (X,Y) + \left(\frac{1-a}{2}\log\left(\frac{\varepsilon_n^2}{\sin^2(\varepsilon_n X)}\right), 0\right).\end{aligned}$$

Let us prove that $\mathcal{G}_{\varepsilon_n}$ is invertible in

$$\Phi^{\iota}_{\varepsilon_n}(\mathcal{R}_n(C)) = \left\{(X,Y)\in\mathbb{C}^2 : \mathrm{Re}\,(\varepsilon_n X) \in \left[\frac{\pi k_n}{10n}, \pi - \frac{\pi k_n}{10n}\right], |\mathrm{Im}(\varepsilon_n X)| \le C\frac{\pi}{n} \text{ and } \frac{1}{C} < |Y| < C\right\}$$

for n big enough. Choose $(X_0,Y_0)\in\Phi^{\iota}_{\varepsilon_n}(\mathcal{R}_n(C))$ and let us show that there is a unique (X,Y) with $\mathrm{Re}\,(\varepsilon_n X)\in(0,\pi)$ such that $\mathcal{G}_{\varepsilon_n}(X,Y) = (X_0,Y_0)$. Since $\mathcal{G}_{\varepsilon_n}(X,Y) = (X+u_n(X),Y)$, with $u_n(X) = \frac{1-a}{2}\log\left(\frac{\varepsilon_n^2}{\sin^2(\varepsilon_n X)}\right)$, we just need to show that, for n large enough, there is a unique X with $\mathrm{Re}\,(\varepsilon_n X)\in(0,\pi)$ such that

$$X + u_n(X) = X_0.$$

By Rouché's theorem, it is enough to prove that $|u_n(X)| < |X - X_0|$ for every $X\in\partial Q_n$ and for n big enough, where

$$Q_n = \left\{X\in\mathbb{C} : \mathrm{Re}\,(\varepsilon_n X) \in \left(\frac{\pi k_n}{20n}, \pi - \frac{\pi k_n}{20n}\right), |\mathrm{Im}(\varepsilon_n X)| < C\frac{\pi}{n} + n\right\}$$

Observe that $Q_n \subset \{X : \mathrm{Re}\,(\varepsilon_n X)\in(0,\pi)\}$ and if $\mathrm{Re}\,(\varepsilon_n X)\in(0,\pi)$ then $X\in Q_n$ for n big enough. Clearly $|X - X_0| \ge \frac{\pi k_n}{20n}$ for every $X\in\partial Q_n$. Moreover, since $|\sin(\varepsilon_n X)| \ge |\sin(\mathrm{Re}\,(\varepsilon_n X))|$ we have that $\frac{\varepsilon_n}{\sin(\varepsilon_n X)} = \mathcal{O}(\frac{1}{k_n})$ for every $X\in\partial Q_n$ so

$$|u_n(X)| = \frac{|1-a|}{2}\left|\log\left(\frac{\varepsilon_n^2}{\sin^2(\varepsilon_n X)}\right)\right| = \mathcal{O}(\log k_n).$$

Hence if n is big enough then $|u_n(X)| < |X - X_0|$ for every $X\in\partial Q_n$, so $\mathcal{G}_{\varepsilon_n}$ is invertible on $\Phi^{\iota}_{\varepsilon_n}(\mathcal{R}_n(C))$, which in turn implies that $(\Phi^{\iota}_{\varepsilon_n})^{-1} = (\widetilde{\Phi}_{\varepsilon_n})^{-1}\circ\mathcal{G}_{\varepsilon_n}^{-1}$ is well-defined on $\Phi^{\iota}_{\varepsilon_n}(\mathcal{R}_n(C))$.

Let us now prove the desired estimate. Since $\mathcal{G}_{\varepsilon_n}(X,Y) = (X+u_n(X),Y)$, we know that

$$\mathcal{G}_{\varepsilon_n}^{-1}(X,Y) = (X + \mathcal{O}(\|u_n\|_\infty), Y).$$

Since $k_n = \lfloor n^\gamma\rfloor$, by the computations above and the maximum principle we have that $|u_n(X)| = \mathcal{O}(\log n)$ for every $(X,Y)\in\Phi^{\iota}_{\varepsilon_n}(\mathcal{R}_n(C))$, so

$$\mathcal{G}_{\varepsilon_n}^{-1}(X,Y) = (X + \mathcal{O}(\log n), Y)$$

and therefore

$$(\Phi^{\iota}_{\varepsilon_n})^{-1}(X,Y) = (\widetilde{\Phi}_{\varepsilon_n})^{-1}\circ\mathcal{G}_{\varepsilon_n}^{-1}(X,Y) = (\widetilde{\Phi}_{\varepsilon_n})^{-1}(X+\mathcal{O}(\log n), Y)$$

so the Proposition follows. □

4.1. **Estimation of the error terms.** In this subsection we will give estimates for

$$A_{\varepsilon_n}(x,y) := w_{\varepsilon_n}(x_1) - w_{\varepsilon_n}(x) - 1 \quad\text{ and }\quad B_{\varepsilon_n}(x,y) := \log\frac{t_{\varepsilon_n}(x_1,y_1)}{t_{\varepsilon_n}(x,y)},$$

where $(x_1,y_1) := g_{\varepsilon_n}(x,y)$. Recall that $n - \pi/\varepsilon_n \to \sigma$, so $\varepsilon_n \sim \pi/n$, $\rho = \mathrm{Re}\,\eta > 3$, $m = \lfloor\rho\rfloor$ and $\gamma\in(1/2,2/3)$ satisfies $\gamma\rho > 2$.

Lemma 4.4. *Take $(x, y) \in \mathcal{R}_n(C)$ and denote $(X, Y) := \Phi^\iota_{\varepsilon_n}(x, y)$. If n is big enough, we have the following estimates:*

$$x = -\varepsilon_n \cot(\varepsilon_n X) + \mathcal{O}\left(\frac{\log n}{n^{2\gamma}}\right) = \mathcal{O}\left(X^{-1}\right) \tag{7}$$

$$\frac{1}{x^2 + \varepsilon_n^2} \sim \frac{\sin^2(\varepsilon_n X)}{\varepsilon_n^2} = \mathcal{O}\left(X^2\right) \tag{8}$$

$$y \asymp \frac{\varepsilon_n^\rho}{\sin^\rho(\varepsilon_n X)} = \mathcal{O}\left(X^{-\rho}\right). \tag{9}$$

Proof. Thanks to Proposition 4.3, we have that if n is big enough then

$$x = -\varepsilon_n \cot\left(\varepsilon_n X + \mathcal{O}\left(\frac{\log n}{n}\right)\right).$$

Since $(x, y) \in \mathcal{R}_n(C)$, we have that $\operatorname{Re}(\varepsilon_n X) \in \left[\frac{\pi k_n}{10n}, \pi - \frac{\pi k_n}{10n}\right]$ and $|\operatorname{Im}(\varepsilon_n X)| \leq C\frac{\pi}{n}$. Then using the mean value inequality we have that for n big enough

$$\begin{aligned} x = -\varepsilon_n \cot\left(\varepsilon_n X + \mathcal{O}\left(\frac{\log n}{n}\right)\right) &= -\varepsilon_n \cot(\varepsilon_n X) + \mathcal{O}\left(\varepsilon_n n^{2-2\gamma}\frac{\log n}{n}\right) \\ &= -\varepsilon_n \cot(\varepsilon_n X) + \mathcal{O}\left(\frac{\log n}{n^{2\gamma}}\right). \end{aligned}$$

We have that $\varepsilon_n \cot(\varepsilon_n X) = \mathcal{O}(X^{-1})$ and moreover, since $\gamma > 1/2$, $n^{-2\gamma} \log n = o(1/n) = o(\varepsilon_n)$ so we get $n^{-2\gamma} \log n = o(X^{-1})$ for n big enough, which proves (7). For (8), using again that $\varepsilon_n \cot(\varepsilon_n X) = \mathcal{O}(X^{-1})$ and $n^{-2\gamma} \log n = o(X^{-1})$ we have

$$x^2 + \varepsilon_n^2 = \varepsilon_n^2 + \varepsilon_n^2 \cot^2(\varepsilon_n X) + o\left(X^{-2}\right) = \frac{\varepsilon_n^2}{\sin^2(\varepsilon_n X)} + o\left(X^{-2}\right)$$

hence

$$\frac{1}{x^2 + \varepsilon_n^2} \sim \frac{\sin^2(\varepsilon_n X)}{\varepsilon_n^2} = \mathcal{O}\left(X^2\right).$$

Finally, using (8) and the fact that $1/C < |Y| < C$ by definition of $\mathcal{R}_n(C)$ we get

$$y = Y(x^2 + \varepsilon_n^2)^{\eta/2} \asymp \frac{\varepsilon_n^\rho}{\sin^\rho(\varepsilon_n X)} = \mathcal{O}\left(X^{-\rho}\right),$$

proving (9). □

Remark 4.5. *Observe that $\mathcal{R}_n(C) \subset U$ for n large enough, where U is the domain of definition of the family (g_ε). If we take $(x, y) \in \mathcal{R}_n(C)$ and denote $(X, Y) := \Phi^\iota_{\varepsilon_n}(x, y)$, then $|\varepsilon_n X| \geq \operatorname{Re}(\varepsilon_n X) \geq \frac{\pi k_n}{10n}$ by definition of $\mathcal{R}_n(C)$, so using (7), (9) and the fact that $\varepsilon_n \sim \pi/n$ we have that $(x, y) \in U$ for n large enough (depending only on $\mathcal{R}_n(C)$ and not on (x, y)).)*

Lemma 4.6. *Consider $(x, y) \in \mathcal{R}_n(C)$ and denote $(x_1, y_1) = g_{\varepsilon_n}(x, y)$. Then if n is big enough*

$$\log \frac{x_1^2 + \varepsilon_n^2}{x^2 + \varepsilon_n^2} = 2x + o\left(\frac{1}{n}\right)$$

Proof. Since $x_1 = x + \left(x^2 + \varepsilon_n^2\right)(1 + \mathcal{O}(x, \varepsilon_n)) + y\mathcal{O}(x, y, \varepsilon_n)$ we have

$$x_1^2 = x^2 + 2x\left(x^2 + \varepsilon_n^2\right) + \left(x^2 + \varepsilon_n^2\right)\mathcal{O}_2(x, \varepsilon_n) + y\mathcal{O}_2(x, y, \varepsilon_n)$$

hence

$$\frac{x_1^2+\varepsilon_n^2}{x^2+\varepsilon_n^2} = 1 + 2x + \mathcal{O}_2(x,\varepsilon_n) + \frac{y}{x^2+\varepsilon_n^2}\mathcal{O}_2(x,y,\varepsilon_n).$$

If $(X,Y) := \Phi^{\iota}_{\varepsilon_n}(x,y)$, we have by definition of $\mathcal{R}_n(C)$ that $X = \mathcal{O}(n)$ and $X^{-1} = \mathcal{O}\left(n^{-\gamma}\right)$. Then using (7) and the fact that $\varepsilon_n \sim \pi/n$ we get $\mathcal{O}_2(x,\varepsilon_n) = \mathcal{O}\left(n^{-2\gamma}\right) = o\left(n^{-1}\right)$ since $2\gamma > 1$. Moreover thanks to (8) and (9) we have that $\left(x^2+\varepsilon_n^2\right)^{-1} = \mathcal{O}\left(n^2\right)$ and $y = \mathcal{O}\left(n^{-\gamma\rho}\right)$, so

$$\frac{y}{x^2+\varepsilon_n^2}\mathcal{O}_2(x,y,\varepsilon_n) = \mathcal{O}\left(\frac{1}{n^{\gamma\rho+2\gamma-2}}\right) = o\left(\frac{1}{n}\right)$$

since $\gamma\rho > 2$. Hence

$$\frac{x_1^2+\varepsilon_n^2}{x^2+\varepsilon_n^2} = 1 + 2x + o\left(\frac{1}{n}\right)$$

and therefore

$$\log\frac{x_1^2+\varepsilon_n^2}{x^2+\varepsilon_n^2} = \log\left(1 + 2x + o\left(\frac{1}{n}\right)\right) = 2x + o\left(\frac{1}{n}\right).$$

□

Lemma 4.7. *If $(x,y) \in \mathcal{R}_n(C)$ then for n big enough*

$$\frac{1}{x \pm i\varepsilon_n} = \mathcal{O}(n) \tag{10}$$

and

$$\frac{y b_{\varepsilon_n}(x,y)}{x \pm i\varepsilon_n} = \frac{bxy}{x \pm i\varepsilon_n} + o\left(\frac{1}{n^2}\right) \tag{11}$$

Proof. If $(X,Y) := \Phi^{\iota}_{\varepsilon_n}(x,y)$, using (7) and the fact that $n^{-2\gamma}\log n = o\left(n^{-1}\right) = o(\varepsilon_n)$ we have

$$\frac{1}{x \pm i\varepsilon_n} = \frac{1}{-\varepsilon_n\cot(\varepsilon_n X) + o(\varepsilon_n) \pm i\varepsilon_n}$$

Since $\frac{1}{\varepsilon_n} = \mathcal{O}(n)$ and $\frac{1}{\cot(\varepsilon_n X)\pm i} = \mathcal{O}(1)$, because $\mathrm{Re}\,(\varepsilon_n X) \in (0,\pi)$ and $\mathrm{Im}(\varepsilon_n X) = \mathcal{O}(1)$ by definition of $\mathcal{R}_n(C)$, we get

$$\frac{1}{x \pm i\varepsilon_n} \sim -\frac{1}{\varepsilon_n}\cdot\frac{1}{\cot(\varepsilon_n X)\pm i} = \mathcal{O}(n),$$

proving (10). Then, to get (11) we just need to prove that $y b_{\varepsilon_n}(x,y) = bxy + o\left(n^{-3}\right)$. Since $b_{\varepsilon_n}(x,y) = bx + \mathcal{O}\left(x^2, y, \varepsilon_n\right)$ we have, using (7), (9) and the fact that $X^{-1} = \mathcal{O}(n^{-\gamma})$,

$$y b_{\varepsilon_n}(x,y) = bxy + \mathcal{O}\left(x^2 y, y^2, \varepsilon_n y\right) = bxy + \mathcal{O}\left(\frac{1}{n^{\gamma\rho+1}}\right),$$

so $y b_{\varepsilon_n}(x,y) = bxy + o\left(n^{-3}\right)$ because $\gamma\rho > 2$. □

Lemma 4.8. *Consider $(x,y) \in \mathcal{R}_n(C)$ and denote $(x_1,y_1) = g_{\varepsilon_n}(x,y)$. Then for n big enough*

$$\frac{y_1}{y} = 1 + \eta x + q\varepsilon_n + o\left(\frac{1}{n}\right).$$

Proof. If $(X,Y) := \Phi^\iota_{\varepsilon_n}(x,y)$, we have by definition of $\mathcal{R}_n(C)$ that $X^{-1} = \mathcal{O}\left(n^{-\gamma}\right)$. Then using (7) and (9) we have

$$\begin{aligned} y_1 &= y + y\left(\eta x + q\varepsilon_n + \mathcal{O}(x^2, y, x\varepsilon_n, \varepsilon_n^2)\right) + \mathcal{O}\left(x^{m+3}\right) + \varepsilon_n \mathcal{O}_{m+1}\left(x, \varepsilon_n\right) \\ &= y\left[1 + \eta x + q\varepsilon_n + o\left(\frac{1}{n}\right) + \frac{1}{y}\mathcal{O}\left(x^{m+3}\right) + \frac{1}{y}\varepsilon_n \mathcal{O}_{m+1}\left(x, \varepsilon_n\right)\right]. \end{aligned}$$

Moreover, using (9)

$$\frac{1}{y} \asymp \frac{\sin^\rho(\varepsilon_n X)}{\varepsilon_n^\rho} = \mathcal{O}\left(X^\rho\right).$$

Then, since $\rho - 1 < m \leq \rho$ we obtain from (7) that

$$\frac{1}{y}\mathcal{O}\left(x^{m+3}\right) = \mathcal{O}\left(\frac{1}{X^{m+3-\rho}}\right) = \mathcal{O}\left(\frac{1}{n^{(m+3-\rho)\gamma}}\right) = o\left(\frac{1}{n}\right)$$

and similarly

$$\frac{1}{y}\mathcal{O}\left(x^{m+1}\varepsilon_n\right) = \mathcal{O}\left(\frac{1}{X^{m+1-\rho}}\,\varepsilon_n\right) = \mathcal{O}\left(\frac{1}{n^{(m+1-\rho)\gamma+1}}\right) = o\left(\frac{1}{n}\right).$$

Finally, using the fact that $X = \mathcal{O}(n)$ by definition of $\mathcal{R}_n(C)$,

$$\frac{1}{y}\mathcal{O}\left(x^{m-j}\varepsilon_n^{2+j}\right) = \mathcal{O}\left(X^{\rho-m+j}\varepsilon_n^{2+j}\right) = \mathcal{O}\left(\frac{1}{n^{2+m-\rho}}\right) = o\left(\frac{1}{n}\right)$$

for every j with $0 \leq j \leq m$. Therefore we have

$$y_1 = y\left[1 + \eta x + q\varepsilon_n + o\left(\frac{1}{n}\right)\right],$$

concluding the proof. □

Proposition 4.9. *We have, for $(x,y) \in \mathcal{R}_n(C)$ and n big enough:*

$$A_{\varepsilon_n}(x,y) = o\left(\frac{1}{n}\right); \quad B_{\varepsilon_n}(x,y) = q\varepsilon_n + o\left(\frac{1}{n}\right).$$

Proof. Denoting $(x_1, y_1) = g_{\varepsilon_n}(x,y)$, we have:

$$\begin{aligned} A_{\varepsilon_n}(x,y) &= w_{\varepsilon_n}(x_1,y_1) - w_{\varepsilon_n}(x,y) - 1 \\ &= \frac{1}{2i\varepsilon_n}\log\left(\frac{x_1 - i\varepsilon_n}{x - i\varepsilon_n}\cdot\frac{x + i\varepsilon_n}{x_1 + i\varepsilon_n}\right) - 1 + \frac{1-a}{2}\log\frac{x_1^2 + \varepsilon_n^2}{x^2 + \varepsilon_n^2} \end{aligned}$$

By Lemma 4.6, we have

$$\frac{1-a}{2}\log\frac{x_1^2+\varepsilon_n^2}{x^2+\varepsilon_n^2} = (1-a)x + o\Big(\frac{1}{n}\Big).$$

On the other hand, we compute:

$$\begin{aligned} \log\frac{x_1 - i\varepsilon_n}{x - i\varepsilon_n} &= \log\frac{x - i\varepsilon_n + (x^2+\varepsilon_n^2)a_{\varepsilon_n} + yb_{\varepsilon_n}}{x - i\varepsilon_n} = \log\left(1 + (x+i\varepsilon_n)a_{\varepsilon_n} + \frac{yb_{\varepsilon_n}}{x - i\varepsilon_n}\right) \\ &= (x+i\varepsilon_n)a_{\varepsilon_n} + \frac{bxy}{x - i\varepsilon_n} - \frac{1}{2}(x+i\varepsilon_n)^2 a_{\varepsilon_n}^2 + \frac{1}{3}x^3 + o\left(\frac{1}{n^2}\right) \end{aligned}$$

using Lemmas 4.4 and 4.7 in the last line (and, as in the previous lemmas, the fact that $X^{-1} = \mathcal{O}\left(n^{-\gamma}\right)$ by definition of $\mathcal{R}_n(C)$, where $(X,Y) := \Phi^\iota_{\varepsilon_n}(x,y)$). Similarly, we also have:

$$\log \frac{x_1 + i\varepsilon_n}{x + i\varepsilon_n} = \log \frac{x + i\varepsilon_n + (x^2+\varepsilon_n^2)a_{\varepsilon_n} + yb_{\varepsilon_n}}{x + i\varepsilon_n} = \log\left(1 + (x - i\varepsilon_n)a_{\varepsilon_n} + \frac{yb_{\varepsilon_n}}{x+i\varepsilon_n}\right)$$
$$= (x - i\varepsilon_n)a_{\varepsilon_n} + \frac{bxy}{x+i\varepsilon_n} - \frac{1}{2}(x-i\varepsilon_n)^2 a_{\varepsilon_n}^2 + \frac{1}{3}x^3 + o\left(\frac{1}{n^2}\right)$$

so that

$$\frac{1}{2i\varepsilon_n}\log\left(\frac{x_1 - i\varepsilon_n}{x - i\varepsilon_n}\cdot\frac{x+i\varepsilon_n}{x_1+i\varepsilon_n}\right) = \frac{1}{2i\varepsilon_n}\left[2i\varepsilon_n a_{\varepsilon_n} + 2i\varepsilon_n\frac{bxy}{x^2+\varepsilon_n^2} - 2i\varepsilon_n x a_{\varepsilon_n}^2 + o\left(\frac{1}{n^2}\right)\right]$$
$$= a_{\varepsilon_n} + \frac{bxy}{x^2+\varepsilon_n^2} - xa_{\varepsilon_n}^2 + o\left(\frac{1}{n}\right)$$
$$= 1 + (a-1)x + o\left(\frac{1}{n}\right)$$

using again Lemma 4.4 in the last line.

Finally, note that we have

$$B_{\varepsilon_n}(x,y) = \log\frac{t_{\varepsilon_n}(x_1,y_1)}{t_{\varepsilon_n}(x,y)} = \log\frac{y_1}{y} - \frac{\eta}{2}\log\frac{x_1^2+\varepsilon_n^2}{x^2+\varepsilon_n^2}$$

Using Lemma 4.8 we have

$$\log\frac{y_1}{y} = \log\left(1 + \eta x + q\varepsilon_n + o\left(\frac{1}{n}\right)\right) = \eta x + q\varepsilon_n + o\left(\frac{1}{n}\right)$$

and using Lemma 4.6 we conclude that $B_{\varepsilon_n}(x,y) = q\varepsilon_n + o\left(n^{-1}\right)$. □

5. Controlling the orbit

The goal of this section is to provide accurate estimates for the position of the orbit $\{g_{\varepsilon_n}^k(x,y) : 0 \le k \le n-N\}$ of a point (x,y) in the parabolic basin of $g := g_0$.

The strategy, similar to that of [ABD+16], is to split this orbit in three regions:

(1) A first one where $g_{\varepsilon_n}^k(x,y)$ approaches $(0,0)$ shadowing closely the orbit $g^k(x,y)$, which we call "approaching the eggbeater"; this will occur for $0 \le k \le k_n$, where $k_n = \lfloor n^\gamma \rfloor$ as in Section 4.

(2) A second one which we call "in the eggbeater", in which $g_{\varepsilon_n}^k(x,y)$ is close to $(0,0)$ and where the effect of the perturbation is relevant. This will be the case for $k_n \le k \le n - k_n$.

(3) A third one where $g_{\varepsilon_n}^k(x,y)$ gets away from $(0,0)$ again shadowing the dynamics of g, which we call "leaving the eggbeater". This will happen for the last $k_n - N$ iterates.

Let us fix some constant $C > 0$. Recall that by Lemma 3.1 and Proposition 3.2, there exists $r^\iota(C) > 0$ such that for all $0 < r \le r^\iota(C)$ small enough the incoming petal

$$P^\iota(r,C) := \left\{(x,y)\in\mathbb{C}^2 : |x+r| < r, \left|\frac{y}{(-x)^\eta}\right| < C\right\}$$

has the property that if $(x,y) \in P^\iota(r,C)$, then $g^k(x,y) \in P^\iota(r,C+1)$ for all $k\in\mathbb{N}$ and moreover the incoming Fatou coordinate Φ^ι is well-defined on $P^\iota(r,C)$. Recall that we have the following asymptotic expansion for Φ^ι as $\mathrm{Re}\,(-1/x) \to +\infty$ in $P^\iota(r,C)$:

$$\Phi^\iota(x,y) =: (w^\iota(x,y), t^\iota(x,y)) = \left(-\frac{1}{x} + (1-a)\log(-x) + o(1), \frac{y}{(-x)^\eta} + o(1)\right).$$

The map

$$\Phi_0^\iota(x,y) := \left(-\frac{1}{x}, \frac{y}{(-x)^\eta}\right)$$

maps $P^\iota(r,C)$ biholomorphically to $\mathbb{H}_R \times \mathbb{D}(0,C)$, where $\mathbb{H}_R := \{X \in \mathbb{C} : \operatorname{Re} X > R\}$ and $R := (2r)^{-1}$.

In a similar way, by Proposition 3.3 there exists $r^o(C) > 0$ such that for all $0 < r \leq r^o(C)$ the outgoing Fatou coordinate Φ^o is well-defined on

$$P^o(r,C) := \left\{(x,y) \in \mathbb{C}^2 : |x - r| < r, \left|\frac{y}{x^\eta}\right| < C\right\}$$

and satisfies

$$\Phi^o(x,y) = \left(-\frac{1}{x} + (1-a)\log x + o\left(1\right), \frac{y}{x^\eta} + o\left(1\right)\right),$$

as $\operatorname{Re}(-1/x) \to -\infty$ inside $P^o(r,C)$. We set

$$\Phi_0^o(x,y) := \left(-\frac{1}{x}, \frac{y}{x^\eta}\right),$$

which maps $P^o(r,C)$ biholomorphically to $-\mathbb{H}_R \times \mathbb{D}(0,C)$, where $-\mathbb{H}_R := \{X \in \mathbb{C} : \operatorname{Re} X < -R\}$.

Throughout this section, we will use the following notations: given (x_0, y_0), we will denote (when defined)

$$(x_j, y_j) := g^j(x_0, y_0), \quad (X_j, Y_j) := \Phi_0^\iota(x_j, y_j), \quad (X_j^o, Y_j^o) := \Phi_0^o(x_j, y_j)$$

$$(x_j^{\varepsilon_n}, y_j^{\varepsilon_n}) := g_{\varepsilon_n}^j(x_0, y_0), \quad (X_j^{\varepsilon_n}, Y_j^{\varepsilon_n}) := \Phi_0^\iota(x_j^{\varepsilon_n}, y_j^{\varepsilon_n}), \quad (X_j^{\varepsilon_n,o}, Y_j^{\varepsilon_n,o}) := \Phi_0^o(x_j^{\varepsilon_n}, y_j^{\varepsilon_n}).$$

We will also denote, for any $t, s \in \mathbb{C}$,

$$A_{t,s}(X,Y) := (X + t, e^{\pi s} Y).$$

Given $C > 0$, we choose $0 < r < 2/5$ small enough (or equivalently $R = (2r)^{-1} > 5/4$ large enough) such that

(R_1) $P^\iota(r,C), P^o(r, Ce^{1+\pi|\operatorname{Re} q|}) \subset U$, where U is the domain of definition of (g_ε), and there exists a constant $K_1 > 0$ such that

$$|X_1 - X - 1| < 1/4, \qquad |Y_1 - Y| < K_1 \frac{1}{|X|^2}$$

for all $(X,Y) \in \Phi_0^\iota(P^\iota(r,C))$, where $(X_1, Y_1) := \Phi_0^\iota \circ g \circ \left(\Phi_0^\iota\right)^{-1}(X,Y)$, and

$$|X_1^o - X - 1| < 1/4, \qquad |Y_1^o - Y| < K_1 \frac{1}{|X|^2}$$

for all $(X,Y) \in \Phi_0^o(P^o(r, Ce^{1+\pi|\operatorname{Re} q|}))$, where $(X_1^o, Y_1^o) := \Phi_0^o \circ g \circ \left(\Phi_0^o\right)^{-1}(X,Y)$.

The existence of such R is guaranteed by the expression (3) for $\Phi_0^\iota \circ g \circ \left(\Phi_0^\iota\right)^{-1}$ and the fact that $\Phi_0^o \circ g \circ \left(\Phi_0^o\right)^{-1}$ has the same expansion.

We will also later in this section need to possibly increase R further to meet some extra conditions (see Definition 5.5). To ensure the lack of circular definitions, we will explicitly mention that several constants appearing in the following computations do not depend on R (but are allowed to depend on C).

Definition 5.1. *We let*

$$U_n(R,C) := \{(X,Y) \in \mathbb{H}_R \times \mathbb{D}(0,C) : |X| < 10k_n\} \subsetneq \Phi_0^\iota\left(P^\iota\left((2R)^{-1}, C\right)\right).$$

Lemma 5.2. *For any $C > 0$ and any R satisfying hypothesis (R_1), the map $G_{\varepsilon_n} := \Phi_0^\iota \circ g_{\varepsilon_n} \circ (\Phi_0^\iota)^{-1}$ is well-defined on $U_n(R, C)$ for all n large enough. Moreover, there exists a constant $C^\iota > 0$ depending on C but not on R (in the sense that it does not increase when we increase R) such that if $(X, Y) \in U_n(R, C)$ and n is large enough then*

$$|X_1^{\varepsilon_n} - X_1| \leq C^\iota \left(\frac{1}{n} + \frac{|X|^2}{n^2}\right) \quad \textit{and} \quad |Y_1^{\varepsilon_n} - Y_1| \leq C^\iota \frac{1}{n}$$

where $(X_1^{\varepsilon_n}, Y_1^{\varepsilon_n}) := G_{\varepsilon_n}(X, Y)$ and $(X_1, Y_1) := \Phi_0^\iota \circ g \circ (\Phi_0^\iota)^{-1}(X, Y)$.

Proof. Let us first prove that G_{ε_n} is well-defined on $U_n(R, C)$ for n large enough. It suffices to show that if $(X, Y) \in U_n(R, C)$ and n is large enough then $\operatorname{Re}(x_1^{\varepsilon_n}) < 0$, where $(x_1^{\varepsilon_n}, y_1^{\varepsilon_n}) := g_{\varepsilon_n} \circ (\Phi_0^\iota)^{-1}(X, Y)$. Set $(x, y) := (\Phi_0^\iota)^{-1}(X, Y)$ and $(x_1, y_1) := g(x, y)$. By the expression (6) of g_{ε_n}, we have

$$x_1^{\varepsilon_n} = x_1 + \mathcal{O}\left(x^2\varepsilon_n, y\varepsilon_n, \varepsilon_n^2\right); \qquad y_1^{\varepsilon_n} = y_1 + \mathcal{O}(y\varepsilon_n) + \varepsilon_n\mathcal{O}_{m+1}(x, \varepsilon_n) \tag{12}$$

so using the fact that $y = \mathcal{O}(x^2)$ because $(x, y) \in P^\iota\left((2R)^{-1}, C\right)$ we get

$$\begin{aligned} -\frac{1}{x_1^{\varepsilon_n}} &= -\frac{1}{x_1 + \mathcal{O}\left(x^2\varepsilon_n, y\varepsilon_n, \varepsilon_n^2\right)} = -\frac{1}{x_1 + \mathcal{O}\left(\frac{1}{n^2}, \frac{x^2}{n}\right)} = -\frac{1}{-\frac{1}{X_1} + \mathcal{O}\left(\frac{1}{n^2}, \frac{1}{nX^2}\right)} \\ &= \frac{X_1}{1 - \mathcal{O}\left(\frac{X_1}{n^2}, \frac{X_1}{nX^2}\right)} = X_1 + \mathcal{O}\left(\frac{X_1^2}{n^2}, \frac{X_1^2}{nX^2}\right). \end{aligned}$$

Since $|X_1 - X - 1| < 1/4$ by our choice of R, we have that $X_1 = \mathcal{O}(X)$ so we get

$$-\frac{1}{x_1^{\varepsilon_n}} = X_1 + \mathcal{O}\left(\frac{1}{n}, \frac{X^2}{n^2}\right).$$

Note that in all the previous computations, the implicit constants in $\mathcal{O}$ depend only on C, not on R, in the sense that the do not increase if we increase R. By definition of $U_n(R, C)$, we have

$$\frac{|X|^2}{n^2} \leq \frac{100k_n^2}{n^2}.$$

Moreover, since $\operatorname{Re} X_1 > \operatorname{Re} X + 3/4$, there exists a constant $C_1 > 0$ independent from R such that

$$\operatorname{Re}\left(-\frac{1}{x_1^{\varepsilon_n}}\right) > \operatorname{Re} X + \frac{3}{4} - C_1\left(\frac{1}{n} + \frac{100}{n^{2-2\gamma}}\right)$$

so $\operatorname{Re}\left(x_1^{\varepsilon_n}\right) < 0$ for n large enough, as desired. This computation also gives the first estimate.

Let us prove the second estimate. By the mean value inequality,

$$\begin{aligned} |Y_1^{\varepsilon_n} - Y_1| &= \left|\frac{y_1^{\varepsilon_n}}{(-x_1^{\varepsilon_n})^\eta} - \frac{y_1}{(-x_1)^\eta}\right| \\ &\leq \sup_{x \in [x_1, x_1^{\varepsilon_n}]} \left|\eta \frac{y_1}{(-x)^{\eta+1}}\right| \cdot |x_1^{\varepsilon_n} - x_1| + \left|\frac{1}{(-x_1^{\varepsilon_n})^\eta}\right| \cdot |y_1^{\varepsilon_n} - y_1| \\ &\leq C_2 \frac{|y_1|}{|x_*|^{\rho+1}} |x_1^{\varepsilon_n} - x_1| + C_2 \frac{1}{|x_*|^\rho} |y_1^{\varepsilon_n} - y_1| \end{aligned}$$

for some $C_2 > 0$, where $|x_*| := \min\{|z|, z \in [x_1, x_1^{\varepsilon_n}]\}$. By the expression of g,

$$x_1 = x\left(1 + \mathcal{O}(x, y)\right) + \mathcal{O}(y^2) = x\left(1 + \mathcal{O}(x)\right),$$

using again the fact that $y = \mathcal{O}(x)$ because $(x, y) \in P^\iota\left((2R)^{-1}, C\right)$. Since R was taken large enough so that $|X_1| < |X| + 2$, we have that we have that

$$\frac{|x_1|}{|x|} > \frac{1}{1+4r},$$

where $r = (2R)^{-1}$. Moreover, using (12) we have that $x_1^{\varepsilon_n} = x_1 + \mathcal{O}\left(x^2\varepsilon_n, \varepsilon_n^2\right)$, so

$$\frac{x_*}{x} = \frac{x_1 + \mathcal{O}\left(x^2\varepsilon_n, \varepsilon_n^2\right)}{x} = \frac{x_1}{x} + \mathcal{O}\left(\frac{1}{n}, \frac{X}{n^2}\right).$$

Since $(X, Y) \in U_n(R, C)$, we have that $|X| < 10k_n$, so we can take n big enough so that

$$\frac{|x_*|}{|x|} > \frac{1}{2}\frac{1}{1+4r}. \tag{13}$$

Hence using the expression of g we have that for n large enough

$$\frac{|y_1|}{|x_*|^{\rho+1}} \leq C_3 \frac{|y_1|}{|x|^{\rho+1}} \leq C_3 \frac{1}{|x|^{\rho+1}} \left|y(1+\mathcal{O}(x,y)) + \mathcal{O}(x^{m+3})\right| \leq C_4 \frac{1}{|x|}$$

for some constants $C_3, C_4 > 0$, so

$$\frac{|y_1|}{|x_*|^{\rho+1}} |x_1^{\varepsilon_n} - x_1| \leq C_4 \frac{1}{|x|} |x_1^{\varepsilon_n} - x_1| \leq C_5 \left(|x|\frac{1}{n} + \frac{|X|}{n^2}\right) \leq C_6 \frac{1}{n}$$

for some constants $C_5, C_6 > 0$, where we used in the last inequality the fact that $|X| < 10k_n$. Finally, using (12) and (13) we have

$$\frac{|y_1^{\varepsilon_n} - y_1|}{|x_*|^\rho} = \frac{\mathcal{O}(y\varepsilon_n) + \varepsilon_n \mathcal{O}_{m+1}(x, \varepsilon_n)}{|x|^\rho} = \mathcal{O}\left(\frac{1}{n}\right)$$

using again in the last identity the fact that $|X| \leq 10k_n$. Putting everything together, we obtain

$$Y_1^{\varepsilon_n} - Y_1 = \mathcal{O}\left(\frac{1}{n}\right),$$

and the Lemma is proved. □

For the outgoing petal, we have the following analogous Lemma:

Definition 5.3. *We let*

$$U_n^o(R, Ce^{1+\pi|\operatorname{Re} q|}) := \{(X, Y) \in -\mathbb{H}_R \times \mathbb{D}(0, Ce^{1+\pi|\operatorname{Re} q|}) : |X| < 10k_n\}.$$

Note that $U_n^o(R, Ce^{1+\pi|\operatorname{Re} q|}) \subsetneq \Phi_0^o\left(P^o\left((2R)^{-1}, Ce^{1+\pi|\operatorname{Re} q|}\right)\right)$.

Lemma 5.4. *For any $C > 0$ and any R satisfying hypothesis (R_1), the map $G_{\varepsilon_n}^o := \Phi_0^o \circ g_{\varepsilon_n} \circ (\Phi_0^o)^{-1}$ is well-defined on $U_n^o(R, Ce^{1+\pi|\operatorname{Re} q|})$ for all n large enough. Moreover, there exists a constant $C^o > 0$ depending on C but not on R (in the sense that it does not increase when we increase R) such that if $(X, Y) \in U_n^o(R, Ce^{1+\pi|\operatorname{Re} q|})$ and n is large enough then*

$$|X_1^{\varepsilon_n,o} - X_1^o| \leq C^o\left(\frac{1}{n} + \frac{|X|^2}{n^2}\right) \quad \text{and} \quad |Y_1^{\varepsilon_n,o} - Y_1^o| \leq C^o\frac{1}{n}$$

where $(X_1^{\varepsilon_n,o}, Y_1^{\varepsilon_n,o}) := G_{\varepsilon_n}^o(X, Y)$ and $(X_1^o, Y_1^o) := \Phi_0^o \circ g \circ (\Phi_0^o)^{-1}(X, Y)$.

Proof. The proof follows essentially the same steps as the previous one. The only modification concerns the argument showing that $\operatorname{Re}(x_1^{\varepsilon_n}) > 0$. In this case, we have

$$\operatorname{Re}\left(-\frac{1}{x_1^{\varepsilon_n}}\right) < \operatorname{Re} X + \frac{5}{4} + C_1\left(\frac{1}{n} + \frac{100}{n^{2-2\gamma}}\right)$$

for some constant $C_1 > 0$. Hence, for n sufficiently large, we have $\operatorname{Re}(-1/x_1^{\varepsilon_n}) < 0$, since by assumption $R > 5/4$. □

We will now state explicitly in which sense R must be taken large enough, or equivalently $r = (2R)^{-1}$ small enough. By Lemma 5.2, there exists $C^\iota > 0$, depending on C but not on R, such that for all $(X, Y) \in U_n(R, C)$ and for n large enough

$$|X_1^{\varepsilon_n} - X_1| \leq C^\iota \left(\frac{1}{n} + \frac{|X|^2}{n^2} \right) \quad \text{and} \quad |Y_1^{\varepsilon_n} - Y_1| \leq C^\iota \frac{1}{n}$$

where $(X_1^{\varepsilon_n}, Y_1^{\varepsilon_n}) := \Phi_0^\iota \circ g_{\varepsilon_n} \circ (\Phi_0^\iota)^{-1}(X, Y)$ and $(X_1, Y_1) := \Phi_0^\iota \circ g \circ (\Phi_0^\iota)^{-1}(X, Y)$. Since $|X| < 10k_n$, we have that for n large enough $|X_1^{\varepsilon_n} - X_1| < 1/4$. Then by our standing assumption (R_1) there exists $M^\iota > 0$ (depending on C but not on R) such that for all $(X, Y) \in U_n(R, C)$ and for n large enough we have

$$|X_1^{\varepsilon_n} - X - 1| < \frac{1}{2}; \qquad |Y_1^{\varepsilon_n} - Y| < M^\iota \left(\frac{1}{n} + \frac{1}{|X|^2} \right). \tag{14}$$

Moreover, by Lemma 5.4, similar estimates also hold on the outgoing petal; more precisely, there exists a constant $M^o > 0$ (also depending on C but not on R) such that for all $(X, Y) \in U_n^o(R, Ce^{1+\pi|\operatorname{Re} q|})$ and for n large enough we have

$$|X_1^{\varepsilon_n, o} - X - 1| < \frac{1}{2}; \qquad |Y_1^{\varepsilon_n, o} - Y| < M^o \left(\frac{1}{n} + \frac{1}{|X|^2} \right) \tag{15}$$

where $(X_1^{\varepsilon_n, o}, Y_1^{\varepsilon_n, o}) := \Phi_0^o \circ g_{\varepsilon_n} \circ (\Phi_0^o)^{-1}(X, Y)$. We let $M := \max(M^\iota, M^o)$.

From now on, we fix a compact $K \subset \mathcal{B}_{U,v}$, where $\mathcal{B}_{U,v}$ is the parabolic basin associated to $v = (1, 0)$. Recall that $\mathcal{B}_{U,v} := \bigcup_{C>0} \bigcup_{n \geq 0} f^{-n}(P^\iota(r, C))$ for any $r > 0$. Then there exists a constant $C > 2$ (depending only on K) such that for any $r > 0$ small enough there exists $n_0 = n_0(r) \in \mathbb{N}$ such that $L := g^{n_0}(K) \Subset P^\iota(r, C - 1)$.

Definition 5.5 (Choice of R). *From now on, we fix $R > 5/2$ (equivalently, $0 < r = (2R)^{-1} < 1/5$ small enough) satisfying hypothesis (R_1) and large enough such that*

- (R_2) $r \leq r^\iota(C)$ *and* $r \leq r^o(Ce^{1+\pi|\operatorname{Re} q|})$.
- (R_3) $M \sum_{j=0}^{\infty} (R + j/2)^{-2} \leq 1/10$.
- (R_4) *For all* $(X, Y) \in \mathbb{H}_{R-1} \times \mathbb{D}(0, C + 1)$

$$\|\Phi^\iota \circ (\Phi_0^\iota)^{-1}(X, Y) - (X - (1 - a) \log X, Y)\| \leq \frac{1}{C^2}$$

and for all $(X, Y) \in -\mathbb{H}_{R-5/2} \times \mathbb{D}(0, Ce^{1+\pi|\operatorname{Re} q|} + 2)$

$$\|\Phi^o \circ (\Phi_0^o)^{-1}(X, Y) - (X - (1 - a) \log(-X), Y)\| \leq \frac{1}{C^2}.$$

Observe that conditions (R_2) and (R_4) are satisfied for R large enough by Propositions 3.2 and 3.3.

Definition 5.6. *Let $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$ denote the projection on the first coordinate. We define*

$$M_C := \left\{ (x, y) \in \mathbb{C}^2 : x \in \pi_1(L) \text{ and } \frac{1}{C - 1} \leq \left| \frac{y}{(-x)^\eta} \right| \leq C - 1 \right\}$$

and

$$N_C := \left\{ (x, y) \in \mathbb{C}^2 : x \in \pi_1(L) \text{ and } \left| \frac{y}{(-x)^\eta} \right| \leq C - 1 \right\}.$$

Observe that, by definition, $M_C \subset N_C$ and $L \subset N_C \subset P^\iota(r, C - 1)$.

5.1. **Approaching the eggbeater.**

Lemma 5.7. *For all n large enough and for all $0 \leq j \leq k_n$ we have that*

$$\Phi_0^\iota \circ g_{\varepsilon_n}^j(M_C) \subset U_n(R,C).$$

In particular, $g_{\varepsilon_n}^j(M_C) \subset P^\iota(r,C)$ for all n large enough and all $0 \leq j \leq k_n$.

Proof. Since $M_C \subset P^\iota(r, C-1)$ is compact, we can assume that n is large enough such that $\Phi_0^\iota(M_C) \subset U_n(R,C)$. Take $(x_0,y_0) \in M_C$, set $(X_0,Y_0) := \Phi_0^\iota(x_0,y_0)$ and denote, for all $j \in \mathbb{N}$ such that it is well-defined, $(X_j^{\varepsilon_n}, Y_j^{\varepsilon_n}) := \Phi_0^\iota \circ g_{\varepsilon_n} \circ (\Phi_0^\iota)^{-1}(X_0,Y_0)$. Set

$$K_n := \min\{j \in \mathbb{N} : (X_j^{\varepsilon_n}, Y_j^{\varepsilon_n}) \notin U_n(R,C)\} > 0$$

and let us prove that $K_n > k_n$.

First, recall that by (14) there exists a constant $M > 0$ such that for all $j \leq K_n$ and for n large enough we have

$$|X_j^{\varepsilon_n} - X_{j-1}^{\varepsilon_n} - 1| < \frac{1}{2}$$

$$|Y_j^{\varepsilon_n} - Y_{j-1}^{\varepsilon_n}| < M\left(\frac{1}{n} + \frac{1}{|X_{j-1}^{\varepsilon_n}|^2}\right).$$

From those inequalities, we get:

$$|X_j^{\varepsilon_n} - X_0 - j| \leq \frac{j}{2} \tag{16}$$

and

$$|Y_j^{\varepsilon_n} - Y_0| \leq \frac{Mj}{n} + M\sum_{k=0}^{j-1} \frac{1}{|X_k^{\varepsilon_n}|^2} \tag{17}$$

for all $0 \leq j \leq K_n$. Using (16) in (17) we get, for all $0 \leq j \leq K_n$,

$$|Y_j^{\varepsilon_n} - Y_0| \leq \frac{Mj}{n} + M\sum_{k=0}^{j-1} \frac{1}{(\operatorname{Re} X_0 + k/2)^2} \leq \frac{Mj}{n} + M\sum_{k=0}^{j-1} \frac{1}{(R + k/2)^2}.$$

Recall that by condition (R_3) we have that $M\sum_{k=0}^{\infty}(R+k/2)^{-2} \leq 1/10$: from now on, we also assume that n is large enough so that $\frac{Mk_n}{n} \leq 1/10$. Suppose by contradiction that $K_n \leq k_n$. By definition $(X_{K_n}^{\varepsilon_n}, Y_{K_n}^{\varepsilon_n}) \notin U_n(R,C)$; but on the other hand, we have

$$|X_{K_n}^{\varepsilon_n} - X_0 - K_n| \leq \frac{K_n}{2} \quad \text{and} \quad |Y_{K_n}^{\varepsilon_n} - Y_0| \leq \frac{2}{10}.$$

Therefore: $|Y_{K_n}^{\varepsilon_n}| < C - 1 + 2/10 < C$, $R + K_n/2 \leq \operatorname{Re}\left(X_{K_n}^{\varepsilon_n}\right)$, and

$$|X_{K_n}^{\varepsilon_n}| \leq |X_0| + \frac{3}{2}K_n \leq \max\left\{\left|-\frac{1}{x}\right| : (x,y) \in M_C\right\} + \frac{3}{2}k_n < 10k_n$$

(again, up to taking n large enough such that $\max\{|-1/x| : (x,y) \in M_C\} + 3k_n/2 < 10k_n$). Therefore $(X_{K_n}^{\varepsilon_n}, Y_{K_n}^{\varepsilon_n}) \in U_n(R,C)$, a contradiction. This proves that $K_n > k_n$, hence that $(X_k^{\varepsilon_n}, Y_k^{\varepsilon_n}) \in U_n(R,C)$ for all $0 \leq k \leq k_n$. ☐

Lemma 5.8. *For n large enough*

$$\Phi^\iota \circ g_{\varepsilon_n}^{k_n}\big|_{M_C} = A_{k_n,0} \circ \Phi^\iota\big|_{M_C} + o(1),$$

where the convergence of the term $o(1)$ is uniform on M_C (recall that $A_{t,s}(X,Y) = (X+t, e^{\pi s}Y)$ for any $t, s \in \mathbb{C}$).

Proof. First note that, by Lemma 5.7, we have $\Phi_0^\iota \circ g_{\varepsilon_n}^j(M_C) \subset U_n(R,C)$ for n large enough and for all $0 \leq j \leq k_n$. In particular, for n large enough and for all $0 \leq j \leq k_n$ we have that $g_{\varepsilon_n}^j(M_C) \subset P^\iota\left((2R)^{-1}, C\right)$, so $\Phi^\iota \circ g_{\varepsilon_n}^j(x_0,y_0)$ is well-defined for any $(x_0,y_0) \in M_C$.

Take $(x_0,y_0) \in M_C$ and denote $(X_0,Y_0) := \Phi_0^\iota(x_0,y_0)$, $(X_1,Y_1) := \Phi_0^\iota \circ g(x,y) \in \mathbb{H}_\mathbb{R} \times \mathbb{D}(0,C)$ and $(X_1^{\varepsilon_n}, Y_1^{\varepsilon_n}) := \Phi_0^\iota \circ g_{\varepsilon_n}(x_0,y_0) \in \mathbb{H}_R \times \mathbb{D}(0,C)$. Set $\Phi_G := \Phi^\iota \circ (\Phi_0^\iota)^{-1} : \mathbb{H}_{R-1} \times \mathbb{D}(0,C+1) \to \mathbb{C}^2$, so

$$\Phi_G(X_1,Y_1) = \Phi_G(X_0,Y_0) + (1,0)$$

and hence

$$\Phi^\iota \circ g_{\varepsilon_n}(x_0,y_0) - \Phi^\iota(x_0,y_0) - (1,0) = \Phi_G(X_1^{\varepsilon_n}, Y_1^{\varepsilon_n}) - \Phi_G(X_1,Y_1).$$

By Proposition 3.2 and condition (R_4), there exists a holomorphic map $u : \mathbb{H}_{R-1} \times \mathbb{D}(0,C+1) \to \mathbb{C}^2$ such that

$$\Phi_G(X,Y) = (X - (1-a)\log X, Y) + u(X,Y)$$

with $\|u(X,Y)\| \leq 1$ for all $(X,Y) \in \mathbb{H}_{R-1} \times \mathbb{D}(0,C+1)$. By Cauchy estimates, we have

$$\|\partial_X u(X,Y)\| \leq 1 \quad \text{and} \quad \|\partial_Y u(X,Y)\| \leq 1$$

for all $(X,Y) \in \mathbb{H}_R \times \mathbb{D}(0,C)$ and hence, by the mean value inequality,

$$\|u(X_1^{\varepsilon_n}, Y_1^{\varepsilon_n}) - u(X_1,Y_1)\| \leq |X_1^{\varepsilon_n} - X_1| + |Y_1^{\varepsilon_n} - Y_1|.$$

On the other hand, also by the mean value inequality, we have

$$|\log X_1^{\varepsilon_n} - \log X_1| < |X_1^{\varepsilon_n} - X_1|.$$

Therefore

$$\begin{aligned}\|\Phi_G(X_1^{\varepsilon_n}, Y_1^{\varepsilon_n}) - \Phi_G(X_1,Y_1)\| &\leq |X_1^{\varepsilon_n} - X_1| + |1-a||\log X_1^{\varepsilon_n} - \log X_1| + |Y_1^{\varepsilon_n} - Y_1| \\ &\quad + \|u(X_1^{\varepsilon_n}, Y_1^{\varepsilon_n}) - u(X_1,Y_1)\| \\ &\leq (2 + |1-a|)\left(|X_1^{\varepsilon_n} - X_1| + |Y_1^{\varepsilon_n} - Y_1|\right).\end{aligned}$$

Then using Lemma 5.2 we obtain that for n large enough

$$\|\Phi^\iota \circ g_{\varepsilon_n}(x_0,y_0) - \Phi^\iota(x_0,y_0) - (1,0)\| \leq K_2\left(\frac{1}{n} + \frac{1}{|x_0|^2 n^2}\right),$$

where $K_2 = 2(2 + |1-a|)C^\iota$. By an immediate induction, we have

$$\|\Phi^\iota \circ g_{\varepsilon_n}^{k_n}(x_0,y_0) - \Phi^\iota(x_0,y_0) - (k_n,0)\| \leq \frac{K_2 k_n}{n} + \frac{K_2}{n^2}\sum_{j=0}^{k_n-1}\frac{1}{|x_j^{\varepsilon_n}|^2},$$

where $(x_j^{\varepsilon_n}, y_j^{\varepsilon_n}) := g_{\varepsilon_n}^j(x_0,y_0)$. By definition of $U_n(R,C)$ we have $|x_j^{\varepsilon_n}|^{-1} < 10k_n$ for all $0 \leq j \leq k_n$; therefore

$$\|\Phi^\iota \circ g_{\varepsilon_n}^{k_n}(x_0,y_0) - \Phi^\iota(x_0,y_0) - (k_n,0)\| \leq \frac{K_2 k_n}{n} + \frac{100 K_2 k_n^3}{n^2}$$

and using the fact that $k_n/n = o(1)$ and $k_n^3/n^2 = o(1)$ the Lemma follows. □

Recall from Section 4 that

$$\Phi_{\varepsilon_n}^\iota(x,y) = (w_{\varepsilon_n}(x), t_{\varepsilon_n}(x,y))$$

where

$$w_{\varepsilon_n}(x) = \frac{1}{\varepsilon_n}\arctan\left(\frac{x}{\varepsilon_n}\right) + \frac{\pi}{2\varepsilon_n} + \frac{1-a}{2}\log(x^2 + \varepsilon_n^2); \quad t_{\varepsilon_n}(x,y) = \frac{y}{(x^2+\varepsilon_n^2)^{\eta/2}}.$$

Lemma 5.9. *For n large enough we have*

$$\Phi^\iota_{\varepsilon_n} \circ g_{\varepsilon_n}^{k_n}\big|_{M_C} = \Phi^\iota \circ g_{\varepsilon_n}^{k_n}\big|_{M_C} + o(1),$$

where the convergence of the term $o(1)$ is uniform on M_C.

Proof. Take $(x_0, y_0) \in M_C$ and let $(x_{k_n}^{\varepsilon_n}, y_{k_n}^{\varepsilon_n}) := g_{\varepsilon_n}^{k_n}(x_0, y_0)$. By Lemmas 5.8, 5.7 and 3.6,

$$x_{k_n}^{\varepsilon_n} = -\frac{1}{k_n} + \mathcal{O}\left(\frac{\log n}{k_n^2}\right) \qquad \text{and} \qquad y_{k_n}^{\varepsilon_n} = t^\iota(x_0, y_0)\frac{1}{k_n^\eta} + o\left(\frac{1}{k_n^\rho}\right),$$

where the terms $\mathcal{O}(k_n^{-2}\log n)$ and $o(k_n^\rho)$ are uniform on M_C. In particular $x_{k_n}^{\varepsilon_n} \sim -1/k_n$ uniformly on M_C and

$$\frac{\varepsilon_n}{x_{k_n}^{\varepsilon_n}} = \frac{\varepsilon_n}{-\frac{1}{k_n} + \mathcal{O}\left(\frac{\log n}{k_n^2}\right)} = -\varepsilon_n k_n + \varepsilon_n \mathcal{O}\left(\log n\right) = o(1).$$

Since $x_{k_n}^{\varepsilon_n} \sim -1/k_n$ uniformly on M_C, for n large enough depending only on M_C we have that $\operatorname{Re}(x_{k_n}^{\varepsilon_n}/\varepsilon_n) < 0$. Then, using the relation $\arctan z + \arctan(1/z) = -\pi/2$ whenever $\operatorname{Re} z < 0$ we have that

$$\begin{aligned} w_{\varepsilon_n}(x_{k_n}^{\varepsilon_n}) &= \frac{1}{\varepsilon_n}\arctan\left(\frac{x_{k_n}^{\varepsilon_n}}{\varepsilon_n}\right) + \frac{\pi}{2\varepsilon_n} + \frac{1-a}{2}\log((x_{k_n}^{\varepsilon_n})^2 + \varepsilon_n^2) \\ &= -\frac{1}{\varepsilon_n}\arctan\frac{\varepsilon_n}{x_{k_n}^{\varepsilon_n}} + \frac{1-a}{2}\log\left((x_{k_n}^{\varepsilon_n})^2 + \varepsilon_n^2\right) \end{aligned}$$

and therefore

$$\begin{aligned} w_{\varepsilon_n}(x_{k_n}^{\varepsilon_n}) &= -\frac{1}{\varepsilon_n}\arctan\frac{\varepsilon_n}{x_{k_n}^{\varepsilon_n}} + \frac{1-a}{2}\log\left((x_{k_n}^{\varepsilon_n})^2 + \varepsilon_n^2\right) \\ &= -\frac{1}{\varepsilon_n}\left(\frac{\varepsilon_n}{x_{k_n}^{\varepsilon_n}} + \mathcal{O}(\varepsilon_n^3 k_n^3)\right) + (1-a)\log\left(-x_{k_n}^{\varepsilon_n}\right) + \frac{1-a}{2}\log\left(1 + \frac{\varepsilon_n^2}{k_n^2}\right) + o(1) \\ &= -\frac{1}{x_{k_n}^{\varepsilon_n}} + (1-a)\log\left(-x_{k_n}^{\varepsilon_n}\right) + o(1) = w^\iota(x_{k_n}^{\varepsilon_n}, y_{k_n}^{\varepsilon_n}) + o(1). \end{aligned}$$

Similarly:

$$\begin{aligned} t_{\varepsilon_n}(x_{k_n}^{\varepsilon_n}, y_{k_n}^{\varepsilon_n}) &= \frac{y_{k_n}^{\varepsilon_n}}{\left((x_{k_n}^{\varepsilon_n})^2 + \varepsilon_n^2\right)^{\eta/2}} = \frac{y_{k_n}^{\varepsilon_n}}{(-x_{k_n}^{\varepsilon_n})^\eta}\left(1 + \frac{\varepsilon_n^2}{(x_{k_n}^{\varepsilon_n})^2}\right)^{-\eta/2} \\ &= \left(t^\iota(x_{k_n}^{\varepsilon_n}, y_{k_n}^{\varepsilon_n}) + o(1)\right)(1 + o(1)) = t^\iota(x_{k_n}^{\varepsilon_n}, y_{k_n}^{\varepsilon_n}) + o(1) \end{aligned}$$

(in the last line, we used the fact that $t^\iota(x_{k_n}^{\varepsilon_n}, y_{k_n}^{\varepsilon_n}) = \mathcal{O}(1)$ by Lemma 5.8). □

5.2. **In the eggbeater.**

Definition 5.10. *Let $\hat{C} := Ce^{1/2+\pi|\operatorname{Re} q|} > C$ and $\tilde{C} := Ce^{1+\pi|\operatorname{Re} q|} > C$ (recall that C was fixed when we fixed the compact $K \subset \mathcal{B}_{U,v}$).*

Lemma 5.11. *For all n large enough, we have $g_{\varepsilon_n}^{k_n}(M_C) \subset \mathcal{R}_n(C)$ (recall that $\mathcal{R}_n(C)$ is the set from Definition 4.1 and that for n large enough $\mathcal{R}_n(C) \subset U$, where U is the domain of definition of (g_{ε_n})).*

Proof. Let $(x_0, y_0) \in M_C$ and $(x_{k_n}^{\varepsilon_n}, y_{k_n}^{\varepsilon_n}) := g_{\varepsilon_n}^{k_n}(x_0, y_0)$. By Lemmas 5.8 and 5.9, we have

$$\Phi^\iota_{\varepsilon_n} \circ g_{\varepsilon_n}^{k_n}(x_0, y_0) = \Phi^\iota(x_0, y_0) + (k_n, 0) + o(1),$$

where the convergence of the term $o(1)$ is uniform on M_C. Therefore

$$t_{\varepsilon_n}(x_{k_n}^{\varepsilon_n}, y_{k_n}^{\varepsilon_n}) = t^\iota(x_0, y_0) + o(1).$$

Since $(C-1)^{-1} \leq |y_0(-x_0)^{-\eta}| \leq C-1$ by definition of M_C and $|t^\iota(x_0,y_0)-y_0(-x_0)^{-\eta}| < 1/C^2 < 1/4$ by condition (R_4), we have that $(C-1)^{-1} - C^{-2} \leq |t^\iota(x_0,y_0)| \leq C - 1/4$ and then $C^{-1} < |t_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}, y^{\varepsilon_n}_{k_n})| < C$ for all n large enough (depending only on M_C and not on the choice of (x_0,y_0)). Similarly

$$w_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}) = w^\iota(x_0,y_0) + k_n + o(1)$$

and therefore, since $\varepsilon_n = \pi/n + o(1/n)$,

$$\varepsilon_n w_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}) = \varepsilon_n k_n + \mathcal{O}(\varepsilon_n) = \frac{\pi k_n}{n} + \mathcal{O}\left(\varepsilon_n\right).$$

where the implicit constants in the $\mathcal{O}$ terms only depend on M_C and not on the choice of (x_0,y_0). In particular, for all n large enough (depending only on M_C)

$$\mathrm{Re}\left(\varepsilon_n w_{\varepsilon_n}(x^{\varepsilon_n}_{k_n})\right) \in \left[\frac{\pi k_n}{10n}, \pi - \frac{\pi k_n}{10n}\right] \quad \text{and} \quad |\mathrm{Im}(\varepsilon_n w_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}))| \leq C\frac{\pi}{n},$$

hence $(x^{\varepsilon_n}_{k_n}, y^{\varepsilon_n}_{k_n}) \in \mathcal{R}_n(C)$ for n large enough. □

Recall that $\Phi^o_{\varepsilon_n} := \Phi^\iota_{\varepsilon_n} - (\pi/\varepsilon_n, 0)$.

Proposition 5.12. *For n large enough we have that $g^{n-k_n}_{\varepsilon_n}(M_C) \subset \mathcal{R}_n(\hat{C})$ and*

$$\Phi^o_{\varepsilon_n} \circ g^{n-k_n}_{\varepsilon_n}\big|_{M_C} = A_{\sigma-2k_n,q} \circ \Phi^\iota_{\varepsilon_n} \circ g^{k_n}_{\varepsilon_n}\big|_{M_C} + o(1),$$

where the convergence of the term $o(1)$ is uniform on M_C.

Proof. Take $(x_0,y_0) \in M_C$ and let $(x^{\varepsilon_n}_j, y^{\varepsilon_n}_j) := g^j_{\varepsilon_n}(x_0,y_0)$. Let us start by proving that for n large enough (depending only on M_C) and for $0 \leq j \leq n-2k_n$ we have $(x^{\varepsilon_n}_{k_n+j}, y^{\varepsilon_n}_{k_n+j}) \in \mathcal{R}_n(\hat{C})$ and

$$w_{\varepsilon_n}(x^{\varepsilon_n}_{k_n+j}) = w_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}) + j + \sum_{k=0}^{j-1} A_{\varepsilon_n}(x^{\varepsilon_n}_{k_n+k}, y^{\varepsilon_n}_{k_n+k})$$

and

$$t_{\varepsilon_n}(x^{\varepsilon_n}_{k_n+j}, y^{\varepsilon_n}_{k_n+j}) = t_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}, y^{\varepsilon_n}_{k_n}) \exp\left(\sum_{k=0}^{j-1} B_{\varepsilon_n}(x^{\varepsilon_n}_{k_n+k}, y^{\varepsilon_n}_{k_n+k})\right)$$

where, as in subsection 4.1,

$$A_{\varepsilon_n}(x,y) := w_{\varepsilon_n}(x_1) - w_{\varepsilon_n}(x) - 1 \quad \text{and} \quad B_{\varepsilon_n}(x,y) := \log\frac{t_{\varepsilon_n}(x_1,y_1)}{t_{\varepsilon_n}(x,y)},$$

with $(x_1,y_1) := g_{\varepsilon_n}(x,y)$.

We argue by induction on j. Indeed, the assertions hold for $j=0$ by Lemma 5.11 and the fact that $\mathcal{R}_n(C) \subset \mathcal{R}_n(\hat{C})$. Moreover, if they hold for some $0 \leq j \leq n-2k_n-1$, then thanks to Proposition 4.9 (applied on $\mathcal{R}_n(\hat{C})$), we have

$$\begin{aligned}
w_{\varepsilon_n}(x^{\varepsilon_n}_{k_n+j+1}) &= w_{\varepsilon_n}(x^{\varepsilon_n}_{k_n+j}) + 1 + A_{\varepsilon_n}(x^{\varepsilon_n}_{k_n+j}, y^{\varepsilon_n}_{k_n+j}) \\
&= w_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}) + j + 1 + \sum_{k=0}^{j} A_{\varepsilon_n}(x^{\varepsilon_n}_{k_n+k}, y^{\varepsilon_n}_{k_n+k}) \\
&= w_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}) + j + 1 + o(1) \\
&= k_n + j + 1 + \mathcal{O}(1),
\end{aligned}$$

where the terms $o(1)$ and $\mathcal{O}(1)$ are uniform, and using in the last line that $w_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}) = w^\iota(x_0,y_0) + k_n + o(1)$ by Lemmas 5.8 and 5.9. Moreover,

$$\begin{aligned} t_{\varepsilon_n}(x^{\varepsilon_n}_{k_n+j+1}, y^{\varepsilon_n}_{k_n+j+1}) &= t_{\varepsilon_n}(x^{\varepsilon_n}_{k_n+j}, y^{\varepsilon_n}_{k_n+j}) e^{B_{\varepsilon_n}(x^{\varepsilon_n}_{k_n+j}, y^{\varepsilon_n}_{k_n+j})} \\ &= t_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}, y^{\varepsilon_n}_{k_n}) \exp\left(\sum_{k=0}^{j} B_{\varepsilon_n}(x^{\varepsilon_n}_{k_n+k}, y^{\varepsilon_n}_{k_n+k})\right) \\ &= t_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}, y^{\varepsilon_n}_{k_n}) e^{qj\varepsilon_n + o(1)}, \end{aligned}$$

where the term $o(1)$ is uniform on M_C. Therefore, using the fact that $n(\varepsilon_n - \pi/n) = \mathcal{O}(\varepsilon_n)$ we have, for n big enough depending only on M_C, $\varepsilon_n w_{\varepsilon_n}(x^{\varepsilon_n}_{k_n+j+1}) = \frac{\pi}{n}(k_n + j + 1 + \mathcal{O}(1))$, hence

$$\frac{\pi k_n}{10n} \leq \mathrm{Re}\,(\varepsilon_n w_{\varepsilon_n}(x^{\varepsilon_n}_{k_n+j+1})) \leq \pi - \frac{\pi k_n}{10n}.$$

And since $\varepsilon_n w_{\varepsilon_n}(x^{\varepsilon_n}_{k_n+j+1}) = \varepsilon_n\left[w_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}) + j + 1 + o(1)\right]$ and $|\mathrm{Im}(\varepsilon_n w_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}))| \leq C\pi/n$ by Lemma 5.11, for n large enough depending only on M_C we get, using the fact that $\varepsilon_n = \pi/n + o(1)$,

$$|\mathrm{Im}(\varepsilon_n w_{\varepsilon_n}(x^{\varepsilon_n}_{k_n+j+1}))| < \hat{C}\frac{\pi}{n}.$$

Moreover, since

$$|t_{\varepsilon_n}(x^{\varepsilon_n}_{k_n+j+1}, y^{\varepsilon_n}_{k_n+j+1})| = |t_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}, y^{\varepsilon_n}_{k_n}) e^{qj\varepsilon_n + o(1)}|,$$

and

$$\frac{1}{C} < |t_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}, y^{\varepsilon_n}_{k_n})| < C$$

for n large enough, by Lemma 5.11, we get that

$$\frac{1}{\hat{C}} < |t_{\varepsilon_n}(x^{\varepsilon_n}_{k_n+j+1}, y^{\varepsilon_n}_{k_n+j+1})| < \hat{C}$$

for n large enough depending only on M_C, and the statement is proved.

Taking $j = n - 2k_n$ we obtain that $(x^{\varepsilon_n}_{n-k_n}, y^{\varepsilon_n}_{n-k_n}) \in \mathcal{R}_n(\hat{C})$ for n large enough and

$$w_{\varepsilon_n}(x^{\varepsilon_n}_{n-k_n}) = w_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}) + n - 2k_n + \sum_{k=0}^{n-2k_n-1} A_{\varepsilon_n}(x^{\varepsilon_n}_{k_n+k}, y^{\varepsilon_n}_{k_n+k})$$

and

$$t_{\varepsilon_n}(x^{\varepsilon_n}_{n-k_n}, y^{\varepsilon_n}_{n-k_n}) = t_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}, y^{\varepsilon_n}_{k_n}) \exp\left(\sum_{k=0}^{n-2k_n-1} B_{\varepsilon_n}(x^{\varepsilon_n}_{k_n+k}, y^{\varepsilon_n}_{k_n+k})\right)$$

so using again Proposition 4.9 and the fact that $\varepsilon_n = \pi/n + o(1/n)$ we get

$$\left(w_{\varepsilon_n}(x^{\varepsilon_n}_{n-k_n}), t_{\varepsilon_n}(x^{\varepsilon_n}_{n-k_n}, y^{\varepsilon_n}_{n-k_n})\right) = \left(w_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}) + n - 2k_n + o(1), e^{\pi q + o(1)} t_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}, y^{\varepsilon_n}_{k_n})\right),$$

where the terms $o(1)$ are uniform on M_C. Finally, since $\Phi^o_{\varepsilon_n}(x,y) = \Phi^\iota_{\varepsilon_n}(x,y) - (\pi/\varepsilon_n, 0)$ we obtain, using the fact that $n - \pi/\varepsilon_n = \sigma + o(1)$,

$$\Phi^o_{\varepsilon_n} \circ g^{n-k_n}_{\varepsilon_n}(x_0,y_0) = \left(w_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}) + \sigma - 2k_n, e^{\pi q} t_{\varepsilon_n}(x^{\varepsilon_n}_{k_n}, y^{\varepsilon_n}_{k_n})\right) + o(1),$$

where the term $o(1)$ is uniform on M_C. □

5.3. **After the eggbeater.** We will now estimate the orbit of g_{ε_n} as it gets away from the origin in the outgoing petal. Recall that $\tilde{C} := Ce^{1+\pi|\operatorname{Re} q|}$ and

$$U_n^o(R,\tilde{C}) := \left\{(X,Y) \in -\mathbb{H}_R \times \mathbb{D}(0,\tilde{C}) : |X| < 10k_n\right\}.$$

Lemma 5.13 (Compare to Lemma 5.9)**.** *For n large enough $\Phi_0^o \circ g_{\varepsilon_n}^{n-k_n}(M_C) \subset U_n^o(R,\tilde{C})$ (so in particular $g_{\varepsilon_n}^{n-k_n}(M_C) \subset P^o(r,\tilde{C})$) and*

$$\Phi_{\varepsilon_n}^o \circ g_{\varepsilon_n}^{n-k_n}\big|_{M_C} = \Phi^o \circ g_{\varepsilon_n}^{n-k_n}\big|_{M_C} + o(1),$$

where the convergence of the term $o(1)$ is uniform on M_C.

Proof. Take $(x_0,y_0) \in M_C$ and let $(x_{n-k_n}^{\varepsilon_n}, y_{n-k_n}^{\varepsilon_n}) := g_{\varepsilon_n}^{n-k_n}(x_0,y_0)$. By Proposition 5.12, we have that $(x_{n-k_n}^{\varepsilon_n}, y_{n-k_n}^{\varepsilon_n}) \in \mathcal{R}_n(\hat{C})$ and

$$w_{\varepsilon_n}(x_{n-k_n}^{\varepsilon_n}) - \frac{\pi}{\varepsilon_n} = w_{\varepsilon_n}(x_{k_n}^{\varepsilon_n}) + \sigma - 2k_n + o(1)$$

so using Lemmas 5.8 and 5.9 we get

$$w_{\varepsilon_n}(x_{n-k_n}^{\varepsilon_n}) - \frac{\pi}{\varepsilon_n} = w^\iota(x_0,y_0) + \sigma - k_n + o(1) = -k_n + \mathcal{O}(1)$$

for n large enough, where the term $\mathcal{O}(1)$ is uniform on M_C. By Lemma 4.4 (applied on $\mathcal{R}_n(\hat{C})$) we have

$$x_{n-k_n}^{\varepsilon_n} = -\varepsilon_n \cot(\varepsilon_n w_{\varepsilon_n}(x_{n-k_n}^{\varepsilon_n})) + \mathcal{O}\left(\frac{\log n}{n^{2\gamma}}\right).$$

Therefore for n large enough

$$x_{n-k_n}^{\varepsilon_n} = -\varepsilon_n \cot\left(\pi - \varepsilon_n k_n + \mathcal{O}(\varepsilon_n)\right) + \mathcal{O}\left(\frac{\log n}{n^{2\gamma}}\right) \sim \frac{1}{k_n}$$

uniformly on M_C. Therefore, for n large enough depending only on M_C we have that

$$\operatorname{Re}\left(\frac{-1}{x_{n-k_n}^{\varepsilon_n}}\right) < -R, \quad \text{and} \quad \left|\frac{1}{x_{n-k_n}^{\varepsilon_n}}\right| < 10k_n.$$

Moreover, since

$$t_{\varepsilon_n}(x_{n-k_n}^{\varepsilon_n}, y_{n-k_n}^{\varepsilon_n}) = \frac{y_{n-k_n}^{\varepsilon_n}}{((x_{n-k_n}^{\varepsilon_n})^2 + \varepsilon_n^2)^{\eta/2}} = \frac{y_{n-k_n}^{\varepsilon_n}}{(x_{n-k_n}^{\varepsilon_n})^\eta}\left(1 + \frac{\varepsilon_n^2}{(x_{n-k_n}^{\varepsilon_n})^2}\right)^{-\frac{\eta}{2}}$$

we obtain, using the fact that $x_{n-k_n}^{\varepsilon_n} \sim 1/k_n$ and the definition of $\mathcal{R}_n(\hat{C})$, that

$$\left|\frac{y_{n-k_n}^{\varepsilon_n}}{(x_{n-k_n}^{\varepsilon_n})^\eta}\right| < \hat{C}e^{1/2} = \tilde{C}$$

for n large enough depending only on M_C, so $\Phi_0^o(x_{n-k_n}^{\varepsilon_n}, y_{n-k_n}^{\varepsilon_n}) \in U_n^o(R,\tilde{C})$ and in particular $(x_{n-k_n}^{\varepsilon_n}, y_{n-k_n}^{\varepsilon_n}) \in P^o(r,\tilde{C})$ for n large enough.

Now, if we denote $(w^o(x,y), t^o(x,y)) := \Phi^o(x,y)$, we want to prove that

$$w_{\varepsilon_n}(x_{n-k_n}^{\varepsilon_n}) - \frac{\pi}{\varepsilon_n} = w^o(x_{n-k_n}^{\varepsilon_n}, y_{n-k_n}^{\varepsilon_n}) + o(1)$$

and

$$t_{\varepsilon_n}(x_{n-k_n}^{\varepsilon_n}, y_{n-k_n}^{\varepsilon_n}) = t^o(x_{n-k_n}^{\varepsilon_n}, y_{n-k_n}^{\varepsilon_n}) + o(1).$$

Since $x^{\varepsilon_n}_{n-k_n} \sim 1/k_n$ uniformly on M_C, for n large enough depending only on M_C we have that $\operatorname{Re}(x^{\varepsilon_n}_{n-k_n}/\varepsilon_n) > 0$. Then, using the relation $\arctan z + \arctan(1/z) = \pi/2$ whenever $\operatorname{Re} z > 0$ and the definition of w_{ε_n} we have that for n large enough

$$\begin{aligned} w_{\varepsilon_n}(x^{\varepsilon_n}_{n-k_n}) - \frac{\pi}{\varepsilon_n} &= \frac{1}{\varepsilon_n}\arctan\left(\frac{x^{\varepsilon_n}_{n-k_n}}{\varepsilon_n}\right) - \frac{\pi}{2\varepsilon_n} + \frac{1-a}{2}\log((x^{\varepsilon_n}_{n-k_n})^2 + \varepsilon_n^2) \\ &= -\frac{1}{\varepsilon_n}\arctan\left(\frac{\varepsilon_n}{x^{\varepsilon_n}_{n-k_n}}\right) + \frac{1-a}{2}\log((x^{\varepsilon_n}_{n-k_n})^2 + \varepsilon_n^2). \end{aligned}$$

Since $\frac{\varepsilon_n}{x^{\varepsilon_n}_{n-k_n}} = o(1)$ because $x^{\varepsilon_n}_{n-k_n} \sim 1/k_n$, we get

$$\begin{aligned} w_{\varepsilon_n}(x^{\varepsilon_n}_{n-k_n}) - \frac{\pi}{\varepsilon_n} &= -\frac{1}{\varepsilon_n}\arctan\left(\frac{\varepsilon_n}{x^{\varepsilon_n}_{n-k_n}}\right) + \frac{1-a}{2}\log((x^{\varepsilon_n}_{n-k_n})^2 + \varepsilon_n^2) \\ &= -\frac{1}{x^{\varepsilon_n}_{n-k_n}} + (1-a)\log(x^{\varepsilon_n}_{n-k_n}) + o(1) \\ &= w^o(x^{\varepsilon_n}_{n-k_n}, y^{\varepsilon_n}_{n-k_n}) + o(1). \end{aligned}$$

using in the second line that $k_n^3/n^2 = o(1)$ and in the last line the asymptotic expansion of w^o. Similarly, by the computation above,

$$t_{\varepsilon_n}(x^{\varepsilon_n}_{n-k_n}, y^{\varepsilon_n}_{n-k_n}) = \frac{y^{\varepsilon_n}_{n-k_n}}{(x^{\varepsilon_n}_{n-k_n})^\eta}\left(1 + \frac{\varepsilon_n^2}{(x^{\varepsilon_n}_{n-k_n})^2}\right)^{-\frac{\eta}{2}} = t^o(x^{\varepsilon_n}_{n-k_n}, y^{\varepsilon_n}_{n-k_n}) + o(1)$$

using in the last line the asymptotic expansion of t^o and the fact that $t^o(x^{\varepsilon_n}_{n-k_n}, y^{\varepsilon_n}_{n-k_n}) = \mathcal{O}(1)$ because $(x^{\varepsilon_n}_{n-k_n}, y^{\varepsilon_n}_{n-k_n}) \in \mathcal{R}_n(\hat{C})$. □

Lemma 5.14 (Compare to Lemma 5.8)**.** *There exists $N_1 > 0$ such that for n large enough $\Phi^o_0 \circ g^{n-N_1}_{\varepsilon_n}(M_C) \subset U^o_n(r, C)$ (so in particular $g^{n-N_1}_{\varepsilon_n}(M_C) \subset P^o(r, \widetilde{C})$) and*

$$\Phi^o \circ g^{n-N_1}_{\varepsilon_n}\big|_{M_C} = A_{k_n-N_1,0} \circ \Phi^o \circ g^{n-k_n}_{\varepsilon_n}\big|_{M_C} + o(1),$$

where the convergence of the term $o(1)$ is uniform on M_C.

Proof. From the asymptotic expansion of $(\Phi^o)^{-1}$ (see Lemma 3.7), Proposition 3.3 and the definitions of Φ^o_0 and $U^o_n(R, \tilde{C})$, there exists a constant $R_0 > 0$ such that

(18) $$(\Phi^o)^{-1}\{(X,Y) \in -\mathcal{H}_{R_0,2} \times \mathbb{D}(0, Ce^{1/2+\pi|\operatorname{Re} q|}) : |X| < 5k_n\} \subset (\Phi^o_0)^{-1}(U^o_n(R, \tilde{C})),$$

where $\mathcal{H}_{R_0,2} := \{X \in \mathbb{H}_{R_0} : |\operatorname{Im} X| < 2\operatorname{Re} X\}$. Up to increasing R_0 if necessary, we assume that $2R_0 \geq \widetilde{K} + 1$, where $\widetilde{K} := \max\{|w^\iota(x,y)| : (x,y) \in M_C\}$. We fix some integer $N_1 > \widetilde{K} + \operatorname{Re}\sigma + 2 + R_0$.

Let us first prove that there exists a constant $K_3 > 0$ such that for n large enough and for all $(x_0, y_0) \in (\Phi^o_0)^{-1}(U^o_n(R, \tilde{C}))$ such that $(x^{\varepsilon_n}_1, y^{\varepsilon_n}_1) := g_{\varepsilon_n}(x_0, y_0) \in P^o((2(R - 3/2))^{-1}, \widetilde{C} + 1)$ we have

(19) $$\|\Phi^o(x^{\varepsilon_n}_1, y^{\varepsilon_n}_1) - \Phi^o(x_0, y_0) - (1, 0)\| \leq K_3\left(\frac{1}{n} + \frac{1}{|x_0|^2 n^2}\right).$$

The computation is analogous to the one in the proof of Lemma 5.8. Fix (x_0, y_0) such that $(X_0, Y_0) := \Phi^o_0(x_0, y_0) \in U^o_n(r, \tilde{C})$ and $(x^{\varepsilon_n}_1, y^{\varepsilon_n}_1) \in P^o((2(R - 3/2))^{-1}, \widetilde{C} + 1)$. Set $\Phi^o_G := \Phi^o \circ (\Phi^o_0)^{-1} : -\mathbb{H}_{R-5/2} \times \mathbb{D}(0, \widetilde{C} + 2) \to \mathbb{C}^2$, so

$$\Phi^o_G(X^o_1, Y^o_1) = \Phi^o_G(X_0, Y_0) + (1, 0),$$

where $(X^o_1, Y^o_1) := \Phi^o_0 \circ g \circ (\Phi^o_0)^{-1}(X_0, Y_0)$. Then

$$\Phi^o(x^{\varepsilon_n}_1, y^{\varepsilon_n}_1) - \Phi^o(x, y) - (1, 0) = \Phi^o_G(X^{\varepsilon_n,o}_1, Y^{\varepsilon_n,o}_1) - \Phi^o_G(X^o_1, Y^o_1)$$

where $(X_1^{\varepsilon_n,o}, Y_1^{\varepsilon_n,o}) := \Phi_0^o(x_1^{\varepsilon_n}, y_1^{\varepsilon_n})$. By Proposition 3.3 and condition (R_4), there exists a holomorphic map $v : -\mathbb{H}_{R-5/2} \times \mathbb{D}(0, \widetilde{C}+2) \to \mathbb{C}^2$ such that

$$\Phi_G^o(X,Y) = (X - (1-a)\log(-X), Y) + v(X,Y),$$

with $\|v(X,Y)\| \le 1$ for all $(X,Y) \in -\mathbb{H}_{R-5/2} \times \mathbb{D}(0,\widetilde{C}+2)$. By Cauchy estimates, we have

$$\|\partial_X v(X,Y)\| \le 1 \quad \text{and} \quad \|\partial_Y v(X,Y)\| \le 1$$

for all $(X,Y) \in -\mathbb{H}_{R-3/2} \times \mathbb{D}(0,\widetilde{C}+1)$ and hence, by the mean value inequality,

$$\|v(X_1^{\varepsilon_n,o}, Y_1^{\varepsilon_n,o}) - v(X_1^o, Y_1^o)\| \le |X_1^{\varepsilon_n,o} - X_1^o| + |Y_1^{\varepsilon_n,o} - Y_1^o|.$$

On the other hand, also by the mean value inequality, we have

$$|\log(-X_1^{\varepsilon_n,o}) - \log(-X_1^o)| < |X_1^{\varepsilon_n,o} - X_1^o|.$$

Therefore

$$\begin{aligned}\|\Phi_G^o(X_1^{\varepsilon_n,o}, Y_1^{\varepsilon_n,o}) - \Phi_G^o(X_1^o, Y_1^o)\| &\le |X_1^{\varepsilon_n,o} - X_1^o| + |1-a||\log(-X_1^{\varepsilon_n,o}) - \log(-X_1^o)| \\ &\quad + |Y_1^{\varepsilon_n,o} - Y_1^o| + \|v(X_1^{\varepsilon_n,o}, Y_1^{\varepsilon_n,o}) - v(X_1^o, Y_1^o)\| \\ &\le (2 + |1-a|)\left(|X_1^{\varepsilon_n,o} - X_1^o| + |Y_1^{\varepsilon_n,o} - Y_1^o|\right).\end{aligned}$$

Then using Lemma 5.4 we obtain (19) for n large enough, with $K_3 = 2(2+|1-a|)C^o$.

Now take $(x_0, y_0) \in M_C$. We will prove by induction on j that for all n large enough and for all $n - k_n \le j \le n - N_1$ we have

(1) $\Phi_0^o(x_j^{\varepsilon_n}, y_j^{\varepsilon_n}) \in U_n^o(R, \tilde{C})$

(2) $\|\Phi^o(x_j^{\varepsilon_n}, y_j^{\varepsilon_n}) - \Phi^o(x_{n-k_n}^{\varepsilon_n}, y_{n-k_n}^{\varepsilon_n}) - (j-(n-k_n), 0)\| \le K_3 \displaystyle\sum_{k=n-k_n}^{j-1} \left(\frac{1}{n} + \frac{1}{|x_k^{\varepsilon_n}|^2 n^2}\right)$.

For $j = n - k_n$, the first statement follows from Lemma 5.13, and there is nothing to prove for the second one. Let $n - k_n \le j < n - N_1$ such that (1) and (2) hold for all $n - k_n \le k \le j$ and denote $(X_j^{\varepsilon_n,o}, Y_j^{\varepsilon_n,o}) := \Phi_0^o(x_j^{\varepsilon_n}, y_j^{\varepsilon_n})$. By equation (15) we have

$$|X_{j+1}^{\varepsilon_n,o} - X_j^{\varepsilon_n,o} - 1| < \frac{1}{2} \quad \text{and} \quad |Y_{j+1}^{\varepsilon_n,o} - Y_j^{\varepsilon_n,o}| < M\left(\frac{1}{n} + \frac{1}{|X_j^{\varepsilon_n,o}|^2}\right).$$

Therefore, by definition of $U_n^o(R, \tilde{C})$, we have for n large enough

$$\operatorname{Re}(X_{j+1}^{\varepsilon_n,o}) < -R + \frac{3}{2}, \quad |Y_{j+1}^{\varepsilon_n,o}| < \widetilde{C} + 1.$$

It follows that $(x_{j+1}^{\varepsilon_n}, y_{j+1}^{\varepsilon_n}) \in P^o((2(R-3/2))^{-1}, \widetilde{C}+1)$ so using (19) and the induction hypothesis we get

$$\|\Phi^o(x_{j+1}^{\varepsilon_n}, y_{j+1}^{\varepsilon_n}) - \Phi^o(x_{n-k_n}^{\varepsilon_n}, y_{n-k_n}^{\varepsilon_n}) - (j+1-(n-k_n), 0)\| \le K_3 \sum_{k=n-k_n}^{j} \left(\frac{1}{n} + \frac{1}{|x_k^{\varepsilon_n}|^2 n^2}\right)$$

and condition (2) is proved.

By definition of $U_n^o(R, \tilde{C})$, we have $|x_k^{\varepsilon_n}|^{-1} < 10k_n$ for all $n - k_n \le k \le j$, therefore

$$\sum_{k=n-k_n}^{j} \left(\frac{1}{n} + \frac{1}{|x_k^{\varepsilon_n}|^2 n^2}\right) \le \frac{k_n}{n} + \frac{100k_n^3}{n^2} = o(1).$$

Therefore, by Lemma 5.13 and Proposition 5.12 we have for n large enough

$$\Phi^o(x_{j+1}^{\varepsilon_n}, y_{j+1}^{\varepsilon_n}) = \Phi_{\varepsilon_n}^\iota(x_{k_n}^{\varepsilon_n}, y_{k_n}^{\varepsilon_n}) + (\sigma + j + 1 - n - k_n, 0) + o(1)$$

so by Lemmas 5.9 and 5.8

$$\Phi^o(x_{j+1}^{\varepsilon_n}, y_{j+1}^{\varepsilon_n}) = (w^\iota(x_0, y_0) + \sigma + j + 1 - n, e^{\pi q} t^\iota(x_0, y_0)) + o(1).$$

Let $(w^o(x, y), t^o(x, y)) := \Phi^o(x, y)$. We then have, for n large enough,

$$\begin{aligned}\mathrm{Re}\,(w^o(x_{j+1}^{\varepsilon_n}, y_{j+1}^{\varepsilon_n})) &\leq \mathrm{Re}\,(w^\iota(x_0, y_0)) + \mathrm{Re}\,\sigma + j + 2 - n \leq \widetilde{K} + \mathrm{Re}\,\sigma + j + 2 - n \\ &\leq \widetilde{K} + \mathrm{Re}\,\sigma + 2 - N_1 < -R_0\end{aligned}$$

and

$$|\mathrm{Im}(w^o(x_{j+1}^{\varepsilon_n}, y_{j+1}^{\varepsilon_n}))| \leq |\mathrm{Im}(w^\iota(x_0, y_0))| + 1 \leq \widetilde{K} + 1 \leq 2R_0 \leq 2\mathrm{Re}\,(-w^o(x_{j+1}^{\varepsilon_n}, y_{j+1}^{\varepsilon_n}))$$

so $\Phi^o(x_{j+1}^{\varepsilon_n}, y_{j+1}^{\varepsilon_n}) \in -\mathcal{H}_{R_0,2}$. Similarly, for n large enough,

$$|w^o(x_{j+1}^{\varepsilon_n}, y_{j+1}^{\varepsilon_n})| \leq |w^\iota(x_0, y_0)| + |\sigma| + n - j \leq \widetilde{K} + |\sigma| + n - j < 5k_n.$$

Finally, using the fact that $|y_0(-x_0)^{-\eta}| \leq C - 1$ by definition of M_C and $|t^\iota(x_0, y_0) - y_0(-x_0)^{-\eta}| < 1$ by condition (R_4), we have that $|t^\iota(x_0, y_0)| \leq C$ and then for n large enough

$$|t^o(x_{j+1}^{\varepsilon_n}, y_{j+1}^{\varepsilon_n})| \leq Ce^{1/2+\pi|\mathrm{Re}\,q|}.$$

Therefore, by (18), $(x_{j+1}^{\varepsilon_n}, y_{j+1}^{\varepsilon_n}) \in U_n^o(R, \tilde{C})$ and (1) is proved. Taking $j = n - N_1$, the Lemma 5.14 is proved. ☐

5.4. **Conclusion.**

Proof of Theorem 3. We have

$$\begin{aligned}\Phi^o \circ g_{\varepsilon_n}^{n-N_1}\big|_{M_C} &= A_{k_n-N_1,0} \circ \Phi^o \circ g_{\varepsilon_n}^{n-n_k}\big|_{M_C} + o(1) \quad \text{by Lemma 5.14} \\ &= A_{k_n-N_1,0} \circ \Phi_{\varepsilon_n}^o \circ g_{\varepsilon_n}^{n-k_n}\big|_{M_C} + o(1) \quad \text{by Lemma 5.13} \\ &= A_{k_n-N_1,0} \circ A_{\sigma-2k_n,q} \circ \Phi_{\varepsilon_n}^\iota \circ g_{\varepsilon_n}^{k_n}\big|_{M_C} + o(1) \quad \text{by Proposition 5.12} \\ &= A_{k_n-N_1,0} \circ A_{\sigma-2k_n,q} \circ \Phi^\iota \circ g_{\varepsilon_n}^{k_n}\big|_{M_C} + o(1) \quad \text{by Lemma 5.9} \\ &= A_{k_n-N_1,0} \circ A_{\sigma-2k_n,q} \circ A_{k_n,0} \circ \Phi^\iota\big|_{M_C} + o(1) \quad \text{by Lemma 5.8} \\ &= A_{\sigma-N_1,q} \circ \Phi^\iota\big|_{M_C} + o(1)\end{aligned}$$

where the convergence of the terms $o(1)$ is uniform in M_C. Therefore

$$g_{\varepsilon_n}^{n-N_1}\big|_{M_C} = (\Phi^o)^{-1} \circ A_{\sigma-N_1,q} \circ \Phi^\iota\big|_{M_C} + o(1)$$

where the convergence of the term $o(1)$ is uniform on M_C. This proves that $g_{\varepsilon_n}^{n-N_1} \to \mathcal{L}_{\sigma-N_1,q}$ uniformly on M_C, where $\mathcal{L}_{\sigma-N_1,q} := (\Phi^o)^{-1} \circ A_{\sigma-N_1,q} \circ \Phi^\iota$.

Let us now explain how we can deduce the same convergence statement first for all $(x, y) \in L$, then for all $(x, y) \in K$. First, assume without loss of generality that the domain U on which the maps g_{ε_n} are defined is a bi-disk $\mathbb{D}(0, \delta)^2$.

Let $\pi_i : \mathbb{C}^2 \to \mathbb{C}$ $(1 \leq i \leq 2)$ denote the projections on each coordinate of $\mathbb{C}^2$. Let $x \in \pi_1(L)$. Recall that r and C are the constants introduced in Definition 5.5. Let

$$M_x(C) := \{y \in \mathbb{C} : (x, y) \in M_C\} = \overline{\mathbb{D}}\,(0, (C-1)|(-x)^\eta|) \setminus \mathbb{D}(0, (C-1)^{-1}|(-x)^\eta|)$$

and

$$N_x(C) := \{y \in \mathbb{C} : (x, y) \in N_C\} = \overline{\mathbb{D}}\,(0, (C-1)|(-x)^\eta|)\,.$$

Let us prove inductively on $0 \leq j \leq n - N_1$ that $g_{\varepsilon_n}^j(N_C) \subset U$. For $j = 0$, there is nothing to prove, since $N_C \subset P^\iota(r, C) \subset U$. Let $0 \leq j \leq n - N_1 - 1$ be such that $g_{\varepsilon_n}^j(N_C) \subset U$, so that $g_{\varepsilon_n}^{j+1}$ is well-defined on N_C.

By the maximum principle, for all $x \in \pi_1(L) = \pi_1(N_C)$, we have

$$\sup_{y \in N_x(C)} |\pi_1 \circ g_{\varepsilon_n}^{j+1}(x,y)| \leq \max_{y \in \partial N_x(C)} |\pi_1 \circ g_{\varepsilon_n}^{j+1}(x,y)| \leq \max_{y \in M_x(C)} |\pi_1 \circ g_{\varepsilon_n}^{j+1}(x,y)| \leq \delta.$$

Similarly, we also have

$$\sup_{y \in N_x(C)} |\pi_2 \circ g_{\varepsilon_n}^{j+1}(x,y)| \leq \max_{y \in \partial N_x(C)} |\pi_2 \circ g_{\varepsilon_n}^{j+1}(x,y)| \leq \max_{y \in M_x(C)} |\pi_2 \circ g_{\varepsilon_n}^{j+1}(x,y)| \leq \delta.$$

Therefore, $g_{\varepsilon_n}^j$ is well-defined on N_C for every $0 \leq j \leq n - N_1$. Moreover, $(g_{\varepsilon_n}^{n-N_1} : N_C \to \mathbb{C}^2)_{n \geq 0}$ is a normal family in the sense of Montel. Let $(n_k)_{k\geq 0}$ be any extracted sequence such that $g_{\varepsilon_{n_k}}^{n_k - N_1}$ converges on N_C to some holomorphic function G. We have proved that $G = \mathcal{L}_{\sigma - N_1, q}$ on M_C, therefore, by the identity principle, $G = \mathcal{L}_{\sigma-N_1,q}$ on all of N_C. Since this is true for any converging subsequence, we conclude that $g_{\varepsilon_n}^{n-N_1} \to \mathcal{L}_{\sigma-N_1,q}$ on all of N_C, hence on L (since $L \subset N_C$).

Let us now prove that $g_{\varepsilon_n}^{n-N} \to \mathcal{L}_{\sigma-N,q}$ on K, where $N := N_1 - n_0$. Indeed, by the definition of L, we have that for all n large enough $g_{\varepsilon_n}^{n_0}(K) \subset L$; therefore, if $(x,y) \in K$ then

$$\lim_{n\to\infty} g_{\varepsilon_n}^{n-N_1+n_0}(x,y) = \lim_{n\to\infty} g_{\varepsilon_n}^{n-N_1} \circ g_{\varepsilon_n}^{n_0}(x,y) = \mathcal{L}_{\sigma-N_1,q} \circ g^{n_0}(x,y) = \mathcal{L}_{\sigma-N_1+n_0,q}(x,y),$$

using in the last two equalities the fact that $g_{\varepsilon_n}^{n_0}$ converges uniformly to g^{n_0} on K and that $\mathcal{L}_{\sigma+1,q} = g \circ \mathcal{L}_{\sigma,q} = \mathcal{L}_{\sigma,q} \circ g$. □

6. Proof of Corollaries 1 and 2

The goal of this section is to prove Corollaries 1 and 2. We start with a general observation about Fatou coordinates for globally defined maps.

Proposition 6.1. *Assume that the germ $g := g_0$ from Theorem 3 extends to a holomorphic self-map of a complex manifold. Then the incoming Fatou coordinate Φ^ι extends to a holomorphic map on the parabolic basin $\mathcal{B}_v$ associated to $v = (1,0)$, and the map $(\Phi^o)^{-1}$ extends to a holomorphic map Ψ^o on $\mathbb{C}^2$. Moreover, if f has discrete fibers, then so do Φ^ι and Ψ^o.*

Proof. Recall that $\mathcal{B}_v = \bigcup_{C>0} \bigcup_{n\geq 0} g^{-n}(P^\iota(r,C))$ for any $0 < r \leq r^\iota(C)$. Then, using the functional equation $\Phi^\iota \circ g = \Phi^\iota + (1,0)$ we can extend Φ^ι to $\mathcal{B}$ by

$$\Phi^\iota(z) := \Phi^\iota \circ g^n(z) - (n,0)$$

for any n such that $g^n(z) \in P^\iota(r,C)$. Moreover, by Proposition 3.3 we have that for any $C > 0$ there exists $r = r^o(C) > 0$ such that

$$\{X \in -\mathbb{H}_{r^{-1}} : |\mathrm{Im}(X)| < 2|\mathrm{Re}\,(X)|\} \times \mathbb{D}(0, C-1) \subset \Phi^o(P^o(r,C))$$

so we can extend $(\Phi^o)^{-1}$ to a map $\Psi^o : \mathbb{C} \times \mathbb{D}(0,C-1) \to \mathbb{C}$ by

$$\Psi^o(X,Y) := g^n \circ (\Phi^o)^{-1}(X-n,Y)$$

for any $n \in \mathbb{N}$ such that $(X-n,Y) \in -\mathbb{H}_{r^{-1}} \times \mathbb{D}(0,C-1)$. Since we can do this for any $C > 0$, the map Ψ^o extends to a holomorphic map on $\mathbb{C}^2$.

Let us now prove the claim about the discreteness of fibers. Take $p \in \mathcal{B}$ and let $F_p := (\Phi^\iota)^{-1}(\{\Phi^\iota(p)\})$ be the corresponding fiber. Let n be large enough such that $g^n(p) \in P^\iota(r,C)$ for some $C > 0$ and $r \leq r^\iota(C)$. Then

$$\Phi^\iota \circ g^n(F_p) = \Phi^\iota(F_p) + (n,0) = \{\Phi^\iota(p)\} + (n,0).$$

Since Φ^ι is injective on $P^\iota(r,C)$, the point $f^n(p)$ is isolated in $f^n(F_p)$. Since g (hence g^n) has discrete fibers, p must also be isolated in F_p.

Similarly, if $(X_0,Y_0) \in \mathbb{C}^2$ then there exists $n \in \mathbb{N}$ such that $(X_0 - n, Y_0) \in \Phi^o(P^o(r,C))$ for some $C > 0$ and $r \leq r^o(C)$. In order to prove that (X_0,Y_0) is isolated in its fiber $F_{(X_0,Y_0)} := (\Psi^o)^{-1}(\{\Psi^o(X_0,Y_0)\})$, it suffices to prove that $(X_0 - n, Y_0)$ is isolated in $F_{(X_0,Y_0)} - (n,0)$. By the relation $g^n \circ \Psi^o(X-n,Y) = \Psi^o(X,Y)$, we have that $F_{(X_0,Y_0)} - (n,0) = (\Psi^o)^{-1}(E_n)$, where $E_n := g^{-n}(\{\Psi^o(X_0,Y_0)\})$. Since g (hence g^n) has discrete fibers, E_n is discrete. Since $(X_0 - n, Y_0) \in \Phi^o(P^o(r,C))$ and Φ^o is injective, $(X_0 - n, Y)$ is therefore isolated in $(\Psi^o)^{-1}(E_n) = F_{(X_0,Y_0)} - (n,0)$. Thus $F_{(X_0,Y_0)}$ is indeed discrete. □

Next, we prove the following Lemma:

Lemma 6.2. *Let f be an endomorphism of $\mathbb{P}^2$ of algebraic degree d, satisfying (H_1) and such that $d > \operatorname{Re}\alpha + 1$. Then for any $q \in \mathbb{C}$ there exists a family $(f_\varepsilon)_{\varepsilon\in\mathbb{D}}$ of endomorphisms of degree d satisfying $(H_1) - (H_3)$, with $f_0 = f$ and q as in Theorem 2.*

Proof. We work in some affine coordinates in which the fixed point of f is the origin and in which the non-degenerate characteristic direction is $[1:0]$ so the non-singular formal invariant curve $\mathcal{C}$ admits a parametrization of the form $\gamma(t) = (t, \zeta(t))$. As in Section 2, we let $\Psi(x,y) := (x, y - J_{m+1}\zeta(x))$, where $J_{m+1}\zeta$ is the jet of order $m+1$ of ζ, where $m := \lfloor \operatorname{Re}\alpha \rfloor + 1$ of ζ, and we let $g := \Psi \circ f \circ \Psi^{-1}$. Then g is of the form

$$g(x,y) = (x + x^2 a(x) + y b(x,y), y(1 + c(x,y)) + \mathcal{O}(x^{m+2})),$$

where $a(0) = 1$ and $b(0,0) = c(0,0) = 0$. We now let

$$g_\varepsilon(x,y) := (x + (x^2 + \varepsilon^2) a(x) + y b(x,y), y(1 + c(x,y) + q\varepsilon) + \mathcal{O}(x^{m+2})),$$

and $\widetilde{f_\varepsilon} := \Phi^{-1} \circ g_\varepsilon \circ \Phi$. By construction, $(\widetilde{f_\varepsilon})_{\varepsilon\in\mathbb{D}}$ satisfies $(H_1) - (H_3)$ and $f_0 = f$. However, the maps $\widetilde{f_\varepsilon}$ need not be endomorphisms of $\mathbb{P}^2$, since Ψ is not an automorphism of $\mathbb{P}^2$; we will therefore need to make one last modification. Let us write $\widetilde{f_\varepsilon}(x,y) = f(x,y) + \varepsilon h(\varepsilon,x,y)$, where $h(\varepsilon,x,y) = \sum_{i,j,k\in\mathbb{N}} a_{i,j,k}\varepsilon^i x^j y^k$ is a holomorphic map defined on some neighborhood of 0 in $\mathbb{C}^3$. Let $h_{m+1}(\varepsilon,x,y) := \sum_{i,j,k\leq m+1} a_{i,j,k}\varepsilon^i x^j y^k$ be the jet of order $m+1$ of h and $f_\varepsilon(x,y) := f(x,y) + \varepsilon h_{m+1}(\varepsilon,x,y)$. It is not difficult to see that conditions $(H_1) - (H_3)$ only depend on the jet of order $m+1$ of f_ε at $(0,0,0)$; therefore, the family of maps $(f_\varepsilon)_{\varepsilon\in\mathbb{D}}$ still satisfies $(H_1) - (H_3)$, and since $m < d$ by assumption, the maps f_ε are endomorphisms of $\mathbb{P}^2$ for all ε in a neighborhood of 0.

Finally, for the fixed points $z_1(\varepsilon) = 1 + 2i\varepsilon + \mathcal{O}(\varepsilon^2)$ and $z_2(\varepsilon) = 1 - 2i\varepsilon + \mathcal{O}(\varepsilon^2)$ we have that

$$\operatorname{Jac} f_\varepsilon(z_1(\varepsilon)) = \begin{pmatrix} 1 + 2i\varepsilon + \mathcal{O}(\varepsilon^2) & \mathcal{O}(\varepsilon) \\ \mathcal{O}(\varepsilon^2) & 1 + q\varepsilon + i\eta\varepsilon + \mathcal{O}(\varepsilon^2) \end{pmatrix}$$

and

$$\operatorname{Jac} f_\varepsilon(z_2(\varepsilon)) = \begin{pmatrix} 1 - 2i\varepsilon + \mathcal{O}(\varepsilon^2) & \mathcal{O}(\varepsilon) \\ \mathcal{O}(\varepsilon^2) & 1 + q\varepsilon - i\eta\varepsilon + \mathcal{O}(\varepsilon^2) \end{pmatrix}.$$

so clearly q is as in Theorem 2 by Lemma 2.4. □

We now complete the proof of Corollary 1, which is essentially the same done by Bianchi in [Bia19b] or the one due to Lavaurs ([Lav89]) in dimension 1:

Proof of Corollary 1. If we take $q \in \mathbb{C}$, the existence of a family $(f_\varepsilon)_{\varepsilon\in\mathbb{D}}$ of endomorphisms of degree d satisfying $(H_1) - (H_3)$ and such that $q := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \frac{\rho_N^1(\varepsilon) + \rho_N^2(\varepsilon) - 2}{\rho_T^1(\varepsilon) + \rho_T^2(\varepsilon)}$ is guaranteed by Lemma 6.2. Now, take $\sigma \in \mathbb{C}$ and $z_0 \in J_1(f, \mathcal{L}_{\sigma,q})$. Without loss of generality, assume that $z_0 = \mathcal{L}_{\sigma,q}^N(p_0)$, for some $N \in \mathbb{N}$ and some $p_0 \in J_1(f)$. Consider a sequence (ε_n) as in Theorem 2. We will find a sequence $z_n \in J_1(f_{\varepsilon_n})$ such that $\lim_{n\to\infty} z_n = z_0$.

By lower semicontinuity of $\varepsilon \mapsto J_1(f_\varepsilon)$, there exists a sequence $p_n \in J_1(f_{\varepsilon_n})$ such that $\lim_{n\to\infty} p_n = p_0$. Let $z_n := f_{\varepsilon_n}^{nN}(p_n) \in J_1(f_{\varepsilon_n})$: by Theorem 2, we have $\lim_{n\to\infty} z_n = \mathcal{L}_{\sigma,q}^N(p_0) = z_0$, and we are done. □

We can now prove Corollary 2. The idea is that thanks to the Main Theorem, we can find a suitable sequence of perturbations f_{ε_n} such that $f_{\varepsilon_n}^n$ maps a point in $\mathcal{B}$ to a repelling periodic point in $J_2(f_{\varepsilon_n})$. Since $J_2(f_{\varepsilon_n})$ is completely invariant, this will mean that $\mathcal{B}$ contains some points in $J_2(f_{\varepsilon_n})$ for all n large enough, thus proving the lack of upper semicontinuity of $\varepsilon \mapsto J_2(f_\varepsilon)$. There is however a technical issue: contrary to the case of rational maps on $\mathbb{P}^1$, not all repelling periodic points belong to $J_2(f_\varepsilon)$.

Proof of Corollary 2. Let $r(0)$ denote a repelling periodic point for f contained in $\Psi^o(\mathbb{C}^2)$. Let $(X_1, Y_1) \in \mathbb{C}^2$ be such that $\Psi^o(X_1, Y_1) = r(0)$. Let $z_0 \in \mathcal{B}$ be such that $\Phi^\iota(z_0) \in \mathbb{C} \times \{0\}$ if $Y_1 = 0$, and $\Phi^\iota(z_0) \in \mathbb{C} \times \mathbb{C}^*$ if $Y_1 \neq 0$. Then there exists a unique $(\sigma, q) \in \mathbb{C}^2$ such that $A_{\sigma,q} \circ \Phi^\iota(z_0) = (X_1, Y_1)$, or in other words, $\mathcal{L}_{\sigma,q}(z_0) = r(0)$. Let $(f_\varepsilon)_{\varepsilon \in \mathbb{D}}$ be the corresponding family of perturbations of f constructed in Lemma 6.2 for that value of q. Let $r(\varepsilon)$ be the repelling periodic point of f_ε, which moves holomorphically with ε for $|\varepsilon|$ small. For $|\varepsilon|$ small, we have $r(\varepsilon) \in J_2(f_\varepsilon)$ by [Bia19a, Lemma 4.9].

Let $\sigma_0 \in \mathbb{C}$ be given by Theorem 2, and let $\varepsilon_n \to 0$ be such that

$$\frac{2i\pi}{\rho_T^1(\varepsilon_n) - 1} = n - (\sigma - \sigma_0) + o(1).$$

Let U be a neighborhood of z_0 in $\mathcal{B}$, small enough such that $\mathcal{L}_{\sigma,q}(z) = r(0)$ and $z \in U$ implies that $z = z_0$. Then the maps $G_n(z) = f_{\varepsilon_n}^n(z) - r(\varepsilon_n)$ converge locally uniformly to $G(z) = \mathcal{L}_{\sigma,q}(z) - r(0)$ by Theorem 2. Since $(0,0)$ is an isolated point in $G^{-1}(\{(0,0)\})$ by Proposition 6.1, there is a sequence $z_n \to (0,0)$ such that $G_n(z_n) = (0,0)$. By the backward invariance of the small Julia set, $z_n \in J_2(f_{\varepsilon_n})$, but $U \cap J_2(f) = \emptyset$ since $U \subset \mathcal{B}$. □

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