HIGHER BIFURCATIONS FOR POLYNOMIAL SKEW PRODUCTS

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We continue our investigation of the parameter space of families of polynomial skew products. Assuming that the base polynomial has a Julia set not totally disconnected and is neither a Chebyshev nor a power map, we prove that, near any bifurcation parameter, one can find parameters where k critical points bifurcate *independently*, with k up to the dimension of the parameter space. This is a striking difference with respect to the one-dimensional case. The proof is based on a variant of the inclination lemma, applied to the postcritical set at a Misiurewicz parameter. By means of an analytical criterion for the non-vanishing of the self-intersections of the bifurcation current, we deduce the equality of the supports of the bifurcation current and the bifurcation measure for such families. Combined with results by Dujardin and Taflin, this also implies that the support of the bifurcation measure in these families has non-empty interior.

As part of our proof we construct, in these families, subfamilies of codimension 1 where the bifurcation locus has non empty interior. This provides a new independent proof of the existence of holomorphic families of arbitrarily large dimension whose bifurcation locus has non empty interior. Finally, it shows that the Hausdorff dimension of the support of the bifurcation measure is maximal at any point of its support.

1. Introduction

Polynomial skew products are regular polynomial endomorphisms of $\mathbb{C}^2$ of the form $f(z,w) = (p(z), q(z,w))$, for p and q polynomials of a given degree $d \geq 2$. *Regular* here means that the coefficient of w^d in q is non zero, which is equivalent to the extendibility of these maps as holomorphic self-maps of $\mathbb{P}^2$. Despite their specific forms, these maps already provided examples of new phenomena with respect to the established theory of one-variable polynomials or rational maps, see for instance [ABD$^+$16, Duj16, Taf17]. We started in [AB18] a detailed study of the parameter space of such maps.

We will denote in what follows by $\mathbf{Sk}(p,d)$ the family of all polynomial skew products of a given degree d over a fixed base polynomial p up to affine conjugacy, and denote by D_d its dimension. Following [BBD18] it is possible to divide the parameter space of the family $\mathbf{Sk}(p,d)$ (identified with $\mathbb{C}^{D_d}$) into two dynamically defined subsets: the *stability locus* and the *bifurcation locus*. The bifurcation locus coincides with the support of $dd^c L_v$, where $L_v(f)$ denotes the vertical Lyapunov function of f, see [Jon99, AB18]. We gave in [AB18] a description of the bifurcation locus and current in terms of natural bifurcation loci and currents associated to the vertical fibres, and a classification of unbounded hyperbolic components in the quadratic case.

For families of rational maps, the study of the self-intersections of the bifurcation current (which are meaningful because of the continuity of its potential) was started in [BB07], see also [Pha05, DF08, Duj11]. A geometric interpretation of the support of

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these currents is the following: the support of $T_{\rm bif}^k := T_{\rm bif}^{\wedge k}$ is the locus where k critical points bifurcate *independently*. Moreover, the current $T_{\rm bif}^k$ is known to equidistribute many kinds of dynamically defined parameters, such as maps possessing k cycles of prescribed multipliers and periods tending to infinity (see, e.g., [BB07, Gau16]). This gives rise to a natural stratification of the bifurcation locus as

$$\operatorname{Supp} T_{\rm bif} \supseteq \operatorname{Supp} T_{\rm bif}^2 f \supseteq \cdots \supseteq \operatorname{Supp} T_{\rm bif}^{k_{\rm max}}$$

where $k_{\rm max}$ is the dimension of the parameter space. The inclusions above are not equalities in general, and are for instance strict when considering the family of all polynomial or rational maps of a given degree (where $k_{\rm max}$ is equal to $d-1$ and $2d-2$, respectively). It is worth pointing out that this stratification is often compared with an analogous stratification for the Julia sets of endomorphisms of $\mathbb{P}^k$ (given by the supports of the self-intersections of the Green current, see for instance [DS10]). We refer to [Duj11] for a more detailed exposition.

In [AB18], the authors have proved the first equidistribution property for the bifurcation current $T_{\rm bif}$ in families of endomorphisms of projective spaces in any dimension, including polynomial skew products: for a generic $\eta \in \mathbb{C}$, the bifurcation current $T_{\rm bif}$ equidistributes the polynomial skew products with a cycle of period tending to infinity and vertical multiplier η. The arguments could easily be adapted to prove a similar statement for the bifurcation currents $T_{\rm bif}^k$: given generic $\eta_1, \dots, \eta_k \in \mathbb{C}$, $k \le k_{\rm max}$, the bifurcation current $T_{\rm bif}^k$ equidistributes skew products having k cycles of periods tending to infinity and respective vertical multipliers $\eta_1, \dots, \eta_k \in \mathbb{C}$. It is then natural to ask whether the supports of the bifurcation currents still give a natural stratification of the bifurcation locus.

The goal of this paper is to show that the situation in families of higher dimensional dynamical systems is completely different from the one-dimensional counterpart. Namely, we establish the following result.

Theorem 1.1. *Let p be a polynomial with Julia set not totally disconnected, which is neither conjugated to $z \mapsto z^d$ nor to a Chebyshev polynomial. Let $\mathbf{Sk}(p,d)$ denote the family of polynomial skew products of degree $d \ge 2$ over the base polynomial p, up to affine conjugacy, and let D_d be its dimension. Then the associated bifurcation current $T_{\rm bif}$ satisfies*

$$\operatorname{Supp} T_{\rm bif} \equiv \operatorname{Supp} T_{\rm bif}^2 \equiv \cdots \equiv \operatorname{Supp} T_{\rm bif}^{D_d}.$$

Theorem 1.1 is stated for the full family $\mathbf{Sk}(p,d)$ of all polynomial skew products of degree d over p (up to affine conjugacy). One could ask whether such a result holds for algebraic subfamilies of $\mathbf{Sk}(p,d)$: clearly, some special subfamilies have to be ruled out, such as the family of trivial product maps of the form $(p,q) : (z,w) \mapsto (p(z), q(w))$. A less obvious example in degree 3 is given by the subfamily of polynomial skew products over the base polynomial $z \mapsto z^3$ of the form

$$f_{a,b} : (z,w) \mapsto (z^3, u^3 + awz^2 + bz^3), \qquad (a,b) \in \mathbb{C}^2.$$

One can check that $f_{a,b}$ is semi-conjugated to the product map

$$g_{a,b} : (z,u) \mapsto (z^3, u^3 + au + b)$$

via the blow-up $\pi : (z,w) \mapsto (z, zw)$, that is, that $f_{a,b} \circ \pi = \pi \circ g_{a,b}$. It follows that $\operatorname{Supp} T_{\rm bif}^2(\Lambda) \subsetneq \operatorname{Supp} T_{\rm bif}(\Lambda)$, where $\Lambda := \{f_{a,b}, (a,b) \in \mathbb{C}^2\}$.

The proof of Theorem 1.1 indeed uses the fact that the family $\mathbf{Sk}(p,d)$ is general enough so that it is possible to perturb a bifurcation parameter to change the dynamical behaviour of a critical point in a vertical fibre without affecting all other fibres. It would be interesting to classify algebraic subfamilies of $\mathbf{Sk}(p,d)$ that, like Λ, are degenerate in the sense that a bifurcation in one fibre implies a bifurcation in all other fibre; for such families, the conclusion of Theorem 1.1 will not hold. Likewise, it would be natural to try to extend Theorem 1.1 to other families with a similar fibred structure, see for instance [DT18]. To do this, one should first ensure that such a family is large enough in the sense above.

The proof of Theorem 1.1 essentially consists of two ingredients, respectively of analytical and geometrical flavours.

The first is an analytical sufficient condition for a parameter to be in the support of T^k_{bif}. This is inspired by analogous results by Buff-Epstein [BE09] and Gauthier [Gau12] in the context of rational maps, and is based on the notion of *large scale condition* introduced in [AGMV19]. It is a way to give a quantified meaning to the *simultaneous independent bifurcation of multiple critical points*, and to exploit this condition to prove the non-vanishing of T^k_{bif}. This part does not require essentially new arguments and is presented in Section 4.

The second ingredient is a procedure to build these multiple independent bifurcations at a common parameter starting from a simple one. The idea is to start with a parameter with a *Misiurewicz* bifurcation, i.e., a non-persistent collision between a critical orbit and a repelling point, and to construct a new parameter nearby where two – and actually, D_d – independent Misiurewicz bifurcations occur simultaneously. Here we say that k Misiurewicz relations are *independent* at a parameter λ if the intersection of the k hypersurfaces given by the Misiurewicz relations has codimension k in $\mathbf{Sk}(p,d)$, see Subsection 2.2, and we denote by Bif^k the closure of such parameters.

This geometrical construction is our main technical result, and the main contribution of this paper. Together with the analytic arguments mentioned above (which give $\text{Bif}^k \subseteq \text{Supp}\, T^k_{\text{bif}}$ for all $1 \le k \le D_d$) and the trivial inclusion $\text{Supp}\, T^{D_d}_{\text{bif}} \subseteq \text{Supp}\, T_{\text{bif}}$, it implies Theorem 1.1.

Theorem 1.2. *Let p be a polynomial with Julia set not totally disconnected, which is neither conjugated to $z \mapsto z^d$ nor to a Chebyshev polynomial. Let $\mathbf{Sk}(p,d)$ denote the family of polynomial skew products of degree $d \ge 2$ over the base polynomial p, up to affine conjugacy, and let D_d be its dimension. Then*

$$\text{Bif} = \text{Bif}^2 = \cdots = \text{Bif}^{D_d}.$$

In order to construct the desired Misiurewicz parameter, we will consider the motion of a sufficiently large hyperbolic subset of the Julia set near a parameter in the bifurcation locus. This hyperbolic set needs to satisfy some precise properties, and this is where the assumptions on p come into play. The construction, presented in Section 3, uses tools from the thermodynamic formalism of rational maps, and more generally of endomorphisms of $\mathbb{P}^k$, as explained in Appendix A. Once the hyperbolic set is constructed, the proof proceed by induction. We show that, given a Misiurewicz relations satisfying a given list of further properties (see Definition 5.1), it is possible to construct, one by one, the extra Misiurewicz relations happening simultaneously. The general construction and the application in our setting are given in Sections 5 and 6, respectively.

Our main theorems and the method developed for their proof have a number of consequences and corollaries. We list here a few of them.

Corollary 1.3. *Let p be a polynomial with Julia set not totally disconnected, which is neither conjugated to $z \mapsto z^d$ nor to a Chebyshev polynomial. Near any bifurcation parameter in $\mathbf{Sk}(p, d)$ there exist algebraic subfamilies M^k of $\mathbf{Sk}(p, d)$ of any dimension $k < D_d$ such that the support of the bifurcation measure of M^k has non-empty interior in M^k.*

These families are given by the maps satisfying a given critical relation. Notice that d (and thus D_d) can be taken arbitrarily large. This result is for instance an improvement of the main result in [BT17], where 1-parameter families with the same property are constructed.

More strikingly, in [Duj17, Taf17], Dujardin and Taflin construct open sets in the bifurcation locus in the family $\mathcal{H}_d(\mathbb{P}^k)$ of all endomorphisms of $\mathbb{P}^k$, $k \geq 1$, of a given degree $d \geq 2$ (see also [Bie19] for further examples). Their strategy also works when considering the subfamily of polynomial skew products (and actually these open sets are built close to this family). Combining Theorem 1.1 with their result we thus get the following consequence.

Corollary 1.4. *Let p be a polynomial with Julia set not totally disconnected, which is neither conjugated to $z \mapsto z^d$ nor to a Chebyshev polynomial. The support of the bifurcation measure in $\mathbf{Sk}(p, d)$ has non empty interior.*

Notice that it is not known whether the bifurcation locus is the closure of its interior (see the last paragraph in [Duj17]). Hence, a priori, the open sets as above could exist only in some regions of the parameter space. The last consequence of our main theorems is a uniform and optimal bound for the Hausdorff dimension of the support of the bifurcation measure, which is a generalization to this setting of the main result in [Gau12].

Corollary 1.5. *Let p be a polynomial with Julia set not totally disconnected, which is neither conjugated to $z \mapsto z^d$ nor to a Chebyshev polynomial. The Hausdorff dimension of the support of the bifurcation measure in $\mathbf{Sk}(p, d)$ is maximal at all points of its support.*

Notice that, in the family of all endomorphisms of a given degree, such a uniform estimate is not known even for the bifurcation locus, see [BB18] for some local estimates.

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2. Notations and preliminary results

2.1. **Notations.** We collect here the main notations that we will use through all the paper. We refer to [AB18] and [Jon99] for more details.

Given a polynomial skew product of degree $d \geq 2$ of the form $f(z,w) = (p(z), q(z,w)) =: (p(z), q_z(w))$, we will write the n-th iterate of f as

$$f^n(z,w) = (p^n(z), q_{p^{n-1}(z)} \circ \cdots \circ q_z(w)) =: (p^n(z), Q_z^n(w)).$$

In particular, if z_0 is a n_0-periodic point for p, the map $Q_{z_0}^{n_0}$ is the return map to the vertical fibre $\{z_0\} \times \mathbb{C}$ and is a polynomial of degree d^{n_0}. For every z in the Julia set J_p of p, we denote by $K_z \subset \mathbb{C}$ the set of of points w such that the sequence $\{Q_z^n\}$ is bounded and by J_z the boundary of K_z. Given a subset $E \subseteq J_p$, we denote $J_E := \overline{\cup_{z \in E} \{z\} \times J_z}$.

Let us now denote by $(f_\lambda)_{\lambda \in M}$ a holomorphic family of polynomial skew products of a given degree $d \geq 2$, that is a holomorphic map $F \colon M \times \mathbb{C}^2 \to \mathbb{C}^2$, where M is a connected complex manifold, such that $f_\lambda := F(\lambda, \cdot)$ is a polynomial skew product of degree d for all $\lambda \in M$. We will denote by $\mathbf{Sk}(p,d)$ the family of all polynomial skew products of degree d with the given polynomial p as first component, up to affine conjugacy. We set $D_d := \dim \mathbf{Sk}(p,d)$. An explicit description of these families in the case $d = 2$ is given in [AB18, Lemma 2.9], the general case is similar.

Lemma 2.1. *Every polynomial skew product of degree $d \geq 2$ over a polynomial p is affinely conjugated to a map of the form*

$$(z,w) \mapsto \big(p(z), w^d + \sum_{j=0}^{d-2} w^j A_j(z)\big) \quad \textit{with} \quad \deg_z A_j = d - j.$$

We are interested in *bifurcations* within families of polynomial skew products. Following [BBD18], the *bifurcation locus* Bif is defined as the support of the $(1,1)-$positive closed current $T_{\mathrm{bif}} := dd^c_\lambda L(\lambda)$ on M, where $L(\lambda)$ is the Lyapunov function associated to f_λ with respect to its measure of maximal entropy. In the case of polynomial skew products, the function L has a quite explicit description. Indeed, by [Jon99] we have $L(\lambda) = L_p(\lambda) + L_v(\lambda)$, where

$$(1)\quad L_p(\lambda) = \log d + \sum_{z \in C_{p_\lambda}} G_{p_\lambda}(z) \text{ and } L_v(\lambda) = \log d + \int \Big(\sum_{w : q'_{\lambda,z}(w)=0} G_\lambda(z,w) \Big) \mu_{p_\lambda}(z).$$

Here $\mu_{p_\lambda}, G_{p_\lambda}, C_{p_\lambda}$ are the measure of maximal entropy, the Green function and the critical set (whose points are counted with multiplicity) of f_λ and p_λ respectively, and $G_\lambda(z,w) := \lim_{n\to\infty} \frac{1}{n} \log^+ \left\| Q^n_{\lambda,z}(w) \right\|$ is the *non-autonomous Green function* for the family $\{Q^n_{\lambda,z}\}_{n \in \mathbb{N}}$. The current $T_p := dd^c_\lambda L_p(\lambda)$ is positive and closed. We proved in [AB18, Proposition 3.1] that $T_v := dd^c_\lambda L_v = T_{\mathrm{bif}} - T_p$ is also positive and closed. This allowed us to define the *vertical bifurcation* in any family of polynomial skew products. This was generalized in [DT18] to cover families of endomorphisms of $\mathbb{P}^k(\mathbb{C})$ preserving a fibration. Of course, when p is constant we have $T_{\mathrm{bif}} = T_v$.

2.2. **Families defined by Misiurewicz relations.** By [BBD18, Bia19] the bifurcation locus of a family $(f_\lambda)_{\lambda\in M}$ coincides with the closure of the set of *Misiurewicz parameters*, i.e., parameters for which we have a non-persistent intersection between some component of the post critical set and the motion of some repelling point. More precisely, in our setting take $\lambda_0 \in \mathbf{Sk}(p,d)$ and let M be any holomorphic subfamily of $\mathbf{Sk}(p,d)$ such that $\lambda_0 \in M$. A *Misiurewicz relation* for f_{λ_0} is an equation of the form $f_{\lambda_0}^{n_0}(z_0,c_0) = (z_1,w_1)$ where (z_1,w_1) is a repelling periodic point of period m for f_{λ_0}, and $q'_{z_0,\lambda_0}(c_0) = 0$.

Assume that c_0 is a simple root of q'_{z_0,λ_0} (this assumption could be removed, but we keep it here for the sake of simplicity). Then there is a unique holomorphic map $\lambda \mapsto c(\lambda)$ defined on a neighbourhood of λ_0 in $\mathbf{Sk}(p,d)$ such that $c(\lambda_0) = c_0$. Similarly, it is possible to locally follow holomorphically the repelling point (z_1,w_1) as $\lambda \mapsto (z_1,w_1(\lambda))$.

The Misiurewicz relation $f_{\lambda_0}^{n_0}(z_0,c_0) = (z_1,w_1)$ is said to be *locally persistent in* M if $f_\lambda^{n_0}(z_0,c(\lambda)) = (z_1,w_1(\lambda))$ for all λ in a neighbourhood of λ_0 in M. If this is not the case, the equation $f_\lambda^{n_0}(z_0,c(\lambda)) = (z_1,w_1(\lambda))$ defines a germ of analytic hypersurface in M at λ_0, which is open inside the algebraic hypersurface of M given by $\{\lambda \in M : \mathrm{Res}_w(q'_{\lambda,z_0}, Q^{n_0+m}_{\lambda,z_0} - Q^m_{\lambda,z_0}) = 0\}$. Here, $\mathrm{Res}_w(P,Q)$ denotes the resultant of two polynomials $P,Q \in A[w]$, where $A := \mathbb{C}[\lambda]$; it is therefore an element of A. Notice that this algebraic hypersurface consists of all $\lambda \in M$ such that some critical point in the fibre at z_0 lands after n_0 iterations on some periodic point of period dividing m. We also say in this case that λ_0 is a *Misiurewicz parameter in* M. If the Misiurewicz relation is non-persistent in M, we denote by $M_{(z_0,c),(z_1,w_1),n_0}$ (or by $M_{(z_0,c_0),(z_1,w_1(\lambda_0)),n_0}$ if we wish to emphasize the starting parameter λ_0) this irreducible component and we call it the locus where the relation is locally preserved. We may avoid mentioning the periodic point if this does not create confusion.

2.3. **The unicritical subfamily $U_d \subset \mathbf{Sk}(p,d)$.** We consider here the *unicritical subfamily* $U_d \subset \mathbf{Sk}(p,d)$ given by

$$U_d := \{f(z,w) = (p(z), w^d + a(z))\}, \quad a(z) \in \mathbb{C}_d[z] \sim \mathbb{C}^{d+1}. \tag{2}$$

Thus, U_d has dimension $d+1$. We parametrize it with $\lambda := (a_0,\dots,a_d)$, where the a_i are the coefficients of $a(z)$. We will write $\lambda(z_0) = 0$ when z_0 is a root of the polynomial $a(z)$ associated to λ, and similarly $\lambda'(z_0) = 0$ when z_0 is a root of $a'(z)$.

We can compactify this parameter space to $\mathbb{P}^{d+1}$ and we denote by $\mathbb{P}^d_\infty$ the hyperplane at infinity. Notice that, unless $p'(z_0) = 0$, $(z_0,0)$ is the only critical point for f_λ in the fibre $\{z = z_0\}$ (this justifies the name chosen for this family, coherently with the name of the one dimensional unicritical family $f_\lambda(z) = z^d + \lambda$).

Lemma 2.2. *There exist two positive constants C_1, C_2 such that, for all $\lambda \in U_d$ and for all $z \in J_p$, we have $K_z(f_\lambda) \subset \mathbb{D}(0, C_1 + C_2|\lambda|^{1/d})$. Moreover, if $\lambda_j \in M$ is a sequence with $|\lambda_j| \to \infty$ and $[\lambda_j] \to [\lambda_\infty]$ for some λ_∞ such that $\lambda_\infty(z_0) \neq 0$, then for all $w_j \in K_{z_0}(f_{\lambda_j})$ we have $|w_j| \asymp |\lambda_j|^{1/d}$ as $j \to \infty$.*

Proof. Set $A(\lambda) := \max_{z\in J_p} |a(z)|$. Observe that we have $A(\lambda) = \mathcal{O}(|\lambda|)$ as $|\lambda| \to \infty$, hence there exists $C_0 > 2$ such that $A(\lambda) \le C_0|\lambda|$ for all $\lambda \in U_d$ large enough. It follows that, if w satisfies $|w| > C_0|\lambda|^{1/d}$, then for any $z \in J_p$ we have $|q_z(w)| > C_0^2|\lambda| - A(\lambda) \ge (C_0^2 - C_0)|\lambda| > C_0|\lambda|$. This proves the first assertion.

For the second assertion, let $a^{(j)}(z)$ be the polynomial associated to λ_j. Take $w_j \in K_{z_0}(f_{\lambda_j})$. Hence, $q_{\lambda_j,z_0}(w_j) \in K_{p(z_0)}(f_{\lambda_j})$. By the first part of the statement, we have $|w_j^d + a^{(j)}(z_0)| = |q_{\lambda_j,z_0}(w)| \le C_1 + C_2|\lambda_j|^{1/d}$. Since $\lambda(z_0) \neq 0$, we have $|a^{(j)}(z_0)| \asymp |\lambda_j|$ as $\lambda_j \to \infty$. Hence, $|w_j^d| \asymp |\lambda_j|$, which gives $|w_j| \asymp |\lambda_j|^{1/d}$. □

Let us now consider the intersection of a Misiurewicz hypersurface in $\mathbf{Sk}(p,d)$ with U_d. This (when not empty) is a Misiurewicz hypersurface in U_d. Since the only critical points for maps in U_d that can give non-persistent Misiurewicz relations in U_d are of the form $(z_0, 0)$ (and these all have multiplicity d), we see that any Misiurewicz hypersurface of U_d has the form

$$Q^n_{\lambda,z_0}(0) = Q^{n+m}_{\lambda,z_0}(0) \text{ for some } m,n \ge 1 \text{ and } z_0 \in J_p \text{ with } p^{n+m}(z_0) = p^n(z_0). \tag{3}$$

For simplicity, we denote by $M_{z_0,n,m}$ the hypersurface defined by (3).

Lemma 2.3. *For any non-empty Misiurewicz hypersurface $M_{z_0,n,m} \subset U_d$ of the form (3), the accumulation on $\mathbb{P}^d_\infty$ of $M_{z_0,n,m}$ is precisely given by $E_{z_0} := \{[\lambda] \colon \lambda(z_0) = 0\}$.*

Proof. It follows from Lemma 2.2 that the accumulation of $M_{z_0,n,m}$ on $\mathbb{P}^d_\infty$ is included in E_{z_0}. On the other hand, the restriction of $M_{z_0,n,m}$ to any 2-dimensional subfamily of U_d cannot be compact. By considering, for every point in E_{z_0}, an affine 2-dimensional subfamily whose line of intersection with $\mathbb{P}^d_\infty$ meets E_{z_0} only in the given point, we see that the inclusion is actually an equality. □

Lemma 2.4. *For any non-empty Misiurewicz hypersurface $M_{z_0,n,m} \subset U_d$ of the form (3), the non-vertical eigenspace of $(df^m_\lambda)_{(z_1,w_1(\lambda))}$ at $(z_1, w_1(\lambda)) := (p^n(z_0), Q^n_{\lambda,z_0}(0))$ is generated by the vector*

$$v_\lambda := \left(1, \frac{\frac{\partial Q^m_{\lambda,z}(w)}{\partial z}\Big|_{(z,w)=(z_1,w_1(\lambda))}}{\frac{\partial Q^m_{\lambda,z}(w)}{\partial w}\Big|_{(z,w)=(z_1,w_1(\lambda))} - (p^m)'(z_1)}\right).$$

In particular, if $z_0 \notin \{p^n(z_0), \ldots, p^{n+m-1}(z_0)\}$, given λ_∞ such that $\lambda'_\infty(p^i(z_0)) \neq 0$ for all $n \le i < n+m$ and a sequence $\lambda_j \in M_{z_0,n,m}$ with $|\lambda_j| \to \infty$ and $[\lambda_j] \to [\lambda_\infty]$, the second component $v^{(2)}_{\lambda_j}$ of v_{λ_j} as above satisfies

$$|v^{(2)}_{\lambda_j}| = \mathcal{O}(|\lambda_j|^{1/d}) \text{ as } j \to \infty. \tag{4}$$

Proof. We have

$$(df^m_\lambda)_{z_1,w_1(\lambda)} = \begin{pmatrix} (p^m)'(z_1) & 0 \\ \frac{\partial Q^m_{\lambda,z}(w)}{\partial z}\Big|_{(z,w)=(z_1,w_1(\lambda))} & \frac{\partial Q^m_{\lambda,z}(w)}{\partial w}\Big|_{(z,w)=(z_1,w_1(\lambda))} \end{pmatrix},$$

from which we deduce the first assertion. A direct computation shows that, for every $\lambda \in M_{z_0,n,m}$,

$$\begin{aligned} \frac{\partial Q^m_{\lambda,z}(w)}{\partial z}\Big|_{(z,w)=(z_1,w_1(\lambda))} &= \sum_{i=0}^{m-1} a'(p^i(z_1)) \prod_{\ell=i+1}^{m-1} q'_{p^\ell(z_1)}(Q^\ell_{\lambda,z_1}(w_1(\lambda))) \\ &= \sum_{i=0}^{m-1} a'(p^i(z_1)) \prod_{\ell=i+1}^{m-1} d\left[Q^\ell_{\lambda,z_1}(w_1(\lambda))\right]^{d-1} \end{aligned} \tag{5}$$

and

$$\frac{\partial Q^m_{\lambda,z}(w)}{\partial w}\Big|_{(z,w)=(z_1,w_1(\lambda))} = \prod_{\ell=0}^{m-1} q'_{p^\ell(z_1)}(Q^j_{\lambda,z_1}(w_1(\lambda))) = \prod_{\ell=0}^{m-1} d\left[Q^\ell_{\lambda,z_1}(w_1(\lambda))\right]^{d-1}$$

where a is the polynomial associated to λ.

Let us now consider the evaluations of the expressions above at a sequence λ_j as in the statement, and let us denote by $a^{(j)}$ the polynomial associated to λ_j. Since $\lambda'_\infty(p^i(z_0)) \neq 0$ for all $n \le i < n+m$, we have $|(a^{(j)})'(p^i(z_0))| \asymp |\lambda_j|$ as $j \to \infty$. Moreover, by Lemma 2.2, for all $0 \le \ell < m$ we have $|Q^\ell_{z_1,\lambda_j}(w_1(\lambda_j))| \asymp |\lambda_j|^{1/d}$. Hence, both the expressions above diverge as $|\lambda_j| \to \infty$ with $[\lambda_j] \to [\lambda_\infty]$, and the largest term in the sum in the last term of (5) is that corresponding to $i = 0$. Hence, as $j \to \infty$, we have

$$\begin{aligned}|v^{(2)}_{\lambda_j}| &\asymp \frac{\left|(a^{(j)})'(z_1)\prod_{\ell=1}^{m-1} d(Q^\ell_{\lambda_j,z_1}(w_1(\lambda_j)))^{d-1}\right|}{\left|\prod_{\ell=0}^{m-1} d(Q^\ell_{\lambda_j,z_1}(w_1(\lambda_j)))^{d-1}\right|} = \frac{|(a^{(j)})'(z_1)|}{d(w_1(\lambda_j))^{d-1}}\\ &= \mathcal{O}\left(\frac{|\lambda_j|}{|\lambda_j|^{(d-1)/d}}\right) = \mathcal{O}(|\lambda_j|^{1/d}),\end{aligned}$$

where in the last steps we used again Lemma 2.2. □

Lemma 2.5. *For any non-empty Misiurewicz hypersurface $M_{z_0,n,m} \subset U_d$ of the form* (3)*, the image of $(df^n_\lambda)_{(z_0,0)}$ is generated by the vector*

$$u_\lambda := \left(1, \frac{\frac{\partial Q^n_{\lambda,z}(w)}{\partial z}\Big|_{(z,w)=(z_0,0)}}{(p^n)'(z_0)}\right).$$

In particular, given λ_∞ such $\lambda'_\infty(p^i(z_0)) \neq 0$ for $0 \le i \le n-1$ and a sequence $\lambda_j \in M_{z_0,n,m}$ with $|\lambda_j| \to \infty$ and $[\lambda_j] \to [\lambda_\infty]$, the second component $u^{(2)}_{\lambda_j}$ of u_{λ_j} as above satisfies

$$|u^{(2)}_{\lambda_j}| \asymp |\lambda_j|^{\frac{n(d-1)+1}{d}} \text{ as } j \to \infty. \tag{6}$$

Proof. Since

$$(df^n_\lambda)_{z_0,0} = \begin{pmatrix} (p^n)'(z_0) & 0 \\ \frac{\partial Q^n_{\lambda,z}(w)}{\partial z}\Big|_{(z,w)=(z_0,0)} & 0 \end{pmatrix},$$

the first part of the statement is immediate. A computation as in Lemma 2.5 gives

$$\frac{\partial Q^n_{\lambda,z}(w)}{\partial z}\Big|_{(z,w)=(z_0,0)} = \sum_{i=0}^{n-1} a'(p^i(z_0)) \prod_{\ell=i+1}^{n-1} d(Q^\ell_{\lambda,z_0}(0))^{d-1}$$

(where again a is the polynomial associated to λ) and, by Lemma 2.2, the above expression diverges as $j \to \infty$ when evaluated at λ_j as in the statement. Moreover, as $j \to \infty$, denoting by $a^{(j)}$ the polynomial associated to λ_j, we have

$$\begin{aligned}|u^2_{\lambda_j}| &\asymp \left|\frac{\partial Q^n_{\lambda_j,z}(w)}{\partial z}\Big|_{(z,w)=(z_0,0)}\right| \asymp \left|(a^{(j)})'(z_0)\prod_{\ell=1}^{n-1} d(Q^\ell_{\lambda_j,z_0}(0))^{d-1}\right|\\ &\asymp |\lambda_j|^{1+\frac{(n-1)(d-1)}{d}} = |\lambda_j|^{\frac{n(d-1)+1}{d}},\end{aligned}$$

where we used the facts that $|(a^{(j)})'(z_0)| \neq 0$ for sufficiently large j, and hence $|(a^{(j)})'(z_0)| \asymp |\lambda_j|$, and that $Q^\ell_{\lambda_j,z_0}(0) \in K(f_{\lambda_j})$, and hence $|Q^\ell_{\lambda_j,z_0}(0)| \asymp |\lambda_j|^{1/d}$ by Lemma 2.2. □

2.4. **Higher bifurcations currents and loci.** Higher bifurcation currents for families of polynomials (or rational maps) in one variable were introduced in [BB07], see also [DF08], with the aim of understanding the loci where simultaneous and independent bifurcations happen, from an analytical point of view. Since the Lyapunov function is continuous with respect to the parameters [DS10], it is indeed meaningful to consider the self-intersections $T^k_{\text{bif}} := T^{\wedge k}_{\text{bif}}$ of the bifurcation current, for every k up to the dimension of the parameter space. The measure obtained by taking the maximal power is usually referred to as the *bifurcation measure*.

While in dimension one it is quite natural to associate a geometric meaning to $\text{Supp}(T^k_{\text{bif}})$ (as, for instance, the points where k independent Misiurewicz relations happens, in a quite precise sense, see, e.g., [Duj11]), in higher dimensions the critical set is of positive dimension and thus this interpretation is far less clear.

The following result gives a first step in the interpretation of the higher bifurcations as average of non-autonomous counterparts of the classical one-dimensional objects, valid in any family of polynomial skew products over a fixed base p. An interpretation of the non-autonomous factors will be the object of Section 4. The case of general polynomial skew products is completely analogous, and the following should be read as a decomposition for the vertical bifurcation $T^k_v = (dd^c L_v)^k$, see Section 2.1. Given $\underline{z} := (z_1, \dots, z_k) \in J^k_p$, we denote by $T_{\underline{z}}$ the current $T_{\underline{z}} = T_{\text{bif}\, z_1} \wedge \cdots \wedge T_{\text{bif}\, z_k}$, where for every $z \in J_p$ we set $T_{\text{bif}\, z} := dd^c_\lambda \Big(\sum_{w: q'_{\lambda,z}(w)=0} G_\lambda(z,w) \Big)$, see [AB18, §2.4].

Proposition 2.6. *Let $(f_\lambda)_{\lambda \in M}$ be a family of polynomial skew products over a fixed base p. Then*

$$T^k_{\text{bif}} = \int_{J^k_p} T_{\underline{z}} \mu^{\otimes k} \quad \text{and} \quad \text{Supp}(T^k_{\text{bif}}) = \overline{\cup_{\underline{z}} \text{Supp}\, T_{\underline{z}}}.$$

Proof. The case $k=1$ follows from the explicit formula for L_v in (1). The first formula in the statement is a consequence of the case $k=1$ and the continuity of the potentials of the bifurcation currents $T_{\text{bif},z}$. The continuity of the potentials (in both z and the parameter) also implies that the currents $T_{\underline{z}}$ are continuous in $\underline{z} \in J^k_p$. We can thus apply the general Lemma 2.7 below to the family of currents $R_a = T_{\underline{z}}$ and $a = \underline{z} \in J^k_p = A$. This concludes the proof. □

Lemma 2.7. *Let A be a compact metric space, ν a positive measure on A and R_a a family of positive closed currents on $\mathbb{C}^N$ depending continuously on $a \in A$. Set $R := \int_A R_a \nu(a)$. Then*

(1) *the support of R_a depends lower semicontinuously from a (in the Hausdorff topology);*
(2) *the support of R is included in $\overline{\cup_a \text{Supp}\, R_a}$;*
(3) *for every $a \in \text{Supp}\, \nu$, we have $\text{Supp}\, R_a \subseteq \text{Supp}\, R$.*

Recall that the current $R = \int_A R_a \nu(a)$ is defined by the identity $\langle R, \varphi \rangle = \int_A \langle R_a, \varphi \rangle \nu(a)$, for φ test form of the right degree.

Proof. The first property is classical and the second is a direct consequence. Let us prove the last item. Fix $a \in A$ and take $x \in \operatorname{Supp} R_a$. There exists an (arbitrarily small) ball B centred at x such that the mass of R_a on B is larger than some $\eta > 0$. By the continuity of R_a, the mass of $R_{a'}$ on B is larger that $\eta/2$ for *every* a' sufficiently close to a. In particular, this is true for all a' in a ball B' centred at a. Since $a \in \operatorname{Supp} \nu$, we have $\nu(B') > \eta'$ for some positive η'. Thus, R has mass $> \eta\eta'/2$ on B, which in turn gives $x \in \operatorname{Supp} R$. □

3. Vertical-like hyperbolic sets and IFSs

Definition 3.1. *Let $f(z) = (p(z), q(z,w))$ be a polynomial skew product of degree ≥ 2 and let H be an f-invariant hyperbolic set. We say that H is* vertical-like *if there exists $\alpha > 0$ such that, for every $(z,w) \in H$, we have $df_{(z,w)}(C_\alpha) \Subset C_\alpha$, where*

$$C_\alpha := \left\{ u \in \mathbb{C}^2 : |\langle u, (0,1) \rangle| > \alpha \|u\| \right\}. \tag{7}$$

Recall that, given any ergodic measure ν supported on a f-invariant hyperbolic set H, by Oseledets theorem one can associate to ν-almost every $x \in H$ a decomposition of the tangent space $T_x\mathbb{C}^2 = E_1 \oplus E_2$, which is invariant under f, with the property that $\lim_{n\to\infty} n^{-1} \log \|df_x^n(v)\| = \chi_i$ for all $v \in E_i$, where χ_1, χ_2 are the Lyapunov exponents of ν. The hyperbolicity of H implies that the decomposition is continuous in x, which in turn implies that it is also independent of ν. Since f is a polynomial skew product, we know that one invariant direction must necessarily coincide with the vertical one. Denoting by $E_v = \langle (0,1) \rangle$ and E_h the two fields of directions, Definition 3.1 implies that E_h is then uniformly far from the vertical direction.

In the case of a periodic cycle, Definition 3.1 can we rephrased as a condition on the eigenvalues of the differential of the return map at the periodic points. Although a periodic point is not an invariant hyperbolic set, we will adopt the following notation for simplicity.

Definition 3.2. *Let $f(z) = (p(z), q(z,w))$ be a polynomial skew product of degree ≥ 2 and let (z_1, w_1) be a m-periodic point for f. Let $A := (p^m)'(z_1)$ and $B := (Q^m_{z_1})'(w_1)$ be the two eigenvalues of $df^m_{(z_1,w_1)}$. We say that (z_1, w_1) is* vertical-like *if $|B| > |A|$.*

Definition 3.3. *Let $z \in \mathbb{C}$. We denote by μ_z the equilibrium measure of J_z.*

By [Jon99, Proposition 2.1], $\mu_z = \frac{1}{2\pi} dd^c G_z$, where $G_z := G(z,\cdot) - G_p(z)$, and G denotes the Green function of the skew-product while G_p denotes the Green function of the base polynomial p. In particular, μ_z has continuous potential, and $z \mapsto \mu_z$ is continuous.

Definition 3.4. *Let f be a polynomial skew product of degree $d \geq 2$. We say that a set $A \subset \mathbb{C}^2$, with $A \cap J \neq \emptyset$, is a* fibred box *for f if A is an open set $A = \cup_{z\in B} \{z\} \times D_z$ where B is an open subset of $\mathbb{C}$ and $D_z \subset \mathbb{C}$ is a topological disk depending continuously on $z \in B$ and such that $\mu_z(D_z)$ is positive and constant in z for all $z \in J_p \cap B$.*

Observe that fibred boxes exist since the family of measures $z \mapsto \mu_z$ is continuous.

Definition 3.5. *Let f be a polynomial skew product of degree $d \geq 2$. A* vertical-like IFS *for f is the datum of a fibred box W and of $m \geq 2$ inverse branches $g_1, \ldots, g_m$ of f^n with $g_i(W) \Subset W$, $1 \leq i \leq m$ and such that:*

(V1) *the limit set is a vertical-like hyperbolic set;*
(V2) *for all* $1 \le i \le m$*, there exists* $i \neq j$ *such that* $\pi_z(g_i(W)) = \pi_z(g_j(W))$*;*
(V3) *there exists* $i \neq j$ *such that* $\pi_z(g_i(W)) \cap \pi_z(g_j(W)) = \emptyset$*.*

Note that due to the skew product structure of f, for all $1, \le i, j \le m$, we automatically have either $\pi_z(g_i(W)) = \pi_z(g_j(W))$ or $\pi_z(g_i(W)) \cap \pi_z(g_j(W)) = \emptyset$. We will consider in the following limit sets of vertical-like IFSs, which are then vertical-like hyperbolic sets (contained in J_{H_p}, for some hyperbolic set $H_p \subset J_p$ for p) as in Definition 3.1 by **(V1)**. Condition **(V2)** ensures that each vertical slice of the limit set is non-trivial (i.e., it is a Cantor set in $\mathbb{C}$), and condition **(V3)** ensures that the limit set is not included in a single vertical fibre.

In order to prove our main results, we will need that our maps admit a vertical-like hyperbolic set. The following result ensures that this requirement is reasonably mild, and explains the assumption on p in our Theorems.

Proposition 3.6. *Let p be a polynomial with Julia set not totally disconnected, which is neither conjugated to $z \mapsto z^d$ nor to a Chebyshev polynomial. Then any polynomial skew product f of the form $f(z, w) = (p(z), q(z, w))$ admits a vertical-like IFS.*

Proof. By a result of Przytycki and Zdunik [PZ20] (see also [Prz85, Zdu90] for previous results in the connected case), since p is neither conjugated to $z \mapsto z^d$ nor to a Chebyshev polynomial, there exists a compact hyperbolic invariant set $\widetilde{H} \subset J_p$, with $\delta := \dim_H \widetilde{H} > 1$ and positive entropy. By the general theory of the thermodynamical formalism, there exists a unique ergodic invariant probability measure $\widetilde{\nu}$ supported on $\widetilde{H}$ that is absolutely continuous with respect to the δ-dimensional Hausdorff measure (see for instance [PU10, Prz18]). By Manning's formula, $L_{\widetilde{\nu}} = \frac{h_{\widetilde{\nu}}}{\delta}$, where $L_{\widetilde{\nu}}$ is the Lyapunov exponent of $\widetilde{\nu}$, and $h_{\widetilde{\nu}}$ its metric entropy. Since $\delta > 1$ and $h_{\widetilde{\nu}} < \log d$, we deduce that $L_{\widetilde{\nu}} < \log d$.

We now consider the measure $\nu := \int_{\widetilde{H}} \mu_z \, d\widetilde{\nu}(z)$, whose support is equal to $J_{\widetilde{H}}$. The existence of the vertical-like IFS as in the statement will follow from the following result. The proof uses tools from the thermodynamical formalism together with quantitative estimates. We give it in Appendix A.

Lemma 3.7. *For every $\varepsilon > 0$ there exists a fibred box A such that, for all n sufficiently large, the exists at least $3d^n$ f^n-inverse branches A_i of A compactly contained in A which are fibred boxes and with the property that, for all i, $f^n : A_i \to A$ is injective and*

$$\frac{1}{n}\log|(p^n)'(x)| < L_{\widetilde{\nu}} + \varepsilon \quad \text{and} \quad \frac{1}{n}\log|(Q_z^n)'(x,y)| > \log d - \varepsilon \quad \text{for all } (x,y) \in A_i. \tag{8}$$

Let A, A_i be given by Lemma 3.7 applied with $\varepsilon < \frac{\log d - L_{\widetilde{\nu}}}{2}$ and n sufficiently large. Since the entropy of $\widetilde{H}$ is smaller than $\log d$, up to removing a small number of A_i's (bounded by d^n) we can assume that for every j there exists $i \neq j$ such that A_i and A_j have the same projection on the first component, giving **(V2)**. The number of remaining A_i's is still bounded below by $2d^n$. Since at most $\sim d^n$ of them can share the same projection on the first coordinate, this also proves **(V3)**. The assertion follows since the inequalities in Lemma 3.7 imply that the limit set is a vertical-like hyperbolic set, giving **(V1)**. □

4. Higher bifurcations: an analytic criterion

In this section we establish the following technical result, which gives an analytic sufficient condition for a point to lie in the support of the higher bifurcation currents. Recall that, given a simple critical point $c(\lambda)$ for q_{λ,z_0} and a repelling point $r(\lambda)$ for f_λ, we denote by $M_{(z_0,c),r,n_0}$ the analytic subset of M given by the equation $f_\lambda^{n_0}(z_0, c(\lambda)) = r(\lambda)$.

Proposition 4.1. *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of polynomial skew products over a given base p. Let $\lambda_0 \in M$ and $z_1, \dots, z_k \in J_p$ satisfy the following properties:*

(1) there exist simple critical points c_i for q_{λ_0,z_i} such that $r_i := f_{\lambda_0}^{m_i}(z_i, c_i)$ is a repelling periodic point for f_{λ_0};

(2) $\operatorname{codim} \cap_{i=1}^k M_{(z_i,c_i),r_i,m_i} = k$.

Then $\lambda_0 \in \operatorname{Supp} T^k_{\text{bif}}(M)$.

In the case of families of rational maps, this result is due to Buff-Epstein [BE09], using transversality arguments. In [Gau12], Gauthier uses different arguments that only require that the intersections are proper, as is the case in Proposition 4.1. A more general condition (called the *generalized large scale condition*) was introduced in [AGMV19] as a sufficient condition for a point to lie in the support of T^k_{bif} (for a family of rational maps). We give an adapted version of this notion in our non-autonomous setting, and deduce that a parameter λ_0 as in the statement satisfies such condition. This will prove Proposition 4.1.

In the following we assume that $z_1, \dots, z_k \in J_p$ and that $c_j(\lambda)$ are holomorphic maps such that $c_j(\lambda)$ is a critical point for q_{λ,z_j} for all $\lambda \in M$. We denote by $\underline{c} : M \to \mathbb{C}^k$ the map $c(\lambda) = (c_1(\lambda), \dots, c_k(\lambda))$. For a $k-$uple $\underline{n} := (n_1, \dots, n_k)$, we define

$$\xi^j_{n_j}(\lambda) := Q^{n_j}_{\lambda,z_j}(c_j(\lambda)) \quad \text{and} \quad \Xi^{\underline{c}}_{\underline{n}}(\lambda) := (\xi^1_{n_1}(\lambda), \dots \xi^k_{n_k}(\lambda)). \tag{9}$$

Notice that $\Xi^{\underline{c}}_{\underline{n}} : M \to \mathbb{C}^k$. We denote by C_j the graph of c_j in $M \times \mathbb{C}$ and by $V_{\underline{n}}$ the graph of $\Xi^{\underline{c}}_{\underline{n}}$ in $M \times \mathbb{C}^k$. We also write $|\underline{n}| := n_1 + \cdots + n_k$ for a k-uple $\underline{n}$ as above.

Definition 4.2 (Fibred large scale condition)**.** *We say that $\lambda_0 \in M$ satisfies the* fibred large scale condition *for the critical points $(z_1, c_1), \dots, (z_k, c_k)$ if there exist $z'_1, \dots z'_k \in J_p$, disks $D_1, \dots, D_k \subset \mathbb{C}$ with $D_i \cap J_{z'_i} \neq \emptyset$, a sequence $\underline{n}_l = (n_{l,1}, \dots, n_{l,k})$ of $k-$uples with $n_{l,i} \to \infty$ and a nested sequence of open subsets Ω_l such that*

- *$\cap_l \overline{\Omega}_l = \{\lambda_0\}$, and*
- *$\Xi^{\underline{c}}_{\underline{n}_l} : \Omega_l \to D_1 \times \cdots \times D_k$ is a proper surjective map.*

Proposition 4.3. *Let $\lambda_0 \in M$ satisfy the fibred large scale condition for some points $(z_1, c_1), \dots, (z_k, c_k)$ with $q'_{z_j}(c_j) = 0$ for every j and such that the z_j are preperiodic for p. Then $\lambda_0 \in \operatorname{Supp} T_{\text{bif}\, z_1} \wedge \cdots \wedge T_{\text{bif}\, z_k}$.*

Proof. The proof follows the same line as that of [AGMV19, Theorem 3.2]. We give here the main steps.

First of all, it is enough to prove the statement in the assumption that the dimension of M is equal to k, see [Gau12, Lemma 6.3]. For every $\underline{n} = (n_1, \dots, n_k)$ with $n_j \geq 0$, we define the map

$$\begin{array}{rcl} F_{\underline{n}} \colon M \times \mathbb{C}^k & \to & M \times \mathbb{C}^k \\ (\lambda, w_1, \dots, w_k) & \mapsto & (\lambda, Q^{n_1}_{\lambda,z_1}(w_1), \dots, Q^{n_k}_{\lambda,z_k}(w_k)). \end{array}$$

and we denote by $\widetilde{\pi}_j \colon M \times \mathbb{C}^k \to M \times \mathbb{C}$ the projection $(\lambda, w_1, \dots, w_k) \mapsto (\lambda, w_j)$. One can prove that, for every $\underline{n}$ as above and Borel set $\Omega \subseteq M$,

$$T_{\mathrm{bif}\, z_1} \wedge \cdots \wedge T_{\mathrm{bif}\, z_k}(\Omega) = d^{-|n|} \int_{\Omega \times \mathbb{C}^k} F_{\underline{n}}^* \Big(\bigwedge_{j=1}^{k} \widetilde{\pi}_j^* (dd^c_{\lambda,w} G_\lambda(z'_j, \cdot)) \Big) \wedge \Big[\bigcap_{j=1}^{k} C_j \Big]$$

$$= d^{-|n|} \int_{\Omega \times \mathbb{C}^k} \Big(\bigwedge_{j=1}^{k} \widetilde{\pi}_j^* (dd^c_{\lambda,w} G_\lambda(z'_j, \cdot)) \Big) \wedge [V_{\underline{n}}],$$

see [AGMV19, Lemma 3.3]. Moreover, with Ω_l and $\underline{n}_l$ as in the statement, we also have (see [AGMV19, Lemma 3.4]) that

$$\liminf_{l \to \infty} \int_{\Omega_l \times \mathbb{C}^k} \Big(\bigwedge_{j=1}^{k} \widetilde{\pi}_j^* (dd^c_{\lambda,w_j} G_\lambda(z'_j, \cdot)) \Big) \wedge [V_{\underline{n}_l}] \geq \prod_{j=1}^{k} (dd^c_w G_{\lambda_0}(z'_j, \cdot))(D_j).$$

We use in this step the second assumption in Definition 4.2. The right hand side of the last expression is strictly positive by the assumption that $D_j \cap J_{z'_j} \neq \emptyset$. This implies that $T_{\mathrm{bif}\, z_1} \wedge \cdots \wedge T_{\mathrm{bif}\, z_k}(\Omega_l) > 0$, for a sequence of integers l going to infinity. Hence $\lambda_0 \in \operatorname{Supp} T_{\mathrm{bif}\, z_1} \wedge \cdots \wedge T_{\mathrm{bif}\, z_k}$, as desired. □

We can now prove Proposition 4.1.

Proof of Proposition 4.1. By Proposition 2.6 it is enough to prove that $\lambda_0 \in \operatorname{Supp} T_{\mathrm{bif}\, z_1} \wedge \cdots \wedge T_{\mathrm{bif}\, z_k}$. By Proposition 4.3 it is thus enough to prove that any λ_0 as in the statement satisfies the fibred large scale condition above. We can also assume that the dimension of M is k.

Denote by s_i the period of the repelling point r_i and set $\underline{s} = (s_1, \dots s_k)$. Set $r_i =: (z'_i, r'_i)$ and similarly let $r_i(\lambda) = (z'_i, r'_i(\lambda))$ be the motion of r_i in a neighbourhood of λ_0 as a periodic point. Fix $\eta > 0$ and an open neighbourhood Ω of λ_0 such that the following properties hold:

1. for all r_i as in the statement, $r'_i(\lambda) \in \mathbb{D}(r'_i, \eta/10)$ for all $\lambda \in \Omega$;
2. for every i and every $\lambda \in \Omega$, the map $Q^{s_i}_{\lambda, z'_i}$ is uniformly expanding on $\mathbb{D}(r'_i(\lambda), \eta)$ (with expansivity factor uniform in λ).

Observe that, for all $\lambda \in \Omega$, we have $\mathbb{D}(r'_i, \eta/2) \subset \mathbb{D}(r'_i(\lambda), \eta)$. We set

$$A_0 := \{(\lambda, w_1, \dots, w_k) \in \Omega \times \mathbb{C}^k \colon w_i \in \mathbb{D}(r'_i(\lambda), \eta)\}.$$

We denote by $g_{\lambda,i} \colon \mathbb{D}(r'_i(\lambda), \eta) \to \mathbb{C}$ the inverse branch of $Q^{s_i}_{\lambda, z'_i}$ such that $g_{\lambda,i}(r'_i(\lambda)) = r'_i(\lambda)$ and by $\underline{G} : A_0 \to \Omega \times \mathbb{C}^k$ the inverse branch of $F_{\underline{s}}$ which agrees on A_0 with the $g_{\lambda,i}$ as above. For $l \in \mathbb{N}$, we set $A_l := \underline{G}^l(A_0)$. Observe that A_l shrinks (exponentially) with $l \to \infty$ to the graph of the product map $\lambda \mapsto (r'_1(\lambda), \dots, r'_k(\lambda))$.

Consider the map $\Phi' \colon \mathbb{C}^k \to \mathbb{C}^k$ defined by

$$\Phi'(w_1, \dots, w_k) = (w_1 - r_1(\lambda), \dots, w_k - r_k(\lambda))$$

and set $\Phi(\lambda, w_1, \dots, w_k) := (\lambda, \Phi'(w_1, \dots, w_k))$. Observe that $\Phi(\lambda, r_1(\lambda), \dots, r_k(\lambda)) = (\lambda, 0, \dots, 0)$. We denote $B_0 := \Phi(A_0) = \Omega \times \mathbb{D}(0, \eta)^k$ and similarly set $B_l := \Phi(A_l)$. We will also need the projections $\pi_M, \underline{\pi}$ of $M \times \mathbb{C}^k$ on M and on $\mathbb{C}^k$, respectively.

For every $\underline{n} = (n_1, \dots, n_k)$ consider the map $H_{\underline{n}} \colon \Omega \to \mathbb{C}^k$ given by $H_{\underline{n}} := \Phi' \circ \Xi^{\underline{c}}_{\underline{n}}$, where $\Xi^{\underline{c}}_{\underline{n}}$ is defined in (9). We claim that the map $H_{\underline{m}}$ is open in a neighbourhood of λ_0, where $\underline{m} = (m_1, \ldots, m_k)$. By [GR12, §3.1.2 and §5.4.3], it is enough to check that the point λ_0 is isolated in $(H_{\underline{m}})^{-1} H_{\underline{m}}(\lambda_0)$. This is precisely given by the second assumption in the statement. The same assumption and the fact that the $q_{\lambda,z}$'s are open imply that, for any $l \in \mathbb{N}$, we also have $\operatorname{codim} \cap_{i=1}^k M_{(z_i,c_i),r_i,m_i+ls_i} = k$. The argument above implies that also the maps $H_{\underline{n}_l}$ are open, where $\underline{n}_l := (m_1 + ls_1, \ldots, m_k + ls_k)$.

By restricting if necessary the Ω as above, we see that the graph Γ_0 of the map $H_{\underline{m}}$ is of dimension k in B_0. We set $\Omega_l := \pi_M(\Gamma_0 \cap B_l)$. The Ω_l's are then open. Since B_l shrinks with l to the constant graph $\{(\lambda, 0, \ldots, 0)\}$, we also have that Ω_l shrinks to $\{\lambda_0\}$ as $l \to \infty$.

Set $D_i := \mathbb{D}(r'_i, \eta/4)$, let Γ_l be the graph of $H_{\underline{n}_l}$ on Ω_l (which, by the above, is also k-dimensional) and recall that $V_{\underline{n}_l}$ denotes the graph of $\Xi^{\underline{c}}_{\underline{n}_l}$. To conclude it is enough to prove that, for all $l \in \mathbb{N}$, $\underline{\pi}(\pi_M^{-1}(\Omega_l) \cap V_{\underline{n}_l}) \supset \prod_{i=1}^k D_i$. We will use the following fact.

Fact. *Let $\Omega' \Subset \Omega$ be an open subset. Let W_v, W_h be two non-empty k-dimensional closed analytic subsets of B_0 with $\pi_M(W_v) \subset \Omega'$ and $\underline{\pi}(W_h) \subset \mathbb{D}(0, \eta/2)^k$. Then $W_h \cap W_v \neq \emptyset$.*

Recall that $r'_i(\lambda) \in \mathbb{D}(r'_i, \eta/10)$ for all i and $\lambda \in \Omega$. Hence the Fact, applied with $\Omega' = \Omega_l$, $W_v = \Gamma_l$ and $W_h = \{(\lambda, y_1 - r'_1(\lambda), \ldots, y_k - r'_k(\lambda))\}$, implies that, for any $\underline{y} = (y_1, \ldots, y_k) \in \prod_{i=1}^k \mathbb{D}(r'_i, \eta/4)$, there exists a $\lambda \in \Omega_l$ such that

$$Q_{z_i}^{m_i+ls_i}(c_i(\lambda)) - r'_i(\lambda) = y_i - r'_i(\lambda) \text{ for all } 1 \leq i \leq k.$$

This implies that $\underline{\pi}(V_{\underline{n}_l}) \supset \prod_{i=1}^k D_i$, as desired. The proof is complete. □

Remark 4.4. *As is the case in* [AGMV19]*, it is enough to make a weaker assumption in Proposition 4.1, namely that the critical orbits fall in the motion of some hyperbolic set. The proof is slightly more involved in that situation (as is the case in* [AGMV19]*). We prefer to state only the simple criterion based on repelling periodic orbits since this simpler version will be enough to deduce our main result.*

5. Creating multiple bifurcations: a geometric method

In this section we develop our method to construct multiple bifurcations (in the form of Misiurewicz parameters) starting from a simple one. In the next section we will ensure the applicability of this method. First, let us introduce the following definition.

Definition 5.1. *Let $(f_\lambda)_{\lambda \in M}$ be a holomorphic family of polynomial skew products over a fixed base polynomial p. We say that M is a* good Misiurewicz family*, or that M has a* persistently good Misiurewicz relation $f_\lambda^{n_0}(z_0, c(\lambda)) = (z_1, w_1(\lambda))$ *if the Misiurewicz relation $f_\lambda^{n_0}(z_0, c(\lambda)) = (z_1, w_1(\lambda))$ (where $(z_1, w_1(\lambda))$ is a repelling periodic point of period m for f_λ) is persistent in M, and if moreover*

- **(G1)** *the vertical eigenvalue $B(\lambda) := (Q^m_{\lambda,z_1})'(w_1(\lambda))$ is non-constant on M;*
- **(G2)** *for all $\lambda \in M$, $(z_1, w_1(\lambda))$ is vertical-like;*
- **(G3)** *$(p^{n_0})'(z_0) \neq 0$ and $z_0 \notin \{p^i(z_1), 1 \leq i \leq m\}$;*
- **(G4)** *for all $\lambda \in M$, $c(\lambda)$ is a simple root of q'_{λ,z_0};*
- **(G5)** *for all λ, if L_λ denotes the unique component of* $\operatorname{Crit}(f_\lambda)$ *passing through $(z_0, c(\lambda))$, then $f_\lambda^{n_0}(L_\lambda)$ is regular at $(z_1, w_1(\lambda))$ and is not tangent to an eigenspace of $df_\lambda^m(z_1, w_1(\lambda))$.*

A parameter $\lambda_0 \in M$ satisfying all the conditions above will be called a good Misiurewicz parameter.

Observe that a good Misiurewicz family in $\mathbf{Sk}(p,d)$ is, in general, an open subset of an algebraic hypersurface of $\mathbf{Sk}(p,d)$. The next Proposition is the key technical result of our argument.

Proposition 5.2. *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of polynomial skew products over a fixed base polynomial p and with a persistently good Misiurewicz relation $(z_1, w_1(\lambda)) := f_\lambda^{n_0}(z_0, c_0(\lambda))$. There exists a dense subset $S \subset M$ such that for all $\lambda_\infty \in S$ and for every (z'', w'') repelling periodic point in the limit set of a vertical-like IFS for f_{λ_∞}, there exists a sequence $\lambda_n \to \lambda_\infty$ such that f_{λ_n} has a Misiurewicz relation of the form $f_{\lambda_n}^{N_n}(y_n, c_n(\lambda_n)) = (z'', w''(\lambda_n))$ (where $(z'', w''(\lambda))$ is the holomorphic motion as repelling periodic point of (z'', w'') in a neighbourhood of λ_∞) which is non-persistent on M and satisfies* **(G2)**, **(G3)**, **(G4)**, *and* **(G5)** *on a neighbourhood of λ_n in $M_{(y_n,c_n(\lambda_n)),(z'',w''(\lambda_n),N_n)} \subset M$.*

Corollary 5.3. *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of polynomial skew products with a persistently good Misiurewicz relation. Then,* $\mathrm{Bif}(M) = M$.

Proof. The assertion follows from the fact that Misiurewicz parameters belong to the bifurcation locus, see [BBD18, Bia19]. □

The remaining part of this section is devoted to proving Proposition 5.2. We start by defining the set S. Recall that $A := (p^m)'(z_1)$ is the non-vertical eigenvalue of the repelling cycle associated to the Misiurewicz relation.

Definition 5.4. *Let M be a good Misiurewicz family, and let $f_\lambda^{n_0}(z_0, c(\lambda)) = (z_1, w_1(\lambda))$ be a persistent Misiurewicz relation satisfying the requirements of Definition 5.1.*

We define the set $S \subset M$ to be the set of $\lambda_\infty \in M$ for which each of the following properties holds:

- **(S1)** $d_\lambda B(\lambda_\infty) \neq 0$;
- **(S2)** $\log B(\lambda_\infty) \notin \mathbb{R} \log A$.

Note that S is open and dense in M. From now on, we fix an arbitrary $\lambda_\infty \in S$, and we choose a one-dimensional disk in local coordinates in M transverse to a level set of B in which $\lambda_\infty = 0$ (hence $\frac{dB}{d\lambda}(0) \neq 0$). The proof of Proposition 5.2 will mostly use local arguments in phase space. Therefore, we will work in local linearizing coordinates near (z_1, w_1); in particular, in the rest of this section we will take $(z_1, w_1) = (0,0)$, and we will assume that $m = 1$ (which we can do up to passing to an iterate).

By item **(S2)** of the definition of S, there are no resonances between the eigenvalues of this fixed point $(0,0)$ for λ close to $\lambda_\infty = 0$. We may therefore assume that the fixed point $(0,0)$ is linearizable for f_λ; moreover the linearizing map can be chosen to depend holomorphically on the parameter. More precisely, we can fix a neighbourhood U of $(0,0)$ such that these linearizing coordinates are defined for $(z,w) \in U$ for all f_λ with $|\lambda|$ small enough. So f_λ acts in those coordinates as the linear map $(z,w) \mapsto (Az, B(\lambda)w)$.

It follows from the Implicit Function Theorem and **(G4)** that there is a unique component of $\mathrm{Crit}(f_\lambda)$ passing through $(z_0, c(\lambda))$, and that this component is smooth and can be locally described as a graph of the form $w = \widetilde{\beta}(z,\lambda)$, for some holomorphic germ $\widetilde{\beta}$. Setting $\beta(z,\lambda) := Q_{\lambda,z}^{N_0}(\widetilde{\beta}(z,\lambda))$, this implies that the graph $w = \beta(z,\lambda)$ is a local parametrization of a component L_λ of $f_\lambda^{N_0}(\mathrm{Crit}(f_\lambda))$. The assumption **(G5)** and our choice of local coordinates imply that the holomorphic map $z \mapsto \beta(z, \lambda)$ is not constantly equal to 0, and moreover that $\beta_1 := \frac{\partial \beta}{\partial z}(0,0) \neq 0$.

Lemma 5.5. *Let $K \subset \mathbb{C}^*$ be a compact set. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in J_p such that $z_k \to 0$, and $(m_k)_{k\in\mathbb{N}}$ be a sequence of integers such that $z_k A^{-m_k} B^{m_k} \in K$ for all k. Set $\varphi_k(\lambda) := \beta(z_k A^{-m_k}, \lambda) B(\lambda)^{m_k}$. Then the sequence $(\varphi_k)_{k\in\mathbb{N}}$ is not normal at $\lambda = 0$.*

Proof. Let us compute the derivative of φ_k at 0:

$$\begin{aligned}\frac{d\varphi_k}{d\lambda}(0) &= \frac{\partial \beta}{\partial \lambda}(z_k A^{-m_k}, 0) B^{m_k} + \beta(z_k A^{-m_k}, 0) B'(0) m_k B^{m_k - 1} \\ &= \mathcal{O}\left(z_k A^{-m_k} B^{m_k}\right) + \beta_1 z_k A^{-m_k} B'(0) m_k B^{m_k-1} + \mathcal{O}\left(z_k^2 A^{-2m_k} B^{m_k} m_k\right).\end{aligned}$$

Since $\beta_1 \neq 0$, by the choice of m_k, we have

$$\frac{d\varphi_k}{d\lambda}(0) = \beta_1 z_k A^{-m_k} B'(0) m_k B^{m_k-1} + \mathcal{O}(1) \asymp m_k,$$

hence $\lim_{k\to+\infty} |\frac{d\varphi_k}{d\lambda}(0)| = +\infty$. This proves the non-normality of (φ_k) at 0. □

The next lemma is the motivation behind the combinatorial requirements in the definition of a vertical-like IFS. It produces two sequences of iterated preimages of a repelling cycle accumulating at two distinct points in the same fibre. We will need these two sequences instead of one to later apply Montel's theorem.

Lemma 5.6. *Let H_0 be the limit set of a vertical-like IFS for f_0. Let $(z', w'), (z'', w'') \in H_0$ be periodic points and let U be an open neighbourhood of (z', w'). There exist $w_1 \neq w_2 \in \mathbb{C}$, both distinct from w' and with $(z', w_1), (z', w_2) \in H_0 \cap U$, and sequences $(z_k, w_{k,i})$ (with $1 \leq i \leq 2$) with $\lim_{k\to+\infty}(z_k, w_{k,i}) = (z', w_i)$ and such that, for all $k \in \mathbb{N}$ and $i \in \{1, 2\}$,*

1. *there exists $n_k \in \mathbb{N}$ such that $f^{n_k}(z_k, w_{k,i}) = (z'', w'')$;*
2. *$(z_k, w_{k,i}) \in H_0 \cap U$;*
3. *$z_k \neq z'$.*

Proof. First, let (z', w_1) and (z', w_2) be two periodic points in $H_0 \cap U$ such that w', w_1, and w_2 are pairwise distinct. Such points exist since $H_0 \cap (\{z'\} \times \mathbb{C})$ contains (the image of) the limit set of a non-trivial IFS in $\mathbb{C}$ by the condition **(V2)** in Definition 3.5. We can also assume that (z', w_1) and (z', w_2) have the same period.

There exists a finite sequence $g_{i_1}, \ldots, g_{i_{\ell_1}}$ with the property that (z', w_1) is the unique fixed point of the finite composition $G_1 := g_{i_1} \circ \cdots \circ g_{i_{\ell_1}}$. Similarly, (z', w_2) is the unique fixed point of a finite composition $G_2 := g_{j_1} \circ \cdots \circ g_{j_{\ell_2}}$. Since (z', w_1) and (z', w_2) have the same period, we can assume that $\ell_1 = \ell_2 = \ell$. Moreover, since (z', w_1) and (z', w_2) belong to the same vertical fibre, the maps G_1 and G_2 agree on the first coordinate.

We now construct the sequences $(z_k, w_{k,i})$. We first assume that z'' does not belong to the orbit of z' under the base polynomial p. In this case, given $k_0 \in \mathbb{N}$ it is enough to set

$$(z_{k,i}, w_{k,i}) := G_i^{k+k_0}(z'', w'') \text{ for all } k \geq 1.$$

Since G_1 and G_2 agree on the first coordinate, we have $z_{k,1} = z_{k,2}$ for all k. We set $z_k := z_{k,1} = z_{k,2}$ for all k. For $i \in \{1, 2\}$, the sequence $(z_k, w_{k,i})$ converges to (z', w_i) as $k \to \infty$ by the definition of G_i. When k_0 is taken sufficiently large, all the points in such sequences belong to $U \cap H_0$. Finally, we have $z_{k,i} \neq z'$ for all k since by assumption z'' is not in the orbit of z'.

Suppose now that z'' belongs to the orbit of z'. Since backwards preimages of (z'', w'') are dense in H_0, there exists at least one such preimage z''' not in the orbit of z'. We choose w''' so that $(z''', w''') \in H_0$ and (z'', w'') is in the orbit of (z''', w'''). It is enough to apply the above argument to (z''', w''') instead of (z'', w''). The proof is complete. □

We are now ready to prove Proposition 5.2.

Proof of Proposition 5.2. We are working in the setting described after Definition 5.4. We fix a periodic point (z'', w'') in the limit set H_0 of a vertical-like IFS as in the statement. Observe that (z'', w'') is vertical-like. We can choose (z'', w'') with z'' not in the post-critical set of p. We also denote by H_λ the holomorphic motion of H_0 as hyperbolic set in a neighbourhood of $\lambda = 0$.

We let $(z_k, w_{k,i})$ be the sequences of preimages of (z'', w'') given by Lemma 5.6 applied to $(z', w') = (0, 0)$ and (z'', w''). Since all of these points belong to H_0, they all move holomorphically as $(z_k, w_{k,i}(\lambda))$ over a common domain in parameter space. Moreover, by the Definition 3.1 and continuity, we may fix a vertical cone $C_{\alpha_0} := \{u \in \mathbb{C}^2 : |\langle u, (0, 1)\rangle| > \alpha_0 \|u\|\}$ such that, for all λ in a neighbourhood of 0 and $(x, y) \in H_\lambda$, we have $(df_\lambda)_{(x,y)}(C_{\alpha_0}) \Subset C_{\alpha_0}$. This implies that the non-vertical Oseledets direction are uniformly bounded away from the vertical direction.

Fix $\varepsilon > 0$. We want to prove that there exists $\lambda \in \mathbb{D}(0, \varepsilon)$ and $i_0 \in \{1, 2\}$ such that $(z_k, w_{k,i_0}(\lambda))$ (and hence $(z'', w''(\lambda))$) is (non-persistently) in the post-critical set of f_λ. To that end, observe that

$$f^{m_k}(z_k A^{-m_k}, \beta(z_k A^{-m_k}, \lambda)) = (z_k, \beta(z_k A^{-m_k}, \lambda)B(\lambda)^{m_k}) = (z_k, \varphi_k(\lambda)) \tag{10}$$

is a post-critical point; it is therefore enough to prove that there exist sequences $\lambda_k \to 0$ and $(i_k) \in \{1, 2\}^{\mathbb{N}}$ such that $w_{k,i_k}(\lambda_k) = \varphi_k(\lambda_k)$.

By Lemma 5.5, the sequence $(\varphi_k)_k$ is not normal at $\lambda = 0$. Therefore, by Montel theorem, the sequence of the graphs of the φ_k's cannot avoid both those of $w_{k,1}$ and $w_{k,2}$. Hence there exist $\lambda_k \to 0$ and $(i_k) \in \{1, 2\}^{\mathbb{N}}$ such that $\varphi_k(\lambda_k) = w_{k,i_k}(\lambda_k)$. Up to a subsequence, we can assume that i_k is constant. By Lemma 5.5 and the Definition 3.5 of a vertical-like IFS, the Misiurewicz relation constructed as above satisfies **(G2)**. By the choice of (z'', w'') at the beginning of the proof, this relation satisfies the first condition in **(G3)**. By taking only k large enough, the second part of the condition is satisfied, too. Condition **(G4)** holds since, by assumption, $c(0)$ is a simple critical point for q'_{0,z_0}, hence the same is true for small λ and z close to z_0. It remains to prove that the relation satisfies **(G5)**.

By the definition (7) of C_α and the choice of α_0 at the beginning of the proof, it is enough to prove that for all k large enough, the tangent space to the component of the postcritical set passing through $(z_k, w_{k,i_0}(\lambda_k))$ and giving the Miriurewicz relation above lies in C_{α_0}. The branch of the postcritical set is locally given by the equation

$$w = \beta(zA^{-m_k}, \lambda_k)B(\lambda_k)^{m_k},$$

and therefore its tangent space is generated by the vector

$$u_k := \left(1, \frac{\partial}{\partial z}_{|z=z_k} \beta(zA^{-m_k}, \lambda_k))B(\lambda_k)^{m_k}\right).$$

Since

$$\frac{\partial}{\partial z}_{|z=z_k}\beta(zA^{-m_k},\lambda_k))B(\lambda_k)^{m_k} = \frac{\partial\beta}{\partial z}(z_kA^{-m_k},\lambda_k)\left(\frac{B(\lambda_k)}{A}\right)^{m_k}$$
$$\sim_{k\to+\infty}\beta_1\cdot\left(\frac{B(\lambda_k)}{A}\right)^{m_k},$$

and $\beta_1 \neq 0$, it follows that, for all k large enough, u_k belongs to C_{α_0}. The proof is complete. □

6. Proof of the main results

In this section, we will first apply inductively Proposition 5.2 in order to prove our main Theorem 1.2, and then deduce from this result and its proof the other results in the Introduction. We start with a few required lemmas.

Lemma 6.1. *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of polynomial skew products over a fixed base p and of a given degree $d \geq 2$, and take $\lambda_0 \in \mathrm{Bif}(M)$. There exists a finite set $E \subset J_p$ such that for all $z_1 \in J_p\backslash E$, if (z_1, w_1) is any repelling periodic point of f_{λ_0} (which we locally follow as $(z_1, w_1(\lambda))$) then, for every $n_0 \in \mathbb{N}$, arbitrarily close to λ_0 there exists $\lambda_1 \in M$ such that f_{λ_1} has a Misiurewicz relation of the form $f^n_{\lambda_1}(z, c) = (z_1, w_1(\lambda_1))$ with $(p^n)'(z) \neq 0$, and $n \geq n_0$, and such that z does not belong to the cycle of z_1.*

Proof. First of all, let us define E as the union of all repelling periodic points in the postcritical set. This set is finite. We fix any repelling periodic point $z_1 \notin E$. By [AB18, Proposition 3.5], we may find λ_0' arbitrarily close to λ_0 for which a critical point of the form (y, c) is active, where y is in the strict backward orbit of z_1 by p and not in its cycle. By Montel's theorem, we can further slightly perturb λ_0' to a λ_1 with the property that some iterate of (y, c) by f_{λ_1} coincides with $(z, w_1(\lambda_1))$. This completes the proof. □

Lemma 6.2. *Take $\lambda_0 \in \mathbf{Sk}(p, d)$, let $z \in J_p$ be a periodic point of period $m > d$, and (z, w_i) ($1 \leq i \leq D_d = \dim \mathbf{Sk}(p, d)$) denote a collection of repelling periodic points of $Q^m_{\lambda_0,z}$ of different periods, which we follow locally as $(z, w_i(\lambda))$ over a domain $U \subset \mathbf{Sk}(p, d)$ containing λ_0. Let $\rho_i(\lambda)$ denote their vertical multipliers, and let $\rho : U \to \mathbb{C}^{D_d}$ denote the map $\rho : \lambda \mapsto (\rho_i(\lambda))_{1\leq i\leq D_d}$. There exists an analytic hypersurface $R \subset U$ such that for all $\lambda \in U\backslash R$, the differential $d\rho_\lambda$ is invertible.*

Proof. First we claim that, since the period m of z satisfies $m > d$, the family of the first returns $(Q^m_{\lambda,z})_{\lambda\in\mathbf{Sk}(p,d)}$ can be mapped to an algebraic subfamily of pure dimension $D_d = \dim \mathbf{Sk}(p, d)$ in the space $\mathrm{Poly}(d^m)$ of monic centred degree polynomials of degree d^m (the fact that the image is given by monic centred polynomials follows from the parametrization of $\mathbf{Sk}(p, d)$ given in Lemma 2.1). Indeed, consider first the map $\varphi_z : \mathbf{Sk}(p, d) \to \mathrm{Poly}(d)^m$ defined by $\varphi_z(\lambda) = (q_{\lambda,z_i})_{1\leq i\leq m}$, where $z_i := p^i(z)$. Since $m > d$ and the coefficients of q_{λ,z_i} are given by polynomials in z_i of degree at most d, the map φ_z is injective. Then, consider the map $C : \mathrm{Poly}(d)^m \to \mathrm{Poly}(d^m)$ defined by $C(q_m, \ldots, q_1) = q_m \circ \ldots \circ q_1$.

Claim 6.3. *The differential of C at $(w^d, \ldots, w^d)$ is injective.*

Proof. For $\varepsilon > 0$, consider the polynomials $q_i = w^d + \varepsilon r_i$, with r_i polynomials in w of degree $\leq d-2$. For every $j \leq m$, we also set $Q_j(\varepsilon, w) := q_j \circ \cdots \circ q_1$. It is enough to check that, for every choice of $r_1, \ldots, r_m$ (not all zero) we have $\frac{\partial Q_m(\varepsilon,w)}{\partial \varepsilon} \neq 0$ (as a polynomial in w) at $\varepsilon = 0$. Since for all $1 \leq j \leq m$ we have $Q_j(0,w) = w^{d^j}$, we can check by induction that

$$\frac{\partial Q_1(\varepsilon,w)}{\partial \varepsilon}\Big|_{\varepsilon=0} = r_1(w);$$

$$\vdots$$

$$\frac{\partial Q_j(\varepsilon,w)}{\partial \varepsilon}\Big|_{\varepsilon=0} = r_j(w^{d^{j-1}}) + d(w^{d^{j-1}\cdot(d-1)}) \cdot \frac{\partial Q_{j-1}(\varepsilon,w)}{\partial \varepsilon}\Big|_{\varepsilon=0};$$

$$\vdots$$

$$\frac{\partial Q_m(\varepsilon,w)}{\partial \varepsilon}\Big|_{\varepsilon=0} = r_m(w^{d^{m-1}}) + d(w^{d^{m-1}\cdot(d-1)}) \cdot \frac{\partial Q_{m-1}(\varepsilon,w)}{\partial \varepsilon}\Big|_{\varepsilon=0}.$$

Since $\deg r_j \leq d-2$ for all j, it follows that, for all $0 \leq j \leq m-1$, $\frac{\partial Q_{j+1}(\varepsilon,w)}{\partial \varepsilon}\Big|_{\varepsilon=0} \neq 0$ as soon as $\frac{\partial Q_j(\varepsilon,w)}{\partial \varepsilon}\Big|_{\varepsilon=0} \neq 0$. Hence, in order to have $\frac{\partial Q_m(\varepsilon,w)}{\partial \varepsilon}\Big|_{\varepsilon=0} = 0$, we must have $\frac{\partial Q_j(\varepsilon,w)}{\partial \varepsilon}\Big|_{\varepsilon=0} = 0$ for all $0 \leq j \leq m$. Since this implies that all the r_j's must be equal to 0, the proof is complete. □

Therefore, since $Q^m_{\lambda,z} = C \circ \varphi_z(\lambda)$, the map $\lambda \mapsto Q^m_{\lambda,z}$ is locally injective near $\lambda := 0$, and so the family $(Q^m_{\lambda,z})_{\lambda \in \mathbf{Sk}(p,d)}$ has indeed dimension D_d.

Once this property is established, the statement follows from a slight adaptation of the main result in [Gor16], which is as follows. For any $D \geq 2$, for any $Q_{\lambda_0} \in \mathrm{Poly}(D)$, let w_i $(1 \leq i \leq D-1)$ be a collection of repelling periodic points for Q_{λ_0}, of distinct periods m_i. Up to passing to a finite branched cover of $\mathrm{Poly}(D)$, we may follow globally those periodic points as functions of the parameter $\lambda \mapsto w_i(\lambda)$. If we denote by $\rho_i(\lambda)$ their respective multipliers $\rho_i(\lambda) := (Q^m_\lambda)'(w_i(\lambda))$ and set $\rho(\lambda) := (\rho_i(\lambda))_{1 \leq i \leq D-1}$, then Gorbovickis proves that there exists a global hypersurface $\mathcal{R}$ such that for all $\lambda \notin \mathcal{R}$, the differential $d\rho_\lambda$ is invertible.

Therefore, it is enough for us to arbitrarily complete our collection of repelling periodic points with some w_i (with $D_d < i < d^m$) and prove that the subfamily $(Q^m_{\lambda,z})_{\lambda \in \mathbf{Sk}(p,d)} \subset \mathrm{Poly}(d^m)$ is not entirely contained in the corresponding algebraic hypersurface $\mathcal{H} \subset \mathrm{Poly}(d^m)$. But this in turn follows from the facts that $w \mapsto w^D$ never belongs to $\mathcal{H}$ ([Gor16, Lemma 2.1]), and that $w \mapsto w^{d^m}$ always belongs to $(Q^m_{\lambda,z})_{\lambda \in \mathbf{Sk}(p,d)} \subset \mathrm{Poly}(d^m)$. The proof is complete. □

Lemma 6.4. *Let $\lambda_0 \in \mathbf{Sk}(p,d)$ and assume that f_{λ_0} has a Misiurewicz relation $f^n_{\lambda_0}(z_0,c_0) = (z_1,w_1)$ satisfying* **(G3)**, *and let m be the period of (z_1,w_1). Let $M_{(z_0,c_0),n} \subset \mathbf{Sk}(p,d)$ denote the local hypersurface of $\mathbf{Sk}(p,d)$ where this Misiurewicz relation is preserved. Then, the set of parameters $\lambda \in M_{(z_0,c_0),n}$ which satisfy* **(G4)** *and* **(G5)** *is open and dense in $M_{(z_0,c_0),n}$.*

Proof. It will be useful to consider the *algebraic* hypersurface $\mathcal{M} \subset \mathbf{Sk}(p,d)$ defined by:

$$\mathcal{M} := \{\lambda \in \mathbf{Sk}(p,d) : \mathrm{Res}(q'_{\lambda,z_0}, Q^{n+m}_{\lambda,z_0} - Q^m_{\lambda,z_0})\}$$

where Res is the resultant. In other words, $\mathcal{M}$ is the set of $\lambda \in \mathbf{Sk}(p,d)$ such that a critical point of the form (z_0, c) lands after n iterations on a periodic point of period dividing m (and that periodic point may or may not be repelling). By definition, $M_{(z_0,c_0),n}$ is a neighbourhood of λ_0 in $\mathcal{M}$.

Let us first prove that the subset of $\mathcal{M}$ where **(G4)** does not hold has codimension at least 1 in $\mathcal{M}$. Let $\Pi := \{(p,q) : q \in \mathrm{Poly}(d)\} \subset \mathbf{Sk}(p,d)$ denote the subfamily of trivial products. Then **(G4)** holds on a dense open subset of $\mathcal{M} \cap \Pi$, and so **(G4)** does not hold on a subset of $\mathcal{M}$ of codimension at least 1 (in fact exactly 1, unless $d = 2$ in which case **(G4)** is always true).

Now let $\lambda_1 \in \mathcal{M}$ be a parameter where **(G4)** holds. Then we may locally follow the critical point (z_0, c_0) as $(z_0, c_0(\lambda))$, and moreover there exists a unique irreducible component L_λ of $\mathrm{Crit}(f_\lambda)$ passing through $(z_0, c_0(\lambda))$ for λ close enough to λ_0, of local equation of the form $w = c(\lambda, z)$. By **(G3)**, the algebraic set $f_\lambda^n(L_\lambda)$ is also locally a graph near $(z_1, w_1(\lambda))$, with local equation given by $w = Q^n_{\lambda,p^{-n}(z)}(c(\lambda, z))$, where p^{-n} denotes the local inverse branch of p^n mapping z_1 to z_0. In particular, it is regular at $(z_1, w_1(\lambda))$ and its tangent space is not vertical.

We now need to prove that the set of $\lambda_1 \in \mathcal{M}$ where the tangent space of $f_\lambda^n(L_\lambda)$ is not an eigenspace of $(df_\lambda^m)_{(z_1,w_1(\lambda))}$ is open and dense in $\mathcal{M}$; again by the algebraicity of the condition, it is in fact enough to prove that this subset is non-empty. Hence, we can restrict ourselves to the unicritical subfamily $U_d \subseteq \mathbf{Sk}(p,d)$ introduced in Section 2.3 and prove the analogous statement there. By Lemmas 2.4 and 2.5, and with the notations as in those lemmas, it is enough to prove that the identity $v_\lambda^{(2)} = u_\lambda^{(2)}$ cannot hold on all of $M \cap U_d$.

By Lemma 2.3, the accumulation on $\mathbb{P}^d_\infty$ of $M \cap U_d$ is equal to $E_{z_0} = \{[\lambda] \colon a(z_0) = 0\}$. We choose λ_∞ such that $[\lambda_\infty] \in E_{z_0}$ and z_0 is the only root in J_p of the derivative a' of the associated polynomial a. Since there exists a sequence $(\lambda_j)_{j\in\mathbb{N}}$ such that for all $j \in \mathbb{N}$, $\lambda_j \in M \cap U_d$ and $[\lambda_j] \to [\lambda_\infty]$, the assertion follows from the estimates (4) and (6). The proof is complete. □

We can now prove Theorem 1.2 and the other results stated in the Introduction.

Proof of Theorem 1.2. Fix $\lambda_0 \in \mathrm{Bif}(\mathbf{Sk}(p,d))$ and $\varepsilon > 0$. Set $M^0 := \mathbf{Sk}(p,d)$, and let H_{λ_0} denote the limit set of a vertical-like IFS for f_{λ_0} (which exists by Proposition 3.6). Let z be a repelling periodic point of period $m > d$ for p that is not in the post-critical set of p and let $(z, w_i(\lambda_0))$ be a collection of repelling periodic points in H_{λ_0} as in Lemma 6.2.

We will prove by induction on $1 \le k \le \dim \mathbf{Sk}(p,d)$ that there exist a parameter λ_k which is $k\varepsilon$-close to λ_0 and a family M^k with $\lambda_k \in M_k$ satisfying the following properties:

- **(I1)** $M^k = \bigcap_{1\le i\le k} M_{(y_i,c_i),(z,w_i(\lambda_k)),n_i}$ is the intersection of k distinct Misiurewicz loci (where a critical point lands after some iterations on one of the periodic points (z, w_i) introduced above);
- **(I2)** M^k has codimension k in $\mathbf{Sk}(p,d)$;
- **(I3)** if $k < \dim \mathbf{Sk}(p,d)$, among the k persistent Misiurewicz relations defining M^k, at least one is good in the sense of Definition 5.1 in a neighbourhood of λ_k.

Recall that each $M_{(y_i,c_i),(z,w_i(\lambda_k)),n_i}$ is a local family; in particular, condition **(I3)** is also local.

Initialization: the case $k = 1$. Recall that by Lemma 6.2, there exists a local hypersurface R of $\mathbf{Sk}(p,d)$ such that outside of that hypersurface, the vertical multipliers of the D_d repelling periodic points $(z, w_i(\lambda))$ (introduced above) define smooth and transverse foliations in parameter space. Up to replacing λ_0 by a first perturbation λ_0', we may assume without loss of generality that $\lambda_0 \in \mathrm{Bif}(\mathbf{Sk}(p,d))\backslash R$. Indeed, the bifurcation locus cannot be locally contained in any proper analytic subset of $\mathbf{Sk}(p,d)$, since the bifurcation current has continuous potential. In the rest of the proof, we will always assume that all perturbations are small enough so that none of the parameters we consider belong to R.

We then apply Lemma 6.1 to find $\lambda' \in \mathbb{B}(\lambda_0, \frac{\varepsilon}{3})$ such that $f_{\lambda'}$ has a Misiurewicz relation of the form $f_\lambda^{n_1}(y_1, c_1) = (z, w_1(\lambda))$, hence $\lambda' \in M_{(y_1,c_1),(z,w_1(\lambda')),n_1}$. Here n_1 can be taken arbitrarily large. We can assume that y_1 satisfies $(p^{n_1})'(y_1) \neq 0$, that it does not belong to the cycle of z and that the period m_1 of z satisfies $m_1 > d$. We need to prove that up to perturbing λ' inside $M_{(y_1,c_1),(z,w_1(\lambda')),n_1}$, we can obtain $\lambda_1 \in M_{(y_1,c_1),(z,w_1(\lambda')),n_1} \cap \mathbb{B}(\lambda_0, \varepsilon)$ which is a good parameter in the sense of Definition 5.1.

Let us first prove that the vertical multiplier $\rho_1(\lambda)$ of $(z, w_1(\lambda))$ is not constant on $M_{(y_1,c_1),(z,w_1(\lambda')),n_1}$. The argument is similar to the one in the proof of Lemma 6.4: we consider the intersection $\widetilde{M} := M_{(y_1,c_1),(z,w_1(\lambda')),n_1} \cap U_d$ with the unicritical subfamily U_d and pick $[\lambda_\infty]$ is the accumulation on $\mathbb{P}^d_\infty$ of $\widetilde{M}$ such that $\lambda_\infty(p^i(z)) \neq 0$ for all $0 \leq i \leq m_1$. Then, by Lemma 2.3, there exists a sequence $(\lambda_j)_{j\in\mathbb{N}}$ such that $\lambda_j \in \widetilde{M}$ for all $j \in \mathbb{N}$, $|\lambda_j| \to +\infty$ and $[\lambda_j] \to [\lambda_\infty]$. By Lemma 2.2, for all $0 \leq i \leq m_1 - 1$, we have $|w_{i,j}| \asymp |\lambda_j|^{1/d}$ where $w_{i,j} := Q^{n_1+i}_{\lambda_j,z}(0)$. In particular, $|\rho_1(\lambda_j)| := \left|(Q^{m_1}_{\lambda_j,z})'(w_{0,j})\right| \asymp |\lambda_j|^{m_1\cdot(d-1)/d}$ and therefore is not constant. This proves **(G1)**.

Observe that **(G3)** follows from the choice of y_1 as in Lemma 6.1. Property **(G2)** is a consequence of the fact that $(z, w_1(\lambda))$ belongs to the limit set of the vertical-like IFS, and it follows from Lemma 6.4 that properties **(G4)** and **(G5)** are generically satisfied in $M_{(y_1,c_1),(z,w_1(\lambda')),n_1}$ as soon as **(G3)** holds. This takes care of **(I1)** and **(I3)**; and **(I2)** is obvious in the case $k = 1$.

Heredity. Assume now that $k < D_d - 1$ is such that there exists λ_k satisfying **(I1)**, **(I2)** and **(I3)**. By the induction hypothesis, there exists $k_0 \in \{1, \ldots, k\}$ such that the vertical multiplier ρ_{k_0} of the repelling cycle from the k_0-th Misiurewicz relation $M_{(y_{k_0},c_{k_0}),(z,w_{k_0}(\lambda))}$ is non-constant on M^k. Consider the germ of analytic subset of M^k defined by $N := \{\lambda \in M^k : \rho_{k_0}(\lambda) = \rho_{k_0}(\lambda_k)\}$. Then N has codimension $k+1$ in $\mathbf{Sk}(p,d)$. We claim that there exists at least one repelling point (z, w_{j_0}) (among all those introduced at the beginning of the proof) with $j_0 \neq k_0$ such that its vertical multiplier ρ_{j_0} is non-constant on N. Indeed, by Lemma 6.2, $\dim \bigcap_{i=1}^{D_d} \{\lambda \in \mathbf{Sk}(p,d) : \rho_i(\lambda) = \rho_i(\lambda_k)\} = 0$, while if $k < D_d - 1$ then $\dim N > 0$. If $j_0 > k$, then we relabel the repelling periodic points $(z, w_i)_{k+1\leq i\leq D_d}$ so that $j_0 = k + 1$.

We now take $\lambda_{k+1,\infty} \in \mathbb{B}(\lambda_k, \frac{\varepsilon}{2})$ to be a point in the dense set S given by Proposition 5.2, and then take $\lambda_{k+1} \in \mathbb{B}(\lambda_{k+1,\infty}, \frac{\varepsilon}{2})$ such that λ_{k+1} has a Misiurewicz relation as in Proposition 5.2, with $(z'', w'') := (z, w_{k+1})$. We consider the associated Misiurewicz locus $M^k \cap M_{(y_{k+1},c_{k+1}),(z,w_{k+1}(\lambda_{k+1})),n_{k+1}}$. By Proposition 5.2, λ_{k+1} already satisfies

(I1) and **(I2)** with

$$M^{k+1} := M^k \cap M_{(y_{k+1},c_{k+1}),(z,w_{k+1}(\lambda_{k+1})),n_{k+1}}.$$

It remains to be proved that at least one of the Misiurewicz relations defining M^{k+1} is good in M^k.

Note that items **(G2)**, **(G3)**, **(G4)**, and **(G5)** are all preserved by restriction, so that they still hold on M^{k+1} for each of the first k Misiurewicz relations $M_{(y_i,c_i),(z,w_i(\lambda_k)),n_i}$ (with $1 \le i \le k$). Moreover, the new Misiurewicz relation $M_{(y_{k+1},c_{k+1}),(z,w_{k+1}(\lambda_{k+1})),n_{k+1}}$ satisfies **(G2)** since (z, w_{k+1}) is vertical-like by definition, and satisfies **(G3)**, **(G4)**, and **(G5)** by Proposition 5.2.

It now only remains to prove that at least one among the $(z, w_i(\lambda))$ (for $1 \le i \le k+1$) has a non-constant vertical multiplier on M^{k+1}, which would give **(G1)**. Recall that there exists $k_0, j_0 \le k+1$ with $k_0 \neq j_0$, such that ρ_{j_0} is non-constant on $\{\lambda \in \mathbf{Sk}(p,d) : \rho_{k_0}(\lambda) = \rho_{k_0}(\lambda_k)\}$. In other words, the level sets $\{\lambda \in M^k : \rho_{k_0}(\lambda) = \rho_{k_0}(\lambda_k)\}$ and $\{\lambda \in M^k : \rho_{j_0}(\lambda) = \rho_{j_0}(\lambda_k)\}$ are two distincts analytic hypersurfaces of M^k. Up to taking λ_{k+1} close enough to λ_k, we may still assume that the same holds at λ_{k+1}. Therefore M^{k+1} (which has codimension 1 in M^k) cannot be contained in

$$\{\lambda \in M^k : \rho_{k_0}(\lambda) = \rho_{k_0}(\lambda_{k+1})\} \cap \{\lambda \in M^k : \rho_{j_0}(\lambda) = \rho_{j_0}(\lambda_{k+1})\},$$

which precisely means that either ρ_{j_0} or ρ_{k_0} is non-constant on M^{k+1}.

Therefore, at least one of the $k+1$ Misiurewicz relations defining M^{k+1} is good in the sense of definition 5.1. This proves **(I3)** and completes the inductive step for all $k < D_d - 1$.

Finally, assume that $k = D_d - 1$, so that $\dim M^k = 1$. By the induction hypothesis, there is at least one good Misiurewicz relation in M^k; therefore we may apply Proposition 5.2 and find $\lambda_{k+1} \in M^k \cap \mathbb{B}(\lambda_k, \varepsilon)$ with a new, non-persistent Misiurewicz relation in M^k. We may then take $M^{k+1} := \{\lambda_{k+1}\}$.

The proof is complete. □

Proof of Theorem 1.1. Let λ_{D_d} be as constructed in the proof of Theorem 1.2. By Proposition 4.1, we have $\lambda_{D_d} \in \operatorname{Supp} T_{\text{bif}}^{D_d}$. This gives $\operatorname{Supp} T_{\text{bif}} = T_{\text{bif}}^{D_d}$, and proves the assertion. □

Proof of Corollary 1.3. By the initialization step in the proof of Theorem 1.2, for every d there exists a Misiurewicz hypersurface of $\mathbf{Sk}(p,d)$ which is good in the sense of Definition 5.1. The result follows from Corollary 5.3. □

Proof of Corollary 1.4. By [Duj17, Taf17], for every $d \ge 2$ the bifurcation locus of the family $\mathbf{Sk}(p,d)$ is not empty. The assertion follows from Theorem 1.1. □

Proof of Corollary 1.5. By Theorem 1.1 it is enough to check that the same property is true for the bifurcation locus. By [AB18, Theorem 3.3], the bifurcation locus associated to the return maps of any periodic fibre is contained in the bifurcation locus of the family $\mathbf{Sk}(p,d)$. By [McM00] the bifurcation loci of the return maps have full Hausdorff dimensions. The assertion follows. □

Appendix A. Proof of Lemma 3.7

We work here in the assumptions of Proposition 3.6. We assume that we are given a hyperbolic set $\widetilde{H}$ in J_p as in the proof of Proposition 3.6, i.e., with positive entropy and $\delta := \dim_H \widetilde{H} > 1$. We will be mainly interested in the following in the dynamics of f on $\widetilde{H} \times \mathbb{C}$. We denote by $\widetilde{m}$ the *conformal measure* associated with the weight $\widetilde{\varphi}(z) := -\log |p'(z)|^\delta$. Recall that this means that $\widetilde{m}$ is an eigenvector (corresponding to some eigenvalue $\widetilde{\lambda}$, which is equal to 1 in our case by the construction of $\widetilde{m}$, since δ is precisely equal to the Hausdorff dimension of the hyperbolic set $\widetilde{H}$) for the dual $\widetilde{\mathcal{L}}^*$ of the Perron-Frobenius operator $\widetilde{\mathcal{L}}$ acting on continuous functions $g\colon \widetilde{H} \to \mathbb{R}$ as

$$\widetilde{\mathcal{L}}_{\widetilde{\varphi}}(g)(x) = \sum_{f(a)=x} e^{\widetilde{\varphi}(a)} g(a).$$

Observe that $\widetilde{\varphi}$ is Hölder continuous on $\widetilde{H}$. This and the hyperbolicity of $\widetilde{H}$ imply the existence and uniqueness of $\widetilde{m}$, see for instance [PU10]. Moreover, $\widetilde{m}$ is equivalent to the δ-dimensional Hausdorff measure. It is also equivalent to the unique *equilibrium state* $\widetilde{\nu}$ for the system $(\widetilde{H}, f)$ associated with $\widetilde{\varphi}$. This means that $\widetilde{\nu}$ is the (unique) maximizer of the *pressure* $P(\widetilde{\varphi}) := \sup_\omega \{h_\omega + \langle \omega, \widetilde{\varphi}\rangle\}$, where the supremum is taken over all invariant probability measures for f and h_ω is the metric entropy of the measure ω. We denote by $\widetilde{\rho}$ the Radon-Nikodym derivative of $\widetilde{\nu}$ with respect to $\widetilde{m}$ (which is strictly positive on the support of $\widetilde{m}$), i.e., set $\widetilde{\nu} = \widetilde{\rho}\widetilde{m}$, and by $L_{\widetilde{\nu}} = \langle \widetilde{\nu}, \log |p'|\rangle$ the Lyapunov exponent for $\widetilde{\nu}$. The function $\widetilde{\rho}$ is also a fixed point of $\widetilde{\mathcal{L}}$, that is, $\widetilde{\mathcal{L}}\widetilde{\rho} = \widetilde{\rho}$. By the construction of $\widetilde{H}$, since $P(\widetilde{\varphi}) = 0$ and $0 < \delta < 1$, we have $0 < L_{\widetilde{\nu}} = \delta^{-1} \log d < \log d$. We also have

$$\lim_{n\to\infty} \sum_{p^n(a)=x} \frac{1}{|(p^n)'(a)|^\delta} g(a) \to \widetilde{\rho}(x)\langle m, g\rangle \tag{11}$$

for all $x \in \widetilde{H}$ and continuous functions $g\colon \widetilde{H} \to \mathbb{R}$, and again we used that $\widetilde{\lambda} = 1$ is the eigenvalue corresponding to $\widetilde{m}$, i.e., $\widetilde{\mathcal{L}}^* \widetilde{m} = \widetilde{m}$.

Recall that $J_{\widetilde{H}} = \overline{\cup_{z\in\widetilde{H}}\{z\} \times J_z}$. We see $(J_{\widetilde{H}}, f)$ as a dynamical system and we can consider the weight on $J_{\widetilde{H}}$ given by $\varphi(z, w) := \widetilde{\varphi}(z) = -\log |p'(z)|^\delta$. Observe that, a priori, φ is not a Hölder continuous weight on J (since the Julia set of the base polynomial p may contain critical points for p), and the system $(J_{\widetilde{H}}, f)$ is not necessarily hyperbolic. Hence, we cannot directly apply the thermodynamical formalism for the system (J, f) and weight φ (see for instance [UZ13, BD20]), nor to the system $(J_{\widetilde{H}}, f)$. However, thanks to the fibred structure we can deduce the following result.

Lemma A.1. *The measures*

$$m := \int_{\widetilde{H}} \mu_z d\widetilde{m}(z) \quad \text{and} \quad \nu := \int_{\widetilde{H}} \mu_z d\widetilde{\nu}(z) = \int_{\widetilde{H}} \mu_z \widetilde{\rho}(z) d\widetilde{m}(z)$$

are the unique conformal measure and an equilibrium state associated with the weight $\varphi(z, w) = -\log |p'(z)|^\delta$ *on the system* $(J_{\widetilde{H}}, f)$*, respectively. The measure* ν *is invariant, mixing, and its metric entropy is strictly larger than* $\log d$*. The Lyapunov exponents of* ν *are equal to*

$$0 < L_{\widetilde{\nu}} < \log d \quad \text{and} \quad L_v(\nu) := \log d + \int \Big(\sum_{w:q'_z(w)=0} G(z, w)\Big)\mu_p(z) \geq \log d.$$

In particular, they are strictly positive.

Proof. We first prove that the measure m satisfies $\mathcal{L}^* m = d \cdot m$. This proves that m is a conformal measure and that d is an eigenvalue for the operator $\mathcal{L}^*$. Let $g\colon J_{\widetilde{H}} \to \mathbb{R}$ be any continuous function. Since $\mu_z = d \cdot (q_z)^* \mu_{p(z)}$ for all $z \in J_p$, we have

$$\begin{aligned}
\langle \mathcal{L}^* m, g \rangle = \langle m, \mathcal{L} g \rangle &= \int \langle \mu_z(\cdot), \mathcal{L} g(z, \cdot) \rangle \widetilde{m}(z) \\
&= \int \sum_{a\colon p(a)=z} \Big(\sum_{b\colon q_a(b)=\cdot} \frac{1}{|p'(a)|^\delta} \langle \mu_z(\cdot), g(a,b) \rangle \Big) \widetilde{m}(z) \\
&= \int \sum_{a\colon p(a)=z} \frac{1}{|p'(a)|^\delta} \langle \mu_z, ((q_a)_* g)(z, \cdot) \rangle \widetilde{m}(z) \\
&= d \cdot \int \sum_{a\colon p^n(a)=z} \frac{1}{|p'(a)|^\delta} \langle \mu_a, g(a, \cdot) \rangle \widetilde{m}(z) \\
&= d \langle m, g \rangle
\end{aligned}$$

where in the last step we used the fact that $\widetilde{\mathcal{L}}^* \widetilde{m} = \widetilde{m}$.

Observe now that, by (11), the sequence $\mathcal{L}^n(\mathbb{1})$ satisfies

$$d^{-n} \mathcal{L}^n \mathbb{1} \to \rho := \widetilde{\rho} \circ \pi_z, \tag{12}$$

where $\mathbb{1}$ denotes the function constantly equal to 1 on $J_{\widetilde{H}}$. Hence, we have $\|\mathcal{L}^n g\|_\infty \lesssim \|g\|_\infty \cdot d^n$ for all continuous functions $g\colon J_{\widetilde{H}} \to \mathbb{R}$. By duality, for all probability measure σ on $J_{\widetilde{H}}$, we have

$$|\langle (\mathcal{L}^n)^* \sigma, g \rangle| \le \|g\|_\infty \langle \sigma, \mathcal{L}^n \mathbb{1} \rangle.$$

Hence, the absolute values of all eigenvalues of $\mathcal{L}^*$ must be $\le d$ and, by the previous part, d is then an eigenvalue of maximal modulus of the operator $\mathcal{L}^*$. Observe that this also implies that $P(\varphi) = \log d$, see for instance [PU10, Propositions 5.2.11 and 5.1.1].

From (12), it follows that ρ satisfies $d^{-1} \mathcal{L} \rho = \rho$, i.e., ρ is an eigenvector of the operator $\mathcal{L}$, of eigenvalue d. Standard arguments (see for instance [BD20, End of the Proof of Theorem 4.1 and Proposition 4.11]) imply the uniqueness of such ρ satisfying the above property, and the uniqueness of the conformal measure m. In particular, for any continuous function $g\colon J_{\widetilde{H}} \to \mathbb{R}$, we have

$$d^{-n} \mathcal{L}^n g \to \langle m, g \rangle \cdot \rho \tag{13}$$

where the convergence is uniform in the norm $\|\cdot\|_\infty$ (for a fixed g).

Let us now consider the measure ν as in the statement. Observe that $\nu = \rho m$. It follows from the definition that ν is f-invariant, and (13) implies that ν is mixing. Indeed, for all continuous functions $g, h\colon J_{\widetilde{H}} \to \mathbb{R}$ with $\langle \nu, g \rangle = \langle \nu, h \rangle = 0$ and all $n \ge 0$ we have

$$\begin{aligned}
\langle \nu, g \cdot h \circ f^n \rangle = \langle \rho \cdot m, g \cdot h \circ f^n \rangle &= d^{-n} \langle (\mathcal{L}^n)^* m, \rho \cdot g \cdot h \circ f^n \rangle \\
&= \langle m, d^{-n} \mathcal{L}^n (\rho \cdot g \cdot h \circ f^n) \rangle = \langle \nu, h \cdot \frac{\mathcal{L}^n(\rho \cdot g)}{\rho d^n} \rangle \to 0
\end{aligned}$$

where the convergence follows from the fact the h is bounded, $\frac{\mathcal{L}^n(\rho \cdot g)}{\rho d^n} \to \langle m, \rho \cdot g \rangle$ uniformly by (13), and $\langle m, \rho \cdot g \rangle = \langle \nu, g \rangle = 0$ by assumption.

It follows as in [PU10, Section 5.6] and [BD20, Section 4.6] that ν is an equilibrium state for the system $(J_{\widetilde{H}}, f)$ associated to the weight φ. In particular, since $P(\varphi) = \log d$, we have

$$\log d = P(\varphi) = h_\nu + \langle \nu, \varphi \rangle = h_\nu - \delta L_{\widetilde{\nu}}.$$

It follows that $h_\nu = \log d + \delta L_{\widetilde{\nu}} > \log d$.

Finally, the expressions for the Lyapunov exponents of ν can be deduced from the fibred structure of ν as done in [Jon99, Theorem 5.3] for the measure μ. □

In order to prove Lemma 3.7, we will give an adapted fibered version of the proof by Briend-Duval [BD99] of the equidistribution of repelling periodic points with respect to the equilibrium measure for endomorphisms of $\mathbb{P}^k$ (which is done by constructing enough contracting inverse branches of a ball centred on the Julia set to itself). Since the method is now standard, we just sketch the overall proof and highlight the differences here (due to the non-constant Jacobian of ν and $\widetilde{\nu}$). More details can be found in [BD20, Section 4.7], where the strategy is adapted to prove the equidistribution of repelling periodic points with respect to the equilibrium state (when the weight satisfies some regularity condition on all of J).

First we need to introduce the natural extension of the system $(J_{\widetilde{H}}, f)$, see [CFS12]. We set $X := J_{\widetilde{H}} \setminus \cup_{m \geq 0} f^{-m}(PC_f)$, where $PC_f := \cup_{n \geq 0} f^n C_f$ is the postcritical set of f. Since, by Lemma A.1, the entropy of ν is strictly larger than $\log d$, we have $\nu(X) = 1$ by [Gro03], see [BD20, Proposition 4.19] for a detailed proof. We then consider the dynamical system $(\hat{X}, \hat{f}, \hat{\nu})$, where $\hat{X} = \{\hat{x} = (\dots x_{-1}, x_0, x_1, \dots) \colon f(x_i) = x_{i+1}\}$ and $\hat{f}(\hat{x}) = (x_{i+1})_{i \in \mathbb{Z}}$ where $\hat{x} = (x_i)_{i \in \mathbb{Z}}$. The measure ν lifts to a measure $\hat{\nu}$ satisfying $(\pi_0)_* \hat{\nu} = \nu$, where $\pi_0 \colon \hat{X} \to X$ is given by $(x_i) \mapsto x_0$. The measure $\hat{\nu}$ is mixing since ν is mixing. For any $\hat{x} \in \hat{X}$ and every n we denote by $f^{-n}_{\hat{x}}$ the inverse branch of f^n in a neighbourhood of x_0 with values in a neighbourhood of x_{-n}. We have the following lemma.

Lemma A.2. *For all $\varepsilon < L_{\widetilde{\nu}}$ there exist measurable functions $r_\varepsilon, L_\varepsilon, T_\varepsilon \colon \hat{X} \to \mathbb{R}^+$ such that, for $\hat{\nu}$-almost all $\hat{x} \in \hat{X}$ and all $n \geq 1$,*

(1) the map $f^{-n}_{\hat{x}}$ is defined on $B(x_0, r_\varepsilon(\hat{x}))$;
(2) $\mathrm{Lip}(f^{-n}_{\hat{x}}) \leq L_\varepsilon e^{-nL_{\widetilde{\nu}} + n\varepsilon}$ on $B(x_0, r_\varepsilon(\hat{x}))$;
(3) $\forall y \in f^{-n}_{\hat{x}}(B(x_0, r_\varepsilon(\hat{x})))$ we have $\left| \frac{1}{n} \log |\mathrm{Jac}\, df^n_y| - (L_{\widetilde{\nu}} + L_v(\nu)) \right| \leq \frac{1}{n} \log T_\varepsilon(\hat{x}) + \varepsilon$;
(4) $\forall y \in f^{-n}_{\hat{x}}(B(x_0, r_\varepsilon(\hat{x})))$ we have $\left| \frac{1}{n} \log \|df^n_y\| - L_v(\nu) \right| \leq \frac{1}{n} \log T_\varepsilon(\hat{x}) + \varepsilon$.

Proof. The statement is a consequence of [BDM08, Theorem 1.4], see also [BD19, Theorem A]. These results are stated for the measure of maximal entropy, but only the strict positivity of the Lyapunov exponents of the measure is needed, see the remark at the end of the Introduction of [BD19]. Hence we can apply [BD19, Theorem A] to ν, since by Lemma A.1 this measure has strictly positive Lyapunov exponents. □

We fix $\varepsilon \ll L_{\widetilde{\nu}}$ in what follows and set $\hat{X}_C := \{\hat{x} \in \hat{X} \colon r_\varepsilon^{-1}, L_\varepsilon, T_\varepsilon < C\}$. We have $\hat{\nu}(\hat{X}_C) \to 1$ as $C \to \infty$. Given a Borel subset $E \subset \mathbb{C}^2$, we set

$$\hat{E} := \pi_0^{-1}(E \cap X), \quad \hat{E}_C := \hat{E} \cap X_C, \quad \text{and} \quad \nu_C = (\pi_0)_*(\hat{\nu}_{|\hat{X}_C}).$$

Fix now a point $x \in X$, a constant C sufficiently large (to be chosen later), and a fibred box $x \in A \subset B(x, 1/(2C))$, with $\nu(A) > 0$. We also fix a subset $A_r := \{y \in A, \mathrm{dist}(y, A^c) > r\}$ such that $\nu(A_r) > 0$.

We call *good component of* $f^{-n}(A)$ any connected component with diameter smaller than $r/2$. Since any good component intersecting A_r is strictly contained in A, to prove the lemma we need to show that (we can choose A, C, r so that) for n sufficiently large, there are at least $3d^n$ good components of $f^{-n}(A)$ intersecting A_r and satisfying the estimates in (8).

Notice that, for any $y \in \hat{A}_C$ the inverse branch $f_{\hat{y}}^{-n}$ is defined on A. Moreover, it follows from Lemma A.2(2) that, for all n sufficiently large all images of such inverse branches have diameter smaller than $r/2$ (uniformly in $\hat{y}$). Hence they are good components.

Since $\hat{\nu}$ is mixing, we have $\hat{\nu}(\hat{f}^{-n}(E_1) \cap E_2) \to \nu(E_1) \cdot \nu(E_2)$ for any Borel subsets $E_1, E_2 \subseteq \hat{X}$. In particular, we have, for all n large enough,

$$\nu(\pi_0(\hat{f}^{-n}(\hat{A}_r)_C) \cap A_r) = \hat{\nu}(\hat{f}^{-n}(\hat{A}_r)_C \cap \hat{A}_r) \geq \frac{1}{2}\hat{\nu}((\hat{A}_r)_C) \cdot \hat{\nu}(\hat{A}_r) = \frac{1}{2}\nu_C(A_r) \cdot \nu(A_r).$$

By the argument above, the LHS of the above expression is less than $\nu(\cup_j A^j)$ where $A^j, 1 \leq j \leq N$, are the good components of $f^{-n}(A)$ intersecting A_r. To get the desired estimate on N, we need to find an upper bound for $\nu(A^j)$ (this bound is immediate when working with the measure of maximal entropy, since this measure has constant Jacobian). We use here the definition of fibred box, letting a be the common μ_z-measure of all the non empty slices $A \cap (z \times \mathbb{C})$, for $z \in J_p$. We then have $\nu(A) = a\widetilde{\nu}(B)$ (where B is the projection of A on the first coordinate) and so $\nu(A^j) = a\widetilde{\nu}(B^j)/d^n$, where B^j is the projection of A^j on the first coordinate.

Since the measure $\widetilde{\nu}$ is not-atomic, the function $M(r) := \sup_{z \in \mathbb{C}} \widetilde{\nu}(B(z, r))$ goes to 0 as $r \to 0$. Since the system $(\widetilde{H}, f)$ is hyperbolic, the diameters of all the B^j tend uniformly to zero as $n \to \infty$. Hence, there exists a function M'_n such that $\widetilde{\nu}(B^j) \leq M'_n$ for all j and $M'_n \to 0$ as $n \to \infty$. Take n large enough so that $M'_n < (\nu_C(A_r) \cdot \nu(A_r))/(6a)$. The above inequalities imply that $N > 3d^n$, as desired. The estimates in (8) follow from items (3) and (4) in Lemma A.2 (up to possibly increasing n).

Remark A.3. *The above actually proves that, for any $x \in J_{\widetilde{H}}$, any fibred box $A \ni x$ in a sufficiently small ball centred at x satisfies the requirements of Lemma 3.7.*

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