BIFURCATION LOCI OF FAMILIES OF FINITE TYPE MEROMORPHIC MAPS

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ABSTRACT. We show that $\mathcal{J}-$ stability is open and dense in natural families of finite type meromorphic maps, that is, meromorphic maps of one complex variable with a finite number of singular values. This extends the results of Mañé-Sad-Sullivan [MSS83] for rational maps of the Riemann sphere and those of Eremenko and Lyubich [EL92] for entire maps of finite type of the complex plane. This result is obtained as a consequence of a detailed study of a new type of bifurcation that arises with the presence poles in addition to essential singularities (namely periodic orbits exiting the domain of definition of the map along a parameter curve), and in particular its relation with the bifurcations in the dynamics of singular values. The presence of these new bifurcation parameters require essentially different methods to those used in previous work for rational or entire maps.

1. Introduction

Structural stability is a key concept in dynamical systems which is attributed to Andronov and Pontryagin in 1937 and even further to Poincaré in the early 1880's. Roughly speaking, a dynamical system is structurally stable (also called *robust*) if its qualitative properties do not change under small perturbations. Somewhat more precisely, a discrete dynamical system $f : X \to X$ (in a certain class of regularity $\mathcal{C}$) is structurally stable, if all maps g sufficiently close to f in $\mathcal{C}$ are topologically conjugate to f, that is, if there exist a homeomorphism $\varphi : X \to X$ such that $\varphi \circ f = g \circ \varphi$, and φ depends continuously on g.

The problem of density of structurally stable systems in the appropriate class is subtle and has been around for a long time. While this property is generically true for $\mathcal{C}^r$ maps, $r \geq 1$, in real dimension one [KSvS07], works of Smale, Williams and Newhouse among others (see e.g. [Sma66, CS70]) concluded that structurally stable systems are not dense in the space of diffeomorphisms on manifolds of real dimension larger or equal than 2.

In the context of holomorphic dynamics, a more natural but closely related notion is $\mathcal{J}$-*stability*, which informally speaking means structural stability in restriction to the Julia set (see Section 2.2 for a precise definition). The seminal work of Mañé, Sad and Sullivan [MSS83] and Lyubich [Lyu84] showed that $\mathcal{J}$-stable systems form an open and dense set in the space of holomorphic maps on the Riemann sphere (i.e. rational maps). Their results therefore solved the problem of density of $\mathcal{J}-$stability for holomorphic maps on *compact* Riemann surfaces.

Date: September 19, 2023.

2020 *Mathematics Subject Classification.* Primary 37F46, 30D05, 37F10, 30D30, 37F44.

† Partially supported by the grant ANR JCJC Fatou ANR-17- CE40-0002-01 and the grant ANR PADAWAN 21-CE-40-0012-01.

* Partially supported by the Indam group GNAMPA, and by PRIN 2017, Real and Complex Manifolds: Topology, Geometry and holomorphic dynamics.

‡ Partially supported by grants PID2020-118281GB-C32 and CEX2020-001084-M (Maria de Maeztu Excellence program) from the Spanish state research agency, and ICREA Acadèmia 2020 from the Catalan government.

One of the key factors in the proof is the renowned $\lambda-$lemma, tying $\mathcal{J}$-stability to the holomorphic movement of periodic orbits which, in a compact manifold, can only fail when periodic orbits collide in a parabolic cycle. With this tool in hand, they provided a complete set of equivalences between possible notions of stability in parameter space: $\mathcal{J}$-stability, stability of critical orbits and stability of periodic cycles. The subset of functions where these properties fail to hold is called the *bifurcation locus*. The study of bifurcation loci is related to some of the deepest questions in holomorphic dynamics: for instance, the bifurcation locus of the quadratic family $\{z \mapsto z^2 + \lambda\}_{\lambda\in\mathbb{C}}$ is exactly the boundary of the famous Mandelbrot set, whose fine description is the object of the principal conjecture in the field (MLC conjecture).

When trying to extend these results to maps on non-compact manifolds (like the complex plane) one must deal with an additional possibility for the failure of periodic orbits being analytically continued, namely the possibility of periodic orbits *exiting the domain* at a certain parameter value. As an example, one can observe this new type of bifurcation occurring at $\lambda_0 = 0$ for the family $f_\lambda(z) = z + \lambda + e^z$, where the fixed points *disappear to infinity* (the essential singularity) when considering curves of parameters converging to 0. Eremenko and Lyubich [EL92] showed that this phenomenon does *not* occur for entire maps of the complex plane with a finite number of *singular values* (points where some branch of the inverse fails to be well-defined), known as *finite type* entire maps. Consequently, they were able to conclude that $\mathcal{J}-$stability is also dense in this class of functions.

In the presence of poles, that is, in the context of finite type meromorphic maps, simple examples like $\lambda \tan z$ for $\lambda_0 = \pi/2$, show that this new type of bifurcations of cycles disappearing to infinity *do* occur and hence obstruct most of the arguments used for the Stability Theorem in [MSS83, Lyu84].

The goal of this paper is to prove that $\mathcal{J}$-stability is dense also in this setting, and we accomplish it by performing a detailed analysis of the new type of bifurcation. In particular we will see how these bifurcation parameters relate to the stability of singular orbits (Theorems A and B) and to parabolic parameters (Theorem D), resulting in a Stability Theorem (Theorem E), from which we conclude that the bifurcation locus has empty interior (corollary F).

One may wonder whether our results might extend beyond finite type meromorphic maps.A simple example (see Section 6) shows that, in general, density of $\mathcal{J}$-stability fails for natural families of entire maps which are not of finite type. Other results in [ER] (unpublished) provide an example of a family of maps with an infinite (but bounded) set of singular values (hence not of finite type) and infinitely many attractors, in the spirit of the Newhouse example [New74]. Nevertheless, results analogous to the ones presented here are likely to hold for general finite type maps in the sense of Epstein (holomorphic finite type maps defined on open subsets of the complex plane) and will be investigated in a subsequent paper. Finally, considering possible generalizations to higher dimensions, structural stability (in the sense of Berteloot, Bianchi and Dupont [BBD18]) is known *not* to be dense in the family of endomorphisms of $\mathbb{P}^k$ for any $k \geq 2$ (see e.g. [Duj17, Taf21, Bie20]).

Statement of results. We start by giving some necessary definitions, starting by the holomorphic families which are the object of our study.

Definition 1.1 (Natural family)**.** Let M be a complex connected manifold. A *natural family* of finite type meromorphic maps is a family $\{f_\lambda\}_{\lambda\in M}$ of the form $f_\lambda = \varphi_\lambda \circ f_{\lambda_0} \circ \psi_\lambda^{-1}$, where $f := f_{\lambda_0}$ is a finite type meromorphic map, and $\varphi_\lambda, \psi_\lambda$ are quasiconformal homeomorphisms depending holomorphically on $\lambda \in M$, and such that $\psi_\lambda(\infty) = \infty$.

Under these conditions, one can check that f_λ depends holomorphically on λ, (i.e. $\lambda \mapsto f_\lambda(z)$ is holomorphic for every fixed $z \in \mathbb{C}$).

Let $S(f)$ denote the set of singular values (critical or asymptotic, see Section 2.1) of a meromorphic map f. If f is of finite type, then it can be embedded in a finite dimensional complex analytic manifold of dimension $\#S(f)+2$ [EL92, Eps93], obtained by allowing φ and ψ to be any pair of homeomorphisms that make $\varphi \circ f \circ \psi^{-1}$ holomorphic (or meromorphic) and satisfy $\psi(\infty) = \infty$. Hence natural families can be viewed as subfamilies in this natural parameter space.

Many simple families are natural, like for example the exponential $E_\lambda(z) = \lambda e^z$, the tangent $T_\lambda(z) = \lambda \tan(z)$ or the quadratic family $Q_\lambda(z) = z^2 + \lambda$. In these three examples, the map φ_λ is conformal, and $\psi_\lambda = \mathrm{Id}$. Notice that the singular values of f_λ are marked points that can be followed holomorphically in $\lambda \in M$, hence their number and their nature (critical or asymptotic) remain constant throughout the entire family. The same is true for their preimages: both critical points and *asymptotic tracts* (logarithmic preimages of punctured neighborhoods of the asymptotic values, (see Section 2.1) can be followed holomorphically in λ and their multiplicity remains constant. One may naturally ask how restrictive is the concept of a natural family. As we show in Theorem 2.6, the answer is that the properties described above are necessary and sufficient conditions for an arbitrary holomorphic family of maps to be locally natural. Hence, since all of our main results are local in parameter space, they still apply if we replace the assumption that $\{f_\lambda\}_{\lambda\in M}$ is natural by the assumption that $S(f_\lambda)$ and $f_\lambda^{-1}(S(f_\lambda))$ both move holomorphically over M.

Next we define the concept of a cycle disappearing to infinity or exiting the domain (both terms will be used indistinctively).

Definition 1.2 (Cycle disappearing to infinity)**.** Let $\{f_\lambda\}_{\lambda\in M}$ be a holomorphic family of meromorphic maps. We say that a cycle of period $m \geq 1$ *disappears to infinity* at $\lambda_0 \in M$ (or *exits de domain* at λ_0) if there exist two continuous curves $t \mapsto \lambda(t)$ and $t \mapsto z(t)$ such that:

(1) for all $t > 0$, $\lambda(t) \in M$ and $z(t) \in \mathbb{C}$, with $f^m_{\lambda(t)}(z(t)) = z(t)$
(2) $\lim_{t\to+\infty} \lambda(t) = \lambda_0 \in M$ and $\lim_{t\to+\infty} z(t) = \infty$.

As mentioned above, Eremenko and Lyubich [EL92] showed that cycles cannot exit the domain for holomorphic families of entire functions.

The phenomenon of cycles disappearing to infinity was previously observed in several particular slices of meromorphic functions starting with the early studies of the tangent family $T_\lambda(z) = \lambda \tan(z)$ by Devaney, Keen and Kotus [KK97, DK89, DK88], see also [KŚ02], followed by several other families with two asymptotic values [CK19, CJK22] and generalized to some dynamically defined one-dimensional families in [FK21]. Following the terminology in the literature we define virtual cycles which, as we will see, describe limits of actual cycles after they disappear at infinity.

Definition 1.3 (Virtual cycle)**.** Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic map. A *virtual cycle* of length n is a finite, cyclically ordered set $z_0, z_1, \ldots, z_{n-1}$ such that for all i, either $z_i \in \mathbb{C}$ and $z_{i+1} = f(z_i)$, or $z_i = \infty$ and z_{i+1} is an asymptotic value for f, and at least one of the z_i is equal to ∞. If $z_i = \infty$ only for one value of i then we say that the virtual cycle has *minimal length n.*

If a virtual cycle remains after perturbation within the family, then it is called *persistent.*

Definition 1.4 (Persistent virtual cycle)**.** Let $\{f_\lambda\}_{\lambda\in M}$ be a holomorphic family of meromorphic maps, let $\lambda_0 \in M$ and assume that f_{λ_0} has a virtual cycle $z_0, \ldots, z_{n-1}$. If there exist holomorphic germs $\lambda \mapsto z_i(\lambda)$ defined near λ_0 in M such that

(1) $z_i(\lambda_0) = z_i$
(2) $z_i(\lambda) \equiv \infty$ if $z_i = \infty$
(3) and $z_0(\lambda), \ldots, z_{n-1}(\lambda)$ is a virtual cycle for f_λ,

then we say that $z_0, \ldots, z_{n-1}$ is a *persistent* virtual cycle.

In particular in a holomorphic family, if $v(\lambda_0)$ is an asymptotic value such that $f^n_{\lambda_0}(v(\lambda_0)) = \infty$ for some $n \geq 0$, then $(v(\lambda_0), f_{\lambda_0}(v(\lambda_0)), \ldots, \infty)$ is a virtual cycle of minimal length $n+1$. (The case $n = 0$ corresponds to the situation where ∞ itself is an asymptotic value). This virtual cycle is persistent if and only if the singular relation $f^n_\lambda(v(\lambda)) = \infty$ is satisfied in all of M. If this is not the case, i.e. if a virtual cycle for λ_0 is non-persistent, we will say that λ_0 is a *virtual cycle parameter.*

Our next and last definition concerns the concept of activity or passivity of a singular value (see [Lev81], [McM94]).

Definition 1.5 (Passive (active) singular value)**.** Let $\{f_\lambda\}_{\lambda \in M}$ be a holomorphic family of finite type rational, entire or meromorphic maps. Let $v(\lambda)$ be a singular value (or a critical point) of f_λ depending holomorphically on λ near some $\lambda_0 \in M$. We say that $v(\lambda)$ is *passive* at λ_0 if there exists a neighborhood V of λ_0 in M such that:

(1) either $f^n_\lambda(v(\lambda)) = \infty$ for all $\lambda \in V$; or
(2) the family $\{\lambda \mapsto f^n_\lambda(v(\lambda))\}_{n \in \mathbb{N}}$ is well-defined and normal on V.

We say that $v(\lambda)$ is *active* at λ_0 if it is not passive.

We are now ready to state our first result, which connects the three concepts defined above: cycles disappearing to infinity, virtual cycles and the activity of a singular value.

Theorem A (Activity Theorem)**.** *Let $\{f_\lambda\}_{\lambda \in M}$ be a natural family of finite type meromorphic maps, and assume that a cycle of period n disappears to infinity at $\lambda_0 \in M$. Then, this cycle converges to a virtual cycle for f_{λ_0}, which contains (at least) either an active* asymptotic value*, or an active* critical point*.*

Note that by definition, activity means here that there exists parameters arbitrarily close to λ_0 for which one of the critical points or asymptotic values in the virtual cycle does not remain in the backward orbit of ∞.

Let us observe how Theorem A implies that cycles cannot disappear at infinity in the finite type entire setting, hence recovering the main theorem [EL92, Theorem 2]. Indeed, because of the lack of poles, it is easy to see that if a cycle of period n disappears at infinity, then every point of the cycle must converge to infinity (and not just one). This means that the limit virtual cycle is of the form $\infty, \ldots, \infty$. In particular, it does not contain any critical points; Theorem A then asserts that ∞ itself must be an active asymptotic value for f_{λ_0}. But this is impossible, since for families of finite type entire maps ∞ is always a *passive* asymptotic value.

Finally, we remark that if the virtual cycle contains an active asymptotic value, then this virtual cycle is non-persistent (as defined in Definition 1.3). It would be interesting to know if there are examples of such limit virtual cycles in which all asymptotic values are passive.

We now state our second result, which in a sense is a converse to Theorem A.

Theorem B (Accessibility Theorem)**.** *Let $\{f_\lambda\}_{\lambda \in M}$ be a natural family of finite type meromorphic maps, and $\lambda_0 \in M$ be such that f_{λ_0} has a non-persistent virtual cycle of minimal length $n+1$*

$$v(\lambda_0), f_{\lambda_0}(v(\lambda_0)), \ldots, f^n(v(\lambda_0)) = \infty.$$

Then there is a cycle of period $n+1$ exiting the domain at λ_0. Moreover, this cycle can be chosen so that its multiplier goes to zero as it disappears to infinity.

In particular, by the definition of a virtual cycle, $v(\lambda_0)$ is an asymptotic value (finite or infinite) and hence λ_0 is a virtual cycle parameter. In the terminology of [FK21], the Accessibility Theorem states that every virtual cycle parameter is also a *virtual center* (since the multiplier of the disappearing cycle is tending to 0 at the limit parameter), and it is accessible from the interior of a component in parameter space for which the analytic continuation of this cycle remains attracting. This proves the main conjecture in [FK21] in much greater generality than originally stated.

Putting together Theorems A and B, we obtain the following immediate corollary.

Corollary C. *Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of finite type meromorphic maps, and assume that this family does not have any persistent virtual cycle. Let $\lambda_0 \in M$. Then a cycle disappears to infinity at λ_0 if and only if $f^n_{\lambda_0}(v(\lambda_0)) = \infty$ for some asymptotic value $v(\lambda_0)$.*

Up to this point we have described the new type of bifurcation that occurs in the presence of poles and asymptotic values. Observe that this phenomenon is *a priori* unrelated to the collision of periodic orbits forming parabolic cycles, in contrast to what occurs for rational or entire maps for which this is the only possible obstruction for the holomorphic motion of the Julia set. Our next result shows that, nevertheless, when an attracting cycle disappears at some parameter value, this can be approximated by parabolic parameters.

Theorem D (Approximation by parabolic parameters)**.** *Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of finite type meromorphic maps, and assume that an attracting cycle of period n disappears to infinity at $\lambda_0 \in M$. Then there exists a sequence $\lambda_k \to \lambda_0$ such that f_{λ_k} has a non-persistent parabolic cycle of period at most n.*

In particular, by Theorem B, this happens at every virtual cycle parameter. Moreover, using other approximation results in Section 5, it follows from Theorem D that parabolic parameters are actually dense in the activity locus (Corollary 5.10).

We now state our last main result, which is an extension of Mañe-Sad-Sullivan's and Lyubich's bifurcation theory in the setting of finite type meromorphic maps. We stress here the fact that the proof relies in a crucial way on both Theorems A and B.

Theorem E (Characterizations of $\mathcal{J}$-stability)**.** *Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of finite type meromorphic maps. Let $U \subset M$ be a simply connected domain of parameters. The following are equivalent:*

(1) *The Julia set moves holomorphically over U (i.e. f_λ is $\mathcal{J}$-stable for all $\lambda \in U$)*
(2) *Every singular value is passive on U.*
(3) *The maximal period of attracting cycles is bounded on U.*
(4) *The number of attracting cycles is constant in U.*
(5) *For all $\lambda \in U$, f_λ has no non-persistent parabolic cycles.*

In view of Theorem E it makes sense to define the *bifurcation locus* of the natural family as

$$\mathrm{Bif}(M) = \{\lambda \in M \mid f_\lambda \text{ is not } \mathcal{J}\text{-stable}\},$$

or equivalently as the set of parameters for which some of the conditions in Theorem E is not satisfied. Since $\mathcal{J}-stable$ parameters form an open set by definition, following the arguments in [MSS83] we obtain the following statement, so well known for rational maps.

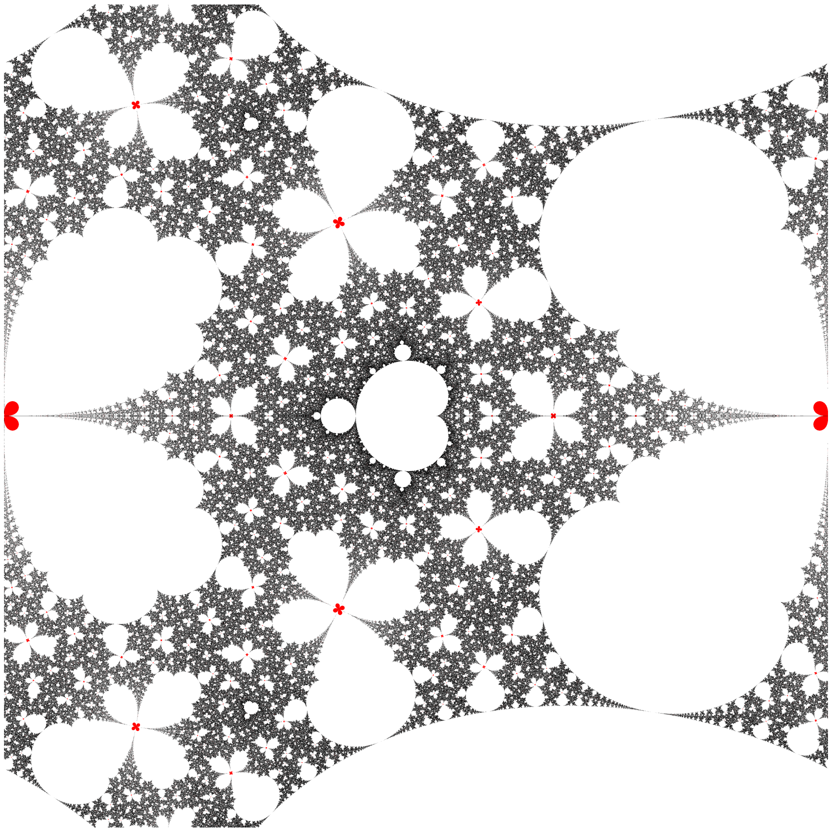


FIGURE 1. Bifurcation locus of the natural family $f_\lambda(z) = \pi \tan^2(z) + \lambda$. These maps have only one critical value and one asymptotic value (λ and $\lambda - \pi$ respectively), which are then both mapped to the same point. Centers of hyperbolic components and Misiurewicz parameters are shown in black. Parameters for which there exist an attracting cycle are shown in white. Virtual cycle parameters are at the center of the red crosses.

Corollary F ($\mathcal{J}$- stable parameters form an open and dense set in M)**.** *If $\{f_\lambda\}_{\lambda \in M}$ is a natural family of finite type meromorphic maps, then* $\mathrm{Bif}(M)$ *has no interior or, equivalently, $\mathcal{J}-$stable parameters are open and dense in M.*

Our last Corollary gives meaning to the notion of hyperbolic components in our setting.

Corollary G. *If $\{f_\lambda\}_{\lambda \in M}$ is a natural family of finite type meromorphic maps, and U is a connected component of $M \setminus \mathrm{Bif}(M)$ containing a parameter λ_0 such that all singular values of f_{λ_0} are captured by attracting cycles, then the same holds for every $\lambda \in U$.*

Structure of the paper. In Section 2 we introduce the concepts of tracts, holomorphic and natural families, holomorphic motions and $\mathcal{J}$-stability. We then show that holomorphic motions of singular values and their preimages imply that a holomorphic family is (locally) natural (Theorem 2.6). We also prove a 'shooting lemma' (Proposition 2.13) which replaces the use of Montel's Theorem when activity of a singular value is due to truncation of its orbits. Section 3 connects cycles disappearing to infinity to virtual cycles, and the latter to active singular values, proving Theorem A. Section 4 shows that virtual cycles are the limits of attracting cycles tending to infinity on which the multiplier tends to zero, making them 'virtual centers' and proving Theorem B. Section 5 contains theorems about the density of several types of parameters in the bifurcation locus of natural families, including the proof of Theorem D. Additionally, all the elements are assembled to prove Theorem E, the meromorphic analogue of Mañé-Sad-Sullivan and Lyubich results, and its corollaries. Finally, Section 6 gives a simple example showing that, without the finite type assumption, the bifurcation locus may be the entire complex plane.

Acknowledgements. We would like to thank Lasse Rempe for helpful discussions, and especially for pointing out to us the statement and proof of Lemma 4.1. We are also grateful to Sarah Koch, Dylan Thurston and Lukas Geyer for interesting conversations and specially to Curt McMullen for his thoughtful comments and suggestions. Finally we are grateful to Bob Devaney, Linda Keen and Janina Kotus for motivating discussions and for their introduction to this beautiful subject.

2. Preliminaries

In this section we state some known results and prove several new tools that will be useful in the proofs of the main theorems.

2.1. Dynamics, asymptotic values and logarithmic tracts.

Given a rational or entire function f, the *Fatou set* $\mathcal{F}(f)$ or *stable set* of f is defined as the largest open set where the family of iterates $\{f^n\}_{n\geq 0}$ is normal in the sense of Montel. However, if $f : \mathbb{C} \to \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is a meromorphic (transcendental) map with at least one non-omitted pole, we need to require additionally that the family of iterates $\{f^n\}_{n\geq 0}$ is first well defined *and* then normal. In both cases, the *Julia set* is the complement of the Fatou set and it is the closure of the repelling periodic points. If the backwards orbit of infinity $\mathcal{O}^-(\infty)$ is an infinite set, the Julia set also coincides with its closure, i.e. $\mathcal{J}(f) = \overline{\mathcal{O}^-(\infty)}$, or equivalently the closure of the set of prepoles of all orders. For background on iteration theory of meromorphic maps we refer to the survey [Ber93] and references therein.

The dynamics of f are determined to a large extent by the dynamics of its *singular values* or points $v \in \widehat{\mathbb{C}}$ for which not all univalent branches of f^{-1} are locally well defined. If f is rational, singular values are always *critical values* $v = f(c)$, being c a *critical point* (i.e. $f'(c) = 0$). If f is transcendental we must also take *asymptotic values* into account, that is values $v = \lim_{t\to\infty} f(\gamma(t))$ where γ is a curve such that $\gamma(t) \to \infty$ as $t \to \infty$. An example is $v = 0$ for the exponential map.

We say that f is of *finite type*, if it has a finite number of singular values, forming the set

$$S(f) = \{v \in \widehat{\mathbb{C}} \mid v \text{ is a critical or an asymptotic value of } f\}.$$

Maps of finite type possess specific dynamical properties which are not satisfied in the general cases. For one, their Fatou set is made exclusively of preperiodic or periodic components, the latter being basins of attraction of attracting or parabolic orbits, or rotation domains (Siegel disks or Herman rings).

In the terminology of [BE08], asymptotic values always have transcendental singularities lying over them. More precisely, we have the following definition.

Definition 2.1 (Tracts)**.** Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic map, let $v \in \hat{\mathbb{C}}$ be an asymptotic value of f, and D be a punctured disk centered at v of radius $r > 0$. We say that a simply connected unbounded set T is a *logarithmic tract above* v if $f : T \to D$ is an infinite degree unbranched covering.

Notice that by decreasing the value of the radius r, we obtain a nested sequence of logarithmic tracts shrinking to infinity. We say that two logarithmic tracts (or sequences thereof) are different, if they are disjoint for sufficiently small values of r. The number of different logarithmic tracts lying over an asymptotic value v is the *multiplicity* of v.

If v is an isolated asymptotic value one can see that there is always a logarithmic tract above v (see e.g. [BE08]). In particular, if f is of finite type all of its asymptotic values have logarithmic tracts lying over them. For this reason, in this paper we will call them simply *tracts*.

2.2. **Holomorphic families, holomorphic motions and $\mathcal{J}$-stability.** We give here the precise definitions of what it means to follow the Julia sets holomorphically, given a family of meromorphic maps.

Definition 2.2 (Holomorphic family)**.** A *holomorphic family* $\{f_\lambda\}_{\lambda\in M}$ *of meromorphic maps* over a complex connected manifold M is a holomorphic map $F : M \times \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ such that $F(\lambda,\cdot) =: f_\lambda$ is a non-constant meromorphic map for every $\lambda \in M$.

Definition 2.3 (Holomorphic motion)**.** A *holomorphic motion* of a set $X \subset \hat{\mathbb{C}}$ over a set $U \subset M$ with basepoint $\lambda_0 \in U$ is a map $H : U \times X \to \widehat{\mathbb{C}}$ given by $(\lambda, x) \mapsto H_\lambda(x)$ such that

(1) for each $x \in X$, $H_\lambda(x)$ is holomorphic in λ,
(2) for each $\lambda \in U$, $H_\lambda(x)$ is an injective function of $x \in X$, and,
(3) at λ_0, $H_{\lambda_0} \equiv \mathrm{Id}$.

A holomorphic motion of a set X *respects the dynamics* of the holomorphic family $\{f_\lambda\}_{\lambda\in M}$ if $H_\lambda \circ f_{\lambda_0} = f_\lambda \circ H_\lambda$ whenever both x and $f_{\lambda_0}(x)$ belong to X.

Note that the continuity of H is not required in the definition. However this property follows as a consequence, as shown in the $\lambda-$Lemma proved in [MSS83].

Theorem 2.4 (The $\lambda-$Lemma [MSS83])**.** *A holomorphic motion H of X as above has a unique extension to a holomorphic motion of $\overline{X}$. The extended map $H : U \times \overline{X} \to \widehat{\mathbb{C}}$ is continuous, and for each $\lambda \in U$, $H_\lambda : \overline{X} \to \widehat{\mathbb{C}}$ is quasiconformal. Moreover, if H respects the dynamics, so does its extension to $\overline{X}$.*

For further results about holomorphic motion and the $\lambda-$Lemma, see for instance [AM01].

Definition 2.5 ($\mathcal{J}$-stability)**.** Consider as above a holomorphic family $F : M \times \mathbb{C} \to \widehat{\mathbb{C}}$ of meromorphic maps. Given $\lambda_0 \in M$, the map f_{λ_0} is $\mathcal{J}$-stable if there exists a neighbourhood $U \subset M$ of λ_0 over which the Julia sets move holomorphically, i.e. there exits a holomorphic motion $H : U \times \mathcal{J}_{\lambda_0} \to \widehat{\mathbb{C}}$ such that $\mathbb{H}_\lambda(\mathcal{J}_{\lambda_0}) = \mathcal{J}_\lambda$ and furthermore it respects the dynamics.

By virtue of the λ-Lemma and the density of periodic points in the Julia set, it is enough to construct a holomorphic motion of the set of periodic points of every period, to obtain one for the entire Julia set .

2.3. **Natural families.** The goal of this section is to prove Theorem 2.6 below. But let us first begin by making a few important observations. Let $f = f_{\lambda_0}$ for some $\lambda_0 \in M$ and let $\{f_\lambda = \varphi_\lambda \circ f \circ \psi_\lambda^{-1}\}_{\lambda\in M}$ be a natural family, as in Definition 1.1. Then ψ_λ maps the critical points of f_{λ_0} to those of f_λ, and φ_λ maps the critical values and asymptotic values of f_{λ_0} to those of f_λ. In particular, as observed in the Introduction, in a natural family, the critical points and singular values always move holomorphically with the parameter and can never collide, while the multiplicity of each singular value remains constant throughout the family.

A converse of this statement, provided by the following theorem, is also true, though not immediate.

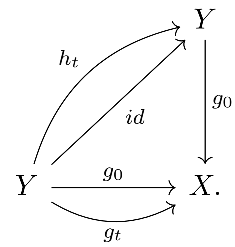
Theorem 2.6 (Characterization of natural families)**.** *Let $\{f_\lambda\}_{\lambda\in M}$ be a holomorphic family of finite type meromorphic maps, on which $S(f_\lambda)$ and $f_\lambda^{-1}(S(f_\lambda))$ both move holomorphically. Then for every $\lambda \in M$, there is a neighborhood V of λ such that $\{f_\lambda\}_{\lambda\in V}$ is a natural family.*

The proof of Theorem 2.6 requires the following lemma.

Lemma 2.7. *Let X, Y be two connected and path connected topological spaces. Suppose that $\{g_t : Y \to X\}_{t\in[0,1]}$, is a homotopy such that g_t is a covering for every $t \in [0,1]$. Then, there exists a homotopy $\{h_t : Y \to Y\}_t$ such that every h_t is a homeomorphism satisfying*

$$g_t = g_0 \circ h_t.$$

Proof. By the homotopy lifting property, since the covering map g_0 can be lifted by the identity map, there exists a unique homotopy $\{h_t : Y \to Y\}_{t\in[0,1]}$ that lifts g_t, hence satisfying the following diagram



It remains to prove that h_t is a homeomorphism for every t.

We first observe that h_t is a local homeomorphism since g_t is a local homeomorphism (and a covering) for every t and the diagram commutes. Next, since h_t is a homotopy to the identity map, we have that $(h_t)_*(\pi_1(Y, y_0)) = (\mathrm{Id})_*((\pi_1(Y, y_0))) = \pi_1(Y, y_0)$, from which it follows that h_t must also be a homeomorphism for every t. □

Proof of Theorem 2.6. Let $\lambda_0 \in M$, and $f := f_{\lambda_0}$. We let $Y := \mathbb{C}\backslash f^{-1}(S(f))$ and $X := \widehat{\mathbb{C}}\backslash S(f)$. Let $B \subset M$ be a small ball centered at λ_0. By Bers-Royden's Harmonic λ-lemma [BR86], we can extend the holomorphic motion of $S(f_\lambda)$ over B to a holomorphic motion $(\lambda, z) \mapsto \varphi_\lambda(z)$ of the whole Riemann sphere, over the ball V of radius one third that of B. Similarly, we extend the holomorphic motion of $f_\lambda^{-1}(S(f_\lambda))$ to a holomorphic motion $(\lambda, z) \mapsto \chi_\lambda(z)$. Notice that both maps are continuous as maps of $V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, holomorphic in λ and quasiconformal in z.

Consider the family of maps $g_\lambda := \varphi_\lambda^{-1} \circ f_\lambda \circ \chi_\lambda$ (in particular, $g_{\lambda_0} = f$). If we could prove that $g_\lambda \equiv f$, the proof would be complete, since f_λ would then be of the required form. This is not quite true, but we will show that in fact $g_\lambda = f \circ h_\lambda$, for some family of homeomorphisms h_λ, which will still be enough to conclude.

Observe that the maps $g_\lambda : \mathbb{C} \setminus f^{-1}(S(f)) \to \widehat{\mathbb{C}} \setminus S(f)$ are covering maps, in general only continuous (both in z and in λ as well as jointly) and not holomorphic. Our first claim is that there exists a continuous family of homeomorphisms $\{h_\lambda : \mathbb{C}\backslash f^{-1}(S(f)) \to \mathbb{C}\backslash f^{-1}(S(f))\}_{\lambda\in M}$ such that

$$\psi_\lambda := \chi_\lambda \circ h_\lambda^{-1} : \mathbb{C} \setminus f^{-1}(S(f)) \to \mathbb{C} \setminus f_\lambda^{-1}(S(f_\lambda))$$

depend holomorphically on λ and the following diagram commutes:

$$\begin{array}{cccc} & \mathbb{C}\setminus f^{-1}(S(f)) & & \\ & \uparrow h_\lambda & \searrow^{g_{\lambda_0}=f} & \\ \psi_\lambda & \mathbb{C}\setminus f^{-1}(S(f)) & \xrightarrow{\ g_\lambda\ } & \widehat{\mathbb{C}}\setminus S(f) \\ & \downarrow \chi_\lambda & & \downarrow \varphi_\lambda \\ & \mathbb{C}\setminus f_\lambda^{-1}(S(f_\lambda)) & \xrightarrow{\ f_\lambda\ } & \widehat{\mathbb{C}}\setminus S(f_\lambda) \end{array}$$

To define h_λ, let us fix $\lambda \in V$ and consider a path $\Lambda : [0,1] \to V$ joining λ_0 and λ. By Lemma 2.7, there exists a homotopy $\{\tilde{h}_t : \mathbb{C} \setminus f^{-1}(S(f)) \to \mathbb{C} \setminus f^{-1}(S(f))\}_t$ composed of homeomorphisms such that

$$g_{\Lambda(t)} = f \circ \tilde{h}_t.$$

We then define $h_\lambda := \tilde{h}_1$. Since V is simply connected, the base points are irrelevant and hence $(\lambda, z) \to (\lambda, h_\lambda(z))$ is continuous.

Since both h_λ^{-1} and χ_λ are homeomorphisms, we have proven that ψ_λ is a homemorphism for every $\lambda \in M$. It is left to prove that for every for every $z \in \mathbb{C} \setminus f^{-1}(S(f))$, $\lambda \mapsto \psi_\lambda(z)$ is holomorphic in λ.

To see this, observe that we have $\varphi_\lambda \circ f(z) = f_\lambda \circ \psi_\lambda(z)$, and that, fixing $z \in \mathbb{C} \setminus f^{-1}(S(f))$, the map $F(\lambda, y) = \varphi_\lambda \circ f(z) - f_\lambda(y)$ is holomorphic. Fix $\lambda_1 \in V$ and $y_1 = \psi_{\lambda_1}(z)$. Then, by construction $f'_{\lambda_1}(y_1) \neq 0$ and hence, by the Implicit Function Theorem, there exists a map $\lambda \mapsto y(\lambda) = \psi_\lambda(z)$ such that $y_1 = y(\lambda_1)$ and $F(\lambda, y(\lambda)) = \varphi_\lambda(f(z)) - f_\lambda(\psi_\lambda(z)) = 0$. It follows that $\lambda \mapsto \psi_\lambda(z)$ is holomorphic locally near every $\lambda_1 \in V$, and so is holomorphic on M.

Finally, we have proved that

(1) for each $\lambda \in V$, $\psi_\lambda : Y \to \mathbb{C}\setminus f_\lambda^{-1}(S(f_\lambda))$ is injective
(2) for every $z \in Y$, $\lambda \mapsto \psi_\lambda(z)$ is holomorphic on Y.

In other words, $(\lambda, z) \mapsto \psi_\lambda(z)$ is a holomorphic motion of $\mathbb{C}\setminus f_\lambda^{-1}(S(f_\lambda))$. By applying again Bers-Royden's Harmonic λ-lemma and replacing again V by a ball of smaller radius, ψ_λ extends to a holomorphic motion of $\widehat{\mathbb{C}}$ (fixing ∞), such that for every $\lambda \in V$, $\psi_\lambda : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasiconformal homeomorphism.

Since $f_\lambda \circ \psi_\lambda = \varphi_\lambda \circ f$, this proves that $\{f_\lambda\}_{\lambda \in V}$ is a natural family. □

Before we end this section, we record here a lemma which illustrates the convenience of working with natural families.

Lemma 2.8. *Let $\{f_\lambda\}_{\lambda \in M}$ be a natural family of finite type meromorphic maps, such that for all $\lambda \in M$, f_λ has finitely many poles. Then the number of poles is independent of λ. Moreover, for all $\lambda \in M$, ∞ is an asymptotic value for f_λ.*

Proof. Let us write $f_\lambda = \varphi_\lambda \circ f \circ \psi_\lambda^{-1}$, with $\psi_\lambda(\infty) = \infty$ and $f = f_{\lambda_0}$ for some $\lambda_0 \in M$, and $\varphi_\lambda, \psi_\lambda$ are both quasiconformal homeomorphisms. Then the set of poles of f_λ is the image under ψ_λ of the set $f^{-1}(\{\varphi_\lambda^{-1}(\infty)\})$. In particular, this set $f^{-1}(\{\varphi_\lambda^{-1}(\infty)\})$ is finite by assumption, so $\varphi_\lambda^{-1}(\infty)$ is a Picard exceptional value for f. By Picard's theorem and connectivity of M, the continuous map $\lambda \mapsto \varphi_\lambda^{-1}(\infty)$ is constant: $\varphi_\lambda^{-1}(\infty) \equiv x \in \hat{\mathbb{C}}$.

Therefore the set of poles of f_λ is given by $\psi_\lambda\left(f^{-1}(\{x\})\right)$, which is a holomorphic motion of $f^{-1}(\{x\})$. In particular, its cardinality is independent of λ.

For the second assertion, it suffices to observe that if f_λ has finitely many poles, then ∞ is a Picard exceptional value, and therefore an asymptotic value [GO08, Chapter 5, Theorem 1.1]. □

2.4. **Quasiconformal distortion.** We state here a well-known distortion estimate for quasiconformal homeomorphisms that we will need in the proof of the main theorems.

Lemma 2.9 (Distortion of small disks)**.** *Let $\{\varphi_\lambda\}_{\lambda\in\mathbb{D}}$ be a holomorphic motion of the Riemann sphere $\hat{\mathbb{C}}$, with $\varphi_0 = \mathrm{Id}$. Let $t \mapsto \lambda(t)$ be a continuous path in $\mathbb{D}$ with $\lim_{t\to+\infty} \lambda(t) = 0$, and $t \mapsto r_t$ a continuous function with $r_t > 0$ and $\lim_{t\to+\infty} r_t = 0$. Let $t \mapsto z_t$ be a path in $\hat{\mathbb{C}}$ and $D_t := \mathbb{D}(z_t, r_t)$. Let $\epsilon > 0$; then for all t large enough:*

$$\mathbb{D}(\varphi_{\lambda(t)}(z_t), r_t^{1+\epsilon}) \subset \varphi_{\lambda(t)}(\mathbb{D}(z_t, r_t)) \subset \mathbb{D}(\varphi_{\lambda(t)}(z_t), r_t^{1-\epsilon})$$

Proof. By Theorem 12.6.3 p. 313 in [AIM08], for all $t > 0$, $\theta \in \mathbb{R}$ and and $r \le 1$, we have :

$$|\varphi_{\lambda(t)}(z_t + re^{i\theta}) - \varphi_{\lambda(t)}(z_t)| \le e^{5(K_{\lambda(t)}-1)} \cdot |\varphi_{\lambda(t)}(z_t) - \varphi_{\lambda(t)}(z_t + e^{i\theta})| \cdot r^{1/K_{\lambda(t)}},$$

where $K_\lambda > 1$ is the dilatation of φ_λ. Since $\varphi_{\lambda(t)} \to \mathrm{Id}$ uniformly on $\hat{\mathbb{C}}$ as $t \to +\infty$, we have that as $t \to +\infty$:

$$\sup_{\theta\in[0,2\pi]} |\varphi_{\lambda(t)}(z_t) - \varphi_{\lambda(t)}(z_t + e^{i\theta})| \to 1.$$

Since (φ_λ) is a holomorphic motion, $K_\lambda \to 1$ as $\lambda \to 0$ and so $K_{\lambda(t)} \to 1$ as $t \to +\infty$. The inclusion $\varphi_{\lambda(t)}(\mathbb{D}(z_t, r_t)) \subset \mathbb{D}(\varphi_{\lambda(t)}(z_t), r_t^{1-\epsilon})$ then follows.

The other inclusion is equivalent to $\varphi_{\lambda(t)}^{-1}(\mathbb{D}(y_t, r_t^{1+\epsilon})) \subset \mathbb{D}(\varphi_{\lambda(t)}^{-1}(y_t), r_t)$, with $y_t := \varphi_{\lambda(t)}(z_t)$. Its proof is essentially the same and is left to the reader. □

We also record here the following well-known property of quasiconformal mappings, that we will need in the proof of Theorem A.

Lemma 2.10 (See [EL92], Lemma 4)**.** *Let $\psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a K-quasiconformal homeomorphism fixing $0, \infty$. Let $\arg\psi(z) - \arg z$ be a uniform branch of the difference of arguments in $\mathbb{C}^*$. Suppose*

$$B^{-1} \le |\psi(z_0)| \le B, \qquad |\arg\psi(z_0) - \arg z_0| \le B,$$

for some $z_0 \in \mathbb{C}$ and $B > 0$. Then for $|z| > |z_0|$ the following estimates hold:

$$C^{-1}|z|^{K_1^{-1}} \le |\psi(z)| \le C|z|^{K_1} \tag{2.1}$$

$$|\arg\psi(z) - \arg z| \le K_1 \ln|z| + C. \tag{2.2}$$

Here K_1, C depend on K, z_0, B but not on z, ψ.

Finally, we close this preliminary subsection with the following lemma, which we will use frequently:

Lemma 2.11. *Let $D \times \widehat{\mathbb{C}} : (\lambda, z) \mapsto \psi_\lambda(z)$ be a holomorphic motion of $\widehat{\mathbb{C}}$ over a domain $D \subset \widehat{\mathbb{C}}$, where the $\psi_\lambda : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ are quasiconformal homeomorphisms. Let $g : D \to \widehat{\mathbb{C}}$ be a non-constant holomorphic function, and assume that there exists $\lambda_0 \in D$ such that for all $\lambda \in D$, $\psi_\lambda(g(\lambda_0)) = g(\lambda_0)$. Let $G(\lambda) := \psi_\lambda^{-1} \circ g(\lambda)$. Then there is a neighborhood $V \subset D$ of λ_0 such $G_{|V} : V \to \widehat{\mathbb{C}}$ is open and discrete.*

Proof. We first observe that $G(\lambda) = y \iff \psi_\lambda(y) - g(\lambda) = 0$. Let $F_y(\lambda) := \psi_\lambda(y) - g(\lambda)$: then F_y is a holomorphic map, which depends continuously on $y \in \hat{\mathbb{C}}$. Moreover, it follows from the assumption on λ_0 that F_{y_0} is non-constant, where $y_0 := g(\lambda_0)$. Therefore, there is an open neighborhood W of $g(\lambda_0)$ such that for all $y \in W$, F_y is non-constant. We let $V := G^{-1}(W)$, which is open since G is continuous.

Let us first prove that G is open. Let $\lambda_1 \in V$ and $y_1 := G(\lambda_1)$. Then by the previous observation, $F_{y_1}(\lambda_1) = 0$. Moreover, since $y \mapsto F_y$ is continuous, Hurwitz's theorem implies that all y close enough to y_1, F_y has a zero λ (close to λ_1); in other words, $G(\lambda) = y$. Therefore G is indeed open.

Moreover, $G^{-1}(\{y\})$ is the set of zeroes of F_y, so it is discrete. □

Remark 2.12. By a result due to Stoilow [Sto32], Lemma 2.11 implies that there exists homeomorphisms h_1 and h_2 defined over respective neighborhoods of λ_0 and $G(\lambda_0)$ such that $h_2 \circ G \circ h_1^{-1}(\lambda) = \lambda^m$, for some $m \geq 1$. In other words, G is locally a finite degree (possibly) branched cover. In particular, it is always possible to lift curves in a neighborhood of $G(\lambda_0)$, although the lift is a priori not unique if $m \geq 2$.

Although we do not require it, it is possible to prove by standard arguments that the map G is in fact quasiregular.

2.5. A shooting Lemma.

In Section 5.1 we shall prove that certain sets of parameters are dense in the bifurcation locus. To that end, we will need the fact that, if some singular value $v(\lambda_0)$ is mapped to infinity in finitely many steps (i.e. $f^n_{\lambda_0}(v(\lambda_0)) = \infty$ for $n \geq 0$ and $f^n_\lambda(v(\lambda)) \not\equiv \infty$ on M), then we can find nearby parameters for which $v(\lambda)$ has some prescribed dynamical behaviour. Similar results can be proven in the rational setting using Montel's Theorem together with the non-normality of the family of iterates of the active singular value. In our setting in which $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a transcendental meromorphic map, and $U \subset \mathbb{C}$ is a domain, the singular value v_λ could be active because its family of iterates $\{f_\lambda(v_\lambda)\}_{n\in\mathbb{N}}$ is not defined in a parameter neighborhood of λ_0 rather than not being normal. As a consequence, one cannot always apply Montel's Theorem as for entire maps or rational maps. Its role will be played by the following statement, which holds for any natural family of meromorphic maps. Notice that here we do not have assumptions on the set of singular values so that a priori functions could be in the general class of meromorphic transcendental functions. Nevertheless, observe that the Proposition is only meaningful when there exists at least one non-omitted pole.

Proposition 2.13 (Shooting Lemma)**.** *Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of meromorphic maps. Let $\lambda_0 \in M$ be such that a singular value v_λ satisfies $f^n_{\lambda_0}(v_{\lambda_0}) = \infty$, but this relation is not satisfied for all $\lambda \in M$. Let $\lambda \mapsto \gamma(\lambda)$ be a holomorphic map such that $\gamma(\lambda_0) \notin S(f_{\lambda_0})$. Then we can find λ' arbitrarily close to λ_0 such that $f^{n+1}_{\lambda'}(v_{\lambda'}) = \gamma(\lambda')$.*

Observe that the map γ, which is a holomorphic map from a neighborhood of λ_0 in M to $\hat{\mathbb{C}}$, is allowed to be constant. In particular, if ∞ is not a singular value of f_{λ_0}, by taking $\gamma(\lambda) \equiv \infty$ we obtain that the parameter λ_0 is a limit of parameters λ_k for which $v(\lambda_k)$ is a prepole of order $n + 1$.

Remark. A weaker version of Proposition 2.13 (allowing $n + 2$ instead of $n + 1$) could be proven by showing that the map $\lambda \mapsto f^{n+1}_\lambda(v(\lambda))$ has an essential singularity at λ_0, and applying the Great Picard's Theorem. This was pointed out to us by R. Roeder.

The proof of Proposition 2.13 uses the following lemmas. The first one can be found in [BFJK18, Lemma 13] (see also [BF15, Lemma 4.6] for a more general statement). In the following, let us denote by $\operatorname{wind}(\sigma(t), P)$ the winding number of a curve $\sigma(t)$ with respect to a point P.

Lemma 2.14 (Computing winding numbers)**.** *Let $\gamma, \sigma : [0,1] \to \mathbb{C}$ be two disjoint closed curves and let $P_\gamma \in \gamma$ and $P_\sigma \in \sigma$ be arbitrary points. Then*

$$\operatorname{wind}(\sigma(t) - \gamma(t), 0) = \operatorname{wind}(\gamma(t), P_\sigma) + \operatorname{wind}(\sigma(t), P_\gamma). \tag{2.3}$$

As a consequence, we obtain the following.

Lemma 2.15 (Fixed point theorem)**.** *Let V be a Jordan domain, and let f, g be holomorphic functions in a neighborhood of $\overline{V}$. Suppose that $g(\overline{V}) \subset f(V)$ and $g(\partial V) \cap f(\partial V) = \emptyset$. Then there exists $\lambda \in V$ such that $f(\lambda) = g(\lambda)$.*

Proof. Consider the map $F(\lambda) = f(\lambda) - g(\lambda)$. Let $\lambda(t), t \in [0,1]$ be a parametrization of ∂V, and notice that $f(\lambda(t))$ and $g(\lambda(t))$ are two disjoint curves and hence $F(\lambda(t)) \neq 0$ for every $t \in [0,1]$. By the Argument Principle, if the winding number of $F(\lambda(t))$ with respect to 0 is positive, then F has at least one zero in V.

Let $P_f = f(\lambda(0))$ and $P_g = g(\lambda(0))$. Applying Lemma 2.14 we get

$$\operatorname{wind}(F(\lambda(t)), 0) = \operatorname{wind}(f(\lambda(t)) - g(\lambda(t)), 0) = \operatorname{wind}(g(\lambda(t)), P_f) + \operatorname{wind}(f(\lambda(t)), P_g).$$

The hypothesis $g(\overline{V}) \subset f(V)$ implies that the curve $g(\lambda(t))$ lies inside a bounded connected component of the complement of $f(\lambda(t))$ from which we deduce that $\operatorname{wind}(g(\lambda(t)), P_f) = 0$. The same hypothesis also implies that $P_g \in f(V)$ which means, again by the Argument Principle, that $\operatorname{wind}(f(\lambda(t)) - P_g), 0) = \operatorname{wind}(f(\lambda(t)), P_g) \geq 1$. Hence $\operatorname{wind}(F(\lambda(t)), 0) > 0$ and the conclusion follows. □

We will also need the following well known fact.

Lemma 2.16 (Shrinking of holomorphic images)**.** *Let $U \subset \mathbb{C}$ be an open set and $a, b \in \mathbb{C}$. Suppose $\{\varphi_n : U \to \mathbb{C} \setminus \{a, b\}\}_{n \in \mathbb{N}}$ is a sequence of holomorphic maps such that $\varphi_n(u_0) \to \infty$ for a certain $u_0 \in U$. Then for every compact set $K \subset U$, the spherical diameter of $\varphi_n(K)$ tends to 0.*

Proof. We claim that $\{\varphi_n\}_{n \in \mathbb{N}}$ converges locally uniformly to ∞. By Montel's Theorem, the sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ admits converging subsequences. Let $\{\varphi_{n_k}\}_{k \in \mathbb{N}}$ be any such subsequence, and let $\varphi : U \to \hat{\mathbb{C}}$ be the limit function. Since by assumption for all $k \in \mathbb{N}$, $\infty \notin \varphi_{n_k}(U)$ and $\varphi(u_0) = \infty$, it follows from Hurwitz's Theorem that $\varphi \equiv \infty$. Since this holds for any converging subsequence, we have $\lim_{n \to \infty} \varphi_n = \infty$, and the lemma follows. □

Observe that if the singular value v_{λ_0} is a prepole of order n, then the map $\lambda \mapsto f_\lambda^n(v_\lambda)$ is a well defined meromorphic map in a sufficiently small neighborhood of λ_0, with an isolated pole at λ_0. Indeed, if a sequence of such parameters of order equal to n were to accumulate at λ_0, by the discreteness of zeros of holomorphic functions we would have that $\lambda \mapsto f_\lambda^k(v_\lambda)$ is identically equal to ∞ for some $k \leq n$, which contradicts the assumption that this is not a persistent condition. Also impossible would be an approximating sequence of likewise parameters of order strictly less than n since, by continuity, the order of λ_0 would also need to be strictly less than n, also a contradiction. As a consequence of this fact, $\lambda \mapsto f_\lambda^{n+1}(v_\lambda)$ has an essential singularity at λ_0.

We are now ready to prove Proposition 2.13.

Proof of Proposition 2.13. First, we pick an arbitrary one-dimensional slice containing λ_0 in the parameter space M on which $\lambda \mapsto f_\lambda^n(v(\lambda))$ is not constant, and we identify M with $\mathbb{D}(\lambda_0, 1) \subset \mathbb{C}$ in the rest of the proof.

By assumption $f_\lambda = \varphi_\lambda \circ f \circ \psi_\lambda^{-1}$ and we may assume without loss of generality that $\varphi_{\lambda_0} = \psi_{\lambda_0} = \mathrm{Id}$ and hence $f = f_{\lambda_0}$. Let D be a disk centered at $\gamma(\lambda_0)$ such that $\overline{D}$ is disjoint from $S(f_{\lambda_0})$ and let $\delta > 0$ be such that $\gamma(\overline{\mathbb{D}(\lambda_0, \delta)}) \subset D$ (see Figure 2).

Decreasing δ if necessary, the function $G(\lambda) := \psi_\lambda^{-1}(f_\lambda^n(v_\lambda))$ satisfies the assumptions of Lemma 2.11 (and of Remark 2.12) on $\mathbb{D}(\lambda_0, \delta)$, with $g(\lambda) := f_\lambda^n(v_\lambda)$ and $g(\lambda_0) = \infty$. It follows that $G(\mathbb{D}(\lambda_0, \delta))$ contains a disk of spherical radius say $\epsilon > 0$ centered at ∞.

Since there are no singular values in $\overline{D}$ and f_{λ_0} has infinite degree, there are infinitely many univalent preimages of D under f_{λ_0} which must accumulate at infinity. Observe that these preimages must miss, for example, a given periodic orbit of period 3 which does not intersect D. Hence, selecting a subset of those preimages if necessary, we may assume (see Lemma 2.16) that they are all bounded and that in fact their spherical diameter tends to 0. Let U be one such preimage contained in $\mathbb{D}_s(\infty, \epsilon)$. Thus $f_{\lambda_0}(U) = D$.

Since U belongs to the image of G, we let V denote a connected component of $G^{-1}(U)$ inside $\mathbb{D}(\lambda_0, \delta)$. If D (and therefore U) is small enough, then V is a Jordan domain as well. Let us now define $F(\lambda) := f_\lambda^{n+1}(v_\lambda)$. Our goal is to show that $\overline{\gamma(V)} \subset F(V)$ so that Lemma 2.15 applied to γ and F gives the result.

In order to see this we write

$$f_\lambda^{n+1}(v_\lambda) = \varphi_\lambda \circ f_{\lambda_0} \circ \psi_\lambda^{-1} \circ f_\lambda^n(v_\lambda) = \varphi_\lambda \circ f_{\lambda_0} \circ G(\lambda),$$

and therefore

$$F(V) = \varphi_\lambda(f_{\lambda_0}(G(V))) = \varphi_\lambda(f_{\lambda_0}(U)) = \varphi_\lambda(D).$$

Now since δ can be taken arbitrarily small, the values of λ can be arbitrarily close to λ_0 and therefore φ_λ is arbitrarily close to the identity. It follows that $F(V) = \varphi_\lambda(D) \simeq D$, while $\gamma(V) \subset \gamma(\overline{\mathbb{D}(\lambda_0, \delta)}) \subset D$. Moreover, $\partial\gamma(\overline{\mathbb{D}(\lambda_0, \delta)})$ separates the boundaries of these two sets, so the hypotheses of Lemma 2.15 can be applied and we are done.

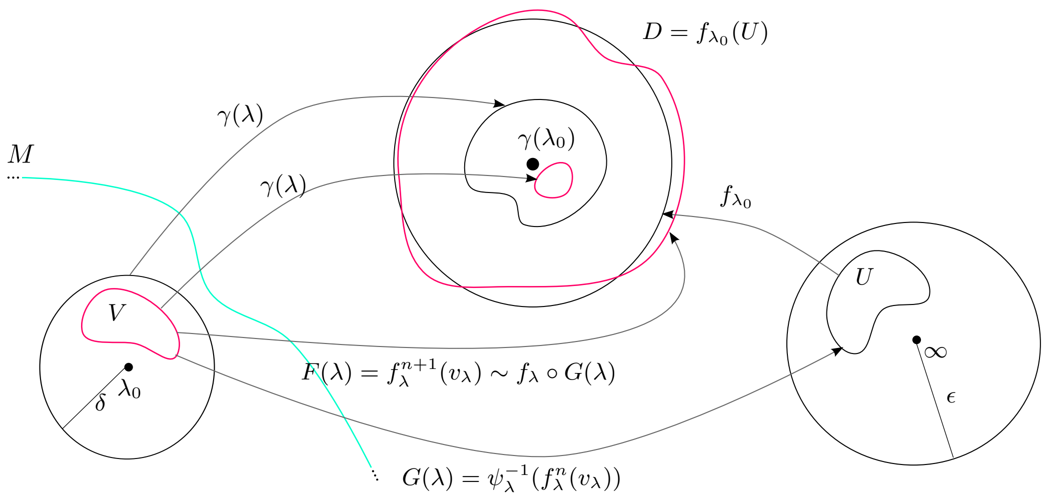


FIGURE 2. An illustration of the proof of Proposition 2.13. The final claim follows by Lemma 2.15, using the fact that $\gamma(V) \subset F(V)$ and $\gamma(\partial V) \cap F(\partial V) = \emptyset$.

□

3. Cycles exiting the domain. Proof of Theorem A.

The goal in this section is to explore the chain of of implications between cycles disappearing to infinity, the existence of virtual cycles (Lemma 3.4), and the activity of at least one asymptotic value or critical point (Subsection 3.3), hence proving Theorem A.

In the process, we shall see that the only possible parameters for which some periodic orbits cannot be followed holomorphically are either parabolic parameters, or those for which a cycle disappears to infinity; or accumulations thereof (Proposition 3.3).

3.1. **Cycles exiting the domain require asymptotic values.**

In this section we show that cycles exiting the domain must create a virtual cycle for the limiting parameter (see Lemma 3.4).

It will be useful first to interpret the concept that $x_i(\lambda) \to \infty$ for $\lambda \to \lambda_0 \in M$ in the following more abstract way (c.f.[MSS83, EL92] and see Figure 3).

Observation 3.1 (The projection map and cycles exiting the domain)**.** For $n \in \mathbb{N}^*$, let

$$P_n := \{(\lambda, z) \in M \times \mathbb{C} : f_\lambda^n(z) = z\},$$

and

$$\pi_1 : P_n \to M$$

be the projection onto the first coordinate. Then, a cycle of period n exits the domain at λ_0 if and only if λ_0 is an asymptotic value of $\pi_1 : P_n \to M$.

In other words, a cycle of period n exits the domain at λ_0 if and only if there exists a continuous curve $t \mapsto (\lambda(t), z(t))$ in P_n such that $\lim_{t\to+\infty} \lambda(t) = \lambda_0$ and $\lim_{t\to+\infty} z(t) = \infty$.

The set P_n is an analytic hypersurface of $M \times \mathbb{C}$, and by the Implicit Function Theorem it is smooth except possibly at points (λ, z) where z is a periodic point of period dividing n with $(f_\lambda^n)'(z) = 1$. Moreover, if $\lambda \in M$ is a critical value of $\pi_1 : P_n \to M$, then f_λ has a parabolic cycle of period dividing n and multiplier 1.

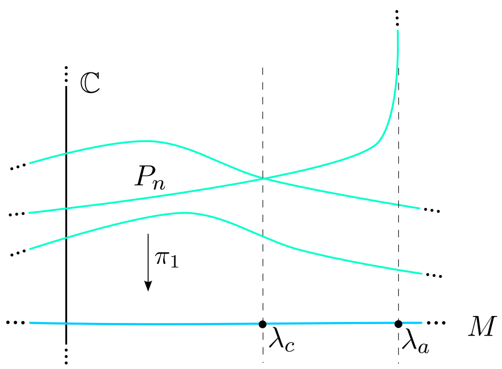


Figure 3. An illustration of the set P_n, the map π_1, and its asymptotic and critical values. Here λ_c is a critical value for π_1, corresponding to the map f_{λ_c} having a parabolic cycle, and λ_a is an asymptotic value for π_1, corresponding to a cycle exiting the domain at λ_a.

Definition 3.2 (Singular value set of π_1)**.** We let X_n denote the singular value set of $\pi_1 : P_n \to M$, which is the closure of the set of critical and asymptotic values of π_1. Let $X := \overline{\bigcup_{n=1}^{\infty} X_n}$.

The next proposition shows that the singular values of $\pi_1 : P_n \to M$ are the only possible obstructions for a holomorphic motion of $\mathrm{Fix}(f_\lambda^n)$ (the fixed points of f_λ^n) to exist, which confirms that X is the appropriate set to study.

Proposition 3.3 ($\mathcal{J}$ moves holomorphically outside X)**.** *Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of meromorphic maps and $U \subset M$ be a simply connected domain. Let X, X_n be as above. Then*

(1) *$U \cap X_n = \emptyset \iff$ the set $\mathrm{Fix}(f_\lambda^n)$ moves holomorphically over U for every $n \geq 1$.*
(2) *$U \cap X = \emptyset \implies$ the Julia set of f_λ moves holomorphically over U.*

Proof. To see (1), suppose $U \cap X_n = \emptyset$. Let $\lambda_0 \in U$, and let $(\lambda_0, z_i)_{i\in\mathbb{N}}$ denote the preimages $\pi_1^{-1}(\lambda_0)$. Then for all $i \in \mathbb{N}$, there exists a holomorphic branch $g_i : U \to P_n$ of π_1^{-1} with $g_i(\lambda_0) = (\lambda_0, z_i)$. In this setting $\lambda \mapsto \pi_2 \circ g_i(\lambda)$ gives the desired holomorphic motion, where π_2 is the projection onto the second coordinate.

For the reverse implication, suppose $\lambda_0 \in U$ is a singular value. If λ_0 is an asymptotic value, there is a fixed point of f_λ^n which escapes to infinity when λ approaches λ_0. Hence any holomorphic motion of the set $\mathrm{Fix}_n(f_{\lambda_0})$ over U could not be surjective. Otherwise, if λ_0 is the image of a critical point (λ_0, z_i), every λ in a neighborhood of λ_0 will have $k > 1$ distinct preimages in P_n splitting off from z_i, hence these periodic points cannot be followed holomorphically either.

Statement (2) follows from (1), the λ-lemma and the fact that (repelling) periodic points are dense in the Julia set. □

In fact, it will follow from our results in Section 5.2 that the converse to item (2) also holds, but for this we shall need the full power of Theorems A and B.

The next lemma shows that any cycle disappearing to infinity requires the help of an asymptotic value which, in the limit, is eventually mapped to infinity, creating a virtual cycle.

Lemma 3.4 (Cycle exiting the domain implies virtual cycle for f_{λ_0})**.** *Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of meromorphic maps. Let $t \mapsto (\lambda(t), z(t))$ be a curve in P_n with $\lim_{t\to+\infty} \lambda(t) = \lambda_0 \in M$ and $\lim_{t\to+\infty} z(t) = \infty$. Then there exists a cyclically ordered set $\infty = a_0, \ldots, a_{n-1} \in \hat{\mathbb{C}}$ such that:*

(1) *for all $0 \leq m \leq n-1$, $a_m = \lim_{t\to+\infty} f_{\lambda(t)}^m(z(t))$;*
(2) *if $a_m \in \mathbb{C}$, then $a_{m+1} = f_{\lambda_0}(a_m)$;*
(3) *if $a_m = \infty$, then a_{m+1} is an asymptotic value of f_{λ_0} (possibly equal to ∞) and a_{m-1} is either ∞ or a pole of f_{λ_0}.*

In other words, the set $a_0, \ldots, a_{n-1}$ is a *virtual cycle* for f_{λ_0}. Notice that the lemma implies that, as $t \to \infty$ (and hence $\lambda(t) \to \lambda_0$), either the whole cycle corresponding to $z(t)$ tends to infinity (in which case ∞ must be an asymptotic value for f_{λ_0}), or there exists at least one finite asymptotic value and one pole in the virtual cycle (possibly more, if there is more than one a_i which equals infinity). Notice that for this lemma, the finite type assumption is not needed.

Proof. To simplify notation, let us denote $x_m(t) := f_{\lambda(t)}^m(z(t))$, and $f = f_{\lambda_0}$. By assumption $\lim_{t\to+\infty} f_{\lambda(t)}^{n-m}(x_m(t)) = \lim_{t\to+\infty} z(t) = \infty$, so any finite accumulation point of the curve $t \mapsto x_m(t)$ must be a pre-pole of f of order at most $n - m$. In particular, the set of finite accumulation points of this curve is discrete, and so $\lim_{t\to+\infty} x_m(t)$ exists (and is possibly ∞). Let $a_m := \lim_{t\to\infty} x_m(t) \in \widehat{\mathbb{C}}$. Item (2) follows easily.

Next, assume that $a_m = \infty$ for some $0 \leq m \leq n-1$. Since $\{f_\lambda\}_{\lambda \in M}$ is a natural family, we have

$$x_{m+1}(t) = f_{\lambda(t)}(x_m(t)) = \varphi_{\lambda(t)} \circ f \circ \psi^{-1}_{\lambda(t)}(x_m(t)),$$

where $f := f_{\lambda_0}$, $\varphi_\lambda, \psi_\lambda : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ are quasiconformal homeomorphisms depending holomorphically on λ, and $\varphi_{\lambda_0} = \psi_{\lambda_0} = \mathrm{Id}$. Therefore, we have

$$f \circ \psi^{-1}_{\lambda(t)}(x_m(t)) = \varphi^{-1}_{\lambda(t)}(x_{m+1}(t)),$$

and $\lim_{t\to+\infty} \psi^{-1}_{\lambda(t)}(x_m(t)) = a_m = \infty$, whereas $\lim_{t\to+\infty} \varphi^{-1}_{\lambda(t)}(x_{m+1}(t)) = a_{m+1}$ since $\varphi^{-1}_{\lambda(t)}$ tends to the identity. Therefore a_{m+1} is indeed an asymptotic value of f.

Finally, still under the assumption that $a_m = \infty$, it follows from item (2) that if a_{m-1} is finite then it is a pole. □

Observe that if the periodic point exiting the domain at λ_0 is actually a fixed point, then the virtual cycle is just $\{\infty\}$ and hence ∞ is a non-persistent asymptotic value of f_{λ_0} (since it cannot be a critical point). This happens e.g. in families where f_{λ_0} is entire but f_λ is not, for values of λ near λ_0, for instance $f_\lambda(z) = \frac{e^z}{1+\lambda e^z}$ and $\lambda_0 = 0$.

The rest of this section is dedicated to showing that either one of the asymptotic values in the virtual cycle must be active, or a critical point is, concluding the proof of Theorem A.

3.2. **Preliminary results.** We first record here several lemmas essentially due to Eremenko and Lyubich, some of them modified for our purposes.

Let $\mathbb{H}$ be the left half plane and T a tract over the asymptotic value v (see Definition 2.1). By uniqueness of the holomorphic universal covering up to biholomorphims, there exists a Riemann map $g : \mathbb{H} \to T$ such that $f \circ g(z) = re^z + v$ for every $z \in \mathbb{H}$.

The following statement is proven in [EL92, Lemma 3] in the case where f is a finite type entire map, but the same proof applies in greater generality, as stated in the following lemma. For this lemma the asymptotic value under consideration is $v = \infty$.

Lemma 3.5 ([EL92], Lemma 3)**.** *Let $R > 0$, and let $T \subset \mathbb{C}$ be a simply connected domain whose boundary is a real-analytic simple curve with both endpoints converging to ∞. Let $f : T \to \mathbb{C} \setminus \overline{\mathbb{D}(0,R)}$ be a holomorphic universal cover, and let* arg *denote a branch of the argument on T. Let $t \mapsto \gamma(t)$ be a continuous curve such that $\lim \gamma(t) = \infty$ and $\gamma(t) \in T$. Then there exists $t_k \to +\infty$ and a constant C independent of k, such that*

$$\ln^2 |f(\gamma(t_k))| + \arg^2 f(\gamma(t_k)) \geq C|\gamma(t_k)| \exp \frac{\arg^2 \gamma(t_k)}{\ln |\gamma(t_k)|}. \tag{3.1}$$

It will be convenient to introduce the following notation, for curves that are not too far apart from each other in log-coordinates, at any given $t > 0$.

Definition 3.6 (Equivalence relation $\asymp$)**.** Let $\gamma_1, \gamma_2 : \mathbb{R}_+ \to \mathbb{C}^*$ be two continuous curves, converging either both to 0 or both to ∞. We will write $\gamma_1 \asymp \gamma_2$ if there exists a constant $C > 1$ such that

$$\frac{1}{C} \ln |\gamma_2(t)| \leq \ln |\gamma_1(t)| \leq C \ln |\gamma_2(t)| \tag{3.2}$$

and

$$|\arg \gamma_1(t) - \arg \gamma_2(t)| \leq C \,|\ln |\gamma_1(t)|| \tag{3.3}$$

Note that this definition makes sense because the arguments $\arg \gamma_i$ are well-defined up to a multiple of $2i\pi$. Also note that $\asymp$ is an equivalence relation.

Remark 3.7. If γ_1, γ_2 are two curves as above and $d \in \mathbb{Z}^*$, then it is easy to see that $\gamma_1 \asymp \gamma_2$ if and only if $\gamma_1^d \asymp \gamma_2^d$, simply because in log coordinates the map $z \mapsto z^d$ becomes $\omega \mapsto d\omega$.

The following lemma can be extracted from arguments present in [EL92]; we include details for the convenience of the reader.

Lemma 3.8 (f^{-1} preserves $\asymp$). *Let $\gamma_1, \gamma_2 : \mathbb{R}_+ \to \mathbb{C}^*$ be two curves, and f be a bounded type meromorphic map. Assume that $\gamma_i(t) \to \infty$ and $f \circ \gamma_i(t) \to \infty$, and that $f \circ \gamma_1 \asymp f \circ \gamma_2$. Then $\gamma_1 \asymp \gamma_2$.*

Proof. Let A be a punctured disk around ∞, and let G denote the union of the tracts T_i such that $f : T_i \to A$ is a universal cover. The set G is non-empty because under the assumptions of the lemma, ∞ is an asymptotic value, and because f has a bounded set of singular values. Let $U := \exp^{-1}(G)$ and

$$\mathbb{H}_R := \exp^{-1}(A) = \{z \in \mathbb{C} : \mathrm{Re}(z) > R\}$$

for some $R > 0$ depending on the radius of A. Then there is a holomorphic map $F : U \to \mathbb{H}_R$ making the following diagram commute:

$$\begin{array}{ccc} U & \xrightarrow{F} & \mathbb{H}_R \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow[f]{} & A \end{array}$$

Let δ_1, δ_2 be two respective lifts of γ_1, γ_2 by exp, chosen to be in the same connected component U_0 of U: then $\delta_j = \ln|\gamma_j| + i \arg \gamma_j$, and $F(\delta_i) = \ln|f \circ \gamma_j| + i \arg f \circ \gamma_j$, for $j = 1, 2$.

Let us denote by I_t the Euclidean segment connecting $F \circ \delta_1(t)$ to $F \circ \delta_2(t)$, by ℓ_{Eucl} the Euclidean length, by $m(t) = \min(\mathrm{Re}(F \circ \delta_1(t)), \mathrm{Re}(F \circ \delta_2(t)))$ and by $M(t) = \max(\mathrm{Re}(F \circ \delta_1(t)), \mathrm{Re}(F \circ \delta_2(t)))$.

By [[EL92], Lemma 1], we have $|F'(z)| \geq \frac{1}{4\pi}(\mathrm{Re}\, F(z) - R)$ and $F : U_0 \to \mathbb{H}_R$ is a conformal isomorphism, hence it has a well defined inverse branch $F_U^{-1} : \mathbb{H}_R \to U$. Therefore

$$\begin{aligned} |\delta_1(t) - \delta_2(t)| &\leq \ell_{\mathrm{Eucl}}(F_U^{-1}(I_t)) \leq \sup_{w \in I_t} |(F_U^{-1})'(w)| \ell_{\mathrm{Eucl}}(I_t) \leq \\ &\leq \frac{4\pi}{m(t) - R} |F \circ \delta_1(t) - F \circ \delta_2(t))| \\ &\leq \frac{2\pi}{m(t) - R} \cdot (2M(t) + |\arg f \circ \gamma_1(t) - \arg f \circ \gamma_2(t)|) \\ &\leq \frac{CM(t)}{m(t) - R} = O(1) \end{aligned}$$

where we used $f \circ \gamma_1 \asymp f \circ \gamma_2$ in the last inequality. Finally, note that $\delta_1 - \delta_2 = \log \gamma_1 - \log \gamma_2 = O(1)$ implies $\gamma_1 \asymp \gamma_2$ (it is in fact much stronger). ☐

Lemma 3.9. *Let f be a meromorphic function of bounded type. Consider a curve $\gamma : \mathbb{R}_+ \to \mathbb{C}^*$ with $\gamma(t) \to \infty$ as $t \to +\infty$ and assume that $f \circ \gamma(t) \to \infty$ as $t \to +\infty$. Let $\{h_t : t \geq 0\}$ be a continuous family of K-qc homeomorphisms satisfying the hypothesis of Lemma 2.10. Then $h_t \circ \gamma \asymp \gamma$.*

Proof. The proof follows directly from Lemma 2.10. ☐

We observe here a technical point which plays an important role in the proof of Theorem A: In Lemma 3.9, it is crucial that $h_t(\infty) = \infty$ for all $t \geq 0$, instead of merely having $\lim_{t\to+\infty} h_t(\infty) = \infty$.

The lemma below is a slightly weaker version of Lemma 5 from [EL92], that will be sufficient for our purposes. We include the proof for the convenience of the reader, since it is very short using Lemmas 3.8 and 3.9.

Lemma 3.10 (Compare [EL92], Lemma 5)**.** *Let f be a meromorphic function with bounded set of singular values. Consider a curve $\gamma : \mathbb{R}_+ \to \mathbb{C}^*$ with $\gamma(t) \to \infty$ as $t \to +\infty$ and assume that $f \circ \gamma(t) \to \infty$ as $t \to +\infty$. Let $\{h_t : t \geq 0\}$ be a continuous family of K-qc homeomorphisms satisfying the hypothesis of Lemma 2.10. Then there exists a curve $\tilde{\gamma} \asymp \gamma$, such that*

$$f \circ \tilde{\gamma}(t) = h_t \circ f \circ \gamma(t). \tag{3.4}$$

Proof. Since $f \circ \gamma(t) \to \infty$, we know that ∞ is an asymptotic value. Hence the existence of a curve $\tilde{\gamma}(t) \to \infty$ satisfying $f \circ \tilde{\gamma} = h_t \circ f \circ \gamma$ follows from the observation that $h_t \circ f \circ \gamma(t) \to \infty$, and that f is a covering over a punctured neighborhood of ∞.

Then, by Lemma 3.9 we have $h_t \circ f \circ \gamma \asymp f \circ \gamma$, so by definition of $\tilde{\gamma}$, we have $f \circ \tilde{\gamma} \asymp f \circ \gamma$. Finally, by Lemma 3.8 we have $\tilde{\gamma} \asymp \gamma$. ☐

3.3. **Proof of Theorem A.**

From now on, we assume that our maps are of finite type.

By assumption, there is a curve $t \mapsto \lambda(t)$ in parameter space with $\lambda(t) \to \lambda_0$, and a cycle of period n exiting the domain along this curve. For simplicity, we will write f_t, φ_t, ψ_t instead of $f_{\lambda(t)}, \varphi_{\lambda(t)}, \psi_{\lambda(t)}$ and f instead of f_{λ_0} or f_0. Analogously, if v is a singular value for f we define $v(t) = \varphi_t(v)$ to be the corresponding value for f_t. The assumptions that the singular set is finite is necessary in order to ensure that any asymptotic value v given by Lemma 3.4 is isolated, and hence that the maps $g_t := \frac{1}{f_t(z)-v(t)}$ are of bounded type, property which is needed in the proof of Lemma 3.13.

Let us introduce some further notations. We denote by $x_1(t), \ldots, x_n(t)$ the points of the cycle of period n for f_t which exits the domain, and assume without loss of generality that $x_n(t) \to \infty$ as $t \to \infty$ (i.e. $\lambda(t) \to \lambda_0$). Recall that by Lemma 3.4 the points $a_i = \lim_{t\to+\infty} x_i(t)$, $i = 1, \ldots, n$ (with indices taken modulo n), form a virtual cycle, hence at least one of them is an asymptotic value.

Therefore, in order to prove Theorem A, we must prove that if the virtual cycle does not contain any active critical point, then at least one asymptotic value in that virtual cycle is active. We assume for a contradiction that all asymptotic relations associated to the limit virtual cycle are preserved (that is, every singular value obtained as a limit of one of the $x_i(t)$ remains in the backward orbit of ∞ for λ in a neighborhood of λ_0, and therefore throughout M), and the same for critical points belonging to the virtual cycle.

In other words, we assume that for *all* $1 \leq i \leq n$ such that $a_i \in S(f)$, we have $f_t^{n-i}(\varphi_t(a_i)) = \infty$ for all $t > 0$. (Recall that if a_i is a singular value of f, then $\varphi_t(a_i)$ is a singular value of the same nature for f_t).

We define a new family of curves $y_1(t), \ldots, y_n(t)$, which record the orbits of all asymptotic values involved in the limit cycle (see Figure 4). More precisely, define

- if $x_i(t) \to \infty$, then $y_i(t) := \infty$
- if $x_{i-1}(t) \to \infty$, then $x_i(t) \to v_i$, where v_i is some asymptotic value of f; then we set $y_i(t) := \varphi_t(v_i) \to v_i$, which is an asymptotic value for f_t.
- if $y_{i-1}(t) \in \mathbb{C}$, then $y_i(t) := f_t(y_{i-1}(t))$.

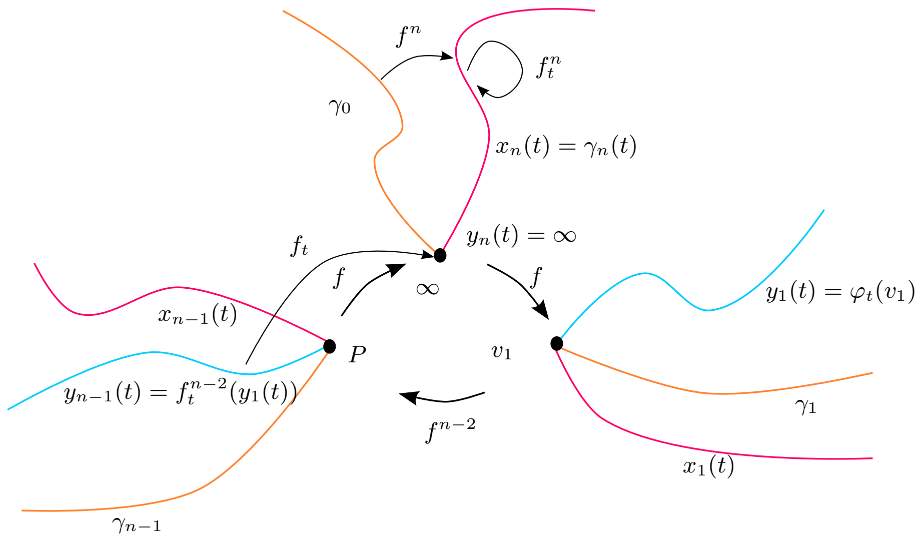


FIGURE 4. An illustration of the proof of Theorem A in a simple case in which there is only one pole P and one asymptotic value v_1 involved. Here $a_n = \infty$, $a_1 = v_1$, and $a_{n-1} = P$. Under the contradiction assumption that the singular relation involving v_1 is persistent we have that $f_t(y_{n-1}(t)) = f_t^{n-1}(\varphi_t(v_1)) = \infty$. This allows to construct the curves γ_i as pullbacks of the curve γ_n, obtaining γ_0 such that $f^n(\gamma_0) = \gamma_n := x_n$ yet $\gamma_0 \asymp \gamma_n$.

The assumption that all singular relations associated to the limit virtual cycle are preserved implies that this definition is coherent, and that $y_1(t), \dots, y_n(t)$ also forms a virtual cycle under f_t for every t. In particular, if $x_{i-1}(t) \to \infty$ and $x_i(t) \to a_i = \infty$, this means that ∞ is an asymptotic value of f , and the assumption forces it to be persistent, i.e. $\varphi_t(\infty) = \infty = y_i(t)$. Note that we also have $\lim_{t\to+\infty} y_i(t) = a_i$.

The idea of the proof is as follows. Each point on the curve $x_n(t)$ is mapped to itself by f_t^n. But since f_t is asymptotically close to f when $t \to \infty$, one could think that these points are mapped "very close to themselves" under f^n when t is large enough, in contradiction with Lemma 3.5. To formalize this idea we consider a third set of curves $\gamma_n(t) := x_n(t), \gamma_{n-1}(t) = f^{-1}(x_n(t)), \dots, \gamma_0(t) = f^{-n}(x_n(t))$, the pull backs of $x_n(t)$ under f (for appropriate branches of the inverse) close to the virtual cycle. We shall show that the $n-$th pullback γ_0 is very close to $\gamma_n = x_n$ (more precisely $\gamma_0 \asymp \gamma_n$) while $f^n(\gamma_0) = \gamma_n = x_n$, which will give a contradiction through Lemma 3.5.

We summarize our goal in the following Lemma.

Lemma 3.11 (Key lemma)**.** *There exist two curves γ_0, γ_n with*

$$f^n(\gamma_0(t)) = \gamma_n(t), \quad \lim_{t\to+\infty} \gamma_0(t) = \lim_{t\to+\infty} \gamma_n(t) = \infty, \quad \text{and} \quad \gamma_0 \asymp \gamma_n.$$

Proof of Theorem A assuming Lemma 3.11. The map f^n is of finite type and since γ_0 and its image γ_n both tend to ∞, it follows that ∞ is an asymptotic value of f^n. Hence there exists a simply connected tract T which maps to a punctured neighborhood of ∞ as a universal covering, and contains the curve $\gamma_0(t)$ for t large enough.

By Lemma 3.5 applied to f^n and $\gamma_0(t)$, we have for all t large enough:

$$\ln^2 |\gamma_n(t_k)| + \arg^2 \gamma_n(t_k) \geq C \exp\left(\ln|\gamma_0(t_k)| \left(1 + \frac{\arg^2 \gamma_0(t_k)}{\ln^2|\gamma_0(t_k)|}\right)\right) \tag{3.5}$$

for some sequence $t_k \to \infty$.

On the other hand, by the assumption that $\gamma_n \asymp \gamma_0$, we have:

$$\ln|\gamma_n(t)| = \ln|\gamma_0(t)| + O(1) \tag{3.6}$$

$$\arg\gamma_n(t) = \arg\gamma_0(t) + O(\ln|\gamma_0(t)|) \tag{3.7}$$

which leads to a contradiction. □

The proof of Lemma 3.11 is done by induction. We start with the curve $\gamma_n := x_n \to a_n = \infty$. Then, given a curve $\gamma_i(t) \to a_i$ with $i = n \dots 1$ we will find a curve $\gamma_{i-1}(t) \to a_{i-1}$ which is an appropriate pullback of γ_i under f. This step is divided into two main cases: the case in which $a_{i-1} \in \mathbb{C}$ (Lemma 3.12) and the case in which $a_{i-1} = \infty$ (Lemma 3.14).

Lemma 3.12. *Let γ_i be a curve such that $\gamma_i(t) \to a_i$ with $\gamma_i(t) \neq a_i$ for all $t > 0$, and assume that $a_{i-1} \in \mathbb{C}$ and that either $\gamma_i(t) - a_i \asymp x_i(t) - y_i(t)$ (if $a_i \in \mathbb{C}$) or $\gamma_i(t) \asymp x_i(t)$ (if $a_i = \infty$). Then there exists a curve γ_{i-1} such that*

(1) $f \circ \gamma_{i-1}(t) = \gamma_i(t)$ and $\gamma_{i-1}(t) \neq a_{i-1}$ for all $t > 0$
(2) $\gamma_{i-1}(t) \to a_{i-1}$
(3) $\gamma_{i-1}(t) - a_{i-1} \asymp x_{i-1}(t) - y_{i-1}(t)$.

Proof. First, we choose γ_{i-1} to be a lift of γ_i by f, such that $\gamma_{i-1}(t) \to a_{i-1}$. Note that if $d_i := \deg(f, a_{i-1}) > 1$, then there are exactly d_i possible choices (since $\gamma_i(t) \neq a_i$ by assumption). This gives (1) and (2).

Next, we claim that $(\gamma_{i-1}(t) - a_{i-1})^{d_i} \asymp \gamma_i(t) - a_i$ if $a_i \in \mathbb{C}$, and that $(\gamma_{i-1}(t) - a_{i-1})^{-d_i} \asymp \gamma_i(t)$ if $a_i = \infty$.

This can be seen easily from the series expansions

$$f(z) - a_i = c \cdot (z - a_{i-1})^{d_i} + o((z - a_{i-1})^{d_i}), \text{ if } a_i \in \mathbb{C}, \text{ or}$$
$$f(z) = c \cdot (z - a_{i-1})^{-d_i} + o((z - a_{i-1})^{-d_i}), \text{ if } a_i = \infty,$$

with $c \neq 0$ (compare Remark 3.7).

Since critical relations are assumed to be persistent along the virtual cycle $a_1, \dots, a_n$, we have $\deg(f_t, y_{i-1}(t)) = d_i = \deg(f, a_{i-1})$. Therefore we also have series expansions of the form

$$f_t(z) - y_i(t) = c(t) \cdot (z - y_{i-1}(t))^{d_i} + o((z - y_{i-1}(t))^{d_i}), \text{ if } a_i \in \mathbb{C}, \text{ or}$$
$$f_t(z) = c(t) \cdot (z - y_{i-1}(t))^{-d_i} + o((z - y_{i-1}(t))^{-d_i}), \text{ if } a_i = \infty,$$

where $c(t) \to c \neq 0$. Since $x_{i+1}(t) = f_t(x_i(t))$, it follows that $(x_{i-1}(t) - y_{i-1}(t))^{d_i} \asymp x_i(t) - y_i(t)$ if $a_i \in \mathbb{C}$, and $(x_{i-1}(t) - y_{i-1}(t))^{d_i} \asymp x_i(t)$ if $a_i = \infty$.

Therefore:

(1) If $a_i = \infty$, then by assumption we have $\gamma_i \asymp x_i$, and we have proved that $(\gamma_{i-1} - a_{i-1})^{d_i} \asymp \gamma_i$ and $(x_{i-1} - a_{i-1})^{d_i} \asymp x_i$; therefore $(\gamma_{i-1} - a_{i-1})^{d_i} \asymp (x_{i-1} - a_{i-1})^{d_i}$, which in turn implies $\gamma_{i-1} - a_{i-1} \asymp x_{i-1} - a_{i-1}$ (see again Remark 3.7).
(2) If $a_i \in \mathbb{C}$, then similarly: by assumption, we have $\gamma_i - a_i \asymp x_i - y_i$, and we have proved that $(\gamma_{i-1} - a_{i-1})^{d_i} \asymp \gamma_i - a_i$ and $(x_{i-1} - y_{i-1})^{d_i} \asymp x_i - y_i$. Therefore we again have $(\gamma_{i-1} - a_{i-1})^{d_i} \asymp (x_{i-1} - a_{i-1})^{d_i}$ and finally $\gamma_{i-1} - a_{i-1} \asymp x_{i-1} - a_{i-1}$.

□

We now turn to the other case, $a_{i-1} = \infty$. Before proving the analogue of Lemma 3.12, namely Lemma 3.14, we will require the following modification of Lemma 3.10 adapted to the case of a finite asymptotic value:

Lemma 3.13. *Let $\gamma(t) \to \infty$ be a curve such that $f_t(\gamma(t)) \to v \in \mathbb{C}$. Then, there exists a curve $\gamma'(t) \to \infty$ such that $\gamma' \asymp \gamma$ and $f(\gamma'(t)) - v = f_t(\gamma(t)) - v(t)$, where $v(t) = \varphi_t(v)$.*

Proof. Let $g_t(z) := \frac{1}{f_t(z)-v(t)}$, and $g(z) := \frac{1}{f(z)-v}$. Let $M_t(z) := \frac{1}{z-v(t)}$. Then observe that $g = M_0 \circ f$, and

$$g_t = M_t \circ f_t = \left(M_t \circ \varphi_t \circ M_0^{-1}\right) \circ g \circ \psi_t^{-1} \tag{3.8}$$

This shows that g_t is a natural family of bounded type meromorphic maps of the form $g_t = \tilde{\varphi}_t \circ g \circ \psi_t^{-1}$, with $\tilde{\varphi}_t := M_t \circ \varphi_t \circ M_0^{-1}$. Moreover, $\tilde{\varphi}_t$ is a quasiconformal homeomorphism of $\hat{\mathbb{C}}$, and $\tilde{\varphi}_t(\infty) = \infty$ (since $M_0^{-1}(\infty) = v$ and $M_t \circ \varphi_t(v) = \infty$).

Since we have $g_t(\gamma(t)) \to \infty$, we may apply Lemma 3.10 to g_t, which gives a curve $\gamma'(t) \to \infty$ such that $\gamma' \sim \gamma$, and $g_t(\gamma(t)) = g(\gamma'(t))$.

It remains to check that $f(\gamma'(t)) - v = f_t(\gamma(t)) - v(t)$. But

$$g_t(\gamma(t)) = g(\gamma'(t))$$
$$\frac{1}{f_t(\gamma(t)) - v(t)} = \frac{1}{f(\gamma'(t)) - v}$$
$$f(\gamma'(t)) - v = f_t(\gamma(t)) - v(t)$$

and the lemma is proved. □

Lemma 3.14. *Let $\gamma_i \to a_i$ be a curve such that either $a_i \in \mathbb{C}$ and $\gamma_i(t) - a_i \asymp x_i(t) - y_i(t)$, or $a_i = \infty$ and $\gamma_i(t) \asymp x_i(t)$. Assume further that $a_{i-1} = \infty$. Then there exists a curve γ_{i-1} such that*

(1) $f \circ \gamma_{i-1}(t) = \gamma_i(t)$ and $\gamma_{i-1}(t) \neq \infty$ for all $t > 0$
(2) $\gamma_{i-1}(t) \to \infty$
(3) $\gamma_{i-1}(t) \asymp x_{i-1}(t)$.

Proof. We will distinguish two cases: $a_i = \infty$ or $a_i \in \mathbb{C}$.

First, assume that $a_i = \infty$. In that case, ∞ is an asymptotic value for f and by assumption it remains an asymptotic value for f_t, so that $\varphi_t(\infty) = \infty$. Moreover, note that $x_i(t) = f_t(x_{i-1}(t)) = \varphi_t \circ f \circ \psi_t^{-1} \circ x_{i-1}(t)$, and that $\psi_t^{-1}(x_{i-1}(t))$ is a curve that tends to ∞. Therefore we can apply Lemma 3.10 with $h_t := \varphi_t$ (since, again, $\varphi_t(\infty) = \infty$) and $\gamma(t) := \psi_t^{-1}(x_{i-1}(t))$. We obtain in this way a curve $\tilde{\gamma}$ such that $\tilde{\gamma}(t) \to \infty$, $\varphi_t \circ f \circ \psi_t^{-1}(x_{i-1}(t)) = x_i(t) = f(\tilde{\gamma}(t))$, and $\tilde{\gamma}(t) \asymp \psi_t^{-1} \circ x_{i-1}(t)$. By Lemma 3.9 we have $x_{i-1}(t) \asymp \psi_t^{-1}(x_{i-1}(t))$ (since $\psi_t^{-1}(\infty) = \infty$). So $\tilde{\gamma}(t) \asymp x_{i-1}(t)$.

Moreover, we have $f(\tilde{\gamma}) = x_i \asymp \gamma_i$ by assumption. Let γ_{i-1} be a lift of γ_i by f: then

$$f \circ \gamma_{i-1} = \gamma_i \asymp x_i = f \circ \tilde{\gamma}_i,$$

so that by Lemma 3.8 we have $\tilde{\gamma} \asymp \gamma_{i-1}$. Finally, we have:

$$\gamma_{i-1} \asymp \tilde{\gamma} \asymp x_{i-1},$$

and we are done in this case.

We now treat the case when $a_i \in \mathbb{C}$. In that case, we apply Lemma 3.13 with $\gamma := x_{i-1}$ and get a curve $\tilde{\gamma}$ such that $\tilde{\gamma} \asymp x_{i-1}$ and $f \circ \tilde{\gamma} - a_i = f_t \circ x_{i-1} - y_i = x_i - y_i$.

Let γ_{i-1} be a lift by f of γ_i, such that $\gamma_{i-1}(t) \to \infty$. It remains to argue as above that $\tilde{\gamma} \asymp \gamma_{i-1}$. But this follows precisely from the same Lemma 3.8 applied to $g := \frac{1}{f-a_i}$ instead of f, since by assumption $\gamma_i - a_i \asymp x_i - y_i$ and therefore

$$f \circ \tilde{\gamma} - a_i = x_i - y_i \asymp f \circ \gamma_{i-1} - a_i.$$

Then finally we also have

$$x_{i-1} \asymp \tilde{\gamma} \asymp \gamma_{i-1}, \tag{3.9}$$

and the lemma is proved. □

We are now finally ready to prove the key Lemma 3.11, which will conclude the proof of Theorem A.

Proof of Lemma 3.11. We define $\gamma_n(t) := x_n(t)$, and then proceed by induction to construct curves γ_i such that $\gamma_i(t) \to a_i$, $f^{n-i}(\gamma_i(t)) = \gamma_n(t)$, and:

- if $a_i \neq \infty$, then $\gamma_i - a_i \asymp x_i - y_i$
- if $a_i = \infty$, then $\gamma_i \asymp x_i$.

Assume γ_i is constructed. We then have two cases: either $a_{i-1} = \infty$ or not. If $a_{i-1} \neq \infty$, then we apply Lemma 3.12. Otherwise, we apply Lemma 3.14. In either case, the induction is proved. □

4. Existence of attracting cycle exiting the domain at virtual cycle parameters. Proof of Theorem B.

The goal in this section is to prove Theorem B, the Accessibility Theorem. We start with a lemma which was kindly pointed out to us by Lasse Rempe.

Lemma 4.1. *Let T be a simply connected hyperbolic domain, ρ_T be the hyperbolic density in T, and let $z, w \in T$. Then*

$$\operatorname{dist}_T(z,w) \geq \frac{1}{2}\left|\ln \frac{\operatorname{dist}(w,\partial T)}{\operatorname{dist}(z,\partial T)}\right|. \tag{4.1}$$

Proof. Let Γ be the collection of all curves $\gamma : [0, t_\gamma] \to T$, parametrized in arclength, for which $\gamma(0) = z$ and $\gamma(t_\gamma) = w$. Notice that by the triangular inequality, given such a curve γ parametrized by a parameter t, for any point $\gamma(t) \in \gamma$ we have

$$\operatorname{dist}(\gamma(t),\partial T) \leq \operatorname{dist}(z,\partial T) + t \quad \text{and in particular} \tag{4.2}$$

$$\operatorname{dist}(w,\partial T) \leq \operatorname{dist}(z,\partial T) + t_\gamma. \tag{4.3}$$

For a curve $\gamma \in \Gamma$ denote by $\ell_T(\gamma)$ its hyperbolic length in T. Then $\operatorname{dist}_T(z,w) = \inf_{\gamma\in\Gamma} \ell_T(\gamma)$. On the other hand, consider any such γ. Using standard estimates for the hyperbolic metric (see [BM07]) as well as (4.2)and (4.3)) we obtain

$$\ell_T(\gamma) = \int_0^{t_\gamma} \rho_T(\gamma(t))dt \geq \int_0^{t_\gamma} \frac{1}{2\operatorname{dist}(\gamma(t),\partial T)}dt \geq$$
$$\int_0^{t_\gamma} \frac{1}{2\operatorname{dist}(z,\partial T)+t}dt = \frac{1}{2}\ln\frac{\operatorname{dist}(z,\partial T)+t_\gamma}{\operatorname{dist}(z,\partial T)} \geq \frac{1}{2}\ln\frac{\operatorname{dist}(w,\partial T)}{\operatorname{dist}(z,\partial T)}.$$

This proves the claim. □

Using Lemma 4.1 and standard hyperbolic estimates, one can show that the Riemann map from the left half plane to a tract, restricted to the negative real axis, cannot contract too much when approaching infinity, no matter what the geometry of the tracts is. In other words, the central curve inside a tract parametrized by the negative half line cannot converge to infinity too slowly.

Lemma 4.2 (Asymptotic derivative of the Riemann map)**.** *Let $\mathbb{H}$ be the left half plane, T be a simply connected hyperbolic domain, $g : \mathbb{H} \to T$ be a Riemann map. Then for every $\alpha > 0$,*

$$\lim_{t\to+\infty} |g'(-t)|e^{\alpha t} = \infty. \tag{4.4}$$

Proof. Let $C = \operatorname{dist}_T(g(-1), \partial T)$. Then (4.1) gives

$$\ln t = \operatorname{dist}_{\mathbb{H}}(-1,-t) = \operatorname{dist}_T(g(-1), g(-t)) \geq \frac{1}{2}\left|\ln \frac{C}{\operatorname{dist}(g(-t), \partial T})\right|. \tag{4.5}$$

By definition of hyperbolic density,

$$\rho_T(g(-t)) = \frac{1}{|g'(-t)|}\rho_{\mathbb{H}}(-t) = \frac{1}{t|g'(-t)|},$$

and by the standard estimates on the hyperbolic density on simply connected sets (see e.g. [BM07]),

$$\rho_T(g(-t)) = \frac{1}{t|g'(-t)|} \leq \frac{2}{\operatorname{dist}(g(-t), \partial T)} \tag{4.6}$$

Suppose first that for a given t, $C \leq \operatorname{dist}(g(-t), \partial T)$; then using (4.6), for every $\beta > 0$ and whenever t is sufficiently large (depending only on C) we obtain that

$$|g'(-t)| \geq \frac{\operatorname{dist}(g(-t), \partial T)}{2t} \geq \frac{C}{2t} \geq e^{-\beta t}.$$

Otherwise, for any t such that $C \geq \operatorname{dist}(g(-t), \partial T)$ we can remove the modulus in the right hand side of (4.5) to obtain

$$2\ln t \geq \ln \frac{C}{\operatorname{dist}(g(-t), \partial T)}$$
$$t^2 \geq \frac{C}{\operatorname{dist}(g(-t), \partial T)} \geq \frac{C}{2t|g'(-t)|}.$$

Hence for every $\beta > 0$ and every t sufficiently large depending only on C

$$|g'(-t)| \geq \frac{C}{2t^3} \geq e^{-\beta t}.$$

□

Proof of Theorem B. To simplify the notations, set $f := f_{\lambda_0}$. Let $v = v_{\lambda_0}$ be an asymptotic value such that $f^n(v) = \infty$. Recall that $f_\lambda = \varphi_\lambda \circ f \circ \psi_\lambda^{-1}$. Let $V := \mathbb{D}^*(v, r)$ be a punctured disk centered at v disjoint from $S(f)$, and T a tract, so that $f : T \to V$ is a universal cover. Let $\Phi : T \to \mathbb{H}$ be a conformal isomorphism, where $\mathbb{H}$ is the left half plane. In particular, $f(z) = v + re^{\Phi(z)}$ for all $z \in T$. See Figure 5.

Let $V_\lambda := \varphi_\lambda(V)$ and $T_\lambda := \psi_\lambda(T)$, so that $f_\lambda : T_\lambda \to V_\lambda$ is also an infinite degree universal cover, and let $\Phi_\lambda := \Phi \circ \psi_\lambda^{-1} : T_\lambda \to \mathbb{H}$. Then $\varphi_\lambda^{-1} \circ f_\lambda : T_\lambda \to V$ is a universal cover, and so for all $z \in T_\lambda$,

$$f_\lambda(z) = \varphi_\lambda\left(v + re^{\Phi_\lambda(z)}\right) \tag{4.7}$$

Now, we wish to find a curve $t \mapsto \lambda(t)$ in parameter space such that

$$\Phi_{\lambda(t)} \circ f^n_{\lambda(t)}(v_{\lambda(t)}) = -t. \tag{4.8}$$

We use the same notations as in the proof of Proposition 2.13: we let $G(\lambda) := \psi_\lambda^{-1} \circ f_\lambda^n(v_\lambda)$ and recall that $G(\lambda_0) = \infty$. Given the definition of Φ_λ, Equation (4.8) is equivalent to

$$\Phi \circ \psi_{\lambda(t)}^{-1} \circ f_{\lambda(t)}^n(v_{\lambda(t)}) = -t, \tag{4.9}$$

or

$$G(\lambda(t)) = \Phi^{-1}(-t). \tag{4.10}$$

Note that $t \mapsto \Phi^{-1}(-t)$ is a curve such that $\lim_{t\to+\infty} \Phi^{-1}(-t) = \infty$. By Remark 2.12, the map G is locally a branched cover over a neighborhood of ∞, and so we can find the desired curve $t \mapsto \lambda(t)$ (defined for t large enough, and possibly not unique). See Figure 5.

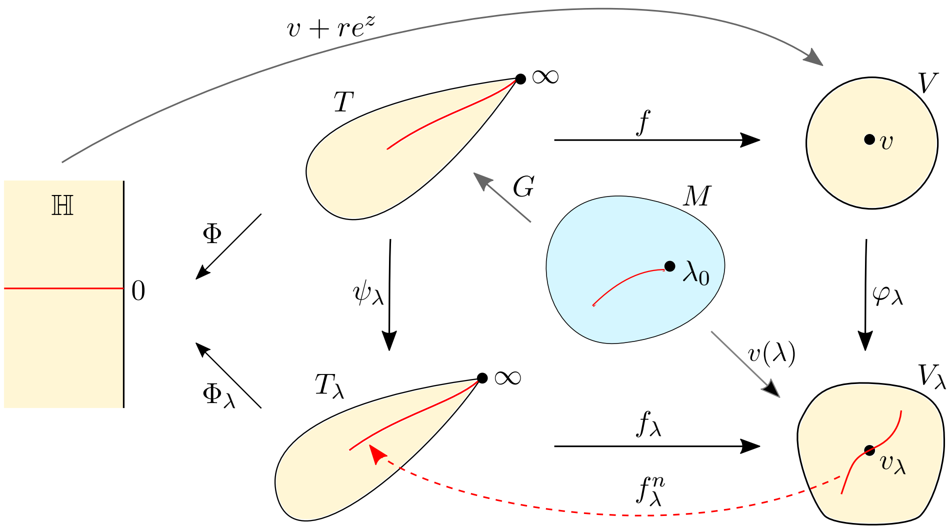


FIGURE 5. Setup of the Proof of Theorem B.

Now let $D_t := \Phi_{\lambda(t)}^{-1}(\mathbb{D}(-t,\pi)) \subset T_{\lambda(t)}$ and let U_t denote the connected component of $f_{\lambda(t)}^{-n}(D_t)$ containing $v_{\lambda(t)}$. We will prove that for all t large enough, $f_{\lambda(t)}(D_t) \Subset U_t$, or equivalently, $f_{\lambda(t)}^{n+1}(U_t) \Subset U_t$; this implies the existence of an attracting fixed point for $f_{\lambda(t)}^n$.

First, let us show that for r small and t large, $f_{\lambda(t)}(D_t)$ is contained in a small disk centered at $v_{\lambda(t)}$, or more precisely,

$$f_{\lambda(t)}(D_t) \subset \mathbb{D}\left(v_{\lambda(t)}, e^{-t(1-\epsilon)}\right). \tag{4.11}$$

By (4.7) we have that for all $z \in \mathbb{H}$.

$$f_\lambda \circ \Phi_\lambda^{-1}(z) = \varphi_\lambda(v + re^z).$$

Since $\mathbb{D}(-t,\pi) \subset \{z \in \mathbb{C} : \Re z < -t + \pi\}$ we have that

$$f_{\lambda(t)}(D_t) \subset \varphi_{\lambda(t)}(\mathbb{D}(v, re^{-t+\pi}))$$

Let $\epsilon > 0$. By Lemma 2.9, we have for all t large enough:

$$f_{\lambda(t)}(D_t) \subset \mathbb{D}\left(v_{\lambda(t)}, (re^\pi)^{1-\epsilon} e^{-t(1-\epsilon)}\right), \tag{4.12}$$

which for r small implies (4.11). See Figure 6.

Now we show that U_t contains a disk centered at $v_\lambda(t)$ whose radius, for t large, is much larger than $e^{-t(1-\epsilon)}$.

Let us first estimate $\mathrm{dist}(f^n_{\lambda(t)}(v_{\lambda(t)}), \partial D_t)$. To lighten the notations, let $g := \Phi^{-1}$; then g is univalent on $\mathbb{H}$ and $D_t = \psi_{\lambda(t)} \circ g(\mathbb{D}(-t, \pi))$. By Koebe's theorem, $g(\mathbb{D}(-t,\pi))$ contains a disk

$$\mathbb{D}(g(-t), C|g'(-t)|).$$

Then, by Lemma 2.9 and (4.9)

$$D_t = \psi_{\lambda(t)} \circ g(\mathbb{D}(-t,\pi)) \supset \mathbb{D}(\psi_{\lambda(t)} \circ g(-t), C^{1+\epsilon}|g'(-t)|^{1+\epsilon}) = \mathbb{D}(f^n_{\lambda(t)}(v_{\lambda(t)}), C^{1+\epsilon}|g'(-t)|^{1+\epsilon}).$$

Now note that as $t \to +\infty$, D_t is arbitrarily close to ∞. In particular, we may assume that for all t large enough $D_t \cap S(f^n_{\lambda(t)}) = \emptyset$; hence since D_t is simply connected, we can define an inverse branch $h_t : D_t \to U_t$ of $f^{-n}_{\lambda(t)}$. In fact, h_t can be extended to some simply connected neighborhood of ∞ independent from t, and as $t \to +\infty$ it converges on that domain to an inverse branch of f^{-n-1}; in particular, its spherical derivative $h_t^{\#}(f^n_{\lambda_t}(v_{\lambda(t)}))$ is bounded independently from t.

Again, Koebe's theorem applied to $h_t : \mathbb{D}\left(f^n_{\lambda(t)}(v_{\lambda(t)}), C^{1+\epsilon}|g'(-t)|^{1+\epsilon}\right) \to U_t$ implies that there exists a constant $C' > 0$ such that

$$\mathbb{D}(v_{\lambda(t)}, C'|g'(-t)|^{1+\epsilon}) \subset U_t. \tag{4.13}$$

See Figure 6. Finally, from equations (4.11) and (4.13), it is enough to prove that

$$\frac{e^{-t(1-\epsilon)}}{C'|g'(-t)|^{1+\epsilon}} \to 0 \quad \text{as } t \to +\infty, \tag{4.14}$$

which follows from Lemma 4.2. This proves that $f^n_{\lambda(t)}(U_t) \Subset U_t$, and the theorem then follows from Schwartz's lemma. Note that (4.14) also implies that the multiplier goes to zero as $t \to +\infty$, since the modulus of $U_t \setminus \overline{f^{n+1}_\lambda(U_t)}$ tends to infinity. □

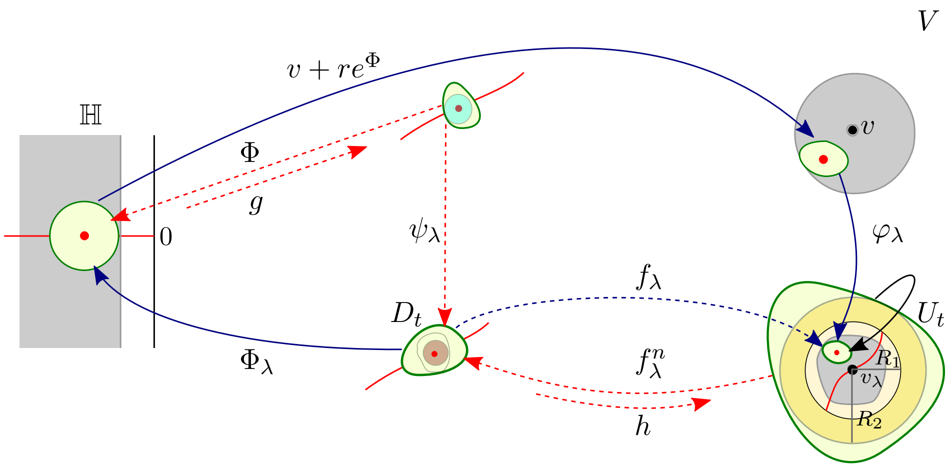


FIGURE 6. Construction of an attracting cycle in the Proof of Theorem B, with $R_1 := e^{-t(1-\epsilon)} \ll C'|g'(-t)|^{1+\epsilon} =: R_2$.

5. Bifurcation locus

In this section we concern ourselves with the set of J-unstable parameters, the *bifurcation locus.* Our goal is to show different characterizations of this set in the spirit of the Mañé-Sad-Sullivan Theorem, leading to the proof that its complement, the set of $\mathcal{J}$-stable parameters, is open and dense in the parameter space.

To that end, we first show several approximation and density results which will be useful and are interesting on their own. Since we have not yet formally linked $\mathcal{J}$-stability to activity of singular values (Theorem E), we will first show density of specific types of parameters in the activity locus, as defined below.

Definition 5.1 (Activity locus)**.** Given a holomorphic family $\{f_\lambda\}_{\lambda\in M}$ we define the *activity locus* $\mathcal{A}$ as the set of parameters in M for which some singular value is active (see Definition 1.5). The activity locus $\mathcal{A}(v_\lambda)$ of a singular value v_λ is the set of parameters in M for which the singular value v_λ is active.

The following Lemma shows that the activity locus of a natural family is a relatively large set in parameter space, a fact which will be useful in the density proofs.

Lemma 5.2. *Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of finite type meromorphic maps, and let $\mathcal{A}(v_\lambda)$ be the activity locus of a marked singular value v_λ. Then $\mathcal{A}(v_\lambda)$ is nowhere locally contained in a proper analytic subset of M. More precisely, if $\lambda_0 \in \mathcal{A}(v_\lambda)\cap H$, where $H \subset M$ is a proper analytic subset, then for every neighborhood U of λ_0 in M, $U \cap (\mathcal{A}(v_\lambda) \setminus H) \neq \emptyset$.*

Proof. Let $\lambda_0 \in \mathcal{A}(v_\lambda)$, H an analytic hypersurface of M containing λ_0, and U be a small polydisk centered at λ_0 in M. Assume for a contradiction that $\mathcal{A}(v_\lambda) \cap U \subset H \cap U$. Let $h_n(\lambda) := f^n_\lambda(v_\lambda)$, wherever this expression is well-defined. Let z_{λ_0} be a repelling periodic point of period at least 3 for f_{λ_0} such that $z_{\lambda_0} \notin S(f_{\lambda_0})$, and let z_λ be the corresponding repelling periodic point for f_λ given by the Implicit Function Theorem. Up to reducing U, we may assume that $\lambda \mapsto z_\lambda$ is defined over U.

Since $\mathcal{A}(v_\lambda) \neq \emptyset$, there is no $N \in \mathbb{N}$ such that $h_N \equiv \infty$ on U; it follows from the definition of the activity locus and our assumptions that if $\lambda \in U$ and $n \in \mathbb{N}$ are such that $h_n(\lambda) = \infty$, then $\lambda \in H$.

We now distinguish two cases:

(1) either there exists $n_0 \in \mathbb{N}$ and $\lambda_1 \in U$ such that $h_{n_0}(\lambda_1) = \infty$;
(2) or for every $n \in \mathbb{N}$, h_n is well-defined over U and $h_n(U) \subset \mathbb{C}$.

Let us first treat case (1). Let D be a holomorphic disk passing through λ_1 and not contained in H. Then by the choice of λ_1 and our previous observation, $h_{n_0}(\lambda_1) = \infty$ and $h_{n_0}(\lambda) \neq \infty$ for every $\lambda \in D \setminus \{\lambda_1\}$. By the Shooting Lemma (Proposition 2.13) applied with $\gamma(\lambda) := z_\lambda$, we find some $\lambda_2 \in D \setminus \{\lambda_1\}$ such that $h_{n_0+1}(\lambda_2) = z_{\lambda_2}$, in other words, v_{λ_2} is Misiurewicz. Therefore $\lambda_2 \in \mathcal{A}(v_{\lambda_2})$, but $\lambda_2 \notin H$, a contradiction.

Case 2 follows from a similar but more classical application of Montel's theorem. □

5.1. Proof of Theorem D and other density theorems.

For the purpose of this section it is convenient to introduce some additional notation.

Definition 5.3 (Truncated parameter)**.** We will say that a parameter $\lambda_0 \in M$ is a *truncated parameter* if there exists a singular value $v(\lambda_0)$ such that $f^n_{\lambda_0}(v(\lambda_0)) = \infty$ for some $n \geq 0$, but this relation does not hold for all λ in a neighborhood of λ_0. The number n is called the *order* of the truncated parameter.

If $v(\lambda_0)$ is an asymptotic value, then λ_0 was already named a *virtual cycle parameter*. If $v(\lambda_0)$ is a critical value, then this means that some critical point for f_{λ_0} is eventually mapped to a pole; we will say in this case that λ_0 is a *critical prepole parameter*. In particular, truncated parameters are virtual cycle parameters, critical prepole parameters, or both.

In a similar vein, if f_{λ_0} has a cycle of multiplier that is a root of unity we will refer to λ_0 as a *parabolic* parameter; if the multiplier does not remain a root of unity under suitable perturbation of λ_0, then we call it *nonpersistent*.

The density of Misiurewicz parameters in the boundary of the Mandelbrot set, or the density of other dynamically (and algebraically) defined parameters are well known consequences of Montel's theorem, as is the accumulation of centers of hyperbolic components on every boundary point. In our setting of natural families of meromorphic maps, we will show that similar results still hold, although the arguments are more involved and require the use of Theorems A and B. As an example, density of parabolic parameters (Corollary 5.10) is not immediate, but rather follows from density of truncated parameters (Proposition 5.5) and approximation of such parameters by parabolic parameters (Theorem D) and by parameters with superattracting cycles (Proposition 5.7).

Before we can prove the useful Proposition 5.5, we first need to deal with the exceptional case of maps without non-omitted poles. The following proposition shows that when the family has no non-omitted poles then everything works as in the rational or entire case.

Proposition 5.4. *Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of finite type meromorphic maps.*

(a) *If for all $\lambda \in M$, f_λ does not have any non-omitted pole, then cycles never disappear to ∞ in the family $\{f_\lambda\}_{\lambda\in M}$.*
(b) *If there exists $\lambda_0 \in M$ such that f_{λ_0} has at least one non-omitted pole, then for all $\lambda \in M$ outside some proper analytic subset of M, f_λ has at least one non-omitted pole.*

Proof. To prove (a), observe first that the assumption implies that for all $\lambda \in M$, f_λ has at most one pole, which (if it exists) is also an omitted value, hence an asymptotic value. By Lemma 2.8, either all the maps f_λ are entire, or they all have exactly one pole, which is also an asymptotic value. By [EL92], in the first case cycles never disappear to infinity and we are done. Let us therefore we are in the second case and denote by v_λ this pole/asymptotic value.

Assume for a contradiction that a cycle disappears at ∞ at some $\lambda_0 \in M$. By Theorem A, the map f_{λ_0} has a virtual cycle which contains either an active asymptotic value or an active critical point. Since v_λ is the only pole of f_λ, that virtual cycle must be either of the form $\infty, \dots, \infty$, or of the form $\infty, v_\lambda, \infty, \dots, v_\lambda$. It is clear that v_λ is always passive, and by Lemma 2.8, ∞ is always an asymptotic value, so it is also always passive. Therefore, the only remaining possibility is that a critical point collides (non-persistently) with v_λ at $\lambda = \lambda_0$.

But by the proof of Lemma 2.8, we have $v_\lambda = \psi_\lambda(v_{\lambda_0})$, and critical points of f_λ are of the form $\psi_\lambda(c)$, where c are critical points of f_{λ_0}. Therefore, if v_{λ_0} is a critical point, then so is v_λ for every $\lambda \in M$, and so it is a passive critical point, contradicting Theorem A.

To show part (b), recall that $f_\lambda = \varphi_\lambda \circ f \circ \psi_\lambda^{-1}$, where $f := f_{\lambda_0}$ and $\varphi_{\lambda_0} = \psi_{\lambda_0} = \mathrm{Id}$. The set of poles of f_λ is the set $\psi_\lambda\left(f^{-1}(\{\varphi_\lambda^{-1}(\infty)\})\right)$. Let us denote by E the set (possibly empty) of (Picard) exceptional values of f.

We will distinguish two cases: either $\lambda \mapsto \varphi_\lambda^{-1}(\infty)$ is constant or it is not. Assume first that it is: $\varphi_\lambda^{-1}(\infty) \equiv \infty \in \hat{\mathbb{C}}$ on M (since $\varphi_{\lambda_0}(\infty) = \infty$). Then the poles of f_λ are the $\psi_\lambda(a_i)$, $a_i \in f^{-1}(\{\infty\})$, and they move holomorphically over M. Let p be the non-omitted pole of f_{λ_0}, and $p_\lambda := \psi_\lambda(p)$. Observe that exceptional values of f_λ (if they exist) are of the form $\varphi_\lambda(v)$, where $v \in E$. In particular, if $p_\lambda \neq \varphi_\lambda(v)$ for every $v \in E$, then f_λ has a non-omitted pole. Since $p_{\lambda_0} \neq \varphi_{\lambda_0}(v)$ for every such v, the set $\bigcup_{v \in E}\{\lambda \in M : p_\lambda = \varphi_\lambda(v)\}$ is either empty or an analytic hypersurface of M.

Let us now assume that $\varphi_\lambda^{-1}(\infty)$ is non-constant. Then for every $\lambda \in M$ such that $\varphi_\lambda^{-1}(\infty) \notin E$, f_λ has infinitely many poles, and in particular it has non-omitted poles. Therefore, outside of the proper analytic set $\bigcup_{v \in E}\{\lambda \in M : \varphi_\lambda(v) = \infty\}$, f_λ has infinitely many poles and in particular some non-omitted poles. □

In view of this result, we will focus in what follows on the case of natural families in which there is at least one λ_* such that f_{λ_*} has at least one non-omitted pole. Indeed, otherwise, it follows from Proposition 5.4 that everything works exactly as for the classical case of rational maps.

Proposition 5.5 (Truncated parameters are dense in the activity locus)**.** *Let $\{f_\lambda\}_{\in \in M}$ be a natural family of finite type meromorphic maps, and assume that there exists $\lambda_* \in M$ such that f_{λ_*} has at least one non-omitted pole. Assume that an asymptotic (resp. critical) value is active at some $\lambda_0 \in M$. Then λ_0 can be approximated by virtual cycle parameters (resp. centers) of arbitrarily large order.*

Proof. By Lemma 5.2 and Proposition 5.4, we can perturb λ_0 if necessary to find some λ_1 that is still in the activity locus, but for which every f_λ for λ in a neighborhood of λ_1 has a non-omitted pole. In the rest of the proof, we still denote it by λ_0.

Then, either there is no neighborhood U of λ_0 for which $\{f_\lambda^n(v_\lambda)\}_n$ is defined for all n and all $\lambda \in U$; or for every neighborhood U of λ_0 where the family $\{f_\lambda^n(v_\lambda)\}_n$ is well defined, it is not normal.

In the first case, λ_0 can be approximated by truncated parameters by definition of those. Moreover these truncated parameters must have unbounded orders or otherwise, there exists $N > 0$ and a sequence of $\lambda_k \to \lambda_0$ such that $f_{\lambda_k}^N(v_{\lambda_k}) = \infty$. By continuity, $f_{\lambda_0}^N(v_{\lambda_0}) = \infty$ and by the identity theorem $f_\lambda^N(v_\lambda) = \infty$ for all $\lambda \in U$ (and in fact for all $\lambda \in M$), which means that v_λ is passive at λ_0, a contradiction.

In the second case, let $p_1(\lambda)$ and $p_2(\lambda)$ be two distinct prepoles varying analytically with λ in U. It follows that the family of maps $g_n(\lambda) = \frac{f_\lambda^n(v_\lambda) - p_1(\lambda)}{p_1(\lambda) - p_2(\lambda)}$ is not normal as well, hence it must take the value $0, -1$ or ∞ for infinitely many different values of n. Since it cannot take the value infinity because the poles are distinct, it follows that it attains 0 or 1 infinitely many times, which correspond to truncated parameters $\lambda \in U$ of order $n+1$ tending to infinity.

To prove the density of truncated parameters in $\mathcal{A}$ it only remains to see that they themselves belong to the activity locus. But this is straightforward from the definition because if λ_0 is a truncated parameter for v_λ, it means that $f_{\lambda_0}^N(v_{\lambda_0}) = \infty$ for some $N \geq 0$, and the relation is non-persistent. Therefore the family $\{f_\lambda^n(v_\lambda)\}_n$ cannot be well defined in any neighborhood of λ_0. □

As a corollary of Proposition 5.5 and Proposition 2.13 (the Shooting Lemma) we obtain the density of other dynamically meaningful parameters, namely Misiurewicz and escaping parameters.

Corollary 5.6 (Density of Misiurewicz and escaping parameters)**.** *Under the assumptions of Proposition 5.5,*

(1) The set of Misiurewicz parameters (for which a singular value is eventually and non-persistently mapped to a repelling periodic point) is dense in $\mathcal{A}$.

(2) If there exists an escaping point which depends holomorphically on λ, the set of parameters for which there is a singular value with orbit converging to infinity is dense (escaping parameters).

Last before the proof of Theorem D, we see two more approximation results, this time from the stable locus. On the one hand Proposition 5.7 will show *centers* (i.e. parameters for which a critical point is periodic) accumulate onto (a subset of) the activity locus (Proposition 5.7). On the other hand, as long as at least one map in the family has non omitted poles, every active parameter is accumulated by parameters for which there is an attracting cycle, of periods increasing to infinity. This last statement is Corollary 5.9, which follows from Theorem B and Proposition 5.7.

Proposition 5.7 (Critical prepole parameters are accumulated by centers)**.** *Let $(f_\lambda)_{\lambda\in M}$ be a natural family of meromorphic maps of finite type. Let λ_0 be a critical prepole parameter of order $n \geq 0$ for the critical value v_{λ_0}, that is $f^n_{\lambda_0}(v_{\lambda_0}) = \infty$. Assume further that v_{λ_0} has a critical preimage c_{λ_0} which is not a Picard exceptional value for f_{λ_0}. Then there exists a sequence of parameters $\lambda_k \to \lambda_0$ as $k \to \infty$, such that each f_{λ_k} has a superattracting cycle of period $n + 3$.*

Proof. Since f_λ is a natural family, critical points can be followed holomorphically with λ. Hence, let $c(\lambda) := \psi_\lambda(c_{\lambda_0})$, so that $c(\lambda)$ is a critical point of f_λ and $c(\lambda_0) = c_{\lambda_0}$.

Up to restricting to a suitable disk in parameter space, we may assume without loss of generality that $M = \mathbb{D}$ and $\lambda_0 = 0$. We let $f := f_0 = f_{\lambda_0}$. We distinguish two cases: either there is at least one preimage a of c_0 which is neither critical nor singular; or there is none.

(1) Let us first assume that there is a preimage a of c_0 which is neither critical nor singular. By the Implicit Function Theorem, there exists a holomorphic map $a(\lambda)$ defined near $\lambda = 0$ such that $f_\lambda(a(\lambda)) = c(\lambda)$, where $c(\lambda) = \psi_\lambda(c_{\lambda_0})$. By Proposition 2.13, there exists a sequence $\lambda_k \to 0$ such that $f^{n+1}_{\lambda_k}(v_{\lambda_k}) = f^{n+2}_{\lambda_k}(c(\lambda_k)) = a(\lambda_k)$. Then $c(\lambda_k)$ is a superattracting periodic point of period $n + 3$, and we are done with this case.

(2) Otherwise : since c_0 is non-exceptional and there are only finitely many singular values, there are infinitely many critical preimages of c_0 which are not singular values, and they must all be critical. Let us pick 3 of them and let us call them a_1, a_2, a_3. The Implicit Function Theorem cannot be applied in this case, but if we let $a_i(\lambda) := \psi_\lambda(a_i)$, then the $a_i(\lambda)$ are critical points of f_λ, and $f_\lambda(a_i(\lambda)) = \varphi_\lambda(c_{\lambda_0}) =: w(\lambda)$ is a critical value. Let us emphasize here that although $w(0) = c(0) = c_{\lambda_0}$, $w(\lambda) \neq c(\lambda)$ in general when $\lambda \neq 0$. In particular, $w(\lambda)$ is a critical value for f_λ but is not necessarily a critical point. By assumption, $f^{n+1}(w(\lambda_0)) = \infty$. Let
$$G(\lambda) := f^{n+2}_\lambda(w(\lambda))$$
and
$$H(\lambda) := \frac{G(\lambda) - a_1(\lambda)}{G(\lambda) - a_2(\lambda)} \frac{a_3(\lambda) - a_2(\lambda)}{a_3(\lambda) - a_1(\lambda)}.$$
The map G has an isolated singularity at $\lambda = 0$, which is an essential singularity by the Shooting Lemma (Proposition 2.13); therefore the same holds for H. By Picard's theorem, H cannot omit $0, 1, \infty$ in any neighborhood of 0; therefore, we can find a sequence $\lambda_k \to 0$ such that $G(\lambda_k) = a_{i_k}(\lambda_k)$, for $1 \leq i_k \leq 3$. This means that the critical point $a_{i_k}(\lambda_k)$ is periodic of period $n + 3$, and we are done in this case as well.

□

Remark 5.8. The assumption that c_{λ_0} is non-exceptional is necessary, as shown by the family $f_\lambda(z) = \tan(\frac{\pi}{4}\lambda z(z-2i))$. Maps in this family have a unique critical point at $c = i$, which is omitted and hence exceptional. The family is natural since it satisfies trivially the conditions of Theorem 2.6 (the critical value is λ). Hence the parameter $\lambda_0 = \pi/2$ is a critical prepole parameter since $f_{\pi/2}(i) = \pi/2$ which is a pole. On the other hand no such map can have a superattracting cycle since the only critical point is omitted.

Corollary 5.9 (Active parameters are accumulated by parameters with attracting cycles)**.** *Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of finite type meromorphic maps, and assume that there exists $\lambda_* \in M$ such that f_{λ_*} has at least one non-omitted pole. Assume that a singular value $v(\lambda_0)$ is active at some $\lambda_0 \in M$. Then λ_0 can be approximated by parameters λ such that f_λ has attracting cycles of arbitrarily large periods.*

Proof. If there exist $n \in \mathbb{N}$ and $v_i(\lambda) \in S(f_\lambda)$ such that $f_\lambda^n(v_i(\lambda)) \equiv \infty$ let

$$N_0 := \max\{n \in \mathbb{N} : f_\lambda^n(v_i(\lambda)) \equiv \infty, v_i(\lambda) \in S(f_\lambda)\}.$$

Otherwise, $N_0 = 0$. Let U be a neighborhood of λ_0 in M, and let $N > N_0$. We will prove that there exists $\lambda \in U$ such that f_λ has an attracting cycle of period at least N. By Proposition 5.5, there exists $\lambda_1 \in U$ such that $f_{\lambda_1}^n(v(\lambda_1)) = \infty$ for some $n \geq N$, and $f_\lambda^n(v(\lambda)) \not\equiv \infty$ on M.

By Theorem B and Proposition 5.7, there is only one remaining case to consider: the case where $v(\lambda_1)$ is a critical value, with $v(\lambda_1) = f_{\lambda_1}(c(\lambda_1))$ and $c(\lambda_1)$ is a critical point which is an exceptional value for f_{λ_1}.

Let $w(\lambda) := \varphi_\lambda(c(\lambda_1))$. Since $c(\lambda_1)$ is an exceptional value, it is also an asymptotic value; therefore $w(\lambda)$ is an asymptotic value for f_λ (see Section 2.3). By our choice of N, we have $f_{\lambda_1}^{n+1}(w(\lambda_1)) = \infty$ but $f_\lambda^{n+1}(w(\lambda)) \not\equiv \infty$. We can then apply Theorem B, to find $\lambda_2 \in U$ such that f_{λ_2} has an attracting cycle of period $n+2$. This concludes the proof.

□

We are now ready to prove Theorem D. Since in the assumptions there is a cycle disappearing at infinity, in view of Theorem A there is at least one function in the family which has at least one non omitted pole.

Proof of Theorem D. We argue by contradiction, and so we assume that there is a neighborhood V of λ_0 in parameter space such that for all $\lambda \in V$, f_λ has no non-persistent parabolic cycles of period up to n. Recall the notations of Section 3: $P_n := \{(\lambda, z) \in M \times \mathbb{C} : z = f_\lambda^n(z)\}$, and $\pi_1 : P_n \to M$ is the projection on the first coordinate. The idea of the proof is the following: first we reduce to the case when V is a one-dimensional disk centered at λ_0. Then, we prove that $\pi_1 : P_n \to V$ restricts to a finite degree branched cover. This allows us (up to passing to a finite branched cover) to construct a single-valued parametrization of the disappearing periodic point, with only a pole at λ_0. Using this parametrization, we show that cycles disappear at infinity from any directions in dynamical space, which finally leads to a contradiction.

Let us make this idea precise. By assumption, there exists a curve $\tilde{\gamma} \subset P_n$ such that $\lim_{t\to+\infty} \tilde{\gamma}(t) = \infty$ and $\lim_{t\to+\infty} \pi_1 \circ \tilde{\gamma}(t) = \lambda_0$, and moreover, if $\tilde{\gamma}(t) =: (\lambda(t), z(t))$, then $z(t)$ is an attracting periodic point for $f_{\lambda(t)}$, of period dividing n. Let S denote the irreducible component of $P_n \cap (V \times \mathbb{C})$ containing $\tilde{\gamma}$. We will start by showing that up to restricting V, we can assume that $\pi_1^{-1}(\{\lambda_0\}) \cap S = \emptyset$.

First, observe that by our assumption on the lack of parabolic cycles, for every $(\lambda, z) \in S$, the point z is an attracting periodic point for f_λ. Indeed, by assumption, the multiplier map

$\rho : (\lambda, z) \mapsto (f_\lambda^n)'(z)$ satisfies $\rho(S) \subset \mathbb{C} \setminus S^1$, and $\rho(S) \cap \mathbb{D} = \emptyset$. Since S is a connected Riemann surface, we therefore have $\rho(S) \subset \mathbb{D}$. But since the maps f_λ are of finite type, there are only finitely many such points; in other words, $\pi_1 : S \to V$ has finite degree. Let $z_0, \dots, z_m$ denote the elements of ${\pi_1}^{-1}(\{\lambda_0\}) \cap S$, if it is not empty. Let $\epsilon > 0$ be small enough that $z_0, \dots, z_m$ all move holomorphically over $\mathbb{B}(\lambda_0, \epsilon)$. Then the points (λ_0, z_i) $(0 \leq i \leq m)$ are not in the same connected component of $P_n \cap (\mathbb{B}(\lambda_0, \epsilon) \times \mathbb{C})$ as the curve $\tilde{\gamma}$. This means that up to replacing V by $\mathbb{B}(\lambda_0, \epsilon)$, we can indeed assume that $\pi_1^{-1}(\{\lambda_0\}) \cap S = \emptyset$, which we do from now on.

In particular, for *every* curve $\tilde{\gamma}(t) = (\lambda(t), z(t)) \subset S$ such that $\pi_1 \circ \tilde{\gamma}(t) = \lambda(t) \to \lambda_0$, we must have $z(t) \to \infty$, since any finite accumulation point would otherwise be an element of $\pi_1^{-1}(\{\lambda_0\}) \cap S$. Let D be an embedded holomorphic disk passing through λ_0 in $V \subset M$, and such that for all $\lambda \in D^* := D \setminus \{\lambda_0\}$, f_λ has no non-persistent singular relation of the form $f_\lambda^k(v(\lambda)) = \infty$, for $0 \leq k \leq n$ and $v(\lambda)$ a singular value, which exists since such parameters form a discrete set. By Theorem A, it follows that no cycle of period up to n can exit the domain at any $\lambda \in D^*$. Therefore, by our assumption on the lack of parabolic cycles, the map $\pi_1 : S_0 \to D^*$ is a covering map, where S_0 is any irreducible component of $S \cap (D^* \times \mathbb{C})$. Moreover, we have already proved that $\pi_1 : S_0 \to D^*$ has finite degree, so there exists a conformal isomorphism $\Phi = (\Phi_1, \Phi_2) : \mathbb{D}^* \to S_0$, such that $\Phi_1 = \pi_1 \circ \Phi : \mathbb{D}^* \to D^*$ is a finite degree cover. Therefore, Φ_1 extends holomorphically to a finite degree branched cover $\Phi_1 : \mathbb{D} \to D$, with $\Phi_1(0) = \lambda_0$. This ends the first part of the proof of reducing to a one dimensional setting.

Now chose any curve $t \mapsto u(t)$ in $\mathbb{D}^*$ such that $u(t) \to 0$, so that $\Phi_1(u(t)) \to \lambda_0$. Then by a previous observation, $\Phi_2(u(t)) \to \infty$, since $\Phi(u(t)) \in S_0$ and $\pi_1 \circ \Phi(u(t)) \to \lambda_0$. (In particular, this means that Φ_2 has a pole at $u = 0$).

In order to lighten the notations, let $z_t := \Phi_2(u(t))$, $f_t := f_{\Phi_1(u(t))}$, $\varphi_t := \varphi_{\Phi_1(u(t))}$ and $\psi_t := \psi_{\Phi_1(u(t))}$, where $f_\lambda = \varphi_\lambda \circ f \circ \psi_\lambda^{-1}$.

Since $f_t(z_t) = \varphi_t \circ f \circ \psi_t^{-1}(z_t) \to v \in \widehat{\mathbb{C}}$ (the next point in the virtual cycle), we have

$$f \circ \psi_t^{-1}(z_t) = \varphi_t^{-1}(v + o(1)) = v + o(1),$$

using the fact that $\varphi_0 = \mathrm{Id}$. Therefore $\psi_t^{-1}(z_t)) \in T$, where T is a tract of f above the asymptotic value v. Let $G(u) := \psi_{\Phi_1(u)}^{-1}(z(\Phi_1(u))) : \mathbb{D} \to \widehat{\mathbb{C}}$. Then G is open by Lemma 2.11, and $G(\mathbb{D})$ is a neighborhood of ∞. But we also proved that $G(\mathbb{D})$ is contained in T a tract of f since $u(t)$ was arbitrary, which is absurd.

□

Theorem D together with Propositions 5.5 and 5.7 implies density of parabolic parameters. Notice that for this corollary it is not necessary to assume the existence of a function in the family which has non omitted poles.

Corollary 5.10. *Let $\{f_\lambda\}_{\lambda \in M}$ be a natural family of finite type meromorphic maps. Then parabolic parameters are dense in the activity locus.*

Proof. Let λ_0 belong to the activity locus and let V be an arbitrary small neighborhood of λ_0. Our goal is to produce at least one parabolic cycle in V.

By Lemma 5.9, for every $k \in \mathbb{N}^*$ large enough, we can find $\lambda_k \in V$ such that f_{λ_k} has an attracting cycle of period at least k. Each of these attracting cycles must capture at least one singular value. But the number of singular values is finite and constant throughout V. Therefore, only finitely many of these attracting cycles can be followed holomorphically over V and remain persistently attracting on V.

This implies either the existence of parabolic parameters or the existence of a parameter λ at which an attracting cycle disappears to infinity. In the first case we are done, and in the second as well by Theorem D.

□

5.2. $\mathcal{J}-$**stable parameters: proof of Theorem E, Corollary F and Corollary G.** From our results above we can now prove Theorem E.

Proof of Theorem E. If for all $\lambda \in M$, f_λ never has any non-omitted pole, then the result is trivial by the classical theory and Proposition 5.4. We therefore assume that the assumptions of Proposition 5.5 are satisfied.

We will prove that (1) ⇔ (2), (1) ⇒ (3) ⇒ (2) ⇔ (5) ⇒ (4) ⇒ (3).

- (1) ⇒ (2): This implication mostly follows the arguments in [MSS83]. If the Julia set moves holomorphically over U, there is $\lambda_0 \in U$ and a holomorphic motion $H : U \times J_{\lambda_0} \to \hat{\mathbb{C}}$ preserving the dynamics. Hence $H_\lambda(J_{\lambda_0}) = J_\lambda$ for all $\lambda \in U$ and
$$H_\lambda(f_{\lambda_0}(z)) = f_\lambda(H_\lambda(z))$$
for all $z \in J_{\lambda_0}$. This means that H_λ maps critical points (resp. values) of f_{λ_0} in J_{λ_0} to critical points (resp. values) of f_λ in J_λ (see e.g. [McM88] for details). Likewise H_λ maps asymptotic values of f_{λ_0} in J_{λ_0} to asymptotic values of f_λ in J_λ, since the latter are locally omitted values.
Hence singular values and their full orbits in the Julia set can be followed by the conjugacy H_λ. Since f_{λ_0} has finitely many singular values, the union of their (forward) orbits is a countable set; but the Julia set is perfect and uncountable, hence we can consider three points z_1, z_2 and z_3 in J_{λ_0} which are disjoint from the forward orbits of the singular values of f_{λ_0}. Consequently, by the injectivity of the holomorphic motion, for all $\lambda \in U$, $H_\lambda(z_i)$, $i = 1, 2, 3$, is disjoint from the forward orbits of the singular values of of f_λ. By Montel's Theorem it follows that the forward orbits of the singular values form normal families, and hence every singular value is passive in U. On the other hand, if a singular orbit lies in the Fatou set of f_{λ_0} then it must remain in the Fatou set of f_λ for every $\lambda \in U$. The orbit then misses all points in the Julia set and the same argument applies.
- (2) ⇒ (1): Assume that the Julia set does *not* move holomorphically over U. Then by Proposition 3.3, either two periodic points in the Julia set collide, or one periodic cycle in the Julia set exits the domain. In the first case, this means that there exists $\lambda_0 \in U$ with a non-persistent parabolic periodic point: there exists $z_{\lambda_0} \in \mathbb{C}$ such that $f^n_{\lambda_0}(z_{\lambda_0}) = z_{\lambda_0}$, $(f^n_{\lambda_0})'(z_{\lambda_0}) = 1$, and $\lambda \mapsto (f^n_{\lambda_0})'(z_{\lambda_0})$ is non-constant on U. Then its parabolic basin must contain at least one singular value v_{λ_0}, and therefore be active. In the second case, a cycle exits the domain at $\lambda_0 \in U$, and by Theorem A, f_{λ_0} has either an active critical value or an active asymptotic value.
- (1) ⇒ (3): Assume that the Julia set moves holomorphically over U, and let H_λ be the conjugating holomorphic motion as above. Then H_λ maps repelling periodic points of f_{λ_0} to repelling periodic points of f_λ in $J(f_\lambda)$ of the same period. Let N be the maximal period of non-repelling cycles for f_{λ_0} (which is finite by Fatou-Shishikura's inequality [EL92]); then for all $\lambda \in U$, cycles of period more than N must be repelling, which implies that attracting cycles have period at most N.
- (3) ⇒ (2): This follows directly from Lemma 5.9.
- (5) ⇒ (4): If there are no non-persistent parabolic parameters in U, there cannot be any $\lambda \in U$ such that f_λ has a non-persistent virtual cycle, and hence no cycle can

exit the domain for a parameter in U by Theorem A. Hence no attracting cycle can disappear to infinity nor change into a repelling cycle, which implies that all attracting cycles remain attracting throughout U.

- (4) $\Rightarrow$ (3) This implication is obvious.

□

We can therefore define the bifurcation locus of the family M, $\mathrm{Bif}(M)$, as the set of $\lambda \in M$ for which the above conditions are not satisfied in any neighborhood of λ. With this notation, Corollary F states:

Corollary F. *Let $\{f_\lambda\}_{\lambda\in M}$ be a natural family of finite type meromorphic maps. Then* $\mathrm{Bif}(\mathring{M}) = \emptyset$.

Proof. By Theorem E, parabolic parameters are dense in the bifurcation locus. Hence whenever a singular value is active at say λ_0, one may perturb λ_0 to make it into a non-persistent parabolic parameter λ_1 whose parabolic basin attracts one of the singular values, say v_λ^1. The parameter λ_1 can be perturbed to λ_1' for which v_λ^1 is attracted to an attracting cycle, persistent in a neighborhood of λ_1'. If this parameter is not in the bifurcation locus, we are done; otherwise, there is another singular value which is active, and we can perturb it to a new parabolic parameter λ_2 while keeping v_λ^1 to be attracted to its attracting cycle, and repeat the reasoning. Since there are only finitely many singular values, this proves that arbitrarily close to any $\lambda_0 \in \mathrm{Bif}(M)$ we may find λ_* such that all singular values of f_{λ_*} are passive (because each is attracted to an attracting cycle), and therefore $\lambda_* \notin \mathrm{Bif}(M)$. □

Finally, we prove Corollary G.

Proof of Corollary G. Let $v(\lambda_0)$ be a singular value of f_{λ_0}, and let m be the period of the attracting cycle it converges to. Let $h_n(\lambda) := f_\lambda^n(v(\lambda))$: then $h_{km}(\lambda_0)$ converges (as $k \to +\infty$) to an attracting periodic point, which we denote by $h(\lambda_0)$. Moreover, this remains true for all λ in a suitably small neighborhood of λ_0: the sequence h_{km} converges to h near λ_0, where $h(\lambda)$ is an attracting periodic point of period m.

By Theorem E, all singular values are passive on U, therefore the sequence h_{km} admits subsequences that converge locally uniformly on U; and any such limit must coincide with h on a neighborhood of λ_0. Therefore, the sequence h_{km} in fact converges on U to a map which we still denote by h. Since $f_\lambda^m(h(\lambda)) - h(\lambda) = 0$ on a neighborhood of λ_0, we have that $h(\lambda)$ is a periodic point of period (dividing) m for all $\lambda \in U$. Moreover, it is clear that this periodic point cannot be repelling: so the multiplier map $\rho(\lambda) := (f_\lambda^m)'(h(\lambda))$ takes values in $\overline{\mathbb{D}}$. By the openness theorem, either ρ is non-constant and open, or ρ is constant equal to $\rho(\lambda_0) \in \mathbb{D}$. In either case, $\rho(\lambda) \in \mathbb{D}$ for every $\lambda \in U$, which proves that $v(\lambda)$ remains captured by an attracting cycle for all $\lambda \in U$.

Since this holds for any singular value, Corollary G is proved. □

6. A simple example with a dense bifurcation locus

In this section we construct a natural family of transcendental entire maps of the form $f_\lambda = f + \lambda$ with $\lambda \in \mathbb{C}$, and f chosen so that f_{λ_n} has a non-persistent parabolic point for all λ_n in a countable dense subset of $\mathbb{C}$. It will follow that the bifurcation locus for this family has nonempty interior. With this construction we have no control on the set of singular values for f, and consequently, for f_λ.

Proposition 6.1 (Bifurcation locus with nonempty interior)**.** *There exists an entire map f such that the natural family $\{f_\lambda = f + \lambda\}_{\lambda \in \mathbb{C}}$ is not $\mathcal{J}$-stable on any open subset of $\mathbb{C}$.*

Proof. Let $\{\lambda_n : n \in \mathbb{N}\}$ be a countable dense subset of the complex plane. By classical results (see e.g. Theorem [Rud87, Theorem 15.13]), there exists en entire map f such that for all $n \in \mathbb{N}$,

(1) $f(n) = n - \lambda_n$
(2) $f'(n) = -1$
(3) $f''(n) = 1$.

It follows that for all $n \in \mathbb{N}$, $f_{\lambda_n}(n) = n$, $f'_{\lambda_n}(n) = -1$, and $f''_{\lambda_n}(n) = 1$. In other words, each f_{λ_n} has a parabolic fixed point of multiplier -1 at the positive integer n. We now show that such fixed points are not persistently parabolic by studying their multiplier map.

By the Implicit Function Theorem, for each $n \in \mathbb{N}$ there exists a holomorphic map $\lambda \mapsto z_n(\lambda)$ defined in a neighborhood U_n of λ_n, such that $f_\lambda(z_n(\lambda)) = z_n(\lambda)$ and $z_n(\lambda_n) = n$.

Differentiating the relation

$$f_\lambda(z_n(\lambda)) - z_n(\lambda) \equiv 0 \text{ on } U_n,$$

we find $\frac{d}{d\lambda}_{|\lambda=\lambda_n} z_n(\lambda) = \frac{1}{2}$.

We now consider the multiplier map defined as $\rho_n(\lambda) := f'_{\lambda_n}(z_n(\lambda))$. We want to show that it is not constant on U_n. We have

$$\frac{d}{d\lambda}_{|\lambda=\lambda_n} \rho_n(\lambda) = f''_{\lambda_n}(z_n(\lambda_n)) \frac{d}{d\lambda}_{|\lambda=\lambda_n} z_n(\lambda) + \frac{d}{d\lambda}_{|\lambda=\lambda_n} f_\lambda(z_n) = 1 \cdot \frac{1}{2} + 1 = \frac{3}{2} \neq 0.$$

Therefore, the map ρ_n is non-constant, and so the fixed point $z_n(\lambda)$ is non-persistently neutral for f_λ. We now show that this implies that the family $\{f_\lambda\}_{\lambda \in \mathbb{C}}$ cannot be $\mathcal{J}$-stable in any neighborhood of λ_n.

Indeed, assume for a contradiction that $\{f_\lambda\}_{\lambda \in \mathbb{C}}$ is $\mathcal{J}$-stable on some disk D centered at λ_n, and let h_λ denote the corresponding holomorphic motion of $\mathcal{J}(f_\lambda)$, which must respect the dynamics. Since $z_n(\lambda_n) = n$ is a parabolic fixed point for λ_n, it lies in $\mathcal{J}(f_{\lambda_n})$. Then the motion $h_\lambda(n)$ must coincide with $z_n(\lambda)$ for $\lambda \in D \cap U_n$, by unicity in the Implicit Function Theorem. In particular, $z_n(\lambda) = h_\lambda(n)$ must be in $\mathcal{J}(f_\lambda)$ for all $\lambda \in U_n \cap D$. On the other hand, since ρ_n is non-constant, there exists λ_* arbitrarily close to λ_n such that $\rho_n(\lambda_*) \in \mathbb{D}$; hence $z(\lambda_*)$ is attracting and must belong to the Fatou set of f_{λ_*}, a contradiction. We are done since $\{\lambda_n : n \in \mathbb{N}\}$ is dense in $\mathbb{C}$. □

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