

# BIFURCATION LOCI OF FAMILIES OF FINITE TYPE MEROMORPHIC MAPS

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**ABSTRACT.** We study bifurcation phenomena in natural families of rational, (transcendental) entire or meromorphic functions of finite type  $\{f_\lambda := \varphi_\lambda \circ f_{\lambda_0} \circ \psi_\lambda^{-1}\}_{\lambda \in M}$ , where  $M$  is a complex connected manifold,  $\lambda_0 \in M$ ,  $f_{\lambda_0}$  is a meromorphic map and  $\varphi_\lambda$  and  $\psi_\lambda$  are families of quasiconformal homeomorphisms depending holomorphically on  $\lambda$  and with  $\psi_\lambda(\infty) = \infty$ . There are fundamental differences compared to the rational or entire setting due to the presence of poles and therefore of parameters for which singular values are eventually mapped to infinity (singular parameters). Under mild geometric conditions we show that singular (asymptotic) parameters are the endpoint of a curve of parameters for which an attracting cycle progressively exits the domain, while its multiplier tends to zero. This proves the main conjecture in [FK21] (asymptotic parameters are virtual centers) in a very general setting. Other results in the paper show the connections between cycles exiting the domain, singular parameters, activity of singular orbits and  $\mathcal{J}$ -unstability, converging to a theorem in the spirit of Mañé-Sad-Sullivan's celebrated result in [MSS83, Lyu84].

## 1. INTRODUCTION

We consider dynamical systems given by the iterates of meromorphic functions (rational or transcendental) in the complex plane, with a finite number of singularities of the inverse map (*finite type*). Given a holomorphic family  $\{f_\lambda\}_{\lambda \in M}$  of such systems, being  $M$  a connected complex manifold, we aim at understanding the nature of the subset of  $M$  for which some sort of structural stability holds; or equivalently, the bifurcations which may occur and how they can be characterized in terms of different dynamical aspects as, for example, possible bifurcations of singular or periodic orbits. Our results show that there are fundamental differences compared to the same problem for holomorphic families of rational or entire functions, successfully addressed by the seminal papers [MSS83, Lyu84] and [EL92] respectively.

Given a rational or entire function  $f$ , the *Fatou set*  $\mathcal{F}(f)$  or *stable set* of  $f$  is defined as the largest open set where the family of iterates  $\{f^n\}_{n \geq 0}$  is normal in the sense of Montel. However, if  $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  is a meromorphic (transcendental) map with at least one non-omitted pole, we need to require additionally that the family of iterates  $\{f^n\}_{n \geq 0}$  is first well defined *and* then normal.

In both cases, the *Julia set* is the complement of the Fatou set and it is the closure of the repelling periodic points. If  $f \in \mathcal{M}$ , the Julia set also coincides with the closure of the backwards orbit of the essential singularity,  $\mathcal{J}(f) = \overline{\mathcal{O}^{-\infty}}$ , or equivalently the closure of the set of prepoles of all orders.

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For a holomorphic family  $\{f_\lambda\}_{\lambda \in M}$  of meromorphic maps, structural stability at a parameter  $\lambda_0 \in M$  is commonly understood as  $\mathcal{J}$ -*stability* or, more precisely, as the Julia set  $\mathcal{J}(f_\lambda)$  moving holomorphically with respect to  $\lambda$  in a neighborhood  $U$  of  $\lambda_0$ . This means that there exists a holomorphic motion  $H : U \times \mathcal{J}(f_{\lambda_0}) \rightarrow \mathcal{J}(f_\lambda)$  respecting the dynamics, which in particular it implies that  $f_{\lambda_0}|_{\mathcal{J}(f_{\lambda_0})}$  is topologically conjugate to  $f_\lambda|_{\mathcal{J}(f_\lambda)}$  for all  $\lambda \in U$ . (See Section 2.1 for details.)

The dynamics of  $f$  are determined to a large extent by the dynamics of its *singular values* or points  $v \in \widehat{\mathbb{C}}$  for which not all univalent branches of  $f^{-1}$  are locally well defined. If  $f$  is rational, singular values are always *critical values*  $v = f(c)$  with  $f'(c) = 0$ . If  $f$  is transcendental we must also take *asymptotic values* into account, that is values  $v = \lim_{t \rightarrow \infty} f(\gamma(t))$  where  $\gamma$  is a curve tending to infinity when  $t \rightarrow \infty$ . An example is  $v = 0$  for the exponential map. Accumulations of critical or asymptotic values are also singular, hence we define the set of singular values or *singular set* as

$$S(f) = \overline{\{v \in \widehat{\mathbb{C}} \mid v \text{ is a critical or an asymptotic value}\}}.$$

Different classes are distinguished depending on the cardinality and location of  $S(f)$ . In this paper we restrict to maps in the *Speiser class*  $\mathcal{S}$ , or maps of *finite type*, consisting on those for which  $\#S(f) < \infty$ , although occasionally we may also refer to the *Eremenko-Lyubich class*  $\mathcal{B}$ , of functions with a bounded set  $S(f)$ .

Maps of finite type hold specific dynamical properties which are not satisfied in the general cases. For one, their Fatou set is made exclusively of preperiodic or periodic components, being the latter basins of attraction of attracting or parabolic orbits, or rotation domains (Siegel disks or Herman rings); at the same time, all asymptotic values  $v$  of a map of finite type are *logarithmic* and hence they have logarithmic *tracts* lying over them (see Section 6.1). Examples of finite type families include all rational maps of a given degree, the exponential family  $\lambda e^z$ , the tangent family  $\lambda \tan(z)$ , maps of the form  $R(z)e^{P(z)}$  with  $R$  rational and  $P$  polynomial or the Weierstrass  $\wp$  function, among many others. All results in this paper hold for families of finite type maps but in some cases, which will be specified, they do for class  $\mathcal{B}$  or even in more generality.

The celebrated results of Mañé, Sad, and Sullivan, and Lyubich, all in the 80's, relate  $\mathcal{J}$ -stability for families of rational maps to sudden changes in the asymptotic dynamics of critical points. To formalize this concept, if  $\{f_\lambda\}_{\lambda \in M}$  is a family of rational maps whose critical values are holomorphic functions of  $\lambda$ , we will say that a critical value  $v_\lambda \in \widehat{\mathbb{C}}$  is *passive* at  $\lambda_0$  if the sequence of holomorphic maps  $\{\lambda \mapsto f_\lambda^n(v_\lambda)\}_{n \in \mathbb{N}}$  is normal in some neighborhood of  $\lambda_0$ . Otherwise  $v_\lambda$  is *active*. A version of the bifurcation theorem for rational maps then reads as follows.

**Theorem 1.1** ([MSS83, Lyu84], c.f.[McM94]). *Let  $\{f_\lambda\}_{\lambda \in M}$  be a holomorphic family of rational maps of degree  $d \geq 2$ , and let  $\lambda_0 \in M$ . Suppose that the critical points of  $f_\lambda$  are holomorphic functions of  $\lambda$ . Then, the following are equivalent.*

- (a) *The Julia set  $\mathcal{J}(f_\lambda)$  moves holomorphically over a neighborhood of  $\lambda_0$ .*
- (b) *All critical values are passive in a neighborhood of  $\lambda_0$ .*
- (c) *The maximal period of attracting cycles is bounded in a neighborhood of  $\lambda_0$ .*

Due to the  $\lambda$ -Lemma ([MSS83] or Theorem 2.3),  $\mathcal{J}$ -stability in an open set of parameters is equivalent to being able to follow and distinguish periodic orbits across  $U$ . A key point in the proof of Theorem 1.1 is that the only obstruction for the existence of a holomorphic motion of the periodic points is the collision of several different periodic orbits, merging in a parabolic

cycle. However, in the presence of an essential singularity at infinity (i.e. for transcendental maps) there is another possible obstruction, namely that a periodic cycle exits the domain of definition of  $f_\lambda$ .

**Definition 1.2** (Periodic cycle exits the domain). Let  $\{f_\lambda\}_{\lambda \in M}$  be a family of meromorphic maps. We say that a cycle *exits the domain* at  $\lambda_0 \in M$  if there exists a curve  $\lambda(t) \rightarrow \lambda_0$  as  $t \rightarrow \infty$  such that a point  $z(\lambda(t))$  in the periodic cycle satisfies that  $\lim_{t \rightarrow \infty} z(\lambda(t)) = \infty$ .

Observe as an example that if  $f_\lambda(z) = \lambda z + e^z$ , all fixed points exit the domain when  $\lambda \rightarrow 0$ .

Eremenko and Lyubich proved that this never occurs in families of entire functions of finite type [EL92, Theorem 2] (although their proof applies also to entire maps in class  $\mathcal{B}$ ).

A fundamental difference in the transcendental meromorphic setting is that, due to the presence of poles, periodic cycles may exit the domain at finite values of the parameter, independently on the number of singular values. Instances of this phenomenon were described in [KK97] for the tangent family  $T_\lambda(z) = \lambda \tan z$  or in [FK21] for more general one-dimensional families, at parameters called *virtual centers*, lying in the boundary of hyperbolic components, and defined as accumulation of parameter values for which a cycle progressively exits the domain while its multiplier tends to zero, thus playing the role of true *centers* in rational dynamics (see also [DK89, CK19, CJK19]). We will prove that, even in a much more general setting, cycles can *only* exit the domain at virtual centers which are moreover always accessible by curves of parameters with these properties (see Theorem A and Corollary A' below).

We claim that all obstructions for  $\mathcal{J}$ -stability are precisely collision of periodic orbits, cycles exiting the domain, or accumulation thereof (see Proposition 3.3). And, as usual, these phenomena occur only with the complicity of the singular values, which returns us to the concept of an active singular value, now generalized to include the transcendental meromorphic setting.

**Definition 1.3** (Active/passive singular value and exceptional family.). Let  $\{f_\lambda\}_{\lambda \in M}$  be a family of rational, entire or meromorphic maps. We say that a singular value  $v_\lambda$  is *passive* at a parameter  $\lambda_0$  if

- (i) there exists a parameter neighborhood  $U \subset M$  of  $\lambda_0$  such that the family  $\{f_\lambda^n(v_\lambda)\}_{n \in \mathbb{N}}$  is well defined *and* normal in  $U$ , or
- (ii) there exists  $N \in \mathbb{N}$  such that  $f_\lambda^N(v_\lambda) \equiv \infty$  in  $U$ .

A singular value  $v_\lambda$  is *active* at a parameter  $\lambda_0$  if it is not passive.

A family is called *exceptional* if there exists a singular value  $v_\lambda$  and some  $N \geq 1$ , such that for all  $\lambda \in M$ ,  $f_\lambda^N(v_\lambda) = \infty$ .

We remark from this definition that a singular value that is persistently mapped to infinity for all values of the parameter is always passive. This includes infinity being itself an asymptotic value, as it is the case for every entire (transcendental) map. Examples of active singular values would be those (non-persistently) escaping to infinity for a parameter  $\lambda_0$ , or converging to a (non-persistent) parabolic cycle, or being preperiodic to a repelling periodic cycle also in a non-persistent fashion.

Thus, the phenomenon that makes the difference in the transcendental meromorphic setting is that of singular values being eventually mapped to infinity, out of the domain of definition of  $f_\lambda$ , or in other words, truncated singular orbits. The special parameters for which this takes place are defined as follows.

**Definition 1.4** (Singular parameter). A parameter  $\lambda$  is *singular* if there is a singular value  $v_\lambda$  of  $f_\lambda$  such that  $f_\lambda^n(v_\lambda) = \infty$  for some  $n \geq 0$  and this property does not persist on all of

$M$ , that is,  $f_\lambda^n(v_\lambda) \neq \infty$  on  $M$ . The integer  $n$  is called the *order* of the singular parameter. The singular parameter  $\lambda$  is *critical* if  $v_\lambda$  is a critical value and it is *asymptotic* if  $v_\lambda$  is an asymptotic value.<sup>1</sup>

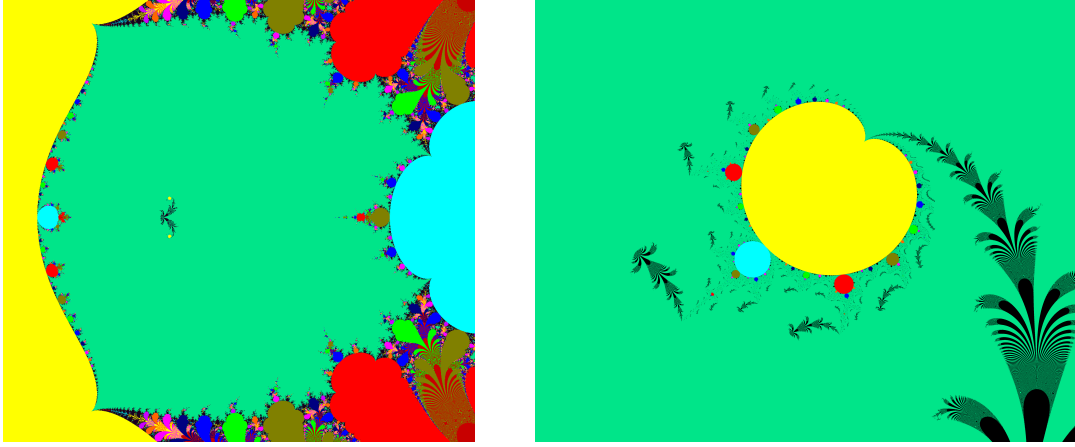


FIGURE 1. Parameter plane of the natural family  $f_a(z) = a \left(1 - \frac{ae^z}{(a+0.5)z+a}\right)$  which has an attracting fixed point at  $z = 0$  with multiplier 0.5, a persistent asymptotic value at infinity, an asymptotic value at  $v_a = a$  and a critical point at  $c_a = \frac{1}{1+2a}$  [FK21]. Left: In the large green central region  $v_a$  is attracted to the origin while  $c_a$  is free, while outside it is the other way around. Asymptotic parameters are dense in the outer boundary of this region. Right: Zoom of part of the central bouquet. In black, parameters for which  $c_a$  escapes to infinity and in color,  $c_a$  is attracted to attracting periodic orbits. The small bouquets are attached at critical parameters, which are accumulated by centers.

At a singular parameter, some singular value is active. Conversely, we will see that if a singular value is active at a parameter  $\lambda_0$ , then we can find singular parameters arbitrarily close to  $\lambda_0$  (see Proposition 5.1).

Asymptotic parameters of order  $n$  have what is known as a *virtual cycle* of period  $n + 1$ : a finite, cyclically ordered set  $a_1, \dots, a_n \in \hat{\mathbb{C}}$  such that for all  $1 \leq i \leq n$ , either  $a_i \in \mathbb{C}$  and  $a_{i+1} = f(a_i)$  or  $a_i = \infty$  and  $a_{i+1}$  is an asymptotic value (c.f. [FK21]).

Our theorems will show that, with great generality, cycles can exit the domain only at asymptotic parameters and moreover, every asymptotic parameter must be a virtual center, thus proving a conjecture in [FK21, Remark 6.11 and Conjecture 6.17]. To do so, we shall need to relate the different concepts introduced so far: cycles existing the domain, active singular values, virtual cycles, singular parameters and  $\mathcal{J}$ -stability, to finally converge into a bifurcation theorem for meromorphic families in the spirit of Theorem 1.1.

**Statement of results.** We work in the setting of *natural families* of meromorphic maps, which consist of compositions of the form

$$\{f_\lambda := \varphi_\lambda \circ f_{\lambda_0} \circ \psi_\lambda^{-1}\}_{\lambda \in M},$$

where  $\lambda_0 \in M$ ,  $f := f_{\lambda_0}$  is a meromorphic map and  $\varphi_\lambda$  and  $\psi_\lambda$  are families of quasiconformal homeomorphisms depending holomorphically on  $\lambda$  and with  $\psi_\lambda(\infty) = \infty$  (see Section 2). Most well known one dimensional families of entire or meromorphic maps of finite type are natural,

<sup>1</sup>In [FK21], asymptotic parameters were called *virtual cycle parameters*.

like for example  $E_\lambda(z) = \exp(z) + \lambda = (z \mapsto z + \lambda) \circ \exp(z) \circ \text{Id}$ , or  $T_\lambda(z) = \lambda \tan(z) = (z \mapsto \lambda z) \circ \tan z \circ \text{Id}$ . In a natural family, singular values are always holomorphic functions of the parameter since they are of the form  $v_\lambda = \varphi_\lambda(v)$  where  $v$  is a singular value of  $f$ . The cardinality (and the nature) of  $S(f_\lambda)$  is independent of  $\lambda$ , as singular values cannot collide and the type of singularity is preserved because of its topological nature. The concept of a natural family is quite general and convenient to study parameter spaces in holomorphic dynamics. It was used, for example, to prove the absence of wandering domains for families of entire maps of finite type, since for every  $f$  finite type map, there exists a natural family of finite type maps  $\text{Def}(f)$  containing  $f$ , which is a complex analytic of dimension  $\#S(f)$  [EL92, Eps93, GK86].

Our first main result shows that cycles cannot exit the domain at a certain parameter value  $\lambda_0 \in M$  unless there is at least one active singular value at  $\lambda_0$  (see Lemma 3.4 and Theorem 3.5).

**Theorem A** (A cycle exiting the domain implies activity). *Let  $(f_\lambda)_{\lambda \in M}$  be a natural family of finite type meromorphic maps, and let  $\lambda_0 \in M$  be such that a cycle of period  $n$  exits the domain at  $\lambda_0$ . Then  $\lambda_0$  is a singular parameter. More precisely, this cycle converges to a virtual cycle for  $f_{\lambda_0}$ , which contains (at least) either an active asymptotic value, or an active critical point.*

Some important remarks are in order.

**Remarks 1.5.**

- (1) By definition, virtual cycles always contain at least one asymptotic value, which may be either active or passive. The theorem asserts that if it does not contain any active critical point, then at least one of those asymptotic values must be active.
- (2) It is possible a priori that the limit virtual cycle contains a critical value but not a critical point. In that case, the theorem still asserts that regardless of whether this critical value is active or not, there must be an additional active asymptotic value in the virtual cycle.
- (3) If every point in the cycle goes to  $\infty$ , then the limit virtual cycle is  $\infty, \dots, \infty$  and therefore cannot contain any critical point; then the theorem asserts that  $\infty$  is an active singular value for  $f_{\lambda_0}$ . In particular, Theorem B generalizes [EL92, Theorem 2], since  $\infty$  is always a passive asymptotic value for families of finite type entire maps.

In Theorem A, the possibility of the cycle exiting the domain because a *critical value* is active is left open if the family is exceptional (i.e. in the presence of an asymptotical value being persistently mapped to infinity). We believe it is plausible that this possibility could be discarded. In any event, when the family is non-exceptional we have the following corollary.

**Corollary A'.** *Let  $(f_\lambda)_{\lambda \in M}$  be a non-exceptional natural family of finite type meromorphic maps, and let  $\lambda_0 \in M$  be such that a cycle of period  $n$  exits the domain at  $\lambda_0$ . Then the limit virtual cycle contains an active asymptotic value, and  $\lambda_0$  is an asymptotic parameter.*

The question of whether a partial converse to Theorem A holds is most natural. Are asymptotic parameters always the result of a cycle that just exited the domain? We prove that the answer is affirmative under a certain mild technical condition **(T)** which requires the asymptotic tracts above the active singular value to have good geometry (see Definition 4.1). In particular if tracts contain a sector, condition **(T)** is always satisfied. Under this technical hypothesis we can prove the following (see Theorem 4.4).

**Theorem B** (Accessibility Theorem). *Let  $(f_\lambda)_{\lambda \in M}$  be a natural family of finite type meromorphic maps, and  $\lambda_0 \in M$  be an asymptotic parameter of order  $n$ . Assume that at least one*

tract above the associated asymptotic value satisfies  $(T)$ . Then there is a cycle of period  $n + 1$  exiting the domain at  $\lambda_0$ , and moreover its multiplier goes to zero as it exits the domain.

In [FK21, Remark 6.11 and Conjecture 6.17] it was conjectured that every asymptotic parameter is a virtual center and hence lies in the boundary of some hyperbolic component (whenever this makes sense). Theorem B proves this conjecture in much greater generality than it was originally stated.

Using an auxilliary shooting result interesting on its own (see Proposition 2.6) we are able to prove that any parameter for which some singular value is active, can be approximated by singular parameters. (see Proposition 5.1 for a stronger statement). With similar techniques, we show additionally that critical parameters of order  $n$  can be approximated by sequences of true centers of period  $n + 2$  or  $n + 3$  (see Proposition 5.3). With these tools at hand and putting together Theorems A and B we obtain the following equivalences.

**Theorem C.** *Let  $(f_\lambda)_{\lambda \in M}$  be a non-exceptional natural family of finite type meromorphic maps whose tracts satisfy  $(T)$ . Let  $U \subset M$  a simply connected domain. The following are equivalent:*

- (1) *there are no asymptotic parameters in  $U$*
- (2) *there are no cycles exiting the domain in  $U$*

*If moreover the maps  $f_\lambda$  have at least one non-omitted pole, then this is also equivalent to*

- (3) *all asymptotic values are passive on  $U$ .*

We are now ready to discuss the bifurcation locus of a natural family. The set of parameters

$$\mathcal{A}(v_\lambda) := \{\lambda_0 \in M : v_\lambda \text{ is active at } \lambda_0\}$$

is called the *activity locus* of  $v_\lambda$ . On the other hand, the *activity locus*  $\mathcal{A}(M)$  for the natural family  $(f_\lambda)_{\lambda \in M}$  is defined as

$$\mathcal{A}(M) := \{\lambda_0 \in M : \text{there exists a singular value which is active at } \lambda_0\}.$$

We can finally conclude with the theorem about  $\mathcal{J}$ -stability that generalizes Theorem 1.1.

**Theorem D** ( $\mathcal{J}$ -stability). *Let  $(f_\lambda)_{\lambda \in M}$  be a natural family of finite type meromorphic maps. Let  $U \subset M$  be a simply connected domain in parameter space. The following are equivalent:*

- (1) *The Julia set moves holomorphically over  $U$*
- (2) *Every singular value is passive on  $U$*

*If moreover the tracts of  $f_\lambda$  satisfy  $(T)$ , then the statements above are also equivalent to*

- (3) *The maximal period of attracting cycles is bounded on  $U$ .*

In view of Theorem D it makes sense to define the *bifurcation locus* of the natural family as

$$\text{Bif}(M) = \{\lambda \in M \mid f_\lambda \text{ is not } \mathcal{J}\text{-stable}\},$$

or equivalently as the set of parameters for which some of the conditions in Theorem D is not satisfied. Since  $\mathcal{J}$ -stable parameters form an open set by definition, following the arguments in [MSS83] we obtain the well known statement for rational maps.

**Corollary D'** ( $\mathcal{J}$ -stable parameters form an open and dense set in  $M$ ). *If  $\{f_\lambda\}_{\lambda \in M}$  is a natural family of finite type meromorphic maps, then  $\text{Bif}(M)$  has no interior or, equivalently,  $\mathcal{J}$ -stable parameters are open and dense in  $M$ .*

We would like to finish by pointing out that the proofs of these theorems are fundamentally different from those in the Mañé, Sad and Sullivan theory for rational maps and also different from the ones for entire functions. For example the proof of (3)  $\implies$  (1) which works for rational or entire maps does not work in the meromorphic setting. Indeed, the fact that the period of attracting orbits is bounded by  $N$  for  $\lambda \in U$ , does not imply that periodic orbits of period higher than  $N$  remain repelling throughout  $U$  (and can therefore be followed holomorphically), since they could be exiting the domain. In the same direction, there does not seem to be any obvious reason for which the absence of non-persistent parabolic cycles should imply that the number of attracting cycles remains constant since, again, attracting cycles might be exiting the domain. Exploring these possible extensions is still work in progress.

The structure of the paper is as follows. In section 2 we state tools and prove preliminary results that will be useful throughout the paper, including the Shooting Lemma. Section 3 deals with the consequences of a cycle exiting the domain and contains the proofs of Theorem A and Corollary A'. The accessibility result, Theorem B, is proven in Section 4, while Section 5 contains the density results and the proofs of Theorems C and D.

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## 2. PRELIMINARIES

In this section we state some known results and prove several new tools that will be useful in the proofs of the main theorems.

### 2.1. Holomorphic families, holomorphic motions and $\mathcal{J}$ -stability.

**Definition 2.1** (Holomorphic family). A *holomorphic family*  $\{f_\lambda\}_{\lambda \in M}$  of meromorphic maps over a complex connected manifold  $M$  is a holomorphic map  $F : M \times \mathbb{C} \rightarrow \hat{\mathbb{C}}$  such that  $F(\lambda, \cdot) =: f_\lambda$  is a non-constant meromorphic map for every  $\lambda \in M$ .

**Definition 2.2** (Holomorphic motion). A *holomorphic motion* of a set  $X \subset \hat{\mathbb{C}}$  over a set  $U \subset M$  with basepoint  $\lambda_0 \in U$  is a map  $H : U \times X \rightarrow \hat{\mathbb{C}}$  given by  $(\lambda, x) \mapsto H_\lambda(x)$  such that

- (1) for each  $x \in X$ ,  $H_\lambda(x)$  is holomorphic in  $\lambda$ ,
- (2) for each  $\lambda \in U$ ,  $H_\lambda(x)$  is an injective function of  $x \in X$ , and,
- (3) at  $\lambda_0$ ,  $H_{\lambda_0} \equiv \text{Id}$ .

A holomorphic motion of a set  $X$  *respects the dynamics* of the holomorphic family  $F$  if  $H_\lambda(f_{\lambda_0}(x)) = f_\lambda(H_{\lambda_0}(x))$  whenever both  $x$  and  $f_{\lambda_0}(x)$  belong to  $X$ .

Note, that continuity of  $H$  is not required in the definition. However this property follows as a consequence, as shown in the  $\lambda$ -Lemma proved in [MSS83].

**Theorem 2.3** (The  $\lambda$ -Lemma [MSS83]). A *holomorphic motion*  $H$  of  $X$  as above has a unique extension to a holomorphic motion of  $\overline{X}$ . The extended map  $H : U \times \overline{X} \rightarrow \hat{\mathbb{C}}$  is continuous, and for each  $\lambda \in U$ ,  $H_\lambda : \overline{X} \rightarrow \hat{\mathbb{C}}$  is quasiconformal. Moreover, if  $H$  respects the dynamics, so does its extension to  $\overline{X}$ .

For further results about holomorphic motion and the  $\lambda$ -Lemma, see for instance [AM01].

**Definition 2.4** ( $\mathcal{J}$ -stability). Consider as above a holomorphic family  $F : M \times \mathbb{C} \rightarrow \hat{\mathbb{C}}$  of meromorphic maps. Given  $\lambda_0 \in M$ , the map  $f_{\lambda_0}$  is  $\mathcal{J}$ -stable if there exists a neighbourhood  $U \subset M$  of  $\lambda_0$  over which the Julia sets move holomorphically, i.e. there exists a holomorphic motion  $H : U \times \mathcal{J}_{\lambda_0} \rightarrow \hat{\mathbb{C}}$  such that  $\mathbb{H}_\lambda(\mathcal{J}_{\lambda_0}) = \mathcal{J}_\lambda$  and furthermore  $H_\lambda \circ f_{\lambda_0} = f_\lambda \circ H_\lambda$ .

In virtue of the  $\lambda$ -Lemma and the density of periodic points in the Julia set, it is enough to construct a holomorphic motion of the set of periodic points of every period, to obtain one for the entire Julia set.

**2.2. Natural families.** We will recall here some basic facts about natural families of finite type maps.

**Definition 2.5** (Natural family). A natural family of meromorphic maps is a holomorphic family  $(f_\lambda)_{\lambda \in M}$ , such that each  $f_\lambda$  is of the form  $f_\lambda = \varphi_\lambda \circ f_{\lambda_0} \circ \psi_\lambda^{-1}$ , for some  $\lambda_0 \in M$  and quasiconformal homeomorphisms  $\varphi_\lambda, \psi_\lambda : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  depending holomorphically on  $\lambda$ , and with  $\psi_\lambda(\infty) = \infty$ .

A simple observation is that  $\psi_\lambda$  maps the critical points of  $f_{\lambda_0}$  to those of  $f_\lambda$ , and that  $\varphi_\lambda$  maps the critical values and asymptotic values of  $f_{\lambda_0}$  to those of  $f_\lambda$ . In particular, in a natural family, the critical points and singular values always move holomorphically and can never collide, while the multiplicity of each singular value remains constant throughout the family. The normalization choice  $\psi_\lambda(\infty) = \infty$  guarantees that the only essential singularity of  $f_\lambda$  remains at  $\infty$ ; but it is possible to choose other normalizations by composing both  $\varphi_\lambda$  and  $\psi_\lambda$  by the same affine map (possibly depending on  $\lambda$ ).

Another useful observation is that if  $\lambda_1 \in M$  is given, then we may assume that  $\varphi_{\lambda_1} = \psi_{\lambda_1} = \text{Id}$  up to changing the base point. More explicitly, noting that

$$f_\lambda = \varphi_\lambda \circ \varphi_{\lambda_1}^{-1} \circ f_{\lambda_1} \circ \psi_{\lambda_1} \circ \psi_\lambda^{-1},$$

we may replace  $\varphi_\lambda$  by  $\tilde{\varphi}_\lambda := \varphi_\lambda \circ \varphi_{\lambda_1}^{-1}$  and  $\psi_\lambda$  by  $\tilde{\psi}_\lambda := \psi_\lambda \circ \psi_{\lambda_1}^{-1}$  without changing  $f_\lambda$ , and we have  $\tilde{\varphi}_{\lambda_1} = \tilde{\psi}_{\lambda_1} = \text{Id}$ .

**2.3. A shooting Lemma.** In the following sections we will need the fact that, if  $\lambda_0$  is a singular parameter, then we can find nearby parameters for which the singular value which is active at  $\lambda_0$  has some prescribed behaviour. Similar results can be proven in the rational setting using Montel's Theorem together with the non-normality of the family of iterates of the active singular value. In our setting in which  $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$  is a transcendental meromorphic map, and  $U \subset \mathbb{C}$  is a domain, the singular value  $v_\lambda$  could be active because its family of iterates  $\{f_\lambda(v_\lambda)\}_{n \in \mathbb{N}}$  is not defined in a parameter neighborhood of  $\lambda_0$  rather than not being normal. As a consequence, one cannot always apply Montel's Theorem as for entire maps or rational maps. Its role will be played by the following statement, which holds for any natural family of maps as long as they have at least one non-omitted pole. Notice that here we do not have assumptions on the set of singular values so that a priori functions could be in class  $\mathcal{B}$  or in the general class of meromorphic transcendental functions.

**Proposition 2.6** (Shooting Lemma). *Let  $(f_\lambda)_{\lambda \in M}$  be a natural family of meromorphic maps in  $\mathcal{M}$ , with at least one pole which is non-omitted. Let  $\lambda_0 \in M$  be a singular parameter of order  $n \geq 0$ , so that a singular value  $v_\lambda$  satisfies  $f_{\lambda_0}^n(v_{\lambda_0}) = \infty$ .*

- (a) *Let  $\lambda \mapsto \gamma(\lambda)$  be a holomorphic map such that  $\gamma(\lambda_0) \notin S(f_{\lambda_0})$ . Then we can find  $\lambda'$  arbitrarily close to  $\lambda_0$  such that  $f_{\lambda'}^{n+1}(v_{\lambda'}) = \gamma(\lambda')$ .*



- (b) For  $i = 1, \dots, 5$ , let  $\lambda \mapsto \gamma_i(\lambda)$  be five holomorphic maps such that  $\{\gamma_i(\lambda_0)\}_{i=1}^5$  are distinct. Then there exists at least one  $i, 1 \leq i \leq 5$  and  $\lambda'$  arbitrarily close to  $\lambda_0$  such that  $f_{\lambda'}^{n+1}(v_{\lambda'}) = \gamma_i(\lambda')$ .

Observe that the maps  $\gamma$  and  $\gamma_i$ , which are holomorphic maps from a neighborhood of  $\lambda_0$  in  $M$  to  $\hat{\mathbb{C}}$ , are allowed to be constant. In particular, if  $\infty$  is not a singular value of  $f_{\lambda_0}$ , by taking  $\gamma(\lambda) \equiv \infty$  we obtain that the singular parameter  $\lambda_0$  is a limit of singular parameters of order  $n + 1$ .

The proof of Proposition 2.6 uses the following lemmas. The first one can be found in [BFJK18, Lemma 13] (see also [BF15, Lemma 4.6] for a more general statement). In the following, let us denote by  $\text{wind}(\sigma(t), P)$  the winding number of a curve  $\sigma(t)$  with respect to a point  $P$ .

**Lemma 2.7** (Computing winding numbers). *Let  $\gamma, \sigma : [0, 1] \rightarrow \mathbb{C}$  be two disjoint closed curves and let  $P_\gamma \in \gamma$  and  $P_\sigma \in \sigma$  be arbitrary points. Then*

$$(2.1) \quad \text{wind}(\sigma(t) - \gamma(t), 0) = \text{wind}(\gamma(t), P_\sigma) + \text{wind}(\sigma(t), P_\gamma).$$

As a consequence, we obtain the following.

**Lemma 2.8** (Fixed point theorem). *Let  $V$  be a Jordan domain, and let  $f, g$  be holomorphic functions in a neighborhood of  $\bar{V}$ . Suppose that  $g(\bar{V}) \subset f(V)$  and  $g(\partial V) \cap f(\partial V) = \emptyset$ . Then there exists  $\lambda \in V$  such that  $f(\lambda) = g(\lambda)$ .*

*Proof.* Consider the map  $F(\lambda) = f(\lambda) - g(\lambda)$ . Let  $\lambda(t), t \in [0, 1]$  be a parametrization of  $\partial V$ , and notice that  $f(\lambda(t))$  and  $g(\lambda(t))$  are two disjoint curves and hence  $F(\lambda(t)) \neq 0$  for every  $t \in [0, 1]$ . By the Argument Principle, if the winding number of  $F(\lambda(t))$  with respect to 0 is positive, then  $F$  has at least one zero in  $V$ .

Let  $P_f = f(\lambda(0))$  and  $P_g = g(\lambda(0))$ . Applying Lemma 2.7 we get

$$\text{wind}(F(\lambda(t)), 0) = \text{wind}(f(\lambda(t)) - g(\lambda(t)), 0) = \text{wind}(g(\lambda(t)), P_f) + \text{wind}(f(\lambda(t)), P_g).$$

The hypothesis  $g(\bar{V}) \subset f(V)$  implies that the curve  $g(\lambda(t))$  lies inside a bounded connected component of the complement of  $f(\lambda(t))$  from which we deduce that  $\text{wind}(g(\lambda(t)), P_f) = 0$ . The same hypothesis also implies that  $P_g \in f(V)$  which means, again by the Argument Principle, that  $\text{wind}(f(\lambda(t)) - P_g, 0) = \text{wind}(f(\lambda(t)), P_g) \geq 1$ . Hence  $\text{wind}(F(\lambda(t)), 0) > 0$  and the conclusion follows.  $\square$

In the proof of part (a) of the Shooting Lemma we will also need the following well known fact.

**Lemma 2.9** (Shrinking of holomorphic images). *Let  $U \subset \mathbb{C}$  be an open set and  $a, b \in \mathbb{C}$ . Suppose  $\{\varphi_n : U \rightarrow \mathbb{C} \setminus \{a, b\}\}_{n \in \mathbb{N}}$  is a sequence of holomorphic maps such that  $\varphi_n(u_0) \rightarrow \infty$  for a certain  $u_0 \in U$ . Then for every compact set  $K \subset U$ , the spherical diameter of  $\varphi_n(K)$  tends to 0.*

*Proof.* We claim that  $(\varphi_n)_{n \in \mathbb{N}}$  converges locally uniformly to  $\infty$ . By Montel's Theorem, the sequence  $(\varphi_n)_{n \in \mathbb{N}}$  admits converging subsequences. Let  $(\varphi_{n_k})_{k \in \mathbb{N}}$  be any such subsequence, and let  $\varphi : U \rightarrow \hat{\mathbb{C}}$  be the limit function. Since by assumption for all  $k \in \mathbb{N}$ ,  $\infty \notin \varphi_{n_k}(U)$  and  $\varphi(u_0) = \infty$ , it follows from Hurwitz's Theorem that  $\varphi \equiv \infty$ . Since this holds for any converging subsequence, we have  $\lim_{n \rightarrow \infty} \varphi_n = \infty$ , and the lemma follows.  $\square$

While regular points as in part (a) always have neighborhoods with infinitely many univalent preimages, singular values may not, so to prove part (b) of Proposition 2.6 we make use of the following Theorem (for a proof see for example [Ber00, Proposition A.1]).

**Theorem 2.10** (Ahlfors Five Islands Theorem). *Let  $f$  be a transcendental meromorphic function in  $\mathbb{C}$  and  $E_1, \dots, E_5$  five simply-connected domains of  $\mathbb{C}$  bounded by analytic Jordan curves and such that the closures of the  $E_i$ , are mutually disjoint. Then for at least one  $i, 1 \leq i \leq 5$ , there are infinitely many bounded simply-connected domains (islands)  $D_n$  in  $\mathbb{C}$  such that  $f$  maps  $D_n$  univalently to  $E_i$ .*

Notice that the statement of [Ber00, Proposition A.1] is slightly different; here we have implicitly applied it to infinitely many domains at once to get infinitely many univalent preimages of at least one of the domains  $E_i$ .

Observe that if  $\lambda_0$  is a singular parameter of order  $n$  for the singular value  $v_\lambda$ , then the map  $\lambda \mapsto f_\lambda^n(v_\lambda)$  is a well defined meromorphic map in a sufficiently small neighborhood of  $\lambda_0$ , with an isolated pole at  $\lambda_0$ . Indeed, if a sequence of singular parameters of order equal to  $n$  were to accumulate at  $\lambda_0$ , by the discreteness of zeros of holomorphic functions we would have that  $\lambda \mapsto f_\lambda^k(v_\lambda)$  is identically equal to  $\infty$  for some  $k \leq n$ , which contradicts the assumption that this is not a persistent condition. Also impossible would be an approximating sequence of singular parameters of order strictly less than  $n$  since, by continuity, the order of  $\lambda_0$  would also need to be strictly less than  $n$ , also a contradiction. As a consequence of this fact,  $\lambda \mapsto f_\lambda^{n+1}(v_\lambda)$  has an essential singularity at  $\lambda_0$ .

We are now ready to prove Proposition 2.6.

*Proof of Proposition 2.6.* We start first with (a). By assumption  $f_\lambda = \varphi_\lambda \circ f \circ \psi_\lambda^{-1}$  and we may assume without loss of generality that  $\varphi_{\lambda_0} = \psi_{\lambda_0} = \text{Id}$  and hence  $f = f_{\lambda_0}$ . Let  $D$  be a disk centered at  $\gamma(\lambda_0)$  such that  $\overline{D}$  is disjoint from  $S(f_{\lambda_0})$  and let  $\delta > 0$  be such that  $\gamma(\overline{\mathbb{D}(\lambda_0, \delta)}) \subset D$  (see Figure 2).

Decreasing  $\delta$  if necessary, the function  $G(\lambda) := \psi_\lambda^{-1}(f_\lambda^n(v_\lambda))$  is a quasiregular map defined in  $\mathbb{D}(\lambda_0, \delta)$  and such that  $G(\lambda_0) = \infty$ . The map  $G$  is therefore either open or constant, and it cannot be constant for otherwise we would have  $f_\lambda^n(v_\lambda) = \infty$  for all  $\lambda \in M$ . We now pick an arbitrary one-dimensional slice containing  $\lambda_0$  in the parameter space  $M$  in which  $G$  is not constant, and we identify  $M$  with  $\mathbb{D}(\lambda_0, 1) \subset \mathbb{C}$  in the rest of the proof. It follows that  $G(\mathbb{D}(\lambda_0, \delta))$  contains a disk of spherical radius say  $\epsilon > 0$  centered at  $\infty$ .

Since there are no singular values in  $\overline{D}$  and  $f_{\lambda_0}$  has infinite degree, there are infinitely many univalent preimages of  $D$  under  $f_{\lambda_0}$  which must accumulate at infinity. Observe that these preimages must miss, for example, a given periodic orbit of period 3 which does not intersect  $D$ . Hence, selecting a subset of those preimages if necessary, we may assume (see Lemma 2.9) that they are all bounded and that in fact their spherical diameter tends to 0. Let  $U$  be one such preimage contained in  $\mathbb{D}_s(\infty, \epsilon)$ . Thus  $f_{\lambda_0}(U) = D$ .

Since  $U$  belongs to the image of  $G$ , we let  $V$  denote a connected component of  $G^{-1}(U)$  inside  $\mathbb{D}(\lambda_0, \delta)$ . If  $D$  (and therefore  $U$ ) is small enough, then  $V$  is a Jordan domain as well. Let us now define  $F(\lambda) := f_\lambda^{n+1}(v_\lambda)$ . Our goal is to show that  $\overline{\gamma(V)} \subset F(V)$  so that Lemma 2.8 applied to  $\gamma$  and  $F$  gives the result.

In order to see this we write

$$f_\lambda^{n+1}(v_\lambda) = \varphi_\lambda \circ f_{\lambda_0} \circ \psi_\lambda^{-1} \circ f_\lambda^n(v_\lambda) = \varphi_\lambda \circ f_{\lambda_0} \circ G(\lambda),$$

and therefore

$$F(V) = \varphi_\lambda(f_{\lambda_0}(G(V))) = \varphi_\lambda(f_{\lambda_0}(U)) = \varphi_\lambda(D).$$

Now since  $\delta$  can be taken arbitrarily small, the values of  $\lambda$  can be arbitrarily close to  $\lambda_0$  and therefore  $\varphi_\lambda$  is arbitrarily close to the identity. It follows that  $F(V) = \varphi_\lambda(D) \simeq D$ , while  $\gamma(V) \subset \gamma(\mathbb{D}(\lambda_0, \delta)) \subset D$ . Moreover,  $\partial\gamma(\mathbb{D}(\lambda_0, \delta))$  separates the boundaries of these two sets, so the hypotheses of Lemma 2.8 can be applied and we are done.

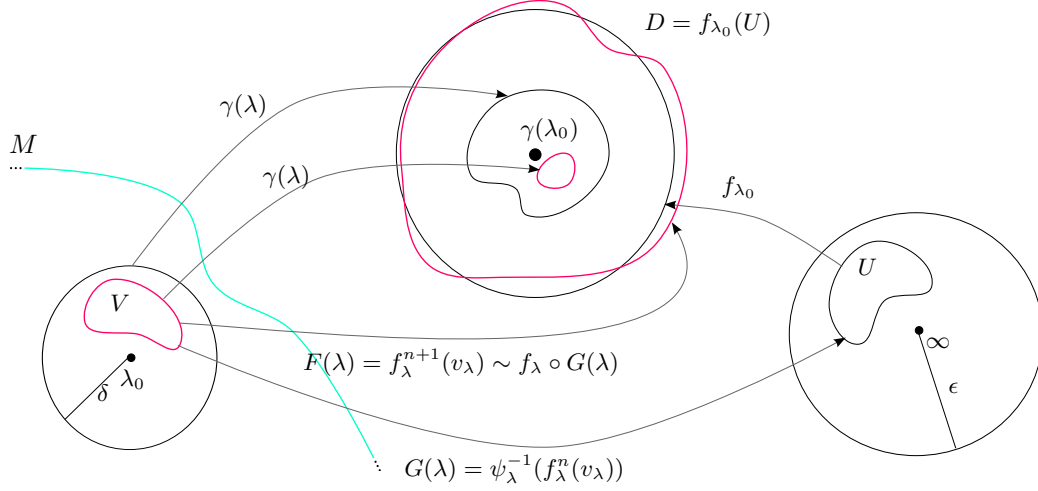


FIGURE 2. An illustration of the proof of Proposition 2.6. The final claim follows by Lemma 2.8, using the fact that  $\gamma(V) \subset F(V)$  and  $\gamma(\partial V) \cap F(\partial V) = \emptyset$ .

We now deal with case (b). We assume here that for  $i = 1, \dots, 5$ ,  $\gamma_i(\lambda_0)$  is a singular value because otherwise we may reduce to case (a).

Choose five disks  $\{D_i\}_{i=1}^5$  with disjoint closures centered at the points  $\gamma_i(\lambda_0)$ . We use Ahlfors's Five Island Theorem (see Lemma 2.10) to obtain one value of  $j$  between 1 and 5 such that  $D_j$  has infinitely many univalent preimages converging to infinity. Renaming  $\gamma = \gamma_j$  and  $D = D_j$ , we may now proceed in the same way as in (a), obtaining the result.  $\square$

**2.4. Quasiconformal distortion.** We state here a well-known distortion estimate for quasiconformal homeomorphisms that we will need later.

**Lemma 2.11** (Distortion of small disks). *Let  $(\varphi_\lambda)_{\lambda \in \mathbb{D}}$  be a holomorphic motion of the Riemann sphere  $\mathbb{P}^1$ , with  $\varphi_0 = \text{Id}$ . Let  $t \mapsto \lambda(t)$  be a continuous path in  $\mathbb{D}$  with  $\lim_{t \rightarrow +\infty} \lambda(t) = 0$ , and  $t \mapsto r_t$  a continuous function with  $r_t > 0$  and  $\lim_{t \rightarrow +\infty} r_t = 0$ . Let  $t \mapsto z_t$  be a path in  $\mathbb{P}^1$  and  $D_t := \mathbb{D}(z_t, r_t)$ . Let  $\epsilon > 0$ ; then for all  $t$  large enough:*

$$\mathbb{D}(\varphi_{\lambda(t)}(z_t), r_t^{1+\epsilon}) \subset \varphi_{\lambda(t)}(\mathbb{D}(z_t, r_t)) \subset \mathbb{D}(\varphi_{\lambda(t)}(z_t), r_t^{1-\epsilon})$$

*Proof.* By Theorem 12.6.3 p. 313 in [AIM08], for all  $t > 0$ ,  $\theta \in \mathbb{R}$  and  $r \leq 1$ , we have :

$$|\varphi_{\lambda(t)}(z_t + re^{i\theta}) - \varphi_{\lambda(t)}(z_t)| \leq e^{5(K_{\lambda(t)}-1)} \cdot |\varphi_{\lambda(t)}(z_t) - \varphi_{\lambda(t)}(z_t + e^{i\theta})| \cdot r^{1/K_{\lambda(t)}},$$

where  $K_\lambda > 1$  is the dilatation of  $\varphi_\lambda$ . Since  $\varphi_{\lambda(t)} \rightarrow \text{Id}$  uniformly on  $\mathbb{P}^1$  as  $t \rightarrow +\infty$ , we have that as  $t \rightarrow +\infty$ :

$$\sup_{\theta \in [0, 2\pi]} |\varphi_{\lambda(t)}(z_t) - \varphi_{\lambda(t)}(z_t + e^{i\theta})| \rightarrow 1.$$

Since  $(\varphi_\lambda)$  is a holomorphic motion,  $K_\lambda \rightarrow 1$  as  $\lambda \rightarrow 0$  and so  $K_{\lambda(t)} \rightarrow 1$  as  $t \rightarrow +\infty$ . The inclusion  $\varphi_{\lambda(t)}(\mathbb{D}(z_t, r_t)) \subset \mathbb{D}(\varphi_{\lambda(t)}(z_t), r_t^{1-\epsilon})$  then follows.

The other inclusion is equivalent to  $\varphi_{\lambda(t)}^{-1}(\mathbb{D}(y_t, r_t^{1+\epsilon})) \subset \mathbb{D}(\varphi_{\lambda(t)}^{-1}(y_t), r_t)$ , with  $y_t := \varphi_{\lambda(t)}(z_t)$ . Its proof is essentially the same and is left to the reader.  $\square$

We also record here the following well-known property of quasiconformal mappings:

**Lemma 2.12** (See [EL92], Lemma 4). *Let  $\psi : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  be a  $K$ -quasiconformal homeomorphism fixing  $0, \infty$ . Let  $\arg \psi(z) - \arg z$  be a uniform branch of the difference of arguments in  $\mathbb{C}^*$ . Suppose*

$$B^{-1} \leq |\psi(z_0)| \leq B, \quad |\arg \psi(z_0) - \arg z_0| \leq B,$$

for some  $z_0 \in \mathbb{C}$  and  $B > 0$ . Then for  $|z| > |z_0|$  the following estimates hold:

$$(2.2) \quad C^{-1}|z|^{K_1^{-1}} \leq |\psi(z)| \leq C|z|^{K_1}$$

$$(2.3) \quad |\arg \psi(z) - \arg z| \leq K_1 \ln |z| + C.$$

Here  $K_1, C$  depend on  $K, z_0, B$  but not on  $z, \psi$ .

### 3. CYCLES EXITING THE DOMAIN. PROOF OF THEOREM A AND COROLLARY A'.

In this section, we let  $(f_\lambda)_{\lambda \in M}$  be a natural family of meromorphic maps of finite type. We study what happens in the parameter space when there is a cycle  $\{x_i(\lambda)\}_{i=0 \dots n}$  one of whose points  $x_i(\lambda)$  tends to the essential singularity as  $\lambda$  tends to some parameter  $\lambda_0 \in M$  (see Definition 1.2). This is a new phenomena that cannot happen in either the rational setting, or families of entire functions with bounded set of singular values (for the latter, see [EL92, Theorem 2]). We saw in the introduction (with the map  $z \mapsto \lambda z + e^z$ ) that periodic points may exit the domain in some families of entire maps, in the absence of further restrictions. We will see in the next sections that in general, many cycles exit the domain in natural families of bounded type meromorphic maps, being the key point that this can occur at parameters  $\lambda_0 \in M$ , while for rational or entire functions, the parameter must be in  $\partial M$ .

An example is given by the tangent family  $T_\lambda(z) = \lambda \tan z$ , where if  $\lambda \rightarrow \frac{\pi i}{2}$  there is both a cycle of period two and another cycle of period 4 which have a point tending to infinity and hence they exit the domain (see [DK89, CJK18] for a detailed study of this and similar parameters with this property.)

A more intricate example is given by the natural family of finite type maps  $f_\lambda(z) := \frac{e^z}{1+\lambda e^z}$ ,  $\lambda \in \mathbb{C}$ , with  $\varphi_\lambda(z) = \frac{z}{1+\lambda z}$ ,  $f(z) = e^z$  and  $\psi_\lambda \equiv \text{Id}$ . It can be checked that for small  $t > 0$ , the map  $f_t$  has a unique real fixed point  $x_t \sim \frac{1}{t}$ , which exits the domain at  $\lambda = 0$ , at the same time that its multiplier is tending to zero. Note that  $\infty$  is a singular value for  $f_0$  (which is entire) but not for  $f_\lambda$  if  $\lambda \neq 0$  (compare Theorem 4.4).

**3.1. Generalities.** It is a crucial point to interpret the concept that  $x_i(\lambda) \rightarrow \infty$  for  $\lambda \rightarrow \lambda_0 \in M$  in the following more abstract way (c.f. [MSS83, EL92], and see Figure 3).

**Definition 3.1** (The projection map and cycles exiting the domain). For  $n \in \mathbb{N}^*$ , let

$$P_n := \{(\lambda, z) \in M \times \mathbb{C} : f_\lambda^n(z) = z\},$$

and

$$\pi_1 : P_n \rightarrow M$$

be the projection onto the first coordinate. We say that a cycle of period  $n$  exits the domain at  $\lambda_0$  if  $\lambda_0$  is an asymptotic value of  $\pi_1 : P_n \rightarrow M$ .

In other words, a cycle of period  $n$  exits the domain at  $\lambda_0$  if and only if there exists a continuous curve  $t \mapsto (\lambda(t), z(t))$  in  $P_n$  such that  $\lim_{t \rightarrow +\infty} \lambda(t) = \lambda_0$  and  $\lim_{t \rightarrow +\infty} z(t) = \infty$ .

The set  $P_n$  is an analytic hypersurface of  $M \times \mathbb{C}$ , and by the Implicit Function Theorem it is smooth except possibly at points  $(\lambda, z)$  where  $z$  is a periodic point of period dividing  $n$  with  $(f_\lambda^n)'(z) = 1$ . Moreover, if  $\lambda \in M$  is a critical value of  $\pi_1 : P_n \rightarrow M$ , then  $f_\lambda$  has a parabolic cycle of period dividing  $n$  and multiplier 1.

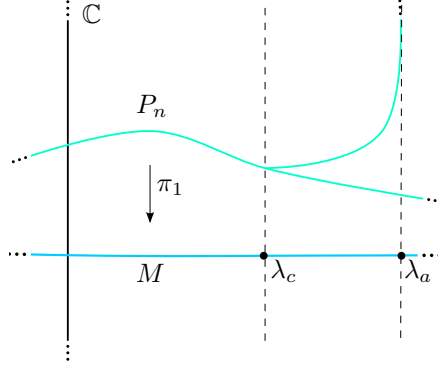


FIGURE 3. An illustration of the set  $P_n$ , the map  $\pi_1$ , and its asymptotic and critical values. Here  $\lambda_c$  is a critical value for  $\pi_1$ , corresponding to the map  $f_{\lambda_c}$  having a parabolic cycle, and  $\lambda_a$  is an asymptotic value for  $\pi_1$ , corresponding to  $f_{\lambda_a}$  having a cycle exiting the domain. Note that for  $n \geq 2$  the function  $f$  has infinitely many periodic points of exact period  $n$  [Ros48], hence  $P_n$  is locally the union of infinitely many connected components, but only one is shown for simplicity.

**Definition 3.2.** We let  $X_n$  denote the singular value set of  $\pi_1 : P_n \rightarrow M$ , which is the closure of the set of critical and asymptotic values of  $\pi_1$ . Let  $X := \bigcup_{n=1}^{\infty} X_n$ .

The next proposition shows that a holomorphic motion of  $\text{Fix}(f_\lambda^n)$ , the fixed points of  $f_\lambda^n$ , can only exist in open sets in absence of singular values of  $\pi_1 : P_n \rightarrow M$ , which confirms that  $X$  is the appropriate set to study.

**Proposition 3.3** ( $\mathcal{J}$  moves holomorphically outside  $X$ ). *Let  $\{f_\lambda\}_{\lambda \in M}$  be a natural family of meromorphic maps and  $U \subset M$  be a simply connected domain. Let  $X, X_n$  be as above. Then*

- (1)  $U \cap X_n = \emptyset \iff$  the set  $\text{Fix}(f_\lambda^n)$ , move holomorphically over  $U$  for every  $n \geq 0$ .
- (2)  $U \cap X = \emptyset \implies$  the Julia set of  $f_\lambda$  moves holomorphically over  $U$ .

*Proof.* To see (1) suppose  $U \cap X_n = \emptyset$ . Let  $\lambda_0 \in U$ , and let  $(\lambda_0, z_i)_{i \in \mathbb{N}}$  denote the preimages  $\pi_1^{-1}(\{\lambda_0\})$ . Then for all  $i \in \mathbb{N}$ , there exists a holomorphic branch  $g_i : U \rightarrow P_n$  of  $\pi_1^{-1}$  with  $g_i(\lambda_0) = (\lambda_0, z_i)$ . In this setting  $\lambda \mapsto \pi_2 \circ g_i(\lambda)$  gives the desired holomorphic motion where  $\pi_2$  is the projection onto the second coordinate.

For the reverse implication, suppose  $\lambda_0 \in U$  is a singular value. If  $\lambda_0$  is an asymptotic value, there is a fixed point of  $f_\lambda^n$  which escapes to infinity when  $\lambda$  approaches  $\lambda_0$ . Hence any holomorphic motion of the set  $\text{Fix}_n(f_{\lambda_0})$  over  $U$  could not be surjective. Otherwise, if  $\lambda_0$  is the image of a critical point  $(\lambda_0, z_i)$ , every  $\lambda$  in a neighborhood of  $\lambda_0$  will have  $k > 1$  distinct preimages in  $P_n$  splitting off from  $z_i$ , hence these periodic points cannot be followed holomorphically either.

Statement (2) follows from (1), the  $\lambda$ -lemma and the fact that (repelling) periodic points are dense in the Julia set.  $\square$

In fact, it will follow from our results in Section 5 that the converse to item (2) also holds.

Our goal in the remaining of this section is to show that cycles exiting the domain require active singular values, thus proving Theorem A. We start by showing (Lemma 3.4) that any such cycle, in the limit, must contain an asymptotic value which eventually maps to infinity. Later on (Theorem 3.5) we show that the asymptotic value involved, or another critical value in the cycle, must be active (i.e. non-persistently mapped to infinity).

**Lemma 3.4.** *Let  $\{f_\lambda\}_{\lambda \in M}$  be a natural family of meromorphic maps. Let  $t \mapsto (\lambda(t), z(t))$  be a curve in  $P_n$  with  $\lim_{t \rightarrow +\infty} \lambda(t) = \lambda_0 \in M$  and  $\lim_{t \rightarrow +\infty} z(t) = \infty$ . Then there exists a cyclically ordered set  $\infty = a_0, \dots, a_{n-1} \in \hat{\mathbb{C}}$  such that:*

- (1) *for all  $0 \leq m \leq n-1$ ,  $a_m = \lim_{t \rightarrow +\infty} f_{\lambda(t)}^m(z(t))$ ;*
- (2) *if  $a_m \in \mathbb{C}$ , then  $a_{m+1} = f_{\lambda_0}(a_m)$ ;*
- (3) *if  $a_m = \infty$ , then  $a_{m+1}$  is an asymptotic value of  $f_{\lambda_0}$  (possibly equal to  $\infty$ ) and  $a_{m-1}$  is either  $\infty$  or a pole of  $f_{\lambda_0}$ .*

In other words, the set  $a_0, \dots, a_{n-1}$  is a *virtual cycle*. Notice that the lemma implies that, as  $t \rightarrow \infty$  (and hence  $\lambda \rightarrow \lambda_0$ ), either the whole cycle corresponding to  $z(t)$  tends to infinity (in which case  $\infty$  must be an asymptotic value for  $f_{\lambda_0}$ ), or there exists at least one finite asymptotic value and one pole in the virtual cycle (possibly more, if there is more than one  $a_i$  which equals infinity).

*Proof.* To simplify notation, let us denote  $x_m(t) := f_{\lambda(t)}^m(z(t))$ . By assumption  $\lim_{t \rightarrow +\infty} f_{\lambda(t)}^{n-m}(x_m(t)) = \lim_{t \rightarrow +\infty} z(t) = \infty$ , so any finite accumulation point of the curve  $t \mapsto x_m(t)$  must be a pre-pole of  $f_{\lambda_0}$  of order at most  $n-m$ . In particular, the set of finite accumulation points of this curve is discrete, and so  $\lim_{t \rightarrow +\infty} x_m(t)$  exists (and is possibly  $\infty$ ). Denote by  $a_m := \lim_{t \rightarrow +\infty} x_m(t) \in \hat{\mathbb{C}}$ . Item (2) follows easily.

Next, assume that  $a_m = \infty$  for some  $0 \leq m \leq n-1$ . Since  $(f_\lambda)_{\lambda \in M}$  is a natural family, we have

$$x_{m+1}(t) = f_{\lambda(t)}(x_m(t)) = \varphi_{\lambda(t)} \circ f \circ \psi_{\lambda(t)}^{-1}(x_m(t)),$$

where  $f := f_{\lambda_0}$ ,  $\varphi_\lambda, \psi_\lambda : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  are quasiconformal homeomorphisms depending holomorphically on  $\lambda$ , and  $\varphi_{\lambda_0} = \psi_{\lambda_0} = \text{Id}$ . Therefore, we have

$$f(y_m(t)) = z_{m+1}(t),$$

where  $y_m(t) := \psi_{\lambda(t)}^{-1}(x_m(t))$  and  $z_{m+1}(t) := \varphi_{\lambda(t)}^{-1}(x_{m+1}(t))$ ; and  $\lim_{t \rightarrow +\infty} y_m(t) = x_m = \infty$ , whereas  $\lim_{t \rightarrow +\infty} z_{m+1}(t) = a_{m+1}$  since  $\varphi_{\lambda(t)}^{-1}$  tends to the identity. Therefore  $a_{m+1}$  is indeed an asymptotic value of  $f$ .

Finally, if  $a_m = \infty$ , it follows from item (2) that if  $x_{m-1}$  is finite then it is a pole.  $\square$

Observe that if  $\lambda_0 \in M$  and  $v_{\lambda_0}$  is a singular value such that  $f_{\lambda_0}^n(v_{\lambda_0}) = \infty$  for some  $n \geq 0$ , then  $v_\lambda$  is passive at  $\lambda_0$  if and only if  $f_\lambda^n(v_\lambda) \equiv \infty$  on  $M$ .

We now state the main result of this section, which corresponds to Theorem A in the introduction.

**Theorem 3.5.** *Let  $(f_\lambda)_{\lambda \in M}$  be a natural family of finite type meromorphic maps. Assume that a cycle of period  $n$  exits the domain at  $\lambda_0 \in M$ . Then either there is an active critical point in the associated virtual cycle, or there is an active asymptotic value.*

Before we proceed with the proof we deduce Corollary A'.

**Corollary 3.6.** *In the hypothesis of Theorem 3.5, if moreover  $(f_\lambda)_{\lambda \in M}$  is not exceptional, then  $\lambda_0$  is actually an asymptotic parameter, and the virtual cycle contains at least one active asymptotic value.*

*Proof.* Observe that either all the points in the cycle go to  $\infty$ , in which case the virtual cycle is  $\infty, \dots, \infty$ , or there is at least one finite asymptotic value in the virtual cycle.

In the first case, there cannot be any critical point in the virtual cycle and therefore there is an active asymptotic value in the virtual cycle. In the second case, the finite asymptotic value must be active by the assumption that the family is not exceptional.  $\square$

The rest of this section is devoted to the proof of Theorem 3.5.

**3.2. Preliminary results.** We first record here several lemmas essentially due to Eremenko and Lyubich, some of them modified for our purposes.

**Lemma 3.7** ([EL92], Lemma 3). *Let  $R > 0$ , and let  $T \subset \mathbb{C}$  be a simply connected domain whose boundary is a real-analytic simple curve with both endpoints converging to  $\infty$ . Let  $f : T \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}(0, R)}$  be a holomorphic universal cover, let  $\arg$  denote a branch of the argument on  $T$ , let  $t \mapsto \gamma(t)$  be a continuous curve such that  $\lim \gamma(t) = \infty$  and  $\gamma(t) \in T$ . Then there exists  $t_k \rightarrow +\infty$  and a constant  $C$  independent of  $k$ , such that*

$$(3.1) \quad \ln^2 |f(\gamma(t_k))| + \arg^2 f(\gamma(t_k)) \geq C |\gamma(t_k)| \exp \frac{\arg^2 \gamma(t_k)}{\ln |\gamma(t_k)|}.$$

The statement is proven in [EL92, Lemma 3] in the case where  $f$  is a finite type entire map, but the same proof applies in the greater generality of Lemma 3.7.

It will be convenient to introduce the following notation.

**Definition 3.8.** Let  $\gamma_1, \gamma_2 : \mathbb{R}_+ \rightarrow \mathbb{C}^*$  be two continuous curves, converging either both to 0 or both to  $\infty$ . We will write  $\gamma_1 \asymp \gamma_2$  if there exists a constant  $C > 0$  such that

$$(3.2) \quad \ln |\gamma_1(t)| - \ln |\gamma_2(t)| = O(1)$$

and

$$(3.3) \quad |\arg \gamma_1(t) - \arg \gamma_2(t)| \leq C |\ln |\gamma_1(t)||$$

Note that this definition makes sense because the arguments  $\arg \gamma_i$  are well-defined up to a multiple of  $2i\pi$ . Also note that  $\asymp$  is an equivalence relation.

**Remark 3.9.** If  $\gamma_1, \gamma_2$  are two curves as above and  $d \in \mathbb{Z}^*$ , then it is easy to see that  $\gamma_1 \asymp \gamma_2$  if and only if  $\gamma_1^d \asymp \gamma_2^d$ , simply because in log coordinates the map  $z \mapsto z^d$  becomes  $\omega \mapsto d\omega$ .

The following lemma can be extracted from arguments present in [EL92]; we include details for the convenience of the reader.

**Lemma 3.10** ( $f^{-1}$  preserves  $\asymp$ ). *Let  $\gamma_1, \gamma_2 : \mathbb{R}_+ \rightarrow \mathbb{C}^*$  be two curves, and  $f$  be a bounded type meromorphic map. Assume that  $\gamma_i(t) \rightarrow \infty$  and  $f \circ \gamma_i(t) \rightarrow \infty$ , and that  $f \circ \gamma_1 \asymp f \circ \gamma_2$ . Then  $\gamma_1 \asymp \gamma_2$ .*

*Proof.* Let  $A$  be a punctured disk around  $\infty$ , and let  $G$  denote the union of the tracts  $T_i$  such that  $f : T_i \rightarrow A$  is a universal cover. The set  $G$  is non-empty because under the assumptions of the lemma,  $\infty$  is an asymptotic value, and because  $f$  has a bounded set of singular values. Let  $U := \exp^{-1}(G)$  and

$$\mathbb{H}_R := \exp^{-1}(A) = \{z \in \mathbb{C} : \operatorname{Re}(z) > R\}$$

for some  $R > 0$  depending on the radius of  $A$ . Then there is a holomorphic map  $F : U \rightarrow \mathbb{H}_R$  making the following diagram commute:

$$\begin{array}{ccc} U & \xrightarrow{F} & \mathbb{H}_R \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow{f} & A \end{array}$$

Let  $\delta_1, \delta_2$  be two respective lifts of  $\gamma_1, \gamma_2$  by  $\exp$ , chosen to be in the same connected component  $U_0$  of  $U$ : then  $\delta_j = \ln |\gamma_j| + i \arg \gamma_j$ , and  $F(\delta_j) = \ln |f \circ \gamma_j| + i \arg f \circ \gamma_j$ , for  $j = 1, 2$ .

Let us denote by  $I_t$  the Euclidean segment connecting  $F \circ \delta_1(t)$  to  $F \circ \delta_2(t)$ , by  $\ell_{\text{Eucl}}$  the Euclidean length, by  $m = \min(\text{Re}(F \circ \delta_1(t)), \text{Re}(F \circ \delta_2(t)))$  and by  $M = \max(\text{Re}(F \circ \delta_1(t)), \text{Re}(F \circ \delta_2(t)))$ .

By [[EL92], Lemma 1], we have  $|F'(z)| \geq \frac{1}{4\pi}(\text{Re } F(z) - R)$  and  $F : U_0 \rightarrow \mathbb{H}_R$  is a conformal isomorphism, hence it has a well defined inverse branch  $F_U^{-1} : \mathbb{H}_R \rightarrow U$ . Therefore

$$\begin{aligned} |\delta_1(t) - \delta_2(t)| &\leq \ell_{\text{Eucl}}(F_U^{-1}(I_t)) \leq \sup_{w \in I_t} |(F_U^{-1})'(w)| \ell_{\text{Eucl}}(I_t) \leq \\ &\leq \frac{4\pi}{m(t) - R} |F \circ \delta_1(t) - F \circ \delta_2(t)| \\ &\leq \frac{8\pi M(t)}{m(t) - R} = O(1), \end{aligned}$$

which implies the desired result.  $\square$

Note that the proof of Lemma 3.10 in fact gives the stronger estimate

$$(3.4) \quad \arg \gamma_1(t) - \arg \gamma_2(t) = O(1),$$

which we will not require.

**Lemma 3.11.** *Let  $f$  be a meromorphic function of bounded type. Consider a curve  $\gamma : \mathbb{R}_+ \rightarrow \mathbb{C}^*$  with  $\gamma(t) \rightarrow \infty$  as  $t \rightarrow +\infty$  and assume that  $f \circ \gamma(t) \rightarrow \infty$  as  $t \rightarrow +\infty$ . Let  $\{h_t : t \geq 0\}$  be a continuous family of  $K$ -qc homeomorphisms satisfying the hypothesis of Lemma 2.12. Then  $h_t \circ \gamma \asymp \gamma$ .*

*Proof.* The proof follows directly from Lemma 2.12.  $\square$

We observe here a technical point which plays an important role in the proof of Theorem 3.5: In Lemma 3.11, it is crucial that  $h_t(\infty) = \infty$  for all  $t \geq 0$ , instead of merely having  $\lim_{t \rightarrow +\infty} h_t(\infty) = \infty$ .

The lemma below is a slightly weaker version of Lemma 5 from [EL92], that will be sufficient for our purposes. We include the proof for the convenience of the reader, since it is very short using Lemmas 3.10 and 3.11.

**Lemma 3.12** (Compare [EL92], Lemma 5). *Let  $f$  be a meromorphic function with bounded set of singular values. Consider a curve  $\gamma : \mathbb{R}_+ \rightarrow \mathbb{C}^*$  with  $\gamma(t) \rightarrow \infty$  as  $t \rightarrow +\infty$  and assume that  $f \circ \gamma(t) \rightarrow \infty$  as  $t \rightarrow +\infty$ . Let  $\{h_t : t \geq 0\}$  be a continuous family of  $K$ -qc homeomorphisms satisfying the hypothesis of Lemma 2.12. Then there exists a curve  $\tilde{\gamma} \asymp \gamma$ , such that*

$$(3.5) \quad f \circ \tilde{\gamma}(t) = h_t \circ f \circ \gamma(t).$$



*Proof.* The existence of a curve  $\tilde{\gamma}$  satisfying  $f \circ \tilde{\gamma} = h_t \circ f \circ \gamma$  follows from the observation that  $h_t \circ f \circ \gamma(t) \rightarrow \infty$ , and that  $f$  is a covering over a punctured neighborhood of  $\infty$ .

Then, by Lemma 3.11 we have  $h_t \circ f \circ \gamma \asymp f \circ \gamma$ , so by definition of  $\tilde{\gamma}$ , we have  $f \circ \tilde{\gamma} \asymp f \circ \gamma$ . Finally, by Lemma 3.10 we have  $\tilde{\gamma} \asymp \gamma$ .  $\square$

**3.3. Proof of Theorem 3.5.** Let us introduce some further notations. By assumption, there is a curve  $t \mapsto \lambda(t)$  in parameter space with  $\lambda(t) \rightarrow \lambda_0$ , and a cycle of period  $n$  exiting the domain along this curve. For the sake of simplicity, we will write  $f_t, \varphi_t, \psi_t$  instead of  $f_{\lambda(t)}, \varphi_{\lambda(t)}, \psi_{\lambda(t)}$  and  $f$  instead of  $f_{\lambda_0}$ .

We denote by  $x_1(t), \dots, x_n(t)$  the points of the cycle of period  $n$  for  $f_t$  which exits the domain, and assume without loss of generality that  $x_n(t) \rightarrow \infty$  as  $t \rightarrow \infty$  (i.e.  $\lambda(t) \rightarrow \lambda_0$ ). Recall that by Lemma 3.4 the points  $a_i = \lim_{t \rightarrow +\infty} x_i(t)$  form a virtual cycle, hence at least one of them is an asymptotic value.

Therefore, in order to prove Theorem 3.5, we must prove that if the virtual cycle does not contain any active critical point, then at least one singular value in that virtual cycle is active. We assume for a contradiction that all asymptotic relations associated to the limit virtual cycle are preserved (that is, every singular value obtained as a limit of one of the  $x_i(t)$  remains in the backward orbit of  $\infty$  for  $\lambda$  in a neighborhood of  $\lambda_0$ , and therefore throughout  $M$ ), and the same for critical points belonging to the virtual cycle.

More precisely, following the notations of Lemma 3.4 for  $i = 1 \dots n$  we define  $a_i := \lim_{t \rightarrow +\infty} x_i(t)$  (with indices are taken modulo  $n$ ). To simplify the notation we set  $f := f_{\lambda_0}$ ,  $f_t := f_{\lambda(t)}$ ,  $\varphi_t = \varphi_{\lambda(t)}$ ,  $\psi_t = \psi_{\lambda(t)}$  (hence  $f_t = \varphi_t \circ f \circ \psi_t^{-1}$ ). We assume for a contradiction that for all  $1 \leq i \leq n$  such that  $a_i \in S(f)$ , we have  $f_t^{n-i}(\varphi_t(a_i)) = \infty$  for all  $t > 0$ . (Recall that if  $a_i$  is a singular value of  $f$ , then  $\varphi_t(a_i)$  is a singular value of same nature for  $f_t$ ).

We define a new family of curves  $y_1(t), \dots, y_n(t)$ , which record the orbit of all asymptotic values involved in the limit cycle (see Figure 4). More precisely, define

- if  $x_i(t) \rightarrow \infty$ , then  $y_i(t) := \infty$
- if  $x_{i-1}(t) \rightarrow \infty$ , then  $x_i(t) \rightarrow v_i$ , where  $v_i$  is some asymptotic value of  $f$ ; then we set  $y_i(t) := \varphi_t(v_i)$ , which is an asymptotic value for  $f_t$ .
- if  $y_{i-1}(t) \in \mathbb{C}$ , then  $y_i(t) := f_t(y_{i-1}(t))$ .

The assumption that all singular relations associated to the limit virtual cycle are preserved implies that this definition is coherent, and that  $y_1(t), \dots, y_n(t)$  also forms a virtual cycle under  $f_t$  for every  $t$ . In particular, if  $x_{i-1}(t) \rightarrow \infty$  and  $x_i(t) \rightarrow a_i = \infty$ , this means that  $\infty$  is an asymptotic value of  $f$ , and the assumption forces it to be persistent, i.e.  $\varphi_t(\infty) = \infty = y_i(t)$ . Note that we also have  $\lim_{t \rightarrow +\infty} y_i(t) = a_i$ .

The idea of the proof is to consider a third set of curves  $\gamma_i(t) := f^{-n+i}(x_n(t))$ , for  $i = n \dots 0$ , and to prove that their distance to the virtual cycle  $a_1, \dots, a_n$  will be close to the distance between the actual cycle  $x_1(t), \dots, x_n(t)$  and the virtual cycle  $y_1(t), \dots, y_n(t)$ . This will ensure that  $\gamma_0 \asymp \gamma_n$  while  $f^n(\gamma_0) = \gamma_n = x_n$ , and will give a contradiction through Lemma 3.7.

**Lemma 3.13** (Key lemma). *There exist two curves  $\gamma_0, \gamma_n$  with  $f^n(\gamma_0(t)) = \gamma_n(t)$ ,  $\lim_{t \rightarrow +\infty} \gamma_0(t) = \lim_{t \rightarrow +\infty} \gamma_n(t) = \infty$ , and  $\gamma_0 \asymp \gamma_n$ .*

*Proof of Theorem 3.5 assuming Lemma 3.13.* By Lemma 3.7 applied to  $f^n$  and with  $\gamma_0(t)$ , we have for all  $t > 0$  large enough:

$$(3.6) \quad \ln^2 |\gamma_n(t_k)| + \arg^2 \gamma_n(t_k) \geq C |\gamma_0(t_k)| \exp \frac{\arg^2 \gamma_0(t_k)}{\ln |\gamma_0(t_k)|}$$

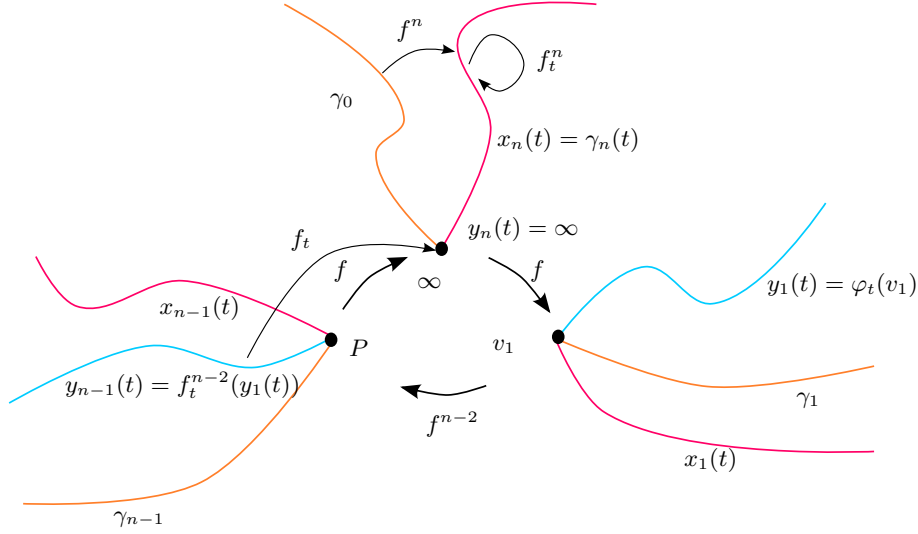


FIGURE 4. An illustration of the proof of Theorem 3.5 in a simple case in which there is only one pole  $P$  and one asymptotic value  $v_1$  involved. Here  $a_n = \infty$ ,  $a_1 = v_1$ , and  $a_{n-1} = P$ . Under the contradiction assumption that the singular relation involving  $v_1$  is persistent we have that  $f_t(y_{n-1}(t)) = f_t^{n-1}(\varphi_t(v_1)) = \infty$ . This allows to construct the curves  $\gamma_i$  as pullbacks of the curve  $\gamma_n$ , obtaining  $\gamma_0$  such that  $f^n(\gamma_0) = \gamma_n$  yet  $\gamma_0 \asymp \gamma_n$ .

for some sequence  $t_k \rightarrow \infty$ .

On the other hand, by the assumption that  $\gamma_n \asymp \gamma_0$ , we have:

$$(3.7) \quad \ln |\gamma_n(t)| = \ln |\gamma_0(t)| + O(1)$$

$$(3.8) \quad \arg \gamma_n(t) = \arg \gamma_0(t) + O(\ln |\gamma_0(t)|)$$

which leads to a contradiction. □

The proof of Lemma 3.13 is done by induction. We start with the curve  $\gamma_n := x_n \rightarrow a_n = \infty$ . Then, given a curve  $\gamma_i(t) \rightarrow a_i$  with  $i = n \dots 1$  we will find a curve  $\gamma_{i-1}(t) \rightarrow a_{i-1}$  which is an appropriate pullback of  $\gamma_i$  under  $f$ . This step is divided into two main cases: the case in which  $a_{i-1} \in \mathbb{C}$  (Lemma 3.14) and the case in which  $a_{i-1} = \infty$  (Lemma 3.16).

**Lemma 3.14.** *Let  $\gamma_i$  be a curve such that  $\gamma_i(t) \rightarrow a_i$  with  $\gamma_i(t) \neq a_i$  for all  $t > 0$ , and assume that  $a_{i-1} \in \mathbb{C}$  and that either  $\gamma_i(t) - a_i \asymp x_i(t) - y_i(t)$  (if  $a_i \in \mathbb{C}$ ) or  $\gamma_i(t) \asymp x_i(t)$  (if  $a_i = \infty$ ). Then there exists a curve  $\gamma_{i-1}$  such that*

- (1)  $f \circ \gamma_{i-1}(t) = \gamma_i(t)$  and  $\gamma_{i-1}(t) \neq a_{i-1}$  for all  $t > 0$
- (2)  $\gamma_{i-1}(t) \rightarrow a_{i-1}$
- (3)  $\gamma_{i-1}(t) - a_{i-1} \asymp x_{i-1}(t) - y_{i-1}(t)$ .

*Proof.* First, we choose  $\gamma_{i-1}$  to be a lift of  $\gamma_i$  by  $f$ , such that  $\gamma_{i-1}(t) \rightarrow a_{i-1}$ . Note that if  $d_i := \deg(f, a_{i-1}) > 1$ , then there are exactly  $d_i$  possible choices (since  $\gamma_i(t) \neq a_i$  by assumption). This gives (1) and (2).

Next, we claim that  $(\gamma_{i-1}(t) - a_{i-1})^{d_i} \asymp \gamma_i(t) - a_i$  if  $a_{i-1} \in \mathbb{C}$ , and that  $(\gamma_{i-1}(t) - a_{i-1})^{d_i} \asymp \gamma_i(t)$  if  $a_i = \infty$ . This can be seen easily from the series expansions

$$\begin{aligned} f(z) - a_i &= c \cdot (z - a_{i-1})^{d_i} + o((z - a_{i-1})^{d_i}), \text{ if } a_i \in \mathbb{C}, \text{ or} \\ f(z) &= c \cdot (z - a_{i-1})^{-d_i} + o((z - a_{i-1})^{-d_i}), \text{ if } a_i = \infty, \end{aligned}$$

with  $c \neq 0$  (compare Remark 3.9).

Since critical relations are assumed to be persistent along the virtual cycle  $a_1, \dots, a_n$ , we have  $\deg(f_t, y_{i-1}(t)) = d_i = \deg(f, a_{i-1})$ . Therefore we also have series expansions of the form

$$\begin{aligned} f_t(z) - y_i(t) &= c(t) \cdot (z - y_{i-1}(t))^{d_i} + o((z - y_{i-1}(t))^{d_i}), \text{ if } a_i \in \mathbb{C}, \text{ or} \\ f_t(z) &= c(t) \cdot (z - y_{i-1}(t))^{-d_i} + o((z - y_{i-1}(t))^{-d_i}), \text{ if } a_i = \infty, \end{aligned}$$

where  $c(t) \rightarrow c \neq 0$ . Since  $x_{i+1}(t) = f_t(x_i(t))$ , it follows that  $(x_{i-1}(t) - y_{i-1}(t))^{d_i} \asymp x_i(t) - y_i(t)$  if  $a_i \in \mathbb{C}$ , and  $(x_{i-1}(t) - y_{i-1}(t))^{d_i} \asymp x_i(t)$  if  $a_i = \infty$ .

Therefore:

- (1) If  $a_i = \infty$ , then by assumption we have  $\gamma_i \asymp x_i$ , and we have proved that  $(\gamma_{i-1} - a_{i-1})^{d_i} \asymp \gamma_i$  and  $(x_{i-1} - a_{i-1})^{d_i} \asymp x_i$ ; therefore  $(\gamma_{i-1} - a_{i-1})^{d_i} \asymp (x_{i-1} - a_{i-1})^{d_i}$ , which in turn implies  $\gamma_{i-1} - a_{i-1} \asymp x_{i-1} - a_{i-1}$  (see again Remark 3.9).
- (2) If  $a_i \in \mathbb{C}$ , then similarly: by assumption, we have  $\gamma_i - a_i \asymp x_i - y_i$ , and we have proved that  $(\gamma_{i-1} - a_{i-1})^{d_i} \asymp \gamma_i - a_i$  and  $(x_{i-1} - y_{i-1})^{d_i} \asymp x_i - y_i$ . Therefore we again have  $(\gamma_{i-1} - a_{i-1})^{d_i} \asymp (x_{i-1} - a_{i-1})^{d_i}$  and finally  $\gamma_{i-1} - a_{i-1} \asymp x_{i-1} - a_{i-1}$ .

□

We now turn to the other case,  $a_{i-1} = \infty$ . Before proving the analogue of Lemma 3.14, namely Lemma 3.16, we will require the following modification of Lemma 3.12 adapted to the case of a finite asymptotic value:

**Lemma 3.15.** *Let  $\gamma(t) \rightarrow \infty$  be a curve such that  $f_t(\gamma(t)) \rightarrow v \in \mathbb{C}$ . Then, there exists a curve  $\gamma'(t) \rightarrow \infty$  such that  $\gamma' \asymp \gamma$  and  $f(\gamma'(t)) - v = f_t(\gamma(t)) - v(t)$ , where  $v(t) = \varphi_t(v)$ .*

*Proof.* Let  $g_t(z) := \frac{1}{f_t(z) - v(t)}$ , and  $g(z) := \frac{1}{f(z) - v}$ . Let  $M_t(z) := \frac{1}{z - v(t)}$ . Then observe that  $g = M_0 \circ f$ , and

$$(3.9) \quad g_t = M_t \circ f_t = (M_t \circ \varphi_t \circ M_0^{-1}) \circ g \circ \psi_t^{-1}$$

This shows that  $g_t$  is a natural family of bounded type meromorphic maps of the form  $g_t = \tilde{\varphi}_t \circ g \circ \psi_t^{-1}$ , with  $\tilde{\varphi}_t := M_t \circ \varphi_t \circ M_0^{-1}$ . Moreover,  $\tilde{\varphi}_t$  is a quasiconformal homeomorphism of  $\hat{\mathbb{C}}$ , and  $\tilde{\varphi}_t(\infty) = \infty$  (since  $M_0^{-1}(\infty) = v$  and  $M_t \circ \varphi_t(v) = \infty$ ).

Since we have  $g_t(\gamma(t)) \rightarrow \infty$ , we may apply Lemma 3.12 to  $g_t$ , which gives a curve  $\gamma'(t) \rightarrow \infty$  such that  $\gamma' \sim \gamma$ , and  $g_t(\gamma(t)) = g(\gamma'(t))$ .

It remains to check that  $f(\gamma'(t)) - v = f_t(\gamma(t)) - v(t)$ . But

$$\begin{aligned} g_t(\gamma(t)) &= g(\gamma'(t)) \\ \frac{1}{f_t(\gamma(t)) - v(t)} &= \frac{1}{f(\gamma'(t)) - v} \\ f(\gamma'(t)) - v &= f_t(\gamma(t)) - v(t) \end{aligned}$$

and the lemma is proved. □

**Lemma 3.16.** *Let  $\gamma_i \rightarrow a_i$  be a curve such that either  $a_i \in \mathbb{C}$  and  $\gamma_i(t) - a_i \asymp x_i(t) - y_i(t)$ , or  $a_i = \infty$  and  $\gamma_i(t) \asymp x_i(t)$ . Assume further that  $a_{i-1} = \infty$ . Then there exists a curve  $\gamma_{i-1}$  such that*

- (1)  $f \circ \gamma_{i-1}(t) = \gamma_i(t)$  and  $\gamma_{i-1}(t) \neq \infty$  for all  $t > 0$
- (2)  $\gamma_{i-1}(t) \rightarrow \infty$
- (3)  $\gamma_{i-1}(t) \asymp x_{i-1}(t)$

*Proof.* We will distinguish two cases:  $a_i = \infty$  or  $a_i \in \mathbb{C}$ .

First, assume that  $a_i = \infty$ . In that case,  $\infty$  is an asymptotic value for  $f$  and by assumption it remains an asymptotic value for  $f_t$ , so that  $\varphi_t(\infty) = \infty$ . Moreover, note that  $x_i(t) = f_t(x_{i-1}(t)) = \varphi_t \circ f \circ \psi_t^{-1} \circ x_{i-1}(t)$ , and that  $\psi_t^{-1}(x_{i-1}(t))$  is a curve that tends to  $\infty$ . Therefore we can apply Lemma 3.12 with  $h_t := \varphi_t$  (since, again,  $\varphi_t(\infty) = \infty$ ) and  $\gamma(t) := \psi_t^{-1}(x_{i-1}(t))$ . We obtain in this way a curve  $\tilde{\gamma}$  such that  $\tilde{\gamma}(t) \rightarrow \infty$ ,  $\varphi_t \circ f \circ \psi_t^{-1}(x_{i-1}(t)) = x_i(t) = f(\tilde{\gamma}(t))$ , and  $\tilde{\gamma}(t) \asymp \psi_t^{-1} \circ x_{i-1}(t)$ . By Lemma 3.11 we have  $x_{i-1}(t) \asymp \psi_t^{-1}(x_{i-1}(t))$  (since  $\psi_t^{-1}(\infty) = \infty$ ). So  $\tilde{\gamma}(t) \asymp x_{i-1}(t)$ .

Moreover, we have  $f(\tilde{\gamma}) = x_i \asymp \gamma_i$  by assumption. Let  $\gamma_{i-1}$  be a lift of  $\gamma_i$  by  $f$ : then

$$f \circ \gamma_{i-1} = \gamma_i \asymp x_i = f \circ \tilde{\gamma}_i,$$

so that by Lemma 3.10 we have  $\tilde{\gamma} \asymp \gamma_{i-1}$ . Finally, we have:

$$\gamma_{i-1} \asymp \tilde{\gamma} \asymp x_{i-1},$$

and we are done in this case.

We now treat the case when  $a_i \in \mathbb{C}$ . In that case, we apply Lemma 3.15 with  $\gamma := x_{i-1}$  and get a curve  $\tilde{\gamma}$  such that  $\tilde{\gamma} \asymp x_{i-1}$  and  $f \circ \tilde{\gamma} - a_i = f_t \circ x_{i-1} - y_i = x_i - y_i$ .

Let  $\gamma_{i-1}$  be a lift by  $f$  of  $\gamma_i$ , such that  $\gamma_{i-1}(t) \rightarrow \infty$ . It remains to argue as above that  $\tilde{\gamma} \asymp \gamma_{i-1}$ . But this follows precisely from the same Lemma 3.10 applied to  $g := \frac{1}{f - a_i}$  instead of  $f$ , since by assumption  $\gamma_i - a_i \asymp x_i - y_i$  and therefore

$$f \circ \tilde{\gamma} - a_i = x_i - y_i \asymp f \circ \gamma_{i-1} - a_i.$$

Then finally we also have

$$(3.10) \quad x_{i-1} \asymp \tilde{\gamma} \asymp \gamma_{i-1},$$

and the lemma is proved.  $\square$

We are now finally ready to prove the key Lemma 3.13, which will conclude the proof of Theorem 3.5.

*Proof of Lemma 3.13.* We define  $\gamma_n(t) := x_n(t)$ , and then proceed by induction to construct curves  $\gamma_i$  such that  $\gamma_i(t) \rightarrow a_i$ ,  $f^{n-i}(\gamma_i(t)) = \gamma_n(t)$ , and:

- if  $a_i \neq \infty$ , then  $\gamma_i - a_i \asymp x_i - y_i$
- if  $a_i = \infty$ , then  $\gamma_i \asymp x_i$ .

Assume  $\gamma_i$  is constructed. We then have two cases: either  $a_{i-1} = \infty$  or not. If  $a_{i-1} \neq \infty$ , then we apply Lemma 3.14. Otherwise, we apply Lemma 3.16. In either case, the induction is proved.  $\square$

#### 4. EXISTENCE OF ATTRACTING CYCLE EXITING THE DOMAIN AT ASYMPTOTIC PARAMETERS. PROOF OF THEOREM B.

The goal in this section is to prove Theorem B, the accessibility theorem. We start by describing a necessary technical condition.

**Definition 4.1** (Technical condition **(T)**). Let  $f$  be a finite type meromorphic map, and let  $v \in \hat{\mathbb{C}}$  be an asymptotic value of  $f$ . Let  $T$  be a tract of  $f$  above  $v$ . Let  $g : \mathbb{H} \rightarrow T$  be a Riemann uniformization, where  $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Re}(z) < 0\}$ . We say that  $T$  satisfies the condition **(T)** if there exists  $\alpha \in (0, 1)$  such that

$$\lim_{t \rightarrow +\infty} g'(-t)e^{\alpha t} = \infty.$$

By Koebe's distortion Theorem, if we fix a constant  $R > 0$ , then  $g(\mathbb{D}(-t, R))$  contains a disk of (spherical) radius comparable to  $|g'(-t)|$ . Therefore, an equivalent formulation of condition **(T)** is

$$(4.1) \quad \lim_{t \rightarrow +\infty} \operatorname{inrad} g(\mathbb{D}(-t, R))e^{\alpha t} = +\infty.$$

**Remark 4.2.** Observe that if  $V$  is a punctured topological disk around  $v$ , and  $T$  is a connected component of  $f^{-1}(V)$ , then the property of satisfying **(T)** or not does not depend on the choice of  $V$ .

Let us remark at this point that if tracts have nice geometry, for example if they contain sectors, then condition **(T)** is satisfied. In section 6.1 we discuss the different situations in which one can ensure that asymptotic tracts contain sectors (see for example Propositions 6.3 and 6.5). One possible conclusion is as follows.

**Lemma 4.3.** *Let  $f$  be a finite type meromorphic map, with finitely many critical points and finitely many tracts. Then, condition **(T)** is satisfied.*

*Proof.* By Proposition 6.3, the boundary of  $T$  is asymptotically close to the boundary of a sector  $S$  at infinity. Without loss of generality we can assume the sectors to be centered at the positive real axis. For such a sector  $S$  one can write explicitly the conformal map  $g_S : \mathbb{H} \rightarrow S$  to deduce that the hyperbolic density satisfies

$$\rho_S(g_S(-t)) \leq C_1 t^{C_2}$$

(the constants  $C_1, C_2$  depends on the angular width and on whether the negative real axis is mapped to the central ray in the sector or not; in fact equality holds if the negative real axis is mapped to the central ray). Now consider the tract  $T$  and recall that the definition of hyperbolic density gives

$$\rho_T(g(-t)) = \frac{1}{|g'(-t)|} \rho_{\mathbb{H}}(-t) = \frac{1}{|g'(-t)|t}$$

Since the boundary of  $T$  converges to the boundary of  $S$ ,  $g(-t) \rightarrow g_S(-t)$  at  $t \rightarrow \infty$ , hence for some  $\epsilon > 0$  we have

$$\frac{1}{|g'(-t)|t} = \rho_T(g(-t)) \leq C_1 t^{C_2 + \epsilon}$$

and the claim follows. □

We now recall the statement of Theorem B.

**Theorem 4.4** (Accessibility Theorem). *Let  $(f_\lambda)_{\lambda \in M}$  be a natural family of finite type meromorphic maps, and  $\lambda_0 \in M$  be an asymptotic parameter of order  $n$ . Assume that at least one tract above the associated asymptotic value satisfies **(T)**. Then there is a cycle of period  $n+1$  exiting the domain at  $\lambda_0$ , and moreover its multiplier goes to zero as it exits the domain.*

*Proof.* Let  $v_{\lambda_0}$  be the associated asymptotic value so that  $f_{\lambda_0}^n(v_{\lambda_0}) = \infty$ . Recall that  $f_\lambda = \varphi_\lambda \circ f_{\lambda_0} \circ \psi_\lambda^{-1}$ . Let  $V_{\lambda_0} := \mathbb{D}^*(v_{\lambda_0}, r)$  be a punctured disk centered at  $v(\lambda_0)$  disjoint from  $S(f_{\lambda_0})$ , and  $T_{\lambda_0}$  a tract, so that  $f_{\lambda_0} : T_{\lambda_0} \rightarrow V_{\lambda_0}$  is a universal cover. Let  $\Phi_{\lambda_0} : T_{\lambda_0} \rightarrow \mathbb{H}$  be a conformal isomorphism, where  $\mathbb{H}$  is the left half plane. In particular,  $f_{\lambda_0}(z) = v_{\lambda_0} + re^{\Phi_{\lambda_0}(z)}$  for all  $z \in T_{\lambda_0}$ .

Let  $V_\lambda := \varphi_\lambda(V_{\lambda_0})$  and  $T_\lambda := \psi_\lambda(T_{\lambda_0})$ , so that  $T_\lambda$  is a tract above  $V_\lambda$ , and let  $\Phi_\lambda := \Phi_{\lambda_0} \circ \psi_\lambda^{-1} : T_\lambda \rightarrow \mathbb{H}$ . Then  $\varphi_\lambda^{-1} \circ f_\lambda : T_\lambda \rightarrow V_{\lambda_0}$  is a universal cover, and so for all  $z \in T_\lambda$ ,

$$(4.2) \quad f_\lambda(z) = \varphi_\lambda \left( v_{\lambda_0} + re^{\Phi_{\lambda_0}(z)} \right)$$

Now, we wish to find a curve  $t \mapsto \lambda(t)$  in parameter space such that

$$(4.3) \quad \Phi_{\lambda(t)} \circ f_{\lambda(t)}^n(v_{\lambda(t)}) = -t$$

We use the same notations as in the proof of Proposition 2.6: we let  $G(\lambda) := \psi_\lambda^{-1} \circ f_\lambda^n(v_\lambda)$ . Given the definition of  $\Phi_\lambda$ , Equation (4.3) is equivalent to

$$\Phi_{\lambda_0} \circ \psi_\lambda^{-1} \circ f_\lambda^n(v_\lambda) = -t,$$

or

$$(4.4) \quad G(\lambda(t)) = \Phi_{\lambda_0}^{-1}(-t).$$

Recall that  $G$  is quasiregular with  $G(\lambda_0) = \infty$ , and note that  $t \mapsto \Phi_{\lambda_0}^{-1}(-t)$  is a curve such that  $\lim_{t \rightarrow +\infty} \Phi_{\lambda_0}(-t) = \infty$ .

The map  $G$  is locally a branched cover over a neighborhood of  $\infty$ , and so we can find the desired curve  $t \mapsto \lambda(t)$  (defined for  $t$  large enough, and possibly not unique).

Now let  $D_t := \Phi_{\lambda(t)}^{-1}(\mathbb{D}(-t, \pi))$  and let  $U_t$  denote the connected component of  $f_{\lambda(t)}^{-n}(D_t)$  containing  $v_{\lambda(t)}$ . We will prove that for all  $t$  large enough,  $f_{\lambda(t)}^{n+1}(U_t) \subseteq U_t$ .

First, let us estimate the diameter of  $f_{\lambda(t)}^{n+1}(U_t) = f_{\lambda(t)}(D_t)$ . Let  $\epsilon > 0$ . From the definition of  $\Phi_\lambda$ , we have for all  $z \in \mathbb{H}$ :

$$f_\lambda \circ \Phi_\lambda^{-1}(z) = \varphi_\lambda(v_{\lambda_0} + re^z)$$

so that

$$f_{\lambda(t)}(D_t) = \varphi_{\lambda(t)} \subset \varphi_{\lambda(t)}(\mathbb{D}(v_{\lambda_0}, re^{-t+\pi}))$$

and by Lemma 2.11, we have for all  $t$  large enough:

$$(4.5) \quad f_{\lambda(t)}(D_t) \subset \mathbb{D}\left(v_{\lambda(t)}, e^{-t(1-\epsilon)}\right).$$

Now we estimate the inner radius of  $U_t$ , or more precisely, the distance  $d(v_{\lambda(t)}, \partial U_t)$  between  $v_{\lambda(t)}$  and the boundary of  $U_t$ .

Let us first estimate  $d(f_{\lambda(t)}^n(v_{\lambda(t)}), \partial D_t)$ . To lighten the notations, let  $g := \Phi_{\lambda_0}^{-1}$ ; then  $g$  is univalent on  $\mathbb{H}$  and  $D_t = \psi_{\lambda(t)} \circ g(\mathbb{D}(-t, 2\pi))$ . By Koebe's theorem,  $g(\mathbb{D}(-t, \pi))$  contains a disk

$$\mathbb{D}(g(-t), C|g'_t(-t)|)$$

for some constant  $C > 0$  independent from  $t$ . Then, by Lemma 2.11,  $D_t = \psi_{\lambda(t)}(g(\mathbb{D}(-t, 2\pi)))$  contains a disk  $\mathbb{D}(\psi_{\lambda(t)} \circ g(-t), C^{1+\epsilon}|g'(-t)|^{1+\epsilon})$ , in other words  $D_t$  contains a disk of the form

$$\mathbb{D}(f_{\lambda(t)}^n(v_{\lambda(t)}), C^{1+\epsilon}|g'(-t)|^{1+\epsilon}).$$

Now note that as  $t \rightarrow +\infty$ ,  $D_t$  is arbitrarily close to  $\infty$ . In particular, we may assume that for all  $t$  large enough  $D_t \cap S(f_{\lambda(t)}^n) = \emptyset$ , and we can define an inverse branch  $h_t : D_t \rightarrow U_t$  of  $f_{\lambda(t)}^{-n}$  (since  $D_t$  is simply connected). In fact,  $h_t$  can be extended to some simply connected neighborhood of  $\infty$  independent from  $t$ , and as  $t \rightarrow +\infty$  it converges on that domain to an inverse branch of  $f_{\lambda_0}^{-n-1}$ ; in particular, its spherical derivative  $h_t^\#(f_{\lambda(t)}^n(v_{\lambda(t)}))$  is bounded independently from  $t$ .

Again, Koebe's theorem applied to  $h_t : \mathbb{D}(f_{\lambda(t)}^n(v_{\lambda(t)}), C^{1+\epsilon}|g'(-t)|^{1+\epsilon}) \rightarrow U_t$  implies that there exists a constant  $C' > 0$  such that

$$(4.6) \quad \mathbb{D}(v_{\lambda(t)}, C'|g'(-t)|^{1+\epsilon}) \subset U_t.$$

Finally, from equations (4.5) and (4.6), it is enough to prove that as  $t \rightarrow +\infty$ :

$$\frac{C^{1+\epsilon}}{C'} \frac{e^{-t(1-\epsilon)}}{|g'(-t)|^{1+\epsilon}} \rightarrow 0.$$

For an appropriate choice of  $\epsilon > 0$ , this follows from Definition 4.1. This proves that  $f_{\lambda(t)}^n(U_t) \Subset U_t$ , and the theorem then follows from Schwartz's lemma. Note that the multiplier does go to zero as  $t \rightarrow +\infty$ , since  $\frac{\text{diam } f_{\lambda(t)}^n(U_t)}{\text{inrad } U_t} \rightarrow 0$ .  $\square$

## 5. BIFURCATION LOCUS. PROOF OF THEOREMS C AND D, AND COROLLARY D'

Before proceeding to the proof of Theorems C and D we prove some approximation results that will be useful, and are interesting on their own.

**Proposition 5.1** (Singular parameters are dense in the activity locus). *Let  $(f_\lambda)_{\lambda \in M}$  be a natural family of meromorphic maps with at least one pole which is not omitted. If  $\lambda_0 \in \mathcal{A}(v_\lambda)$  for some singular value  $v_\lambda$ , then  $\lambda_0$  is the limit point of a sequence of singular parameters for  $v_\lambda$  of order tending to infinity. Additionally, singular parameters belong to  $\mathcal{A}$  and hence they are dense in  $\mathcal{A}$ .*

*Proof.* Let  $\lambda_0 \in \mathcal{A}(v_\lambda)$  for some  $v_\lambda$ . Then either there is no neighborhood  $U$  of  $\lambda_0$  for which  $\{f_\lambda^n(v_\lambda)\}_n$  is defined for all  $n$  and all  $\lambda \in U$ ; or for every neighborhood  $U$  of  $\lambda_0$  where the family  $\{f_\lambda^n(v_\lambda)\}_n$  is well defined, it is not normal.

In the first case,  $\lambda_0$  can be approximated by singular parameters by definition of those. Moreover these singular parameters must have unbounded orders or otherwise, there exists  $N > 0$  and a sequence of  $\lambda_k \rightarrow \lambda_0$  such that  $f_{\lambda_k}^N(v_{\lambda_k}) = \infty$ . By continuity,  $f_{\lambda_0}^N(v_{\lambda_0}) = \infty$  and by the identity theorem  $f_\lambda^N(v_\lambda) = \infty$  for all  $\lambda \in U$  (and in fact for all  $\lambda \in M$ ), which means that  $v_\lambda$  is passive at  $\lambda_0$ , a contradiction.

In the second case, let  $p_1(\lambda)$  and  $p_2(\lambda)$  be two distinct prepoles varying analytically with  $\lambda$  in  $U$ . It follows that the family of maps  $g_n(\lambda) = \frac{f_\lambda^n(v_\lambda) - p_1(\lambda)}{p_1(\lambda) - p_2(\lambda)}$  is not normal as well, hence it must hit 0, 1 or  $\infty$  for infinitely many different  $n$ 's. Since it cannot hit infinity because the poles are distinct, it follows that it attains 0 or 1 infinitely many times, which correspond to singular parameters  $\lambda \in U$  of order  $n + 1$  tending to infinity.

To prove the density of singular parameters in  $\mathcal{A}$  it only remains to see that they themselves belong to the activity locus. But this is straightforward from the definition because if  $\lambda_0$  is

a singular parameter for  $v_\lambda$ , it means that  $f_{\lambda_0}^N(v_{\lambda_0}) = \infty$  for some  $N \geq 0$ , and the relation is non-persistent. Therefore the family  $\{f_\lambda^n(v_\lambda)\}_n$  cannot be well defined in any neighborhood of  $\lambda_0$ .  $\square$

**Remark 5.2.** It follows from the proof that asymptotic (resp. critical) parameters are accumulated by other asymptotic (resp. critical) parameters of orders tending to infinity.

**Proposition 5.3** (Critical parameters are accumulated by centers). *Let  $(f_\lambda)_{\lambda \in M}$  be a natural family of meromorphic maps of finite type. Let  $\lambda_0$  be a critical parameter of order  $n \geq 0$  for the critical value  $v_{\lambda_0}$ . Assume further that  $v_{\lambda_0}$  has a critical preimage  $c_{\lambda_0}$  which is not an exceptional value for  $f_{\lambda_0}$ . Then there exists a sequence of parameters  $\lambda_k \rightarrow \lambda_0$  as  $k \rightarrow \infty$ , such that for each  $f_{\lambda_k}$ ,  $v_{\lambda_k}$  is a superattracting periodic point of period  $n+2$  (if  $c_{\lambda_0}$  is not a critical value) or  $n+3$  (otherwise).*

*Proof.* Since  $c_{\lambda_0}$  is not exceptional, it is not an asymptotic value.

If  $c_{\lambda_0}$  is not itself a critical value, then it is not a singular value. In this case, let  $\gamma(\lambda) := c_\lambda$  be the analytic continuation of the critical point  $c_{\lambda_0}$ . By Lemma 2.6, given a neighborhood  $U$  of  $\lambda_0$  we can find  $\lambda_1 \in U$  such that  $f_{\lambda_1}^{n+1}(v_{\lambda_1}) = \gamma(\lambda_1) = c_{\lambda_1}$ . Hence  $v_{\lambda_1}$  is periodic of period  $n+2$  and  $c_{\lambda_1}$  belongs to the periodic orbit, thus  $\lambda_1$  is a center of period  $n+2$ . By taking successively smaller neighborhoods around  $\lambda_0$  we obtain a sequence of parameters  $\lambda_k \rightarrow \lambda_0$  with the same property.

If otherwise  $c_{\lambda_0}$  is a critical value, let  $a_1, \dots, a_5$  be five distinct preimages of  $c_{\lambda_0}$ , which exist because  $c_{\lambda_0}$  is not exceptional. For  $i = 1, \dots, 5$ , define  $\gamma_i(\lambda)$  as the analytic continuation of  $a_i$ , with  $\gamma_i(\lambda_0) = a_i$ . Again by Lemma 2.6 part (b), there exists  $i \in \{1, \dots, 5\}$  and  $\lambda_1$  arbitrarily close to  $\lambda_0$  such that  $f_{\lambda_1}^{n+1}(v_{\lambda_1}) = a_i(\lambda_1)$ . This means that  $f_{\lambda_1}^{n+3}(v_{\lambda_1})$  and its orbit contains a critical point. Therefore  $\lambda_1$  is a center of period  $n+3$ . Again, by obtaining parameters  $\lambda_k$  successively closer to  $\lambda_0$  we obtain a sequence of centers of order  $n+3$  approximating  $\lambda_0$ .  $\square$

We are now ready to prove the main results in this section.

**Theorem 5.4.** *Let  $(f_\lambda)_{\lambda \in M}$  be a non-exceptional natural family of finite type meromorphic maps whose tracts satisfy (T). Let  $U \subset M$  a simply connected domain. The following are equivalent:*

- (1) *there are no asymptotic parameters in  $U$*
- (2) *there are no cycles exiting the domain in  $U$*

*If moreover the maps  $f_\lambda$  have at least one pole that is not omitted, then this is also equivalent to*

- (3) *all asymptotic values are passive on  $U$ .*

*Proof.* The equivalence between (1) and (2) follows directly Theorems 3.5 and 4.4 respectively. The equivalence between (1) and (3) is given by Proposition 5.1.  $\square$

**Theorem 5.5.** *Let  $(f_\lambda)_{\lambda \in M}$  be a natural family of finite type meromorphic maps. Let  $U \subset M$  be a simply connected domain in parameter space. The following are equivalent:*

- (1) *The Julia set moves holomorphically over  $U$*
- (2) *Every singular value is passive on  $U$*

*If moreover the tracts of  $f_\lambda$  satisfy (T), then the statements above are also equivalent to*

- (3) *The maximal period of attracting cycles is bounded on  $U$ .*



*Proof.* We will prove that  $(1) \Leftrightarrow (2)$ ,  $(1) \Rightarrow (3)$  and, if moreover the tracts satisfy **(T)**, then  $(3) \Rightarrow (1)$ .

- $(1) \Rightarrow (2)$ : This implication mostly follows the arguments in [MSS83]. If the Julia set moves holomorphically over  $U$ , there is  $\lambda_0 \in U$  and a holomorphic motion  $H : U \times J_{\lambda_0} \rightarrow \hat{\mathbb{C}}$  preserving the dynamics. Hence  $H_\lambda(J_{\lambda_0}) = J_\lambda$  for all  $\lambda \in U$  and

$$H_\lambda(f_{\lambda_0}(z)) = f_\lambda(H_\lambda(z))$$

for all  $z \in J_{\lambda_0}$ . This means that  $H_\lambda$  maps critical points (resp. values) of  $f_{\lambda_0}$  in  $J_{\lambda_0}$  to critical points (resp. values) of  $f_\lambda$  in  $J_\lambda$  (see e.g. [McM88] for details). Likewise  $H_\lambda$  maps asymptotic values of  $f_{\lambda_0}$  in  $J_{\lambda_0}$  to asymptotic values of  $f_\lambda$  in  $J_\lambda$ , since the latter are locally omitted values.

Hence singular values and their full orbits in the Julia set can be followed by the conjugacy  $H_\lambda$ . Since  $f_{\lambda_0}$  has finitely many singular values, the union of their (forward) orbits is a countable set; but the Julia set is perfect and uncountable, hence we can consider three points  $z_1, z_2$  and  $z_3$  in  $J_{\lambda_0}$  which are disjoint from the forward orbits of the singular values of  $f_{\lambda_0}$ . Consequently, by the injectivity of the holomorphic motion, for all  $\lambda \in U$ ,  $H_\lambda(z_i)$ ,  $i = 1, 2, 3$ , is disjoint from the forward orbits of the singular values of  $f_\lambda$ . By Montel's Theorem it follows that the forward orbits of the singular values form normal families, and hence every singular value is passive in  $U$ . On the other hand, if a singular orbit lies in the Fatou set of  $f_{\lambda_0}$  then it must remain in the Fatou set of  $f_\lambda$  for every  $\lambda \in U$ . The orbit then misses all points in the Julia set and the same argument applies.

- $(2) \Rightarrow (1)$ : Assume that the Julia set does *not* move holomorphically over  $U$ . Then by Lemma 3.3, either two periodic points in the Julia set collide, or one periodic cycle in the Julia set exits the domain. In the first case, this means that there exists  $\lambda_0 \in U$  with a non-persistent parabolic periodic point: there exists  $z_{\lambda_0} \in \mathbb{C}$  such that  $f_{\lambda_0}^n(z_{\lambda_0}) = z_{\lambda_0}$ ,  $(f_{\lambda_0}^n)'(z_{\lambda_0}) = 1$ , and  $\lambda \mapsto (f_\lambda^n)'(z_{\lambda_0})$  is non-constant on  $U$ . Then its parabolic basin must contain at least one singular value  $v_{\lambda_0}$ , and therefore be active. In the second case, a cycle exits the domain at  $\lambda_0 \in U$ , and by Theorem 3.5,  $f_{\lambda_0}$  has either an active critical point or an active asymptotic value.
- $(1) \Rightarrow (3)$ : Assume that the Julia set moves holomorphically over  $U$ , and let  $H_\lambda$  be the conjugating holomorphic motion as above. Then  $H_\lambda$  maps repelling periodic points of  $f_{\lambda_0}$  to repelling periodic points of  $f_\lambda$  in  $J(f_\lambda)$  of the same period. Let  $N$  be the maximal period of non-repelling cycles for  $f_{\lambda_0}$  (which is finite by Fatou-Shishikura's inequality); then for all  $\lambda \in U$ , cycles of period more than  $N$  must be repelling, which implies that attracting cycles have period at most  $N$ .
- $(3) \Rightarrow (1)$ : Assume now that tracts of  $f_\lambda$  satisfy **(T)**. Suppose by contraposition that there is a singular value  $v_\lambda$  active at  $\lambda_0$ . Then by Proposition 5.1, there exists a sequence of singular parameters  $\lambda_k \rightarrow \lambda_0$  of order  $n_k \rightarrow \infty$ . Moreover, if  $v_\lambda$  is an asymptotic (resp. critical) value the parameters  $\lambda_k$  are asymptotic (resp. critical) parameters. In the case where  $\lambda_k$  are asymptotic parameters, they are limits of curves consisting of parameters which have an attracting cycle of period  $n_k + 1 \rightarrow \infty$ , by Theorem 4.4. In the case where  $\lambda_k$  are critical parameters, each of them is accumulated by centers whose superattracting cycles have period at least  $n_k + 2 \rightarrow \infty$ , by Proposition 5.3. In both cases, there are parameters with attracting cycles of period tending to infinity in  $U$ , a contradiction.

□

We can therefore define  $\text{Bif}(M)$  as the set of  $\lambda \in M$  for which the above conditions are not satisfied in any neighborhood of  $\lambda$ . In this notation, we have

**Corollary 5.6.**  $\text{Bif}^\circ(M) = \emptyset$ .

*Proof.* The proof follows the same argument as in [MSS83]: whenever a singular value is active, one may perturb the parameter to make it captured by an attracting cycle (either by Proposition 5.3 for the case of critical values, or by Theorem 4.4 for the case of a singular value). Since there are only finitely many singular values, this proves that arbitrarily close to any  $\lambda_0 \in \text{Bif}(M)$  we may find  $\lambda_1$  such that all singular values of  $f_{\lambda_1}$  are passive, and therefore  $\lambda_1 \notin \text{Bif}(M)$ .  $\square$

## 6. APPENDIX: GEOMETRY OF TRACTS FOR MEROMORPHIC FUNCTIONS WITH FINITELY MANY SINGULARITIES OF $f^{-1}$

**6.1. Singularities and logarithmic tracts.** In this section we recollect results from [Nev32], [Hil76], [Elf34] in order to deduce that the tracts for meromorphic functions with finitely many singularities are asymptotically close to sectors at infinity. See Section 6.2 for a precise statement.

We start with precise definitions of singular values and singularities.

A value  $v$  is called a *regular value* for  $f$  if there exists a neighborhood  $U$  of  $v$  such that for every connected component  $V$  of  $f^{-1}(U)$  we have that  $f : V \rightarrow U$  is univalent; it is called a *singular value* otherwise. Every function  $\varphi : U \rightarrow V$  satisfying  $f \circ \varphi = \text{Id}$  is called a regular branch of  $f^{-1}$ .

The precise behaviour of the branches of  $f^{-1}$  which are not regular gives the following classification of singularities, which then implies a classification for the singular values themselves ([BE95]; compare with [HY98], p.66).

**Definition 6.1** (Singularities of  $f^{-1}$ ). Let  $v \in \hat{\mathbb{C}}$  and denote by  $\mathbb{D}(r, v)$  the disk of radius  $r$  (in the spherical metric) centered at  $v$ . For every  $r > 0$  choose a component  $U_r$  of the preimage  $f^{-1}(\mathbb{D}(r, v))$  in such a way that  $r_1 < r_2$  implies  $U_{r_1} \subset U_{r_2}$ . Note that the function  $U : r \rightarrow U_r$  is completely determined by its germ at 0. Two possibilities can occur:

- (1)  $\bigcap_{r>0} U_r = \{z\}$ ,  $z \in \mathbb{C}$ . Then  $v = f(z)$ . If  $v \in \mathbb{C}$  and  $f'(z) \neq 0$  or if  $v = \infty$  and  $z$  is a simple pole then  $z$  is called a *regular point*. If  $v \in \mathbb{C}$  and  $f'(z) = 0$  or if  $v = \infty$  and  $z$  is a multiple pole of  $f$  then  $z$  is called a *critical point* and  $v$  is called a critical value. We say that the choice  $r \rightarrow U_r$  defines an *algebraic singularity* of  $f^{-1}$ . We also say that the critical point  $z$  lies over  $v$ .
- (2)  $\bigcap_{r>0} U_r = \emptyset$ . Then we say that our choice  $U : r \mapsto U_r$  defines a *transcendental singularity* of  $f^{-1}$ , which lies over  $v$ . If  $f : U_r \rightarrow \mathbb{D}(r, v)$  is an (infinite degree) unbranched covering,  $U$  is called a *logarithmic singularity*, and the sets  $U_r$  are called *logarithmic tracts*.

A transcendental singularity  $U$  over  $v$  is called *direct* if  $v \notin f(U_r)$  for  $r$  small, and *indirect* otherwise.

The value  $v$  is called an *asymptotic value* if there exists a curve  $\gamma : \mathbb{R}^+ \rightarrow \mathbb{C}$  such that  $\gamma(t) \rightarrow \infty$  and  $f(\gamma(t)) \rightarrow v$ .

If  $U$  is a transcendental singularity, then  $v$  is an asymptotic value (see [BE95], right after the definition of singularity), and to every asymptotic value corresponds at least one transcendental singularity.

With this definition,  $\infty$  is an asymptotic value for  $e^z$ , with an asymptotic tract laying over it; instead, for  $z \sin(z)$ , there is a transcendental singularity lying over  $\infty$  but no logarithmic tract. An example of indirect singularities is the singularity defined for the value 0 for the map  $\frac{\sin(z)}{z}$ . For more on the relation between critical points and indirect singularities see [BE95], p.357. Additional explicit examples can be found in [Nev70], Chapter XI. Singular values can be both critical and asymptotic, depending on the branch of the inverse under consideration. One can see that singular values are the closure of the set of critical and asymptotic values.

Note that, while logarithmic singularities are always direct (because of the covering property), the reverse is not true: In [BE08] one can find an example of a function whose set of direct singularities over 0 has the power of continuum, but none of these singularities is logarithmic.

If  $v \in \hat{\mathbb{C}}$  be an asymptotic value,  $V$  a punctured disk centered at  $v$ . If  $V$  does not contain singular values other than  $v$ , any unbounded connected component  $T$  of  $f^{-1}(V)$  is a logarithmic tract; let  $T$  be such a tract. Since  $f : T \rightarrow V$  is a universal cover, by the basic theory of (holomorphic) coverings there exists a biholomorphism  $\varphi : \mathbb{H} \rightarrow T$  such that  $f = \exp \circ \varphi^{-1}$ .

In the paper, we refer to logarithmic tracts simply as tracts. Technically, to each transcendental singularity correspond infinitely many nested tracts, depending on  $r$ . However we implicitly consider them all equivalent if they correspond to the same transcendental singularity. So in fact, when we say that  $f$  has finitely many tracts, we really mean finitely many equivalent classes of tracts, that is finitely many logarithmic singularities.

## 6.2. Geometry of tracts.

**Definition 6.2.** We say that a set  $U$  contains a sector at infinity if there exists  $\theta_0 \in [0, 2\pi]$ ,  $R > 0$ , and  $\alpha \in [0, \pi]$  such that  $U$  contains the set

$$\{z = |z|e^{2\pi i\theta} : |z| > R, |\theta - \theta_0| < \alpha\}$$

The goal for this appendix is to use results by Nevanlinna and Elfving ([Nev32] and [Elf34]) to deduce the following result. Previous successful applications of this approach in complex dynamics include the results in [BT98], [Ere04], [DK89].

**Proposition 6.3.** *Let  $f$  be a meromorphic functions with finitely many transcendental singularities and finitely many critical points. Let  $a \in \hat{\mathbb{C}}$  be an asymptotic value for  $f$ ,  $T$  be a tract over  $a$ . Then  $T$  contains a sector at infinity.*

One can replace the condition that  $f$  has finitely many transcendental singularities with the condition that it has finitely many (logarithmic) tracts. Indeed, finitely many critical points imply that any transcendental singularity is in fact logarithmic, hence corresponds to a logarithmic tract.

On the other hand, the hypothesis that  $f$  finitely many tracts (or in alternative finite order, see Proposition 6.5) cannot be replaced by the hypothesis that  $f$  has finitely many singular values. For example the function  $e^{e^z}$  has only two singular values but its infinitely many tracts over 0 asymptotically contain strips, not sectors.

The hypothesis that  $f$  has finitely many transcendental singularities can be replaced by the hypothesis that  $f$  has finite order. The fact that *entire* functions of finite order have finitely many asymptotic values is a consequence of the classical Denjoy-Carleman-Ahlfors Theorem, but the proof thereof in fact shows the following more precise result ([HY98], p.67; for the second part under the assumption that  $f$  has finitely many critical values see [BE95]).

**Theorem 6.4** (Denjoy-Carleman-Ahlfors Theorem tract version). *Let  $f$  be a meromorphic function of order  $\rho < \infty$ , and set  $p := \max(2\rho, 1)$ . Then  $f$  has at most  $p$  direct singularities. If in addition  $f$  has only finitely many critical values, then  $f$  has at most  $p$  transcendental singularities.*

Since all logarithmic singularities are also direct, under the hypothesis of the theorem there are only finitely many logarithmic singularities, or equivalently finitely many (logarithmic) tracts; since every transcendental singularity lies over an asymptotic value, if  $f$  has finite order and finitely many critical values, then  $f$  has at most  $p$  asymptotic values.

Observe that there exist meromorphic functions of finite order with infinitely many asymptotic values [Val25], and even for which each point in the Riemann sphere is an asymptotic value [Erë78]. According to Theorem 6.4, all but finitely many of them are indirect.

So in view of the Denjoy-Carleman-Ahlfors Theorem we have that Proposition 6.3 implies the following alternative version.

**Proposition 6.5.** *Let  $f$  be a meromorphic function of finite order  $\rho \leq p/2$  with  $p \in \mathbb{N}$  and assume that  $f$  has finitely many critical points. Let  $a$  be an asymptotic value for  $f$ ,  $T$  be a tract over  $a$ . Then  $T$  contains a sector at infinity.*

We will need to relate the singularities of the function  $f^{-1}$  to the theory of Riemann surfaces elaborated by Nevanlinna in [Nev32], in order to deduce results about the asymptotic properties of  $f$  applying results from [Nev32], [Elf34]. The following correspondence is well known (see for example [Nev70], Chapter XI and [Zhe10], Chapter 6).

**Theorem 6.6** (Riemann surfaces and singularities). *Let  $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$  be a meromorphic function with finitely many critical points and finitely many transcendental singularities (all of which must hence be logarithmic). Then there exists a Riemann surface  $S$  such that  $f : \mathbb{C} \rightarrow S$  is conformal. This is the Riemann surface associated to  $f^{-1}$ ; it has as many branching points of finite order as the critical points for  $f$  (i.e. the algebraic singularities of  $f$ ), and as many branching points of infinite order as the (logarithmic) tracts for  $f$  (i.e. the logarithmic singularities for  $f$ ).*

**6.3. Deducing Proposition 6.3 from results by Nevanlinna and Elfving.** Let  $f$  be a meromorphic function and consider its Schwarzian derivative, defined as

$$(6.1) \quad S_f = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2.$$

A direct computation ([Elf34], p. 36) shows that  $S_f$  (which is a meromorphic function) only has poles at critical points for  $f$  (recall that multiple poles for  $f$  do count as critical points), so if  $f$  has no critical points,  $S_f$  is entire. See [Elf34], p. 36, for more general relations between the local/asymptotic expansion of  $f$  and the local/asymptotic expansion of  $S_f$ .

The following proposition and its proof can be found in [Nev32], page 341-343 if  $f$  has no critical points, and can be modified to work also in the case that  $f$  has finitely many critical points ([Elf34], p.39).

**Proposition 6.7** (Rational Schwarzian). *Let  $f$  be a meromorphic function. If the Riemann surface associated to  $f$  has  $p < \infty$  branching points of infinite order and no branching points of finite order then  $S_f(z)$  is a polynomial function of degree  $p - 2 > 0$ . If in addition  $f$  also has  $q < \infty$  branching points of finite order,  $S_f(z)$  is a rational function  $R_f(z)$  which only has poles of order exactly 2, and has degree at most  $2q + p - 2$ . If  $f$  is transcendental the function  $S_f(z)$  has the asymptotic expansion  $S_f(z) \sim z^m$  for  $z \rightarrow \infty$ , with  $m \geq 0$ .*

Notice that the initial bound on the degree in [Nev32] is  $2p$ , and the better bound of  $p - 2$  is proven only later on. There are no transcendental functions with only one asymptotic value and no critical points. Indeed, the Schwarzian of such a function would be a poly of degree at least 0, hence has  $p = \text{degree} + 2 = 2$  asymptotic values. Here we are using also the sltns of the differential eqtn

For the fact that  $R_f(z) \sim az^m$  with  $m \geq -1$  integer if  $f$  is meromorphic see [BT98], Chapter 2, and the references therein [Lai93], paragraph 5. In fact, as it will be shown by Theorem 6.10, the case in which  $p$  is not even cannot occur if  $f$  is meromorphic, but only if  $f$  is a multivalued function (for example, functions of the form  $e^{z^{p/2}}$ , with  $p$  odd). See for example the remark in [Elf34], p.532, paragraph 40, for  $p = 1$ . The same discussion shows that functions with noninteger orders cannot have both finitely many logarithmic singularities and finitely many critical points. Notice however that Nevanlinna's methods work for  $m \geq -1$ , not just  $m \geq 0$ , which is the reason why sometimes in Proposition 6.7 the bound  $m \geq -1$  appears.

At this point we are not able yet to deduce a relationship between  $m$  (the asymptotics of the rational function  $S_f$  as  $z \rightarrow \infty$ ) and  $p$  (the number of tracts), but it will turn out from the sequel that in fact,  $m = p - 2$ .

In view of the relation between singularities and Riemann surfaces given by Theorem 6.6, Proposition 6.7 can be rewritten as follows. Compare with Theorem 8.1 in [?], keeping in mind that in their statement, asymptotic values are counted 'with multiplicity', so what they call number of asymptotic values is, with our notation, the number of non-equivalent asymptotic tracts.

**Proposition 6.8.** *If  $f$  has  $p < \infty$  tracts and no critical points its Schwarzian  $S_f(z)$  is a polynomial function of degree  $p - 2$ . If in addition  $f$  also has  $q < \infty$  critical points,  $S_f(z)$  is a rational function  $R_f(z)$  which only has poles of order exactly 2, and has degree at most  $2q + p - 2$ . If  $f$  is transcendental the function  $S_f(z)$  has the asymptotic expansion  $S_f(z) \sim z^m$  for  $z \rightarrow \infty$ , with  $m \geq 0$ .*

As in the previous section, using Denjoy-Carleman-Ahlfors Theorem the assumption that  $f$  has  $p$  tracts can be replaced by the assumption that  $f$  has finite order. Notice that if the order is zero,  $f$  may have one tract.

Nevanlinna has proven the following result for solutions of (6.1) under the assumption that  $S_f(z)$  polynomial (see pages 352-353 of [Nev32], referring to sectors defined in p. 351). See also Theorem 5.1 in [HY98].

**Theorem 6.9** (Asymptotic distribution of tracts for Polynomial Schwarzian). *Every solution  $f$  of (6.1) with  $S(f)$  polynomial of degree  $p - 2$ ,  $p \geq 2$  is a meromorphic function of order exactly  $p/2$ . Moreover, there are  $p$  disjoint sectors  $\{W_i\}_{1 \leq i \leq p}$  of angular width  $\frac{2\pi}{p}$  and  $p$  (non necessarily distinct) values  $\{a_i\}_{1 \leq i \leq p} \subset \hat{\mathbb{C}}$  such that  $f \rightarrow a_i$  uniformly on any proper subsector of  $W_i$  as  $z \rightarrow \infty$ . In particular, poles and zeroes of  $f$  are concentrated in the neighborhoods of the boundaries of the sectors  $W_i$ .*

In other words, if  $f$  is a meromorphic function whose Schwarzian is a polynomial, it can be seen as a solution of the differential equation (6.1) and hence satisfies Theorem 6.9.

If the Schwarzian is a rational functions rather than a polynomial one,

the methods by Hille that were used by Nevanlinna ([Hil76], Chapter 5 and Chapter 10) only give *local* meromorphic solutions defined on simply connected domains where  $R_f(z)$  has no poles (for example, a slit neighborhood of infinity, since rational functions have finitely many poles). In other words, while (6.1) always has globally defined meromorphic solutions if  $S_f$  is

a polynomial, this is no longer true if  $S_f$  is rational; this problem is investigated in [Elf34]. For the local solutions, analogous properties as the ones in Theorem 6.9 can be deduced on the aforementioned simply connected sets. However, if you start with a global meromorphic solution, then it is possible to obtain information on the asymptotic behaviour on the entire neighborhood of infinity.

Indeed, the following Theorem can be deduced ([Elf34], Theorem on p.54). Another version of this theorem appeared in [Erë93]. Eremenko's statement covers also cases with infinitely many critical points, under the assumption that the number of critical points which are contained in the disk of radius  $r$  grows slowly with respect to the order of the function. This is expressed precisely in terms of quantities from Nevanlinna theory:  $N_1(r) = o(T(r, f))$ .

**Theorem 6.10** (Asymptotic distribution of tracts for rational Schwarzian). *Let  $f$  be a global (meromorphic) solution of (6.1) with  $S_f(z)$  rational,  $S_f(z) \sim az^m$  as  $z \rightarrow \infty$ . Then  $f$  has order exactly  $\frac{m+2}{2}$ . Then  $m \geq -1$ , and there are  $m+2$  sectors  $\{W_i\}_{1 \leq i \leq m+2}$  of angular width  $\frac{2\pi}{m+2}$  and  $m+2$  (non necessarily distinct) values  $\{a_i\}_{1 \leq i \leq m+2} \subset \hat{\mathbb{C}}$  such that  $f \rightarrow a_i$  uniformly on any proper subsector of  $W_i$  as  $z \rightarrow \infty$ . Moreover, there are no asymptotic paths except for the ones contained in the sectors  $W_i$ .*

*Let  $d_i$  denote the straight rays which form the boundaries of the  $W_i$ . Then poles and zeroes of  $f$  are concentrated in sector neighborhoods of the directions  $d_i$ , and  $f$  takes every value except possibly  $0, \infty$  infinitely many times in any sector neighborhood of the directions  $d_i$ .*

If  $f$  is as in Theorem 6.6 or Proposition 6.3,  $f$  had exactly  $p$  tracts to start with, hence we deduce that in fact  $m+2 = p$ . While Elfving states explicitly that there cannot be other asymptotic paths except for the ones in the sectors  $W_i$ , this is already implicit in Nevanlinna's results.

As corollary of Theorems 6.9 and 6.10 we obtain Proposition 6.3.

*Proof of Proposition 6.3.* Since  $f$  has finitely many transcendental singularities and finitely many critical points, the asymptotic values for  $f$  are isolated singular values, hence they are logarithmic singularities. By Theorem 6.6 and Theorem 6.7,  $S_f$  is rational and  $S_f \sim z^m$ , with  $m \geq 0$ . Since  $f$  is a global transcendental meromorphic solution of the Schwarzian equation it satisfies the hypothesis of Theorem 6.9 or 6.10. Hence on each sector which is compactly contained in  $W_i$ ,  $f$  converges to the asymptotic value  $a_i$  as  $z \rightarrow \infty$ , hence, any such sector is contained in a logarithmic tract over  $a_i$ , and moreover  $f$  has no other tracts.  $\square$

## REFERENCES

- [AIM08] Kari Astala, Tadeusz Iwaniec, and Gaven Martin, *Elliptic partial differential equations and quasi-conformal mappings in the plane (pms-48)*, Princeton University Press, 2008.
- [AM01] K. Astala and G. J. Martin, *Holomorphic motions*, Papers on analysis, Rep. Univ. Jyväskylä Dep. Math. Stat., vol. 83, Univ. Jyväskylä, Jyväskylä, 2001, pp. 27–40. MR 1886611
- [BE95] Walter Bergweiler and Alexandre Eremenko, *On the singularities of the inverse to a meromorphic function of finite order*, Rev. Mat. Iberoamericana **11** (1995), no. 2, 355–373. MR 1344897
- [BE08] ———, *Direct singularities and completely invariant domains of entire functions*, Illinois J. Math. **52** (2008), no. 1, 243–259. MR 2507243
- [Ber00] Walter Bergweiler, *The role of the ahlfors five islands theorem in complex dynamics*, Conformal Geometry and Dynamics of the American Mathematical Society **4** (2000), no. 2, 22–34.
- [BF15] Anna Miriam Benini and Núria Fagella, *A separation theorem for entire transcendental maps*, Proc. Lond. Math. Soc. (3) **110** (2015), no. 2, 291–324. MR 3335280
- [BFJK18] Krzysztof Barański, Núria Fagella, Xavier Jarque, and Bogusław Karpińska, *Connectivity of Julia sets of Newton maps: a unified approach*, Rev. Mat. Iberoam. **34** (2018), no. 3, 1211–1228. MR 3850285

- [BT98] Walter Bergweiler and Norbert Terglane, *On the zeros of solutions of linear differential equations of the second order*, J. London Math. Soc. (2) **58** (1998), no. 2, 311–330. MR 1668183
- [CJK18] Tao Chen, Yunping Jiang, and Linda Keen, *Cycle doubling, merging and renormalization in the Tangent family*, Conformal Geometry and Dynamical Systems. **22** (2018), no. 5, 271–314. MR MR3880593
- [CJK19] ———, *Slices of parameter space for meromorphic maps with two asymptotic values*, ArXiv: 1908.06028, 2019.
- [CK19] Tao Chen and Linda Keen, *Slices of parameter spaces of generalized Nevanlinna functions*, Discrete and Dynamical Systems **39** (2019), no. 10, 5659–5681.
- [DK89] Robert L. Devaney and Linda Keen, *Dynamics of meromorphic maps: maps with polynomial Schwarzian derivative*, Ann. Sci. École Norm. Sup. (4) **22** (1989), no. 1, 55–79.
- [EL92] A. È. Erëmenko and M. Yu. Lyubich, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier (Grenoble) **42** (1992), no. 4, 989–1020. MR 1196102
- [Elf34] G. Elfving, *Über eine Klasse von Riemannschen Flächen und ihre Uniformisierung*, Acta Soc. Sci. Fenn. Ser.A **2** (1934), 1–60.
- [Eps93] Adam Lawrence Epstein, *Towers of finite type complex analytic maps*, ProQuest LLC, Ann Arbor, MI, 1993, Thesis (Ph.D.)—City University of New York. MR 2690048
- [Erë78] A. E. Erëmenko, *The set of asymptotic values of a finite order meromorphic function*, Mat. Zametki **24** (1978), no. 6, 779–783, 893.
- [Erë93] A. Erëmenko, *Meromorphic functions with small ramification*, Indiana Univ. Math. J. **42** (1993), no. 4, 1193–1218. MR 1266090
- [Ere04] A. Eremenko, *Geometric theory of meromorphic functions*, In the tradition of Ahlfors and Bers, III, Contemp. Math., vol. 355, Amer. Math. Soc., Providence, RI, 2004, pp. 221–230. MR 2145064
- [FK21] Núria Fagella and Linda Keen, *Stable components in the parameter plane of transcendental functions of finite type*, J. Geom. Anal. **31** (2021), no. 5, 4816–4855. MR 4244887
- [GK86] Lisa R. Goldberg and Linda Keen, *A finiteness theorem for a dynamical class of entire functions*, Ergodic Theory Dynam. Systems **6** (1986), no. 2, 183–192.
- [Hil76] Einar Hille, *Ordinary differential equations in the complex domain*, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1976, Pure and Applied Mathematics. MR 0499382
- [HY98] Xin-Hou Hua and Chung-Chun Yang, *Dynamics of transcendental functions*, Asian Mathematics Series, vol. 1, Gordon and Breach Science Publishers, Amsterdam, 1998. MR 1652248
- [KK97] Linda Keen and Janina Kotus, *Dynamics of the family  $\lambda \tan z$* , Conformal geometry and dynamics **1** (1997), 28–57.
- [Lai93] Ilpo Laine, *Nevanlinna theory and complex differential equations*, De Gruyter Studies in Mathematics, vol. 15, Walter de Gruyter & Co., Berlin, 1993.
- [Lyu84] M. Yu. Lyubich, *Investigation of the stability of the dynamics of rational functions*, Teor. Funktsii Funktsional. Anal. i Prilozhen. (1984), no. 42, 72–91, Translated in Selecta Math. Soviet. **9** (1990), no. 1, 69–90. MR 751394
- [McM88] Curt McMullen, *Braiding of the attractor and the failure of iterative algorithms*, Invent. Math. **91** (1988), no. 2, 259–272.
- [McM94] Curtis T. McMullen, *Complex dynamics and renormalization*, Annals of Mathematics Studies, vol. 135, Princeton University Press, Princeton, NJ, 1994. MR 1312365 (96b:58097)
- [MSS83] R. Mañé, P. Sad, and D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. École Norm. Sup. (4) **16** (1983), no. 2, 193–217. MR 732343 (85j:58089)
- [Nev32] Rolf Nevanlinna, *Über Riemannsche Flächen mit endlich vielen Windungspunkten*, Acta Math. **58** (1932), no. 1, 295–373. MR 1555350
- [Nev70] ———, *Analytic functions*, Translated from the second German edition by Phillip Emig. Die Grundlehren der mathematischen Wissenschaften, Band 162, Springer-Verlag, New York-Berlin, 1970. MR 0279280
- [Ros48] Rosenbloom, *L'itération des fonctions entières.*, Ann. Sci. École Norm. Sup. (4) (1948), no. 227, 382–383.
- [Val25] G. Valiron, *Sur les valeurs asymptotiques de quelques fonctions méromorphes*, Rend. Circ. Matem. Palermo **49** (1925), 415–421.
- [Zhe10] Jianhua Zheng, *Value distribution of meromorphic functions*, Tsinghua University Press, Beijing; Springer, Heidelberg, 2010. MR 2757285

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