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Domaines errants et bifurcations en dynamique holomorphe

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Introduction

Complex dynamics is the study of orbits $z, f(z), \dots f^n(z), \dots$ of points in a complex manifold X under the iteration of a holomorphic self-map $f : X \to X$. Following the pioneering work of Fatou and Julia in the early 20th century, one may partition X into different sets, one stable (the Fatou set) and one unstable or chaotic (the Julia set). The study of the global dynamics of f involves both its Fatou and Julia set, using different tools and approaches. In higher dimensions, two classes of complex dynamical systems are especially studied: polynomial automorphisms of $\mathbb{C}^2$, including the so-called Hénon maps, and endomorphisms of projective spaces $\mathbb{P}^k = \mathbb{P}^k(\mathbb{C})$; with some exceptions, much of the work described in this manuscript is focused on the setting of endomorphisms of $\mathbb{P}^k$ $(k \geq 1)$.

On the one hand, it is a fundamental question to classify dynamics in the Fatou set. For rational self-maps of $\mathbb{P}^1$, this classification was essentially achieved thanks to the seminal work of Sullivan [**Sul85**], who proved that every connected component of the Fatou set is eventually periodic. Together with a sharp upper bound on the number of possible periodic components by Shishikura [**Shi87**] and a classification of periodic components (by works of Fatou, Siegel, Herman and others), dynamics in the Fatou set of rational maps on $\mathbb{P}^1$ is therefore very well understood. Despite recent progress, the picture is far from being as clear in higher dimension.

On the other hand, the dynamics on the Julia set (or small Julia set) is chaotic, and is typically studied via tools from pluripotential and ergodic theory. Notably, endomorphisms of $\mathbb{P}^k$ admit a unique measure of maximal entropy, which enjoys good properties: it is ergodic, mixing, and it has positive Lyapunov exponents, with sharp lower bounds. In that respect, the ergodic-theoretic aspect of higher-dimensional complex dynamics is particularly successful and generally better understood than its Fatou set counterpart.

Bifurcations. A fundamental type of questions in complex dynamics relates to parameter spaces: given a holomorphic family $(f_\lambda)_{\lambda \in M}$ of self-maps $f_\lambda : X \to X$, how are global dynamics of f_λ affected by a variation of the parameter λ? Of particular interest is the set of parameters $\lambda \in M$ for which the global dynamics is stable under perturbation of the parameter in some sense; this is called the *stability* locus, and its complement is the *bifurcation* locus. Again, for families of rational maps on $\mathbb{P}^1$ of given degree, much is understood: in the foundational works [**MSS83**], [**Lyu83**], [**DeM01**], it has been proven that several possible notions of stability coincide (in terms of continuous motion of the Julia set, stability of periodic orbits, harmonicity of Lyapunov exponents and support of a bifurcation current T_{bif}). Moreover, there is a natural stratification

$$\mathrm{Bif} = \mathrm{Bif}^1 \supsetneq \mathrm{Bif}^2 \supsetneq \dots \supsetneq \mathrm{Bif}^{2d-2}$$

of the bifurcation locus into "bifurcation loci of order k", which may be defined as supports of the currents T^k_{bif}, $1 \leq k \leq 2d-2$. Heuristically, Bif^k describes the closure of the set of parameters which exhibit bifurcation phenomena of codimension at least k: k non-persistent neutral cycles, or k critical orbits bifurcating independantly. An important consequence of Mañé-Sad-Sullivan-Lyubich's results is the lack of *robust bifurcations* in one-dimensional rational dynamics: the bifurcation locus always has empty interior. However, we proved in [**AGMV19**] that the maximal bifurcation locus Bif^{2d-2} is as large as it possibly could, in the sense that it has positive Lebesgue volume in the moduli space of degree $d \geq 2$ rational maps on $\mathbb{P}^1$.

This theory has been generalized to higher dimensions only recently: for polynomial automorphisms in [**DL15**], and in parallel, for projective endomorphisms by Berteloot, Bianchi and Dupont [**BBD18, Bia19a**]. A significant new challenge in understanding higher-dimensional bifurcation loci is the presence of *robust bifurcations*, by works of Bianchi, Biebler, Dujardin and Taflin ([**BT17a**], [**Duj17**], [**Taf21**], [**Bie19**], see also [**GTV23**]). Remarkably, all these constructions of robust bifurcations (excepting [**Bie19**]) start from a special type of map preserving a foliation (skew-product).

With the aim of understanding more precisely the bifurcation locus of a toy-model family (keeping in mind the role that the quadratic family plays in one-dimensional rational dynamics), Bianchi and I studied in detail families of skew-products in [**AB23**], and in particular quadratic skew-products. We obtained a geometric and quantitative description of the bifurcation locus near infinity in parameter space. We also proved that for skew-products, hyperbolicity is preserved within stable families (which remains an open question for endomorphisms of $\mathbb{P}^k$): in particular, the notion of hyperbolic component is well-defined. We then give a classification of unbounded hyperbolic components, using a new topological invariant. In the sequel paper [**AB22**], we prove a perhaps surprising theorem:

THEOREM 0.1. *Let p be a polynomial map with Julia set not totally disconnected, which is neither conjugated to $z \mapsto z^d$ nor to a Chebyshev polynomial. Let $\mathbf{Sk}(p, d)$ denote the family of polynomial skew-products of degree $d \geq 2$ over the base polynomial p, up to affine conjugacy, and let D_d be its dimension. Then*

$$\mathrm{Bif} = \mathrm{Bif}^1 = \mathrm{Bif}^2 = \ldots = \mathrm{Bif}^{D_d}.$$

In other words, in the family $\mathbf{Sk}(p, d)$, it is impossible to isolate the bifurcation of a single critical point: if one critical point bifurcates near a parameter, then many others must do so as well. This contrast with the picture in dimension one may be intuitively understood by the fact that critical points now form a hypersurface of $\mathbb{P}^2$ instead of a finite subset, and are not isolated anymore. It seems very likely that this type of phenomenon is not restricted to skew-products, but also occurs in the general family of endomorphisms of $\mathbb{P}^k$ of a given degree.

Going back to complex dimension one, one may also consider non-algebraic holomorphic dynamical systems, such as entire or meromorphic maps $f : \mathbb{C} \to \mathbb{P}^1$ with an essential singularity at ∞. In this setting, the potential and ergodic side of the theory vanishes: transcendental maps do not have the equivalent of a measure of maximal entropy (the topogical entropy being always infinite), Lyapunov exponent or Green function. Despite this, there are some connections between transcendental dynamics in dimension one and algebraic dynamics in dimension 2: for instance, transcendental arguments appear when studying unstable manifolds of Hénon maps (see e.g. [**DL15**]), or parabolic dynamics in $\mathbb{C}^2$ ([**ABD**$^+$**16**], [**ABTP23**], [**AT22**]).

For transcendental meromorphic maps, in addition to the bifurcation of critical values, a new type of singular value (asymptotic values) must also be considered; and with respect to the bifurcations of periodic points, a new type of phenomenon also appears, namely collisions between periodic points and the essential singularity. Since transcendental entire or meromorphic maps can be quite badly behaved in general, many works in transcendental dynamics place some restrictions on the set of singular values. In the joint work [**ABF21**] with A.M. Benini and N. Fagella, we study bifurcations for families of *finite type* meromorphic maps, i.e. maps with only finitely many singular values. By a result of Eremenko and Lyubich ([**EL92**]), collisions between periodic points and the essential singularity are not possible for finite type *entire* maps; as a result, their bifurcations work essentially in the same way as those of rational maps. However, for finite type *meromorphic* maps, the presence of poles makes collisions between periodic points and ∞ possible. We analyze this phenomenon in detail and prove that it is closely related to the bifurcation of asymptotic values.

Parabolic dynamics. A holomorphic self-map has a *parabolic cycle* if it admits a periodic point with at least one multiplier which is a root of unity. Parabolic cycles are a mechanism of bifurcations, as parameters with parabolic cycles are dense in the bifurcation locus. While in dimension one the local dynamics near a parabolic cycle are well-understood, by work of Leau and Fatou at the beginning of 20th century, in higher dimension the local dynamics near parabolic cycles started developing more recently (see e.g., [**Hak98**], [**Aba01**]) and is an active subject of research (e.g., [**LHRRSS19**] and [**LHRSSV20**]). A major tool in one-dimensional complex dynamics is the study of local dynamics near perturbations of parabolic maps: these techniques, called *parabolic implosion*, have led to major results such as the fact that the boundary of the Mandelbrot set has full Hausdorff dimension [**Shi98**], or the construction of polynomial Julia sets of positive Lebesgue measure [**BC12**]. Some analoguous results have started to appear in higher dimension: in [**BSU17**], Bedford, Smillie and Ueda study perturbations of *semi-parabolic* germs of $\mathbb{C}^2$, i.e. germs of diffeomorphisms of $\mathbb{C}^2$ fixing the origin with one multiplier equal to 1 and the other in the unit disk. Bianchi [**Bia19b**] partially extended that theory to a class of germs tangent to the identity in $\mathbb{C}^2$. Following an original idea of Lyubich, the main technical ingredient for the construction of wandering domains in [**ABD$^+$16**],[**ABTP23**] and [**AT22**] is an adaptation of parabolic implosion techniques to skew-products. Parabolic implosion has also been used in [**DL15**] to prove the density of homoclinic tangencies in bifurcation loci of polynomial automorphisms.

Wandering domains. A long-standing open question was to know whether Sullivan's Theorem could be extended to either endomorphisms of $\mathbb{P}^k$ ($k \geq 2$) or polynomial automorphisms of $\mathbb{C}^2$ (Hénon maps); it turns out that the answer is negative in both cases. The first examples of *wandering domains* (non eventually periodic Fatou components) have been constructed for endomorphisms of $\mathbb{P}^2$ in [**ABD$^+$16**], and the first examples of wandering domains for polynomial automorphisms of $\mathbb{C}^2$ have been constructed in [**BB23**] by Berger and Biebler, through completely different methods.

These recent breakthroughs lead to many natural questions, such as, to cite a few:

(1) Can wandering domains occur in low degree?
(2) What can be said about the limits of iterates restricted to a wandering Fatou component? More precisely, can there exist non-constant limits in dimension 2?
(3) What can be said about the accumulation set of a wandering Fatou component?
(4) How often do wandering domains occur in parameter space?

Part of my work these last few years has been motivated by these questions. Question (4) in particular remains largely unanswered. However, for instance, the following theorem proved in [**AT22**] provides an answer to (1) and (2):

THEOREM 0.2. [**AT22**] *Let* $b := \frac{1}{4} + \frac{\pi^2}{(\ln \alpha)^2}$, *where* $\alpha \in \mathbb{N}^*$ *and* $\alpha \geq 2$. *Let*

$$f_b([z : w : t]) := [zt + z^2, wt + w^2 + bz^2 : t^2].$$

Then f_b *has infinitely many distinct grand orbits of wandering Fatou components, each admitting non-constant limit maps taking values in the line* $z = 0$.

As previously mentionned, the proof of Theorem 0.2 is based on parabolic implosion, but it also involves studying a one-parameter family of non-algebraic finite type maps and ideas from [**ABF21**].

Outline. This manuscript is composed of three chapters. The first one is dedicated to finite type maps in the sense of A. Epstein, introduced in [**Eps93**]. Finite type maps are a class of holomorphic maps in one complex variable which contains e.g. rational maps, entire or meromorphic maps with only finitely many singular values, and horn maps of rational maps. They appear in several places in my work: for instance in the proof of Theorem 0.2, but also more directly in [**Ast22**]. The article [**ABF21**] is also strongly motivated by and linked to finite type maps. Chapter 1 is intended as a sort of survey, gathering in one place several unpublished results (including reasonnably self-contained proofs) which may be known to some experts but are probably not widely known in the community. Moreover, there are no publicly available reference for some of them. Most of the material comes from A. Epstein's PhD thesis ([**Eps93**]), the unpublished manuscript [**Eps09**], or private communications with A. Epstein or X. Buff, which I both thank; however, a few proofs are original.

We start by developping the Fatou/Julia theory for finite type maps (density of repelling cycles in the Julia set, lack of Baker or exotic Fatou components, absence of wandering Fatou components and Fatou-Sullivan's classification); this part of Chapter 1 is more or less directly reproduced from [**Eps93**]. We then introduce Epstein's deformation space, which forms a natural parameter space akin to that introduced by Eremenko and Lyubich in [**EL92**], or Goldberg and Keen in [**GK86**], and the Teichmüller space of a finite type map. We finish with a generalization of a theorem of McMullen-Sullivan ([**MS98**][Theorem 7.4]), which is the only original result from Chapter 1.

Chapter 2 is devoted to the topic of bifurcations, in various settings: one-dimensional rational maps ([**AGMV19**]), endomorphisms of $\mathbb{P}^k$ (with a strong emphasis on families of polynomial skew-products on $\mathbb{P}^2$: [**AB23**], [**AB22**]), and finally finite type meromorphic maps in dimension one ([**ABF21**]).

Finally, Chapter 3 is devoted to parabolic dynamics and wandering domains in dimension 2 ([**ABTP23**] and [**AT22**]).

List of works presented

Here is the list of works that are presented in this manuscript:

(1) [**AB22**]*Hyperbolicity and bifurcations in holomorphic families of polynomial skew-products*, with F. Bianchi. Amer. J. Math. 145 (2023), no. 3, 861–898.
(2) [**AB22**] *Higher bifurcations for polynomial skew-products*, with F. Bianchi. Journal of Modern Dynamics, vol. 18, n.3 (2022)
(3) [**ABTP23**]*Wandering domains arising from Lavaurs maps with Siegel disks*, with L. Boc Thaler and H. Peters. Anal. PDE 16 (2023), no. 1, 35–88.
(4) [**AGMV19**]*Collet, Eckmann and the bifurcation measure*, with T. Gauthier, N. Mihalache and G. Vigny. Inventiones mathematicae, 217(3), 749-797 (2019)
(5) [**AT22**] *Dynamics of skew-products tangent to the identity*, with L. Boc Thaler (2022). Submitted
(6) [**ABF21**]*Bifurcation loci of families of finite type meromorphic maps*, with A. M. Benini and N. Fagella (2022). Submitted

The articles [**ABD**$^+$**16**], [**Ast17**] and [**Ast22**] are mentionned for exposition purposes, but are not presented in detail since most of the work was done during my PhD thesis.

I have also chosen not to include the paper [**Ast20**], which was written after my PhD but is not directly related to the main themes of this manuscript, namely parabolic dynamics and bifurcations.

CHAPTER 1

Finite type maps

1. First definitions

1.1. Singular values, tracts, island property. By definition, we require Riemann surfaces to be connected; if not, we will use the term "complex 1-manifold".

DEFINITION 1.1. *Let X be a Riemann surface, and W be a complex 1-manifold. Let $f : W \to X$ be a holomorphic map. The singular value set of f, denoted by $S(f)$, is the smallest subset of X such that $f : W \setminus f^{-1}(S(f)) \to X \setminus S(f)$ is a covering map (when restricted to each connected component of W).*
We will use the following notation: $X^* := X \setminus S(f)$ and $W^* := W \setminus f^{-1}(S(f))$.

A covering map is surjective by definition, so $X \setminus f(W) \subset S(f)$.

DEFINITION 1.2. *Let X be a compact Riemann surface, and let $W \subset X$ be a non-empty open set. Let $f : W \to X$ be a holomorphic map. We say that f is a finite type map on X if*

(1) f is non-constant on every connected component of W
(2) f has no removable singularities
(3) $S(f)$ is finite.

Let us emphasize the importance of the assumption that X is compact. By the previous remark, if $f : W \to X$ is a finite type map, then $X \setminus f(W)$ is finite, possibly empty.

DEFINITION 1.3. *Let W, X be complex 1-manifolds and $f : W \to X$ be a holomorphic map. Let $v \in X$.*

(1) We say that $v \in X$ is a critical value for f if there exists $c \in W$ such that $f'(c) = 0$ and $f(c) = v$.
(2) We say that $v \in X$ is an asymptotic value for f if there exists a continuous curve $\gamma : \mathbb{R}_+ \to W$ such that $\gamma(t) \to \partial W$ and $\lim_{t\to+\infty} f \circ \gamma(t) = v$.

We will denote by $\mathrm{A}(f)$ the set of asymptotic values of f, and by $\mathrm{CV}(f)$ the set of its critical values.

More explicitly, $\gamma(t) \to \partial W$ means that for every compact $K \subset W$, there exists $t_K > 0$ such that for all $t > t_K$, $\gamma(t) \notin K$. An important point is that this does not in general imply that $\lim_{t\to+\infty} \gamma(t)$ exists in X.

REMARK 1.1. *If $f : S_1 \to S_2$ is a holomorphic map between Riemann surfaces, then $\mathrm{A}(f)$ depends on the range S_2. For instance, the identity map $i : \mathbb{D} \to \mathbb{D}$ has no asymptotic values, but the inclusion map $j : \mathbb{D} \hookrightarrow \mathbb{P}^1$ satisfies $\mathrm{A}(j) = S^1$. Neither are finite type maps: the first one because the range $\mathbb{D}$ is not compact, and the second one because $S(f)$ is not finite.*

On the other hand, the map $\exp : \mathbb{C} \to \mathbb{P}^1$ is a finite type map with $S(f) = \{0, \infty\} = \mathrm{A}(f)$, and $\mathrm{CV}(f) = \emptyset$. Rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ are also finite type maps, with $\mathrm{A}(f) = \emptyset$ and $S(f) = \mathrm{CV}(f)$.

PROPOSITION 1.1. *Let $f : W \to X$ be a finite type map. Then $S(f) = \mathrm{A}(f) \cup \mathrm{CV}(f)$.*

PROOF. Let $v \in S(f)$, and let $D = D(v, r)$ be a disk centered at v with $r > 0$ small enough that $D \cap S(f) = \{v\}$. Let $(U_i)_{i \in I}$ be the family of connected component of $f^{-1}(D)$, and let $U_i^* := U_i \setminus f^{-1}(\{v\})$, and $D^* := D \setminus \{v\}$. Then for every $i \in I$, $f : U_i^* \to D^*$ is a covering map. Since $v \in S(f)$, there is at least one $i_0 \in I$ such that $f : U_{i_0} \to D$ is not an isomorphism. Therefore, either $f : U_{i_0}^* \to D^*$ is a degree $d \geq 2$ covering map, or it is a universal cover. In the first case, v is a critical value; in the second, v is an asymptotic value. Indeed, there exists $r > 0$ and a conformal isomorphism $\phi : \mathbb{H} \to U_{i_0}$ such that $f(z) = v + re^{i\phi^{-1}(z)}$ for $z \in U_{i_0}$; then one can take $\gamma(t) := \phi(it)$. $\square$

REMARK 1.2. *In fact, the following more general statement holds: for every surjective holomorphic map $f : W \to X$, where W is a complex 1-manifold and X is a Riemann surface, $\mathrm{A}(f) \cup \mathrm{CV}(f)$ is dense in $S(f)$.*

DEFINITION 1.4 (Tract). *Let $f : W \to X$ be a holomorphic map, and let $v \in \mathrm{A}(f)$. Let D be a simply connected domain of X containing v. We say that an open subset U of W is a logarithmic tract above D if $f : U \to D \setminus \{v\}$ is a universal cover.*

The proof of Proposition 1.1 shows that finite type maps always have logarithmic tracts above their asymptotic values. Since we will only consider logarithmic tracts, from now on we will just use the term "tract".

LEMMA 1.1. *Let $f : W \to X$ be a finite type map, and let $x \in \partial W$. There exists $y \in X \setminus S(f)$ and a sequence $w_k \to x$ in W such that $f(w_k) \to y$.*

PROOF. Let $x_k \to x$ by any sequence converging to x in W. By compactness of X, up to extracting a subsequence, we may assume that $f(x_k) \to y_0 \in X$. If $y_0 \notin S(f)$, we are done.

Otherwise, let D be a Jordan domain containing y_0, small enough that $\overline{D} \cap S(f) = \{y_0\}$, and let $\epsilon > 0$. For all $k \in \mathbb{N}$ large enough, $f(x_k) \in D$, and we let U_k denote the connected component of $f^{-1}(D)$ which contains x_k. Since $x_k \to x \in \partial W$ and $U_k \subset W$, for all k large enough, we have $\mathbb{D}(x, \epsilon) \cap \partial U_k \neq \emptyset$. If $f : U_k \to D$ is a branched cover (which is the case if $y_0 \notin \mathrm{A}(f)$), then f maps ∂U_k to ∂D. If we choose $w_k \in \partial U_k \cap \mathbb{D}(x, \epsilon)$, then $f(w_k) \in \partial D$, and by choosing smaller and smaller values of ϵ, we construct a sequence $w_k \to x$ such that for all $k \in \mathbb{N}$, $f(w_k) \in \partial D$. Up to extracting a subsequence, we have $f(w_k) \to y \in \partial D$ and $y \notin S(f)$ by the choice of D.

It finally remains to treat the case where $y_0 \in \mathrm{A}(f)$ and for all $k \geq 0$ large enough, U_k is a tract above D. This case is a little bit more delicate, because then we have $\partial U_k \cap \partial W \neq \emptyset$, so it is not true anymore that f maps ∂U_k to ∂D. However, by [[**Eps93**], Lemma 58 p. 85], it holds that $f^{-1}(\partial D)$ is dense in ∂U_k, so that we may still pick $w_k \in \mathbb{D}(x, \epsilon) \cap \partial U_k$, and then argue as above. $\square$

PROPOSITION 1.2 (Island property; [**Eps93**], Proposition 9 p. 88). *Let $f : W \to X$ be a finite type map, and let $z \in \partial W$ and U a neighborhood of z in X. Let $D \subset X$ be a Jordan domain whose closure does not intersect $S(f)$. Then there exists a domain $\Omega \Subset U \cap W$ such that $f : \Omega \to D$ is a conformal isomorphism.*

PROOF. By definition, $f : W^* \to X^*$ is a covering map. If X^* is not hyperbolic, then we are in one of the following cases:

(1) either X is the Riemann sphere and $\operatorname{card} S(f) \leq 2$
(2) or X is a complex torus and $S(f) = \emptyset$.

In case (1), W^* must be isomorphic to either $\mathbb{C}$ or $\mathbb{C}^*$. Then, there exists m_1, m_2 two automorphisms of $\mathbb{P}^1$ such that $m_1 \circ f \circ m_2^{-1}$ is either $\exp$ or $z \mapsto z^d$ (with $d \geq 1$), and the lemma is either obvious (in the case of $\exp$) or vacuously true.

In case (2), we must have $W^* = X = X^*$, and f is an endomorphism of the complex torus. Then $\partial W = \emptyset$ and the lemma is also vacuously true.

We therefore assume from now on that both X^* and W^* are hyperbolic (that is, every connected component of W^* is hyperbolic). Let $x \in \partial W \cap U$ and let $w_k \to x$ in W given by Lemma 1.1, with $y = \lim_{k \to +\infty} f(w_k) \in X^*$. Up to replacing D by a larger simply connected domain (also chosen to be relatively compact in X^*), we may assume without loss of generality that $y \in D$.

Let $r > 0$ denote the hyperbolic diameter of D in X^* (which is finite by assumption). Let $(U_i)_{i \in I}$ be the connected components of $f^{-1}(D)$. Since $f : W^* \to D^*$ is a covering, for every $i \in I$ we have that $f : U_i \to D$ is a conformal isomorphism and $\mathrm{diam}_{W^*}(U_i) = r$. Moreover, by definition of y, for every k large enough there exists $i_k \in I$ such that $w_k \in U_{i_k}$. Then, since $w_k \to x$ and $\mathrm{diam}_{W^*}(U_{i_k}) = r$, $U_{i_k} \Subset U \cap W$ for all k large enough.

□

1.2. Basic Fatou/Julia theory.

DEFINITION 1.5. *Let $f : W \to X$ be a finite type map. The Fatou set $F(f)$ of f is defined as the union of all open subsets $U \subset W$ such that*

(1) either there exists $n \in \mathbb{N}^*$ such that $f^n(U) \cap W = \emptyset$

(2) or $f^n(U) \subset W$ for all $n \in \mathbb{N}$, and $\{f^n_{|U} : U \to X : n \in \mathbb{N}\}$ is normal.

The Julia set is $J(f) := X \setminus F(f)$.

Observe that by this definition, we have $\partial W \subset J(f)$, where ∂W denotes the boundary of W as a subset of X.

LEMMA 1.2. *Let $W_\infty := \mathrm{int} \bigcap_{n \geq 0} f^{-n}(W)$. Assume that either W_∞ is empty, or that all its connected components are hyperbolic. Then $J(f) = \overline{\bigcup_{n \geq 0} f^{-n}(\partial W)}$.*

PROOF. The inclusion $\overline{\bigcup_{n \geq 0} f^{-n}(\partial W)} \subset J(f)$ is always true by definition. Conversely, if $W_\infty \neq \emptyset$, W_∞ is completely invariant, and $f : W_\infty \to W_\infty$ is non-increasing for the hyperbolic metric. Therefore $W_\infty \subset F(f)$.

This means that for any open set U intersecting $J(f)$, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap (X \setminus W) \neq \emptyset$; and moreover, we must have $f^n(U) \cap \partial W \neq \emptyset$, for otherwise by definition we would have $U \subset F(f)$. In other words, $\bigcup_{n \geq 0} f^{-n}(\partial W)$ is indeed dense in $J(f)$. □

LEMMA 1.3 ([**Eps93**], Lemma 68 p. 99). *Let $f : U \to V$ be a polynomial-like map with exactly one critical value v, and assume that $v \notin \overline{U}$. Then f has a repelling fixed point.*

PROOF. Let γ be a simple curve joining v to ∂V, while avoiding $\overline{U}$. Then the slit region $V^- := V \setminus \gamma$ is simply connected; let U^- be a simply connected preimage of V^-. Then $f : U^- \to V^-$ is a conformal isomorphism, and $U^- \Subset V^-$. Therefore f has a repelling periodic point inside U^-. □

REMARK 1.3. *Using Douady-Hubbard's straightening theorem, it is not difficult to see that the assumption on the critical values is unnecessary. However, the proof given above is elementary and the statement of Lemma 1.3 is sufficient for our purposes.*

Here is the first fundamental result we prove for finite type maps:

THEOREM 1.1 (Epstein, [**Eps93**] p. 100). *Let $f : W \to X$ be a finite type map which is not an automorphism of X. Then repelling cycles are dense in $J(f)$.*

PROOF. If W_∞ is non-hyperbolic, then W_∞ is isomorphic to a complex torus, $\mathbb{P}^1$, $\mathbb{C}^*$ or $\mathbb{C}$. Then f is either an endomorphism of a complex torus (of degree $d \geq 2$), a rational map (of degree $d \geq 2$), a transcendental self-map of $\mathbb{C}^*$ or a transcendental entire map. In all these cases, Theorem 1.1 is classical and we will not prove it here. We therefore only deal with the case where Lemma 1.2 applies.

Let $z \in J(f)$. Since $J(f)$ is perfect, we may assume without loss of generality that $z \notin S(f)$; and by Lemma 1.2, we may also assume that $f^n(z) \in \partial W$ for some $n \in \mathbb{N}$. Let D be a some small disk centered at z, small enough that $f^n : D \to f^n(D)$ is a branched cover, ramified only possibly at $f^n(z)$. Let $\Omega \Subset f^n(D)$ be the simply connected domain given by the Island Property (Proposition 1.2): then $f^{n+1} : \Omega \to f^n(D)$ is a branched cover with at most one critical value $f^n(z)$, which lies outside of $\overline{\Omega}$. Therefore, by Lemma 1.3, f^{n+1} has a repelling fixed point in Ω, which means that D contains a repelling cycle. □

REMARK 1.4. *We remark here that the proof above shows that in non-exceptionnal cases, f admits infinitely many repelling cycles of every period $n \geq 2$. We will implicitly use this fact later on, by choosing a repelling 3-cycle for an arbitrary finite type map.*

We now turn to the classification of periodic Fatou components. A priori, there could be new types of non-compactly contained in W periodic Fatou components on which the iterates accumulate the boundary of W; the following theorem rules out this possibility.

THEOREM 1.2 ([**Eps93**], Proposition 15 p. 105). *Let $f : W \to X$ be a finite type map, and let U be a fixed Fatou component such that $f^n_{|U} \to \partial U$. Then U is a parabolic basin.*

We start with the following lemma:

LEMMA 1.4 ([**Eps93**], Lemma 28 p. 49). *Let $f : W \to X$ be a finite type map, and let U be a fixed Fatou component such that $f^n_{|U} \to \partial U$ and $f^n_{|U}$ has at least a limit point in W. Then U is a parabolic basin.*

PROOF OF LEMMA 1.4. By assumption, there exists $a_0 \in U$ and a sequence $m_k \to +\infty$ with $f^{m_k}(a_0) \to x_0 \in \partial U \cap W$. Let $\gamma : [0,1] \to U$ be a C^1 curve such that $f(\gamma(0)) = \gamma(1)$ and $\gamma(0) = a_0$. We will still denote by $\gamma : \mathbb{R}^+ \to U$ its unique extension to $\mathbb{R}^+$ such that $f \circ \gamma(t) = \gamma(t+1)$ for all $t \geq 0$. Let

$$L := \{x \in X : \gamma(t_k) \to x \text{ for some sequence } t_k \to +\infty\}.$$

If $\operatorname{card} L > 1$ then L has no isolated points. Let $x \in L$ and $t_k \to +\infty$ such that $\gamma(t_k) \to x$. Let $n_k + s_k = t_k$, where $n_k \in \mathbb{N}$ and $s_k \in [0,1)$, and let $w_k := \gamma(s_k)$: then $f^{n_k}(w_k) \to x$. Up to extracting a subsequence, we may assume that $w_k \to a \in \gamma([0,1])$ and that $f^{n_k}_{|U} \to h$. Then $h(a) = x \in \partial U$, so h is constant and $h \equiv a$. In particular, $\lim_{k\to+\infty} f^{n_k}(\gamma(0)) = x = \lim_{k\to+\infty} f^{n_k}(\gamma(1))$, so if $x \in W$ then $f(x) = x$. In particular, $W \cap L$ consists of isolated points.

Since by assumption $x_0 \in W \cap L$, we conclude that $L = \{x_0\}$. Then by the Snail Lemma (see [**Mil06**], Lemma 16.2), U is a parabolic basin. □

We can now turn to the proof of Theorem 1.2.

PROOF OF THEOREM 1.2. Let U be a fixed Fatou component of f such that $f^n_{|U} \to \partial U$, and let $\gamma : \mathbb{R}^+ \to U$ be an invariant piecewise C^1 curve. By Lemma 1.4, it is enough to prove that γ accumulates on at least one point in $\partial U \cap W$.

For any hyperbolic domain $V \subset X$, we will denote by $\ell_V(n)$ the hyperbolic length in V of the segment $\gamma([n, n+1])$. We will let $V^* := V \setminus S(f)$ and $V^\times := V \setminus f^{-1}(S(f))$, and we will denote by ρ_V the density of the hyperbolic metric on V.

As before, if W_∞ is non-hyperbolic, then f is a rational map, a transcendental entire map, a transcendental self-map of $\mathbb{C}^*$ or an affine toral endomorphism (this last case is in fact impossible). In the rational case, $\partial W = \emptyset$ and so we are done; and in the two remaining cases, Theorem 1.2 is proved in [**EL92**]. Therefore, we will assume from now on that W_∞ is hyperbolic, in particular, U is also hyperbolic. Similarly, we have seen that if X^* is non-hyperbolic, then f is holomorphically conjugated to either $z^{\pm d}$, an affine toral endomorphism or a meromorphic map with exactly two asymptotic values; therefore, we may assume without loss of generality that X^* is hyperbolic as well.

We have $0 < \ell_{X^*}(n) \leq \ell_{U^*}(n)$ and $\ell_U(n) \geq \ell_U(n+1)$. For all $t > 0$ large enough, $\gamma(t)$ stays away from $S(f) \cap U$, so $\rho_{U^*}(\gamma(t)) \sim_{t \to +\infty} \rho_U(\gamma(t))$; let $n_0 \in \mathbb{N}$ be large enough that for all $n \geq n_0$, $\ell_{U^*}(n) \leq 2\ell_U(n)$. Then, for all $n \geq n_0$:

$$\ell_{X^*}(n) \leq \ell_{U^*}(n) \leq 2\ell_U(n) \leq 2\ell_U(n_0).$$

In particular, $(\ell_{X^*}(n))_{n \in \mathbb{N}}$ is bounded.

Let L denote the accumulation set of γ, and assume for a contradiction that $L \cap W = \emptyset$. As before, L is either a point or a continuum in ∂W; assume first that L is not a point. First, by [[**Eps93**], Proposition 1 p. 7], we have $\rho_{W^\times}(\gamma(t)) \sim_{t \to +\infty} \rho_{X^*}(\gamma(t))$. Next, since γ accumulates on a continuum component of ∂W, we have

$$\frac{\rho_{W^\times}(\gamma(t))}{\rho_{X^*}(\gamma(t))} \to +\infty$$

(see [**Eps93**], Corollary 1 p.7). Therefore, for all n large enough:

$$\ell_{X^*}(n+1) = \ell_{W^\times}(n) \geq \ell_{X^*}(n)$$

which contradicts the boundedness of $\ell_{X^*}(n)$.

It finally remains to treat the case where $\gamma(t) \to x \in \partial W$. Then $f \circ \gamma(t) = \gamma(t+1) \to x$ as well, therefore x is an asymptotic value. Let D be a small disk around x; then for all $t > 0$ large enough, $\gamma(t)$ lies in a tract T above $D^* = D \setminus \{x\}$. Then

$$\ell_{D^*}(n+1) = \ell_T(n) \geq 2\ell_{D^*}(n),$$

and since $\rho_{X^*}(\gamma(t)) \sim \rho_{D^*}(\gamma(t))$ as $t \to +\infty$, we again have $\ell_{X^*}(n) \to +\infty$, a contradiction.

Therefore we have proved that L contains at least one point in W, and therefore we are done by Lemma 1.4. $\square$

Finally, a consequence of Theorem 1.2 is that the usual Fatou-Sullivan classification of fixed Fatou components also applies to finite type maps. We will not give the proof here, as it is the same as in the rational case; the key point is the absence of so-called Baker or exotic domains, i.e. periodic Fatou components U such that $f^{np}_{|U}$ converges to the boundary of W. Note that Baker domains may in fact occur for entire maps that are not of finite type: for instance, Fatou gave the example of the map $f(z) = z + 1 + e^{-z}$, which has a Baker domain containing a right half-plane.

COROLLARY 1.1. *Let $f : W \to X$ be a finite type map, and let U be a fixed Fatou component. Then U is one of the 5 standard types:*

(1) attracting basin
(2) super-attracting basin
(3) Siegel disk
(4) Herman ring
(5) parabolic basin.

2. Background on Teichmüller spaces

From now on, for the sake of simplicity, we will restrict ourselves to the case of finite type maps $f : W \to \mathbb{P}^1$. We will assume that the reader is already familiar with quasiconformal analysis and Teichmüller spaces, and in particular with objects such as Beltrami coefficients, quasiconformal vector fields, and quadratic differentials. We start by recalling the definitions of these objects and a few key properties, without proofs. To keep the presentation elementary, we have often opted for definitions using coordinates; more formal, intrinsic definitions are also possible.

2.1. Beltrami coefficients, isotopies and Teichmüller space.

DEFINITION 2.1. *We let $\mathrm{Bel}(\mathbb{P}^1)$ denote the space of Beltrami forms on $\mathbb{P}^1$, that is: $\mu \in \mathrm{Bel}(\mathbb{P}^1)$ if μ is written in coordinates as $\mu(z)\frac{d\bar{z}}{dz}$, where $\mu \in L^\infty(\mathbb{P}^1)$ and $\|\mu\|_\infty < 1$. We let $\mathrm{bel}(\mathbb{P}^1)$ denote the space of Beltrami* differentials*, obtained by replacing the condition $\|\mu\|_\infty < 1$ by the condition $\|\mu\|_\infty < \infty$.*

REMARK 2.1. *If $\mu \in \mathrm{bel}(\mathbb{P}^1)$, then $|\mu(z)|$ is a well-defined function in $L^\infty(\mathbb{P}^1, \mathbb{R})$, independent from the choice of coordinates. Therefore, the definition above makes sense.*

We think of $\mathrm{bel}(\mathbb{P}^1)$ as the tangent space to $\mathrm{Bel}(\mathbb{P}^1)$, which is itself the unit ball of an infinite-dimensional complex Banach space.

Recall that by the Measurable Riemann Mapping Theorem, for every $\mu \in \mathrm{Bel}(\mathbb{P}^1)$, there exists a quasiconformal homeomorphism $h_\mu : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\frac{\bar\partial h_\mu}{\partial h_\mu} = \mu$. We say that h_μ integrates the Beltrami form μ. Moreover, h_μ is unique up to post-composition by an automorphism of $\mathbb{P}^1$. If $Z \subset \mathbb{P}^1$ is a set of cardinal 3, we will denote by h_μ^Z the unique quasiconformal homeomorphism $h_\mu^Z : \mathbb{P}^1 \to \mathbb{P}^1$ which fixes Z pointwise and such that $\frac{\bar\partial h_\mu^Z}{\partial h_\mu^Z} = \mu$. Recall as well that for every $z \in \mathbb{P}^1$, the map $\mu \mapsto h_\mu^Z(z)$ is holomorphic.

DEFINITION 2.2. *Let $A \subset \mathbb{P}^1$ be a closed set with* $\mathrm{card}\, A \geq 3$. *We say that a quasiconformal homeomorphism $h : \mathbb{P}^1 \to \mathbb{P}^1$ is uniformly isotopic to the identity relative to A if there exists a continuous map $H : [0,1] \times \mathbb{P}^1 \to \mathbb{P}^1$ and $0 < k < 1$ such that*

(1) $H(0, \cdot) = \mathrm{id}_{\mathbb{P}^1}$ and $H(1, \cdot) = h$
(2) for all $t \in [0,1]$, $h_t := H(t, \cdot)$ is a quasiconformal homeomorphism of dilatation less than k, which fixes A pointwise.

DEFINITION 2.3. *Let $A \subset \mathbb{P}^1$ with* $\mathrm{card}\, A \geq 3$. *We say that a quasiconformal homeomorphism $h : \mathbb{P}^1 \to \mathbb{P}^1$ is the identity relative to the ideal boundary of A if there exists a lift $\tilde{f} : \mathbb{H} \to \mathbb{H}$ of f by a universal cover $\pi : \mathbb{H} \to \mathbb{P}^1 \setminus A$ which extends continuously to $\mathbb{R}$ by the identity.*

THEOREM 2.1 ([**EM88**]). *Let $A \subset \mathbb{P}^1$ with $\operatorname{card} A \geq 3$, and let $h : \mathbb{P}^1 \to \mathbb{P}^1$ be a quasiconformal homeomorphism. The following properties are equivalent:*

(1) h is uniformly quasiconformally isotopic to the identity relative to A
(2) h is isotopic to the identity relative to the ideal boundary of A
(3) h is isotopic to the identity through an isotopy fixing pointwise $\mathbb{P}^1 \setminus A$.

We let $\mathrm{QC}_0(A)$ denote the group of quasiconformal homeomorphisms which satisfy one of the equivalent properties above. It acts on $\mathrm{Bel}(\mathbb{P}^1)$ by pullback, in the following sense: if $\mu \in \mathrm{Bel}(\mathbb{P}^1)$ and $h : \mathbb{P}^1 \to \mathbb{P}^1$ is a quasiconformal homeomorphism, then $h^*\mu$ is defined as the Beltrami form associated to $h_\mu \circ h$, where h_μ integrates μ.

Recall the definition of the Teichmüller space of a finite type Riemann surface:

DEFINITION 2.4. *Let $A \subset \mathbb{P}^1$ be a finite set with $\operatorname{card} A \geq 3$. The Teichmüller space of $\mathbb{P}^1$ with marked set A is:*

$$\mathrm{Teich}(\mathbb{P}^1, A) := \mathrm{Bel}(\mathbb{P}^1)/\mathrm{QC}_0(\mathbb{P}^1, A).$$

THEOREM 2.2. *There is a complex structure on $\mathrm{Teich}(\mathbb{P}^1, A)$, for which the quotient map $\pi : \mathrm{Bel}(\mathbb{P}^1) \to \mathrm{Teich}(\mathbb{P}^1, A)$ admits local holomorphic sections. Equipped with this complex structure, $\mathrm{Teich}(\mathbb{P}^1, A)$ is a complex manifold of dimension $\operatorname{card} A - 3$.*

2.2. Quasiconformal vector fields. We refer the reader to [**GL00**] for more background on quasiconformal vector fields.

DEFINITION 2.5. *Let ξ be a continuous vector field on $\mathbb{P}^1$. We say that ξ is a quasiconformal vector field if $\overline{\partial}\xi \in \mathrm{Bel}(\mathbb{P}^1)$ (in the sense of distributions, in local coordinates): $\xi = h(z)\frac{d}{dz}$, where $\overline{\partial} h \in L^\infty$.*

Every Beltrami differential μ on $\mathbb{P}^1$ may be written $\mu = \overline{\partial}\xi$ for some quasiconformal vector field ξ, which is unique up to adding a holomorphic vector field.

It can be proved that quasiconformal vector fields are α-Hölder for every $0 < \alpha < 1$, but not Lipschitz in general. In fact, their modulus of continuity is controlled by $C\epsilon \ln \epsilon^{-1}$.

We will make frequent use of the following fact:

LEMMA 2.1. *Let $\lambda \mapsto \mu_\lambda$ be a holomorphic map from $\mathbb{D}$ to $\mathrm{Bel}(\mathbb{P}^1)$ with $\mu_0 = 0$, and let $Z \subset \mathbb{P}^1$ be a set of cardinal 3. Then $\xi := \frac{d}{d\lambda}_{|\lambda=0} h^Z_{\mu_\lambda}$ is a quasiconformal vector field, and*

$$\frac{d}{d\lambda}_{|\lambda=\lambda_0} \mu_\lambda = \overline{\partial}\xi.$$

The tangent space $T_\tau \mathrm{Teich}(\mathbb{P}^1, A)$ (for $\tau \in \mathrm{Teich}(\mathbb{P}^1, A)$) may be canonically identified with $\mathrm{bel}(\mathbb{P}^1)/\{\overline{\partial}\xi : \xi$ is a qc vector field and $\xi_{|A} = 0\}$, in the sense that

$$\ker d\pi_A = \{\overline{\partial}\xi : \xi \text{ is a qc vector field and } \xi_{|A} = 0\},$$

where $\pi : \mathrm{Bel}(\mathbb{P}^1) \to \mathrm{Teich}(\mathbb{P}^1, A)$ is the quotient map.

Finally, we will also use:

THEOREM 2.3 ([**Ast17**], Theorem A). *Let Ω be a hyperbolic open subset of $\mathbb{P}^1$ and ξ be a quasiconformal vector field on Ω. We denote by $\rho_\Omega(\xi)$ the hyperbolic length of the vector field ξ. The following properties are equivalent:*

i) We have $\rho_\Omega(\xi) \in L^\infty(\Omega)$.
ii) We have $\|\rho_\Omega(\xi)\|_{L^\infty(\Omega)} \leq 4\|\overline{\partial}\xi\|_{L^\infty(\Omega)}$.

iii) *There exists a quasiconformal extension $\hat{\xi}$ of ξ on all of $\mathbb{P}^1$ with $\hat{\xi}=0$ on $\partial\Omega$.*
iv) *The extension $\hat{\xi}$ defined by $\hat{\xi}(z)=\xi(z)$ if $z\in\Omega$ and 0 else is quasiconformal on $\mathbb{P}^1$, and $\overline{\partial}\hat{\xi}(z)=0$ almost everywhere in the complement of Ω.*

2.3. Quadratic differentials.

DEFINITION 2.6. *A meromorphic quadratic differential on $\mathbb{P}^1$ is a meromorphic section of $T^*\mathbb{P}^1\otimes T^*\mathbb{P}^1$. If $A\subset\mathbb{P}^1$, we let $\mathcal{Q}(A)$ denote the vector space of meromorphic quadratic differentials with at worst simple poles, all of which are in A. We endow $\mathcal{Q}(A)$ with the L^1 norm $\|q\|_1:=\int_{\mathbb{P}^1}|q|$.*

REMARK 2.2. *The condition that $q\in\mathcal{Q}(A)$ has at most simple poles ensures that $\|q\|_1<\infty$. Moreover, $\|q\|_1$ is independant from the choice of coordinates.*

The cotangent space $T^*_\tau\mathrm{Teich}(\mathbb{P}^1,A)$ may be identified with $\mathcal{Q}(A)$. The pairing between $T^*_\tau\mathrm{Teich}(\mathbb{P}^1,A)$ and $T_\tau\mathrm{Teich}(\mathbb{P}^1,A)$ is given by

$$\langle q,[\mu]\rangle=\int_{\mathbb{P}^1}q\cdot\mu=\int_{\mathbb{P}^1}q(z)\mu(z)dzd\overline{z}=2i\pi\sum_{z_i\in A}\mathrm{Res}(q_i\cdot\xi,z_i),$$

where $\mu=\overline{\partial}\xi$ is any representative of $[\mu]$ in

$$\mathrm{bel}(\mathbb{P}^1)/\{\overline{\partial}\xi:\xi\text{ is a qc vector field and }\xi_{|A}=0\}.$$

DEFINITION 2.7. *Let A,B be finite subsets of $\mathbb{P}^1$ with $A\subset B$. We define $\varpi:\mathrm{Teich}(\mathbb{P}^1,B)\to\mathrm{Teich}(\mathbb{P}^1,A)$ as the "forgetful map" induced by the inclusion $\mathrm{QC}_0(B)\hookrightarrow\mathrm{QC}_0(A)$.*

LEMMA 2.2. *The map $\varpi:\mathrm{Teich}(\mathbb{P}^1,B)\to\mathrm{Teich}(\mathbb{P}^1,A)$ is an analytic submersion. Its codifferential $d^*\varpi_{[0]}$ at the basepoint $[0]$ is the inclusion map $\mathcal{Q}(A)\hookrightarrow\mathcal{Q}(B)$.*

PROOF. The analyticity of ϖ follows from the existence of local sections of the quotient map $\pi_B:\mathrm{Bel}(\mathbb{P}^1)\to\mathrm{Teich}(\mathbb{P}^1,B)$, and from the commutative diagram

$$\begin{tikzcd}\mathrm{Bel}(\mathbb{P}^1) \arrow[r, "\mathrm{id}"] \arrow[d, "\pi_B"'] & \mathrm{Bel}(\mathbb{P}^1) \arrow[d, "\pi_A"] \\ \mathrm{Teich}(\mathbb{P}^1,B) \arrow[r, "\varpi"'] & \mathrm{Teich}(\mathbb{P}^1,A)\end{tikzcd}$$

The differential $d\varpi_{[0]}:T_{[0]}\mathrm{Teich}(\mathbb{P}^1,B)\to\mathrm{Teich}(\mathbb{P}^1,A)$ is given by the natural linear map

$$\mathrm{Bel}(\mathbb{P}^1)/\ker d\pi_B\to\mathrm{Bel}(\mathbb{P}^1)/\ker d\pi_A.$$

Recall that $\ker d\pi_B=\{\overline{\partial}\xi:\xi_{|B}=0\}$. Therefore, $d\varpi_{[0]}$ is surjective, so ϖ is a submersion. Finally, by duality, $d^*\varpi_{[0]}$ is the inclusion map $\mathcal{Q}(A)\hookrightarrow\mathcal{Q}(B)$. $\square$

DEFINITION 2.8. *Let $f:W\to\mathbb{P}^1$ be a finite type map, and let $A\subset\mathbb{P}^1$ be a finite set of cardinal at least 3. Let $q\in\mathcal{Q}(A)$. The pushforward f_*q is defined by:*

$$f_*q(z)=\sum_{f(x_i)=z}\frac{q(x_i)}{f'(x_i)^2}$$

for every $z\in\mathbb{P}^1\setminus(f(A)\cup S(f))$.

LEMMA 2.3. *Let $f:W\to\mathbb{P}^1$ be a finite type map, and let $A\subset\mathbb{P}^1$ be a finite set of cardinal at least 3. Let $q\in\mathcal{Q}(A)$. Then $f_*q\in\mathcal{Q}(f(A)\cup S(f))$, and $\|f_*q\|_1\leq\|q\|_1$.*

PROOF. It is clear from the definition that f_*q is holomorphic on $\mathbb{P}^1 \setminus (f(A) \cup S(f))$. Let $U \subset \mathbb{P}^1$ be a dense simply connected domain of full measure, contained in $\mathbb{P}^1 \setminus (f(A) \cup S(f))$ (for instance, one can choose $v_0 \in S(f)$ and a collection Γ of pairwise disjoint simple curves joining v_0 to each other element of $f(A) \cup S(f)$, and then take $U := \mathbb{P}^1 \setminus \Gamma$). Let $g_i : U \to U_i$ denote the set of univalent inverse branches of f defined on U: then

$$f_*q(z) = \sum_{i \in I} q \circ g_i(z) g_i'(z)^2 = \sum_{i \in I} g_i^* q(z),$$

for all $z \in U$. In particular:

$$\int_{\mathbb{P}^1} |f_*q| = \int_U |f_*q| = \int_U \left| \sum_{i \in I} g_i^* q(z) \right| \leq \sum_{i \in I} \int_U |g_i^* q| \leq \sum_{i \in I} \int_{U_i} |q| \leq \|q\|_1 \tag{1}$$

which proves that $\|f_*q\|_1 \leq \|q\|_1 < +\infty$. Finally, a holomorphic function is integrable in a punctured neighborhood of an isolated singularity if and only if that singularity is a simple pole; therefore f_*q has at worst simple poles. □

DEFINITION 2.9. *Let $A \subset \mathbb{P}^1$ be a finite set, and let $\mathcal{Q}(A)$ denote the vector space of integrable quadratic differentials on $\mathbb{P}^1$ with at worst simple poles, all of which are located in A. Let $\nabla_f : \mathcal{Q}(A) \to \mathcal{Q}(A \cup S(f))$ be defined by $\nabla_f q = q - f_*q$.*

LEMMA 2.4. *If f is not a flexible Lattès map nor an automorphism of $\mathbb{P}^1$, then ∇_f is injective on $\mathcal{Q}(A)$ and in fact, $\|f_*q\|_1 < \|q\|_1$.*

PROOF. Lemma 2.4 is well-known in the case of a rational map (see e.g. [**McM94**]). Therefore, we will only deal with the case where f has infinite degree. If $q = f_*q$ for some non-zero $q \in \mathcal{Q}(A)$, then $\|f_*q\|_1 = \|q\|_1$, so the chain of inequalities in (1) are equalities. By the equality case of the triangular inequality, this means that for all $i \in I$, there exists a function $\alpha_i : U \to \mathbb{R}_+$ such that $g_i^* q = \alpha_i f_*q$. Moreover, since both $g_i^* q$ and f_*q are holomorphic on U, the function α_i is meromorphic on U, hence constant; and since $\sum_{i \in I} g_i^* q = f_*q$, we have $\sum_{i \in I} \alpha_i = 1$. In particular, there exists a sequence i_k such that $\alpha_{i_k} \to 0$.

Finally, observe that for every $i \in I$,

$$\alpha_i f^* f_* q = f^*(\alpha_i f_* q) = f^* g_i^* q = q.$$

But then, with $i = i_k$ and $k \to +\infty$, we find $q = 0$, a contradiction. □

REMARK 2.3. *If f is a rational map of degree $d \geq 2$, then a similar argument proves that $q = f_*q$ implies that $f^* f_* q = dq$ (see [**BE09**]), and with additional work, one can prove that this happens only for flexible Lattès maps.*

2.4. Holomorphic motions of subsets of the Riemann sphere.

DEFINITION 2.10. *Let $E \subset \mathbb{P}^1$ and let $h : M \times E \to \mathbb{P}^1$, where M is a connected complex manifold. We say that h is a holomorphic motion of the set E over M if*

(1) there exists $\lambda_0 \in M$ such that $h(\lambda_0, \cdot)$ is the injection $i : E \hookrightarrow \mathbb{P}^1$
(2) for every $\lambda \in M$, $h(\lambda, \cdot)$ is injective
(3) for every $x \in E$, $h(\cdot, x)$ is holomorphic on M.

We will also commonly use the notation $h_\lambda := h(\lambda, \cdot)$. The two classical results below will be needed, to extend holomorphic motions:

THEOREM 2.4 (Mañé-Sad-Sullivan's λ-Lemma, [**MSS83**]). *Let $E \subset \mathbb{P}^1$ and $h : M \times E \to \mathbb{P}^1$ be a holomorphic motion of E over M. Then h extends uniquely to a holomorphic motion $\tilde{h}$ of $\overline{E}$, the closure of E. This extension is continuous on $M \times \overline{E}$.*

THEOREM 2.5 (Slodkowski's λ-lemma, [**Slo91**]). *Let $E \subset \mathbb{P}^1$ and $h : \mathbb{D} \times E \to \mathbb{P}^1$ be a holomorphic motion of E over the unit disk $\mathbb{D}$. Then h extends to a holomorphic motion $\tilde{h} : \mathbb{D} \times \mathbb{P}^1 \to \mathbb{P}^1$ which is continuous on $\mathbb{D} \times \mathbb{P}^1$, and such that for all $\lambda \in \mathbb{D}$, $\tilde{h}_\lambda = \tilde{h}(\lambda, \cdot) : \mathbb{P}^1 \to \mathbb{P}^1$ is a quasiconformal homeomorphism.*

We note that the extension given by Slodkowski's theorem is not unique; additionally, it requires that the holomorphic motion is parametrized by a one-dimensional disk. The next and last result that we will need about holomorphic motions has none of these drawbacks:

DEFINITION 2.11. *Let $E \subset \mathbb{P}^1$ be a closed set with $\operatorname{card} E > 2$, and let $\mu \in \operatorname{Bel}(\mathbb{P}^1)$. We will say that μ is harmonic on $\mathbb{P}^1 \setminus E$ if locally near every $z \in \mathbb{P}^1 \setminus E$, μ is of the form*

$$\mu = \frac{\overline{q}}{\rho^2} = \frac{\overline{q(z)}}{\rho(z)^2} \frac{d\overline{z}}{dz}$$

where q is a holomorphic quadratic differential, and $\rho(z)^2 dz d\overline{z}$ is the area element of the hyperbolic metric on $\mathbb{P}^1 \setminus E$.

THEOREM 2.6 (Bers-Royden's Harmonic λ-lemma, [**BR86**]). *Let $E \subset \mathbb{P}^1$ and $h : B \times E \to \mathbb{P}^1$ be a holomorphic motion of E over the open unit ball B of a complex Banach space, where E is closed set with $\operatorname{card} E > 2$. Let $B' := \frac{1}{3}B$. Then there is a* unique *holomorphic motion $\tilde{h} : B' \times \mathbb{P}^1 \to \mathbb{P}^1$ which coincides with h on $B' \times E$ and such that for all $\lambda \in B'$, $\tilde{h}_\lambda : \mathbb{P}^1 \to \mathbb{P}^1$ is a quasiconformal homeomorphism whose Beltrami form is harmonic on $\mathbb{P}^1 \setminus E$.*

With a slight abuse of terminology, we will refer to $\tilde{h}$ as an extension of the holomorphic motion h to $B' \times \mathbb{P}^1$ (even though $B \times E$ is not contained in $B' \times \mathbb{P}^1$).

REMARK 2.4. *If U, V are hyperbolic domains of $\mathbb{P}^1$, $f : U \to V$ is a holomorphic covering map and μ is a harmonic Beltrami form on V, then $f^*\mu$ is a harmonic Beltrami form on U. Indeed,*

$$f^*\mu = \frac{\overline{f^*q}}{f^*\rho_V^2} = \frac{\overline{f^*q}}{\rho_U^2}$$

since $f : U \to V$ is a local isometry for the hyperbolic metrics on U and V, and f^*q is a holomorphic quadratic differential on U.

3. Epstein's deformation space

3.1. Pullback of Beltrami forms and Thurston's pullback map. If $f : W \to \mathbb{P}^1$ is a finite type map, and $\mu \in \operatorname{bel}(\mathbb{P}^1)$ (resp. $\operatorname{Bel}(\mathbb{P}^1)$), the pullback $f^*\mu$ is by definition the Beltrami differential (resp. form) on $\mathbb{P}^1$ defined by

$$f^*\mu(z) = \begin{cases} \mu(f(z))\frac{\overline{f'(z)}}{f'(z)} & \text{if } z \in W \\ 0 & \text{else.} \end{cases}$$

Note that when W has full measure in $\mathbb{P}^1$, as is the case e.g. for rational maps or entire maps, this is the usual definition of the pullback. We denote by $\operatorname{Bel}(f)$ (resp. $\operatorname{bel}(f)$) the space of pullback-invariant Beltrami forms (resp. differentials) on $\mathbb{P}^1$.

The Beltrami form $f^*\mu$ is characterized by the following diagram:

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "h_{f^*\mu}"'] & \mathbb{P}^1 \arrow[d, "h_\mu"] \\ h_{f^*\mu}(W) \arrow[r, "f_\mu"'] & \mathbb{P}^1 \end{tikzcd}$$

where h_μ and $h_{f^*\mu}$ integrate μ and $f^*\mu$ respectively, and $f_\mu : h_{f^*\mu}(W) \to \mathbb{P}^1$ is holomorphic. More precisely, if μ and ν are Beltrami forms, then $h_\mu \circ f \circ h_\nu^{-1}$ is holomorphic on $h_\nu(W)$ if and only if $\nu = f^*\mu$ a.e. on W.

The next lemma states that the pushforward operator on quadratic differentials is dual to the pullback operator on Beltrami differentials:

LEMMA 3.1. *Let $A \subset \mathbb{P}^1$ be a finite set, $q \in \mathcal{Q}(A)$, and $\mu \in \mathrm{bel}(f)$. Let $f : W \to \mathbb{P}^1$ be a finite type map. Then*

$$\int_{\mathbb{P}^1} f_* q \cdot \mu = \int_{\mathbb{P}^1} q \cdot f^*\mu.$$

PROOF. Let $U \subset \mathbb{P}^1 \setminus S(f)$ be a simply connected domain of full Lebesgue measure. Let $(g_i)_{i \in I}$ denote the set of univalent branches of f^{-1} defined on U. Then

$$\begin{aligned} \int_{\mathbb{P}^1} f_* q \cdot \mu &= \int_U f_* q \cdot \mu \\ &= \int_U \sum_{i \in I} g_i^* q \cdot \mu \\ &= \int_U \sum_{i \in I} g_i^* q \cdot (g_i^* f^* \mu) \\ &= \sum_{i \in I} \int_U g_i^* (q \cdot f^*\mu) \\ &= \sum_{i \in I} \int_{g_i(U)} q \cdot f^*\mu \\ &= \int_{\mathbb{P}^1} q \cdot f^*\mu. \end{aligned}$$

In the last equality, we used the fact that $\bigcup_{i \in I} g_i(U)$ has full Lebesgue measure in W, and that $f^*\mu = 0$ a.e. on $\mathbb{P}^1 \setminus W$. $\square$

LEMMA 3.2. *Let $Z \subset A \subset B$ be finite subsets of $\mathbb{P}^1$. Assume that $f(A) \subset B$, $S(f) \subset B$ and $\mathrm{card}\, Z = 3$. Let $\mu \in \mathrm{Bel}(\mathbb{P}^1)$, and let ϕ, ψ denote the quasiconformal homeomorphisms integrating μ and $f^*\mu$ respectively and fixing Z pointwise. If $\phi \in \mathrm{QC}_0(B)$, then there exists quasiconformal isotopies $(\phi_t)_{t \in [0,1]}$ and $(\psi_t)_{t \in [0,1]}$ such that*

(1) $\phi_0 = \psi_0 = \mathrm{id}$ and $\phi_1 = \phi$, $\psi_1 = \psi$
(2) $\phi_{t|B} = \mathrm{id}$ and $\psi_{t|(\mathbb{P}^1 \setminus W) \cup f^{-1}(B)} = \mathrm{id}$, for all $t \in [0,1]$
(3) $\phi_t \circ f = f \circ \psi_t$ for all $t \in [0,1]$.

Moreover, if $\mu = f^*\mu$, then we can take $\psi_t = \phi_t$.

PROOF. Let $(\phi_t)_{t\in[0,1]}$ be an isotopy of ϕ to id such that $\phi_{t|B} = \mathrm{id}$ for all $t \in [0,1]$ (see Theorem 2.1).

By [[**ABF21**], Lemma 2.7], there exists an isotopy $(\tilde{\psi}_t)_{t\in[0,1]}$ such that $\tilde{\psi}_t : W \setminus f^{-1}(B) \to W \setminus f^{-1}(B)$ is a homeomorphism, $\phi_t \circ f = f \circ \tilde{\psi}_t$, and $\tilde{\psi}_0 = \mathrm{id}$. Moreover, the relation $\phi_t \circ f = f \circ \tilde{\psi}_t$ implies that $\tilde{\psi}_t : W \setminus f^{-1}(B) \to W \setminus f^{-1}(B)$ is in fact a quasiconformal homeomorphism, of same dilatation as ϕ_t. By Theorems 4.2 and 4.3 of [**MS98**], $\tilde{\psi}_t$ extends to an isotopy ψ_t of $\mathbb{P}^1$ which fixes E pointwise, where $E := \mathbb{P}^1 \setminus (W \setminus f^{-1}(B)) = (\mathbb{P}^1 \setminus W) \cup f^{-1}(B)$. We will prove that $\psi = \psi_1$.

Indeed $\psi_1 : \mathbb{P}^1 \to \mathbb{P}^1$ is a quasiconformal homeomorphism, which satisfies the following two properties:

(1) $\phi \circ f \circ \psi_1^{-1} = f$ is holomorphic on $\psi_1(W) = W$
(2) and $\overline{\partial}\psi_1 = 0$ a.e. on $\mathbb{P}^1 \setminus W$ (since $\psi_{1|\mathbb{P}^1\setminus W} = \mathrm{id}$).

Therefore, the Beltrami coefficient of ψ_1 is exactly $f^*\mu$, and we are done. □

In view of Lemma 3.2, we may define:

DEFINITION 3.1. *Let $f : W \to \mathbb{P}^1$ be a finite type map, and A, B be finite subsets of $\mathbb{P}^1$ with $A \subset B$, $f(A) \subset B$ and $S(f) \subset B$. We define the pullback map*

$$\sigma_f : \mathrm{Teich}(\mathbb{P}^1, B) \to \mathrm{Teich}(\mathbb{P}^1, A)$$

as the natural map induced by $\mu \mapsto f^*\mu$ on $\mathrm{Bel}(\mathbb{P}^1)$.

LEMMA 3.3. *The map $\sigma_f : \mathrm{Teich}(\mathbb{P}^1, B) \to \mathrm{Teich}(\mathbb{P}^1, A)$ is holomorphic, and its codifferential $d^*(\sigma_f)_{[0]}$ of the map σ_f at the basepoint $[0]$ is the pushforward operator*

$$f_* : \mathcal{Q}(A) \to \mathcal{Q}(B).$$

PROOF. The operator $f^* : \mathrm{Bel}(\mathbb{P}^1) \to \mathrm{Bel}(\mathbb{P}^1)$ is $\mathbb{C}$-linear, hence holomorphic. The analyticity of σ_f then follows from the existence of local sections of the quotient map $\pi_B : \mathrm{Bel}(\mathbb{P}^1) \to \mathrm{Teich}(\mathbb{P}^1, B)$, and from the commutative diagram

$$\begin{tikzcd} \mathrm{Bel}(\mathbb{P}^1) \arrow[r, "f^*"] \arrow[d, "\pi_B"'] & \mathrm{Bel}(\mathbb{P}^1) \arrow[d, "\pi_A"] \\ \mathrm{Teich}(\mathbb{P}^1, B) \arrow[r, "\sigma_f"'] & \mathrm{Teich}(\mathbb{P}^1, A) \end{tikzcd}$$

We have $d\sigma_{f\,[0]} \circ d\pi_{B|0} = d\pi_{A|0} \circ f^*$, which means that

$$d\sigma_{f\,[0]}([\mu]_B) = [f^*\mu]_A$$

where $[\mu]_A$ and $[\mu]_B$ denote the equivalence class of μ in $T_{[0]}\mathrm{Teich}(\mathbb{P}^1, A)$ (resp. $T_{[0]}\mathrm{Teich}(\mathbb{P}^1, B)$). □

DEFINITION 3.2 (Epstein's deformation space). *Let $A \subset B$ be two finite subsets of $\mathbb{P}^1$, such that* $\mathrm{card}\, A \geq 3$*, $A \subset B$, $f(A) \subset B$ and $S(f) \subset B$. We let*

$$\mathrm{Def}_A^B(f) := \{[\mu] \in \mathrm{Teich}(\mathbb{P}^1, B) : \varpi([\mu]) = \sigma_f([\mu])\}.$$

Let us explain this definition. Let $[\mu] \in \mathrm{Def}_A^B(f)$, and let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a quasiconformal homeomorphism integrating a representative μ of $[\mu] \in \mathrm{Teich}(\mathbb{P}^1, B)$, normalized so that it fixes three points $a_1, a_2, a_3 \in A$. Let $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ be a quasiconformal homeomorphism integrating a representative of $\sigma_f([\mu]) \in \mathrm{Teich}(\mathbb{P}^1, A)$, also fixing a_1, a_2, a_3. Then $g := \phi \circ f \circ \psi^{-1} : \psi(W) \to \mathbb{P}^1$ is holomorphic, and it is then easy to see that it is a finite type map, with $S(g) = \phi(S(f))$. We will say that the triple (ϕ, ψ, g) represents $[\mu]$ in $\mathrm{Def}_A^B(f)$. The condition that $\varpi([\mu]) = \sigma_f([\mu])$ means that ϕ and ψ are isotopic to each other relative to A. In particular, $\phi_{|A} = \psi_{|A}$, so if for instance A is a finite union of cycles for f, then $\phi(A) = \psi(A)$ is also a finite union of cycles for g. However, as ϕ is allowed to differ from ψ outside A, the multipliers of those cycles can be different.

PROPOSITION 3.1. *If (ϕ_1, ψ_1, g_1) and (ϕ_2, ψ_2, g_2) represent the same $\tau \in \mathrm{Def}_A^B(f)$, then g_1 and g_2 are conjugated by an automorphism of $\mathbb{P}^1$.*

PROOF. Assume without loss of generality that ϕ_i and ψ_i fix the same 3 points $a_1, a_2, a_3 \in A$. By assumption, we have $\phi_1 \circ \phi_2^{-1} \in \mathrm{QC}_0(B)$; if σ denotes the Beltrami form of $\phi_1 \circ \phi_2^{-1}$, then observe that $f^*\sigma$ is the Beltrami form of $\psi_1 \circ \psi_2^{-1}$. By Lemma 3.2, we have:

$$(\phi_1 \circ \phi_2^{-1}) \circ f = f \circ (\psi_1 \circ \psi_2^{-1})$$

from which we deduce $g_1 = g_2$. □

We can therefore think of $\mathrm{Def}_A^B(f)$ as a "natural parameter space" parametrizing a family of finite type maps f_τ, which are locally of the form $\phi_\tau \circ f \circ \psi_\tau^{-1}$, and which have the same combinatorics as f on the finite set A.

The following result is proved in [**Eps09**] in the case of rational maps:

THEOREM 3.1. *Assume that f is not a flexible Lattès map nor an automorphism. Then $\mathrm{Def}_A^B(f)$ is a smooth complex manifold of dimension* $\mathrm{card}\,(B \setminus A)$.

PROOF. This follows immediately from Lemmas 2.2, 3.3, 2.4 and the submersion lemma. □

3.2. No wandering domains. We now prove our last important result on Fatou/Julia theory of finite type maps. The presentation is closely based on the proof of Sullivan's theorem given in [**Ast17**], which is itself based on an infinitesimal argument due to McMullen. The core of the proof is essentially that the Teichmüller space of a finite type map can be embedded in a deformation space $\mathrm{Def}_A^B(f)$, and that the presence of a wandering domain would make this Teichmüller space infinite-dimensional, which is absurd as $\mathrm{Def}_A^B(f)$ is a finite-dimensional manifold. However, it is not formally necessary to define the Teichmüller space of f and the embedding into $\mathrm{Def}_A^B(f)$ for this argument to work: it is enough to consider the differential of the embedding, and show that if f had a wandering domain, then this linear map would map injectively an infinite-dimensional vector space into a finite-dimensional one, which is of course a contradiction. This is what is done below.

THEOREM 3.2 (Epstein, [**Eps93**], Theorem 7 p. 148). *Let $f : W \to \mathbb{P}^1$ be a finite type map. Then f has no wandering domains.*

In the setting of rational maps or transcendental entire functions of finite type, a lemma due to Baker asserts that given any wandering domain $U_0 \to U_1 \ldots \to U_n \to \ldots$, the Fatou component U_n must eventually be simply connected, which simplifies considerably the proof. Unfortunately, the proof of this lemma doesn't go through in the more general setting of finite type maps, which will force us to consider the case of multiply connected wandering domains, as in the original paper of Sullivan [**Sul85**]. We start with a series of lemma designed to analyse this situation.

DEFINITION 3.3. *Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of hyperbolic Riemann surfaces, and $f_n : U_n \to U_{n+1}$ a sequence of covering maps. We say that a Riemann surface U_∞ represents the direct limit of $(f_n)_{n\in\mathbb{N}}$ if for every $n \in \mathbb{N}$, there exists a holomorphic covering map $\pi_n : U_n \to U_\infty$ such that $\pi_{n+1} \circ f_n = \pi_n$ and for all $x, y \in U_n$, $\pi_n(x) = \pi_n(y)$ if and only if there exists $m \geq n$ such that $f_m \circ f_{m-1} \circ \ldots \circ f_n(x) = f_m \circ f_{m-1} \circ \ldots \circ f_n(y)$.*

LEMMA 3.4 ([**Sul85**], Prop. 1). *Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of hyperbolic Riemann surfaces, and assume that $\pi_1(U_0)$ is not abelian. Then there exists a Riemann surface U_∞ representing the direct limit of $(f_n)_{n\in\mathbb{N}}$, and either*

(1) all f_n are eventually isomorphisms
(2) or $\pi_1(U_\infty)$ is not finitely generated

LEMMA 3.5 (Compare [**Sul85**], Prop. 4). *Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of wandering domains of a finite type map, and $f_n := f : U_n \to U_{n+1}$. Assume without loss of generality that $S(f) \cap U_n = \emptyset$ for all $n \in \mathbb{N}$. Then either*

(1) all U_n are annuli and $d_n \to +\infty$, where $d_n := \deg(f^n : U_0 \to U_n)$
(2) or there exists a Riemann surface U_∞ representing the direct limit of $(f_n)_{n\in\mathbb{N}}$ which is not of finite type (i.e. not a compact Riemann surface with a finite number of punctures).

PROOF. Since $S(f) \cap U_n = \emptyset$, all maps $f : U_n \to U_{n+1}$ are coverings.

Up to further relabeling the U_n, we may reduce without loss of generality to one of the two following situations: either for all $n \in \mathbb{N}$, U_n is simply connected and $f : U_n \to U_{n+1}$ is a conformal isomorphism, or U_0 is not simply connected. In the first situation, U_0 represents the direct limit of the $(f_n)_{n\in\mathbb{N}}$, so we are in case (2) of Lemma 3.5 (indeed, $U_0 \subset \mathbb{P}^1 \setminus U_1$ so it is not a finite type Riemann surface). From now on we assume that $\pi_1(U_0)$ is non-trivial.

If $\pi_1(U_0)$ is not abelian, then either all maps $f : U_n \to U_{n+1}$ are isomorphisms, and then $U_\infty \simeq U_n$ is not of finite type; or U_∞ is still not of finite type by Lemma 3.4.

It therefore suffices to prove that if $\pi_1(U_0)$ is abelian, then all U_n are annuli. Indeed, if it is the case, then $\pi_1(U_0) \simeq \mathbb{Z}$ and U_0 is biholomorphic to $\mathbb{D}^*$, $\mathbb{C}^*$ or an annulus. The first two cases are impossible for a Fatou component, therefore U_0 is an annulus, and then all U_n are annuli as well since they are covered by U_0. □

LEMMA 3.6. *Case (1) in Lemma 3.5 is not possible.*

PROOF. Assume for a contradiction that there is a sequence of wandering domains $(U_n)_{n\in\mathbb{N}}$ as in Case (1) of Lemma 3.5, that is, each U_n is an annulus and $f^n : U_0 \to U_n$ is a covering map of degree $d_n \to +\infty$. Let $\gamma_0 \subset U_0$ be a geodesic that is a non-trivial simple closed curve, and let $\gamma_n := f^n(\gamma_0)$. Then $\gamma_n \subset U_n$ is also a simple closed curve, and $f^n : \gamma_0 \to \gamma_n$ is a degree d_n covering map.

We first claim that any limit h of a convergent subsequence of $(f^n : U_0 \to \mathbb{P}^1)_{n\in\mathbb{N}}$ must be constant. Indeed, the U_n are disjoint open subsets of $\mathbb{P}^1$, so their spherical area must tend to 0; therefore, if $f^{n_k} \to h$ on U_0, we must have that h is constant.

In particular, $\mathrm{diam}(\gamma_n) \to 0$ for the spherical metric in $\mathbb{P}^1$. By Jordan's theorem, $\mathbb{P}^1 \setminus \gamma$ has two simply connected components, one of which has small diameter when n is large. We denote by B_n the component with small diameter and we refer to it as the interior of γ_n, the other component being the exterior (this terminology is well-defined for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$; up to relabeling, we assume $n_0 = 0$).

Let C_n denote the connected component of $f^{-1}(B_{n+1})$ whose closure meets γ_n. For n large enough, we must have $C_n \subset B_n$. Since $S(f)$ is finite and $\mathrm{diam}(B_n) \to 0$, for all n large enough, $B_n \cap S(f)$ contains at most one point. Therefore C_n is simply connected, and then $C_n = B_n$.

In particular, $B_n \subset W$, and then by Montel' theorem, B_n must be in the Fatou set of f. But this contradicts the assumption that U_n is a Fatou component isomorphic to an annulus. $\square$

LEMMA 3.7. *Let U_∞ be a Riemann surface which is not of finite type. Let $\pi : \mathbb{H} \to U_\infty$ a universal cover, and let*

$$N := \{\mu \in \mathrm{bel}(U_\infty) : \exists \xi \text{ qc and hyperbolically bounded on } \mathbb{H},\ \pi^*\mu = \overline{\partial}\xi\}.$$

Then there exists an infinite dimensional vector subspace $V \subset \mathrm{bel}(U_\infty)$ such that $V \cap N = \{0\}$.

PROOF. The quotient vector space $\mathrm{bel}(U_\infty)/N$ is exactly the tangent space to the Teichmüller space $T_{[0]}\mathrm{Teich}(U_\infty)$.

Moreover, since U_∞ is not of finite type, $\mathrm{Teich}(U_\infty)$ is infinite-dimensional. $\square$

LEMMA 3.8. *Let $\mathfrak{X}(W)$ denote the vector space of meromorphic vector fields on W. If $\overline{\partial}\xi \in \mathrm{bel}(f)$, then $f^*\xi - \xi \in \mathfrak{X}(W)$. Moreover, if $\xi_{|B} = 0$ for some finite set B containing $S(f)$ and at least 3 other points, then $\xi = f^*\xi$.*

PROOF. If $z \in W$ is not a critical point, then there is a neighborhood U of z in W such that $f : U \to f(U)$ is a conformal isomorphism. Then we have $\overline{\partial} f^*\xi = f^*\overline{\partial}\xi$ on U, and since $f^*\overline{\partial}\xi = \overline{\partial}\xi$, we have $\overline{\partial}\eta = 0$ on U. Moreover, if $c \in \mathrm{Crit}(f)$, then in local coordinates we have $|f^*\xi(z) - \xi(z)| = \mathcal{O}(|z-c|^{-\nu})$, where $\nu \geq 1$ is the multiplicity of the critical point c. This proves that $f^*\xi - \xi$ is meromorphic on W, with poles contained in $\mathrm{Crit}(f)$ (of order at most the multiplicity of the critical points).

If $\xi_{|S(f)} = 0$, then ξ is hyperbolically bounded on X^* by Theorem 2.3, and therefore so is $f^*\xi$ on W^*. So $f^*\xi$ extends to $\mathbb{P}^1$ as a quasiconformal vector field that vanishes outside of W^*; let us denote by χ this extension. So $\chi - \xi$ is a quasiconformal vector field on $\mathbb{P}^1$, which is holomorphic on W, and such that $\overline{\partial}(\chi - \xi) = 0$ on $\mathbb{P}^1 \setminus W$ (since $\overline{\partial}\xi = 0$ outside of W). By Weil's lemma, $\chi - \xi$ is therefore holomorphic on $\mathbb{P}^1$, and it vanishes on 3 points, so it vanishes everywhere. $\square$

PROOF OF THEOREM 3.2. Assume that f has a wandering domain U_0, and let $U_n := f^n(U_0)$ for $n \geq 0$. Up to relabeling, we may assume without loss of generality that $U_n \cap S(f) = \emptyset$ for all $n \geq 0$, so that each map $f : U_n \to U_{n+1}$ is a covering map. By Lemmas 3.5 and 3.6, there exists a Riemann surface U_∞ representing the direct limit of the maps $f : U_n \to U_{n+1}$, and U_∞ is not of finite type.

Let V be the vector space given by Lemma 3.7. We define a linear map $J : V \to \mathrm{bel}(f)$ in the following way: if $\mu \in V$, we let $\sigma := J(\mu) := \pi_n^*\mu$ on U_n for all $n \geq 0$ ($\pi_n : U_\infty \to U_n$ are the maps from the definition of the direct limit), and $\sigma := (f^n)^*\pi_0^*\mu$ on $f^{-n}(U_0)$. We then extend σ by 0 outside of $\bigcup_{n\in\mathbb{Z}} f^n(U_0)$.

Let $\mu \in V$ and let ξ be a quasiconformal vector field on $\mathbb{P}^1$ such that $\sigma = \overline{\partial}\xi$. Assume that $\xi = f^*\xi$ on W; we will prove that this implies that $\mu = 0$. Indeed, if $\xi = f^*\xi$ then ξ must vanish on every repelling periodic point, therefore $\xi = 0$ on $J(f)$ by Theorem 1.1. In particular, ξ vanishes on ∂U_0, hence ξ is hyperbolically bounded on U_0 by Theorem 2.3. Let $p : \mathbb{H} \to U_0$ be a universal cover, so that $p_\infty := \pi_0 \circ p : \mathbb{H} \to U_\infty$ is a universal cover of U_∞. On $\mathbb{H}$, we have

$$p^*\sigma = p^*\overline{\partial}\xi = \overline{\partial}p^*\xi$$

Since ξ is hyperbolically bounded on U_0 and since $p : \mathbb{H} \to U_0$ is a (universal) cover, $p^*\xi$ is hyperbolically bounded on $\mathbb{H}$. Therefore, by definition of V, $\mu = 0$.

Let $Z \subset \mathbb{P}^1 \setminus S(f)$ be any set of cardinal 3 and let us introduce the map

$$\mathcal{I} : \mathrm{bel}(f) \to \bigoplus_{v \in S(f)} T_v\mathbb{P}^1$$

defined by $\mathcal{I}(\mu) = (\xi(v))_{v \in S(f)}$, where ξ is the unique quasiconformal vector field on $\mathbb{P}^1$ vanishing on Z such that $\mu = \overline{\partial}\xi$. With Lemma 3.8 we have proved the following: the linear map $\mathcal{I} \circ J : V \to \bigoplus_{v \in S(f)} T_v\mathbb{P}^1$ is injective. But this is absurd, since V is infinite-dimensional but $\bigoplus_{v \in S(f)} T_v\mathbb{P}^1$ is not. Therefore f cannot have wandering domains. $\square$

4. Teichmüller spaces of finite type maps

We denote by $\mathrm{QC}(f)$ the group of quasiconformal homeomorphisms $h : \mathbb{P}^1 \to \mathbb{P}^1$ with $h(W) = W$ and which commute with f. Let $\mathrm{QC}_0(f)$ the subgroup of quasiconformal homeomorphisms commuting with f and *uniformly quasiconformally isotopic to the identity*. This means that $h \in \mathrm{QC}_0(\mathbb{P}^1)$ if and only if there is $0 < k < 1$ and a continuous map $H : [0,1] \times \mathbb{P}^1 \to \mathbb{P}^1$ such that

(1) for all $t \in [0,1]$, $h_t := H(t, \cdot) \in \mathrm{QC}(f)$
(2) h_0 is the identity on $\mathbb{P}^1$, and $h_1 = h$
(3) for all $t \in [0,1]$, the quasiconformal dilatation of h_t is less than k.

The group $\mathrm{QC}_0(f)$ acts on $\mathrm{Bel}(f)$ by pullback.

DEFINITION 4.1. *The Teichmüller space of f, denoted by* $\mathrm{Teich}(f)$*, is defined as the quotient* $\mathrm{Bel}(f)/\mathrm{QC}_0(f)$.

DEFINITION 4.2. *We say that a singular value is acyclic if it is not preperiodic. We say that two acyclic singular values lie in the same foliated acyclic class if the closure of their grand orbits are the same.*

THEOREM 4.1. *The Teichmüller space* $\mathrm{Teich}(f)$ *can be naturally identified with the product of a polydisk and of a product of Teichmüller spaces of punctured spheres and tori. It makes* $\mathrm{Teich}(f)$ *into a finite-dimensional complex manifold of dimension*

$$\dim \mathrm{Teich}(f) = n_H + n_J + n_f - n_p$$

where n_H is the number of Herman rings of f, n_J is the number of ergodic line fields of f, n_f is the number of foliated acyclic critical classes lying in the Fatou set, and n_p is the number of parabolic cycles.

In order to prove Theorem 4.1, we could adapt the approach taken in [**Ast17**], replacing the moduli space $\mathcal{M}_d$ of degree d rational maps by a deformation space $\mathrm{Def}_A^B(f)$. This would give a construction from first principles of the complex structure on $\mathrm{Teich}(f)$. However, we choose to follow here the original approach of McMullen-Sullivan ([**MS98**]), which uses heavier machinery but is quicker and more precise (they prove that $\mathrm{Teich}(f)$ is isomorphic to a finite product of disks and Teichmüller spaces of marked tori and spheres).

Now that we know that repelling cycles are dense in the Julia set of finite type maps (Theorem 1.1) and that we dispose of the Fatou-Sullivan classification of periodic Fatou components (Corollary 1.1), the proof is essentially the same as in [**MS98**]. We give the ideas below, but we mostly refer to [**MS98**].

Let us first introduce and recall some notations. Let Ω be a hyperbolic complex 1-manifold; let

(1) $\mathrm{Bel}(\Omega)$ denote the space of Beltrami forms on Ω
(2) $\mathrm{QC}_0(\Omega)$ denote the group of quasiconformal homeomorphisms isotopic to id relative to the ideal boundary of Ω
(3) $\mathrm{Teich}(\Omega) = \mathrm{Bel}(\Omega)/\mathrm{QC}_0(\Omega)$.

Note that Ω is not assumed to be connected; we will use this notation in a context where Ω has finitely many connected components Ω_i, in which case

$$\mathrm{Teich}(\Omega) \simeq \prod_i \mathrm{Teich}(\Omega_i).$$

If additionnally there is a holomorphic endomorphism $g : \Omega \to \Omega$, we let

(1) $\mathrm{Bel}(\Omega, g)$ denote the space of g-invariant Beltrami forms on Ω
(2) $\mathrm{QC}(\Omega, g)$ denote the group of quasiconformal homeomorphisms commuting with g
(3) $\mathrm{QC}_0(\Omega, g)$ denote the group of quasiconformal homeomorphisms isotopic to id relative to the ideal boundary of Ω in $\mathrm{QC}(\Omega, g)$
(4) $\mathrm{Teich}(\Omega, g) = \mathrm{Bel}(\Omega, g)/\mathrm{QC}_0(\Omega, g)$.

DEFINITION 4.3. *Let $f : W \to \mathbb{P}^1$ be a finite type map, and let Λ_f denote the closure of the grand orbits of the union of $S(f)$ and all periodic points. Let $\Omega_f = W_\infty \setminus \Lambda_f$ (recall that $W_\infty = \bigcap_{n\in\mathbb{N}} f^{-n}(W)$). Let $M_1(J, f)$ denote the subspace of $\mathrm{Bel}(f)$ of invariant Beltrami forms supported in J.*

The open set Ω_f is a completely invariant subset of the Fatou set, such that $f : \Omega_f \to \Omega_f$ is a covering without periodic points. By the Fatou-Sullivan classification (Corollary 1.1) and the absence of wandering domains (Theorem 3.2), we have the following description of Ω_f: it is the union of all periodic Fatou components and their preimages, minus a discrete subset of W (iterated preimages of cycles and singular values contained in attracting and parabolic basins) and a countable collection of real-analytic Jordan curves (closures of the grand orbits of singular values contained in super-attracting basins, Herman rings or Siegel disks). We let Ω^{dis} denote the subset of Ω_f contained in attracting and parabolic basins and their preimages, and Ω^{fol} denote the subset of Ω_f contained in super-attracting basins, Herman disks and Siegel disks, so that $\Omega_f = \Omega^{\mathrm{dis}} \sqcup \Omega^{\mathrm{fol}}$.

This choice of notation comes from the fact that the grand orbit relation (that is, $x \sim y$ if and only if there exists $(n, m) \in \mathbb{N}^2 : f^n(x) = f^m(y)$) on Ω^{dis} is discrete, while in Ω^{fol} the closure of grand orbit equivalence classes gives a foliation by real analytic curves.

LEMMA 4.1 (See [**MS98**], Theorem 6.2). *Let $f : W \to X$ be a finite type map. We have*

$$\mathrm{Teich}(f) \simeq M_1(J, f) \times \mathrm{Teich}(\Omega^{\mathrm{dis}}/f) \times \mathrm{Teich}(\Omega^{\mathrm{fol}}, f)$$

PROOF. First, observe that $\mathrm{Bel}(f) = M_1(J, f) \bigoplus \mathrm{Bel}(\Omega^{\mathrm{dis}}, f) \bigoplus \mathrm{Bel}(\Omega^{\mathrm{fol}}, f)$. Indeed, any $\mu \in \mathrm{Bel}(f)$ must be supported in W_∞; and by the discussion above, $\mathrm{int}(W_\infty)$ and Ω_f differ only by a set of zero Lebesgue measure. Additionnally, the boundary of W_∞ is contained in the Julia set.

Next, observe that any $h \in \mathrm{QC}_0(f)$ must fix Λ_f pointwise. Therefore, by [[**MS98**], Theorem 4.3], we have

$$\mathrm{QC}_0(f) \simeq \mathrm{QC}_0(\Omega^{\mathrm{dis}}, f) \times \mathrm{QC}_0(\Omega^{\mathrm{fol}}, \mathbb{P}^1)$$

which proves that

$$\mathrm{Teich}(f) \simeq M_1(J,f) \times \mathrm{Teich}(\Omega^{\mathrm{dis}}, f) \times \mathrm{Teich}(\Omega^{\mathrm{fol}}, f).$$

Finally, the quotient Ω^{dis}/f is a finite union of Riemann surfaces (since $f : \Omega^{\mathrm{dis}} \to \Omega^{\mathrm{dis}}$ is a covering map without periodic points), corresponding to the finitely many attracting and parabolic basins of f (there are at most $\operatorname{card} S(f)$ of them). By the discussion at the end of the proof of [[**MS98**], Theorem 6.2], we therefore have $\mathrm{Teich}(\Omega^{\mathrm{dis}}, f) \simeq \mathrm{Teich}(\Omega^{\mathrm{dis}}/f)$. □

LEMMA 4.2 ([**MS98**], Theorem 6.5). *The quotient space Ω^{dis} is a finite union of Riemann surfaces, one for each cycle of attracting or parabolic components of the Fatou set of f. An attracting basin contributes an n-times punctured torus to Ω^{dis}, while a parabolic basin contributes an $(n+2)$-times punctured sphere, where $n \geq 1$ is the number of grand orbits of singular values landing in the corresponding basin.*

We simply replaced the words "critical points" by "singular value" in the statement above. The proof of Lemma 4.2 is exactly the same as in [**MS98**], and we do not reproduce it completely here. We just note that Ω^{dis} can be written as a disjoint union of completely invariant open subsets Ω_i, each of which is contained either in an attracting or a parabolic basin. In the case of an attracting basin, the fact that Ω_i/f is isomorphic to a punctured torus can be seen by constructing a fundamental domain in a linearizing coordinate. In the case of a parabolic basin, we use a Fatou coordinate instead to prove that Ω_i/f is isomorphic to a cylinder with n punctures, which is itself isomorphic to a sphere with $n+2$ punctures.

LEMMA 4.3 ([**MS98**], Theorem 6.8). *The space $\mathrm{Teich}(\Omega^{\mathrm{fol}}, f)$ is a finite-dimensional polydisk, whose dimension is given by the number n_H of cycles of Herman rings plus the number n_{fol} of foliated equivalence classes of acyclic singular values landing in Siegel disks, Herman rings or superattracting basins.*

Again, we simply present here the idea of the proof. We can write Ω^{fol} as a disjoint union of completely invariant open subsets Ω_i^{fol}, where the Ω_i^{fol} each intersect exactly one cycle of Siegel disks, Herman rings or a super-attracting basin; and we have $\mathrm{Teich}(\Omega^{\mathrm{fol}}, f) = \prod_i \mathrm{Teich}(\Omega_i^{\mathrm{fol}}, f)$. The space $\mathrm{Teich}(\Omega_i^{\mathrm{fol}}, f)$ is a polydisk of dimension the number of grand orbits of annular components in Ω_i^{fol}: each such annulus can be stretched and twisted, which gives 2 real parameters or 1 complex parameter of quasiconformal deformations. Let n_i denote the number of foliated acyclic equivalence classes of singular values contained in Ω_i^{fol}. If Ω_i^{fol} intersects a cycle of Siegel disks, then it has n_i grand orbits of annuli components; if Ω_i^{fol} intersects a cycle of Herman rings, then it has $n_i + 1$ grand orbits of annuli components; and if Ω_i^{fol} intersects a super-attracting basin, then it has n_i grand orbits of annuli components.

PROOF OF THEOREM 4.1. Now that we have the identification given by Lemma 4.1, it remains to describe each factor and count its dimension.

First, the space $M_1(J,f)$ is a polydisk of dimension n_J, where n_J is the number of ergodic line fields. Lemma 4.2 provides a description of $\mathrm{Teich}(\Omega^{\mathrm{dis}}/f)$. Since the Teichmüller space of a sphere with n punctures is $n-3$ and the dimension of the Teichmüller space of a torus with n punctures is n, the dimension of $\mathrm{Teich}(\Omega^{\mathrm{dis}}/f)$ is $n_{\mathrm{dis}} - n_p$, where n_{dis} is the number of grand orbits of singular values captured by either attracting or parabolic cycles.

Lastly, Lemma 4.3 asserts that $\mathrm{Teich}(\Omega^{\mathrm{fol}}, f)$ is a polydisk of dimension $n_{\mathrm{fol}}+n_H$, where n_{fol} is the number of foliated acyclic equivalence classes of singular values in Ω^{fol}. The dimension count of Theorem 4.1 then follows from the observation that $n_f = n_{\mathrm{dis}} + n_{\mathrm{fol}}$. □

DEFINITION 4.4. *Let $A \subset \mathbb{P}^1$ be a finite set of cardinal at least 3, such that A is a union of cycles and of pieces of singular orbits $\bigcup_{0\leq j\leq n} f^j(v)$, $v \in S(f)$. Let $B := A \cup f(A) \cup S(f)$. Then $\mathrm{QC}_0(f) \subset \mathrm{QC}_0(A)$, so that there is a natural map $\Upsilon_{A,B} : \mathrm{Teich}(f) \to \mathrm{Def}_A^B(f)$.*

THEOREM 4.2. *The map $\Upsilon_{A,B}$ is a holomorphic injection.*

PROOF. This is a direct consequence of Lemma 3.2. □

COROLLARY 4.1. *Let $f : W \to \mathbb{P}^1$ be a finite type map. With the same notations as in Theorem 4.1, we have*

$$n_H + n_J + n_f - n_p \leq \mathrm{card}\, S(f).$$

PROOF. It suffices to choose A to be a finite union of cycles of cardinal at least 3 disjoint from $S(f)$, and $B := A \cup S(f)$. Then A and B satisfy the conditions of definition 4.4, and $\dim \mathrm{Def}_A^B(f) = \mathrm{card}\,(B \setminus A) = \mathrm{card}\, S(f)$. Finally, by Theorem 4.2, we have $\dim \mathrm{Teich}(f) \leq \dim \mathrm{Def}_A^B(f)$. □

4.1. No singular relations and lift to the Teichmüller space.

DEFINITION 4.5. *A natural family of finite type maps on $\mathbb{P}^1$ is a triple (f, ϕ, ψ), where $f : W \to \mathbb{P}^1$ is a finite type map on $\mathbb{P}^1$, and $\phi, \psi : M \times \mathbb{P}^1 \to \mathbb{P}^1$ are holomorphic motions of the Riemann sphere parametrized by a complex manifold M, such that for all $\lambda \in M$, $\overline{\partial}\psi_\lambda = 0$ a.e. on $\mathbb{P}^1 \setminus W$.*

We will still use the notation $(f_\lambda)_{\lambda \in M}$, where $f_\lambda := \phi_\lambda \circ f \circ \psi_\lambda^{-1}$.

If $\tau_0 \in \mathrm{Teich}(f)$ and $A \subset \mathbb{P}^1$ is a 3-cycle of f, then the choice of a local section $\sigma : U \to \mathrm{Bel}(\mathbb{P}^1)$ at τ_0 of the quotient map $\pi : \mathrm{Bel}(f) \to \mathrm{Teich}(f)$ defines a natural family in the sense of the previous definition, by taking $\phi_\tau = \psi_\tau$ to be the unique quasiconformal homeomorphism integrating $\sigma(\tau)$ and normalized to fix A pointwise. Moreover, the map $f_\tau := \phi_\tau \circ f \circ \phi_\tau^{-1}$ depends only on τ, A and f, and not on the choice of section.

We have thus defined a map $\Psi^A : \mathrm{Teich}(f) \ni \tau \mapsto f_\tau$, which locally induces natural families of finite type maps in the sense of Definition 4.5.

DEFINITION 4.6. *A singular value relation for a finite type map f is a relation of the form*

$$f^m(v_1) = f^n(v_2)$$

where v_1, v_2 are either singular values or critical points, and $m, n \in \mathbb{N}$ with either $v_1 \neq v_2$ or $m \neq n$, or of the form

$$f^n(v_1) \in \mathbb{P}^1 \setminus W.$$

If $(f_\lambda)_{\lambda \in M}$ is a natural family of finite type maps, then a singular value relation can be persistent or not in the family, in the obvious sense.

The next theorem is an adaptation to the setting of finite type maps of [[**MS98**], Theorem 7.4]. It will be applied later in the manuscript.

THEOREM 4.3. *Let B denote the unit ball of a complex Banach space, and let $B' := \frac{1}{3}B$. Let $(f_\lambda)_{\lambda \in B}$ be a natural family of finite type maps on $\mathbb{P}^1$, and assume that both of the following properties hold:*

1. *all singular relations are persistent on B*
2. *and for every $v \in S(f_0)$, $f_0^n(v) \in W$ for all $n \geq 0$.*

Then there is a holomorphic map $j : B' \to \mathrm{Teich}(f_0)$*, mapping* 0 *to* $[0]$*, such that for all* $\lambda \in B'$*,*

$$\Psi^A \circ j(\lambda) = f_\lambda.$$

Assumption (2) is probably not necessary, but it simplifies the proof and it will be enough for our purposes. The idea of the proof is that the absence of non-persistent singular relations will allow us to construct a dynamical holomorphic motion h_λ respecting the dynamics on W, and such that $\overline{\partial} h_\lambda = 0$ a.e. on $\mathbb{P}^1 \setminus W$. Compared to the case of rational maps, the added difficulties are this new requirement that $\overline{\partial} h_\lambda = 0$ a.e. on $\mathbb{P}^1 \setminus W$, and the presence of asymptotic values which makes pulling back holomorphic graphs by the dynamics more delicate. We begin with the following lemmas:

LEMMA 4.4. *Assume that the hypotheses of Theorem 4.3 are satisfied. Let* $\gamma_1, \gamma_2 : B \to \mathbb{P}^1$ *be holomorphic maps such that for all* $\lambda \in B$*,*

$$f_\lambda^m \circ \gamma_1(\lambda) = f_\lambda^m \circ \gamma_2(\lambda) = f_\lambda^n(v(\lambda)),$$

where $v(\lambda) \in S(f_\lambda)$ *and* $m, n \in \mathbb{N}$*. Then either* $\gamma_1 \equiv \gamma_2$*, or for all* $\lambda \in M$*,* $\gamma_1(\lambda) \neq \gamma_2(\lambda)$*.*

PROOF OF LEMMA 4.4. We prove this by induction on m. For $m = 0$, there is nothing to prove.

Let us now assume $m = 1$:

$$f_\lambda \circ \gamma_1(\lambda) = f_\lambda \circ \gamma_2(\lambda) = f_\lambda^n(v(\lambda)),$$

and let us assume for a contradiction that there exists $\lambda_0 \in B$ such that $\gamma_1(\lambda_0) = \gamma_2(\lambda_0)$, but $\gamma_1 \not\equiv \gamma_2$. Then $c(\lambda_0) := \gamma_i(\lambda_0)$ is a critical point for f_{λ_0}, so we have $f_\lambda(c(\lambda)) = f_\lambda^n(v(\lambda))$ for all $\lambda \in B$ by the persistence of singular relations. But on the other hand, $c(\lambda)$ has constant multiplicity as a critical point; and the fact that $\gamma_1 \not\equiv \gamma_2$ implies that there is a loss of local degree at $c(\lambda)$ for some $\lambda \neq \lambda_0$, a contradiction.

Let $m \geq 1$, and assume that the lemma is proved for m. Let $\gamma_1, \gamma_2 : B \to \mathbb{P}^1$ such that for all $\lambda \in B$,

$$f_\lambda^{m+1} \circ \gamma_1(\lambda) = f_\lambda^{m+1} \circ \gamma_2(\lambda) = f_\lambda^n(v(\lambda)).$$

If $\gamma_1(\lambda) \neq \gamma_2(\lambda)$, then we are done, so assume that $\gamma_1(\lambda_0) = \gamma_2(\lambda_0)$. Let $\tilde{\gamma}_i := f_\lambda \circ \gamma_i$, $1 \leq i \leq 2$. Applying the induction hypothesis to $\tilde{\gamma}_i$, we get $\tilde{\gamma}_1 = \tilde{\gamma}_2$. We now argue as before: if $\gamma_1 \not\equiv \gamma_2$, then $c(\lambda_0) := \gamma_i(\lambda_0)$ is a critical point for f_{λ_0}. Therefore, $f_{\lambda_0}^{m+1}(c(\lambda_0)) = f_{\lambda_0}^n(v(\lambda_0))$, and by the persistence of singular relations, $f_\lambda^{m+1}(c(\lambda)) = f_\lambda^n(v(\lambda))$ for all $\lambda \in B$. We then obtain a contradiction by the same local degree consideration.

Therefore $\gamma_1 = \gamma_2$, and the lemma is proved. $\square$

LEMMA 4.5. *Assume that the hypotheses of Theorem 4.3 are satisfied. Let* $\gamma : B \to \mathbb{P}^1$ *be a holomorphic map such that* $\gamma(\lambda)$ *is in the grand orbit of* $S(f_\lambda)$*, i.e. such that there exists* $m, n \in \mathbb{N}$*,* $f_\lambda^m(\gamma(\lambda)) = f_\lambda^n(v(\lambda))$ *where* $v(\lambda) \in S(f_\lambda)$*. Let* $x \in f^{-1}(\{\gamma(0)\})$*. Then there is a holomorphic map* $\tilde{\gamma} : B \to \mathbb{P}^1$ *such that* $f_\lambda \circ \tilde{\gamma} = \gamma$ *and* $\tilde{\gamma}(0) = x$*.*

PROOF OF LEMMA 4.5. Let $Z := \{(\lambda, z) : f_\lambda(z) = \gamma(\lambda)\}$, and let Z_0 denote the connected component of Z containing $(0, x)$: it is an analytic hypersurface of the open set

$$U := \{(\lambda, \psi_\lambda(z)) : (\lambda, z) \in B \times W\} \subset B \times \mathbb{P}^1.$$

Let $\pi : Z_0 \to B$ denote the projection on the first coordinate.

Assume first that $\pi : Z_0 \to B$ is not locally invertible near (λ_0, z_0). Then, by the implicit function theorem, we must have $f'_{\lambda_0}(z_0) = 0$, i.e. z_0 is a critical point of f_{λ_0} and $\gamma(\lambda_0) =: v_1(\lambda_0)$ is a critical value, so f_{λ_0} has a singular relation

$$f^m_{\lambda_0}(v_1(\lambda_0)) = f^n_{\lambda_0}(v(\lambda_0)).$$

By assumption, this relation is persistent, so that

$$f^m_\lambda(v_1(\lambda)) = f^n_\lambda(v(\lambda)) = f^m_\lambda(\gamma(\lambda))$$

for all $\lambda \in B$. Then we must have $\gamma(\lambda) = v_1(\lambda)$ for all $\lambda \in B$ by Lemma 4.4, so if we let $v_1 := v_1(0)$, we have $\gamma(\lambda) = \phi_\lambda(v_1)$. In particular, $f(x) = \gamma(0) = v_1$, and we can just take $\tilde{\gamma}(\lambda) := \psi_\lambda(x)$.

Assume now that $\pi : Z_0 \to B$ has an asymptotic value λ_0. Then there is a curve $t \mapsto (\lambda_t, z_t) \in Z_0$ such that $\lambda_t \to \lambda_0$ as $t \to 0$, but $t \mapsto z_t$ has no accumulation point in $W(f_{\lambda_0})$ as $t \to 0$. Using the expression $f_\lambda = \phi_\lambda \circ f \circ \psi_\lambda^{-1}$ and the definition of Z_0, we obtain:

$$f \circ \psi_{\lambda_t}^{-1}(z_t) = \phi_{\lambda_t}^{-1} \circ \gamma(\lambda_t)$$

and $\phi_{\lambda_t}^{-1} \circ \gamma(\lambda_t) \to \phi_{\lambda_0}^{-1} \circ \gamma(\lambda_0)$ as $t \to 0$, while $\psi_{\lambda_t}^{-1}(z_t) = z_t + o_{t\to 0}(1)$ has no accumulation point in $W(f_{\lambda_0})$. Therefore, $v_1 := \phi_{\lambda_0}^{-1} \circ \gamma(\lambda_0)$ is an asymptotic value for f; but then $v_1(\lambda) := \phi_\lambda(v_1)$ is an asymptotic value for f_λ for all $\lambda \in B$; in particular, $\gamma(\lambda_0) = v_1(\lambda_0)$, and by the persistence of singular relations, we must have

$$f^m_\lambda(v_1(\lambda)) = f^n_\lambda(v(\lambda)) = f^n_\lambda(\gamma(\lambda))$$

for all $\lambda \in B$. Then, by Lemma 4.4, we have $\gamma(\lambda) = v_1(\lambda)$ for all $\lambda \in B$.

In particular, $f(x) = \gamma(0) = v_1$, and we can just define $\tilde{\gamma}(\lambda) := \psi_\lambda(x)$.

Finally, it remains to treat the case where $\pi : Z_0 \to B$ has no asymptotic values nor critical values. Then $\pi : Z_0 \to B$ is a covering map, and since B is simply connected, it has a well-defined inverse $g : B \to Z_0$. We can then define $\tilde{\gamma} := \pi_2 \circ g$, where $\pi_2 : Z_0 \to \mathbb{P}^1$ is the projection on the second coordinate. $\square$

LEMMA 4.6. *Assume that the hypotheses of Theorem 4.3 are satisfied. Let $\gamma : B \to \mathbb{P}^1$ be a holomorphic map such that $\gamma(\lambda)$ avoids in the grand orbit of $S(f_\lambda)$, i.e. such for all $\lambda \in B$ and all $m, n \in \mathbb{N}$, $f^m_\lambda \circ \gamma(\lambda) \neq f^n_\lambda(v(\lambda))$, where $v(\lambda) \in S(f_\lambda)$. Let $x \in f^{-1}(\{\gamma(0)\})$. Then there is a holomorphic map $\tilde{\gamma} : B \to \mathbb{P}^1$ such that $f_\lambda \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = x$.*

PROOF OF LEMMA 4.6. The proof is similar to that of Lemma 4.5, except that the first two cases are now ruled out by the assumption on γ.

Indeed, let Z and Z_0 be as above. By the proof of Lemma 4.5, if $\pi : Z_0 \to B$ has a critical value or an asymptotic value at λ_0, then $\gamma(\lambda_0)$ is a critical value or an asymptotic value for f_{λ_0}, which contradicts our hypothesis on γ. Therefore $\pi : Z_0 \to B$ is a covering map, hence an isomorphism, and we can take $\gamma := \pi_2 \circ \pi^{-1}$ as before. $\square$

PROOF OF THEOREM 4.3. Let $f := f_0$. Recall that since $(f_\lambda = \phi_\lambda \circ f \circ \psi_\lambda^{-1})_{\lambda \in B}$ is a natural family, we have a holomorphic motion $\phi_\lambda(S(f))$ of the singular values and a holomorphic motion ψ_λ of the critical points. Moreover, the absence of persistent singular value relation implies that these motions are compatible; we denote it by h_λ, i.e. $h_\lambda(x) = \psi_\lambda(x)$ if $x \in \operatorname{Crit}(f)$ and $h_\lambda(x) = \phi_\lambda(x)$ if $x \in S(f)$.

Let P denote the post-singular set of f, that is,

$$P := \bigcup_{n\in\mathbb{N}} f^n(S(f))$$

(which is well-defined by assumption (2)). We first extend h_λ to P by

$$h_\lambda(f^n(v)) := f_\lambda^n(v_\lambda) = f_\lambda^n(\phi_\lambda(v))$$

for any $v \in S(f)$. Again, the absence of non-persistent singular relations implies that this defines a compatible holomorphic motion, well-defined and without collisions. Moreover, for all $\lambda \in B$ and all $x \in P$, we have

$$h_\lambda \circ f(x) = f_\lambda \circ h_\lambda(x).$$

Using Lemmas 4.5 and 4.4, we then extend h_λ to $\hat{P} := \bigcup_{n\in\mathbb{N}} f^{-n}(P) \subset W$. Next, we observe that the holomorphic motion h_λ (defined so far on P^*) and ψ_λ restricted to $\mathbb{P}^1 \setminus W$ are compatible by assumption (2). We may therefore extend h_λ to $\mathbb{P}^1 \setminus W$ by ψ_λ, which gives us a holomorphic motion $h : B \times (\hat{P} \cup (\mathbb{P}^1 \setminus W)) \to \mathbb{P}^1$. Moreover, by construction, h respects the dynamics on $\hat{P}$: $h_\lambda \circ f(x) = f_\lambda \circ h_\lambda(x)$, for all $x \in \hat{P}$. Finally, we use Bers-Royden Harmonic λ-lemma (Theorem 2.6, [**BR86**]) to extend h to a holomorphic motion $h : B' \times \mathbb{P}^1 \to \mathbb{P}^1$ which is harmonic outside of $E := \hat{P} \cup (\mathbb{P}^1 \setminus W)$; we observe moreover that since by construction $h_\lambda = \psi_\lambda$ on $\mathbb{P}^1 \setminus W$, and $\overline{\partial}\psi_\lambda = 0$ a.e. on $\mathbb{P}^1 \setminus W$, this holomorphic motion is actually harmonic on $\mathbb{P}^1 \setminus \hat{P}$, with $\mu_\lambda = 0$ a.e. on $\mathbb{P}^1 \setminus W$, where μ_λ is the Beltrami form of h_λ.

On the other hand, applying Lemma 4.6 with $\gamma(\lambda) := h_\lambda \circ f(x)$, $x \notin E = \hat{P} \cup (\mathbb{P}^1 \setminus W)$, we define a second holomorphic motion $\tilde{h}$ on $B' \times \mathbb{P}^1$ satisfying

$$f_\lambda \circ \tilde{h}_\lambda(x) = h_\lambda \circ f(x)$$

for every $x \notin E$. We also define $\tilde{h}_\lambda(x) := h_\lambda(x)$ for $x \in E$: these two definitions are compatible, and patch together to define a holomorphic motion $\tilde{h} : B' \times \mathbb{P}^1 \to \mathbb{P}^1$ satisfying

$$f_\lambda \circ \tilde{h}_\lambda = h_\lambda \circ f$$

on W.

Let μ_λ, ν_λ denote the Beltrami forms of h_λ and $\tilde{h}_\lambda$ respectively. By construction, $h_\lambda \circ f \circ \tilde{h}_\lambda^{-1} = f_\lambda$ is holomorphic on $W_\lambda = h_\lambda(W) = \tilde{h}_\lambda(W)$, and $\nu_\lambda = 0$ on W. Therefore $\nu_\lambda = f^*\mu_\lambda$, and since $f : W \setminus \hat{P} \to \mathbb{P}^1 \setminus \hat{P}$ is a covering map, the Beltrami form ν_λ is harmonic on $W \setminus \hat{P}$, and in fact on $\mathbb{P}^1 \setminus \hat{P}$ since $\nu_\lambda = 0$ on $\mathbb{P}^1 \setminus W$.

Therefore, by unicity in Bers-Royden's Harmonic λ-lemma, $\nu_\lambda = \mu_\lambda = f^*\mu_\lambda$. We may then define $j : B' \to \mathrm{Teich}(f)$ by $j(\lambda) := \pi(\mu_\lambda)$, where $\pi : \mathrm{Bel}(f) \to \mathrm{Teich}(f)$ is the quotient map. □

CHAPTER 2

Bifurcations

1. The case of rational maps on $\mathbb{P}^1$

1.1. A short overview of the topic. We begin with a short overview of the theory of bifurcations in one-dimensional rational dynamics. Since the topic has been the focus of an intense research activity over the last decades, we will not be able to cover all important results, and we refer the reader to surveys such as [**BC11**] or [**Duj21**].

1.1.1. *Mañé-Sad-Sullivan and Lyubich's theorem.* Let Rat_d denote the space of rational maps of degree d on $\mathbb{P}^1$, that is, $f \in \mathrm{Rat}_d$ if $f = \frac{p}{q}$, where p and q are complex polynomials with no common factors and $\max(\deg p, \deg q) = d$. The space Rat_d can by naturally identified with a Zariski open subset of $\mathbb{P}^{2d+1}$.

A holomorphic family of rational maps of degree d is a holomorphic map

$$F : M \times \mathbb{P}^1 \to M \times \mathbb{P}^1$$

such that for all $\lambda \in M$, $F(\lambda, \cdot) =: f_\lambda \in \mathrm{Rat}_d$. In what follows, we will use the notation $(f_\lambda)_{\lambda \in M}$ and refer to M as the parameter space.

The automorphism group $\mathrm{PSL}_2(\mathbb{C})$ acts on Rat_d by conjugacy; we denote by $\mathcal{M}_d := \mathrm{Rat}_d/\mathrm{PSL}_2(\mathbb{C})$ the *moduli space* of rational maps of degree d. It is an orbifold of dimension $2d - 2$.

DEFINITION 1.1. *Let $(f_\lambda)_{\lambda \in M}$ be a holomorphic family of degree d rational maps, and assume that there is a holomorphic map $c : M \to \mathbb{P}^1$ such that for all $\lambda \in M$, $f'_\lambda(c(\lambda)) = 0$ (we say that c is a marked critical point). We say that c is passive over M if*

$$\{\lambda \mapsto f_\lambda^n(c(\lambda)) : n \in \mathbb{N}\}$$

is normal. If $\lambda_0 \in M$ and if there exists a neighborhood U of λ_0 in M such that c is passive on U, then we say that c is passive at λ_0.

When f_{λ_0} has only simple critical points, all critical points can be marked locally near λ_0 by the implicit function theorem; and in general, we can always reduce to the case of a holomorphic family where all critical points are marked, up to passing to a finite degree cover in the parameter space. Thus it always makes sense to talk about activity or passivity of critical points at a parameter $\lambda_0 \in M$.

The following theorem, proved independantly by Mañé-Sad-Sullivan ([**MSS83**]) and Lyubich ([**Lyu83**]), is the basis of the theory of bifurcations in one-dimensional complex dynamics:

THEOREM 1.1. *Let $(f_\lambda)_{\lambda \in M}$ be a holomorphic family of rational maps of degree $d \geq 2$. Let $\lambda_0 \in M$ and let $U \subset M$ be a simply connected neighborhood of λ_0. The following properties are equivalent:*

(1) the map $\lambda \mapsto J(f_\lambda)$ is continuous on U for the Hausdorff distance

(2) there exists a dynamical holomorphic motion $h : U \times J(f_{\lambda_0}) \to \mathbb{P}^1$ of the Julia set over U, satisfying $h_\lambda \circ f_{\lambda_0} = f_\lambda \circ h_\lambda$
(3) the number of non-repelling cycles of f_λ is constant over U
(4) the maximal period of attracting cycles is bounded on U
(5) all critical values are passive on U.

If λ_0 satisfies any of the equivalent conditions above, we say that λ_0 is in the stable locus; otherwise, we say that it is in the bifurcation locus. When there is no ambiguity about which family we are working with, we will denote the stability locus by Stab and the bifurcation locus by Bif. An essential consequence of Theorem 1.1 is the genericity of stabiliy:

COROLLARY 1.1. *Let $(f_\lambda)_{\lambda \in M}$ be a holomorphic family of rational maps of degree $d \geq 2$. Then the bifurcation locus is a closed set with empty interior in M.*

We will see later that this is a feature unique to the one-dimensional setting.

1.1.2. *Bifurcation currents.* Theorem 1.1 was later complemented by a potential-theoretic approach to bifurcations introduced by DeMarco in [**DeM01**]. Let us first recall some results on the ergodic properties of rational maps. First, a rational map on $\mathbb{P}^1$ has a unique measure of maximal entropy; this result was first obtained by Brolin for polynomials and then by Lyubich for rational maps:

THEOREM 1.2 ([**Bro65**], [**Lju83**],). *Let f be a rational map on $\mathbb{P}^1$ of degree $d \geq 2$. There exists a unique probability measure μ_f of maximal entropy $\log d$. Moreover, $\operatorname{supp} \mu_f = J(f)$.*

We now state DeMarco's result:

THEOREM 1.3. [**DeM01**] *Let $(f_\lambda)_{\lambda \in M}$ be a holomorphic family of rational maps of degree $d \geq 2$. Let $L : M \to \mathbb{R}$ be the Lyapunov exponent, defined by*

$$L(\lambda) := \int_{J(f_\lambda)} \log \|df_\lambda(z)\| d\mu_\lambda(z).$$

Then L is continuous and plurisubharmonic. Moreover, it is pluriharmonic exactly on the stability locus in M.

The fact that L is plurisubharmonic on M means that $dd^c L$ is a closed positive current on M of bidegree $(1,1)$, denoted by T_{bif}. Therefore Theorem 1.3 adds another equivalent characterization of the bifurcation/stabiliy dichotomy: $\lambda \in \mathrm{Bif}$ if and only if $\lambda \in \operatorname{supp} T_{\mathrm{bif}}$.

Let us also note that the important particular case of quadratic polynomials had been investigated by Levin prior to this. In terms of DeMarco's more general setting, his results may be reformulated as:

THEOREM 1.4. [**Lev82**] *In the quadratic family $(z \mapsto z^2 + \lambda)_{\lambda \in \mathbb{C}}$, Bif is exactly the boundary of the Mandelbrot set $\mathbf{M}$ and T_{bif} is exactly the harmonic measure of $\mathbf{M}$.*

1.1.3. *Higher bifurcation currents.* Bassanelli and Berteloot introduced in [**BB07**] higher degree currents T_{bif}^k, detecting higher codimensional bifurcation phenomena. Since T_{bif} has continuous potential, the intersections

$$T_{\mathrm{bif}}^k := T_{\mathrm{bif}} \wedge \ldots \wedge T_{\mathrm{bif}}$$

are well-defined closed positive currents of bidegree (k,k). When $k > \dim M$, $T_{\mathrm{bif}}^k = 0$. When $k = \dim M$, T_{bif}^k is a positive measure called the bifurcation measure and denoted by μ_{bif}.

Two families are particularly important: the family Poly_d of (monic centered) degree d polynomials, and the total family induced by Rat_d. In the latter case, with a slight abuse of notations, μ_{bif} will denote instead the unique positive measure on $\mathcal{M}_d$ such that $\mu_{\mathrm{bif}} = \pi^*(T_{\mathrm{bif}}^{2d-2})$, where $\pi : \mathrm{Rat}_d \to \mathcal{M}_d$ is the quotient map.

THEOREM 1.5 ([**BB09**], [**DF08**]). *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of rational maps of degree d. Then*

$$\mathrm{supp}\ T_{\mathrm{bif}}^k \subset \overline{\{\lambda \in M : f_\lambda \text{ has } k \text{ periodic critical points }\}}.$$

DEFINITION 1.2. *Let $f \in \mathrm{Rat}_d$ and $1 \leq k \leq 2d-2$. We say that f is a k-Misiurewicz map if there exists a compact hyperbolic set $K \subset J$ and critical points $c_1, \dots, c_k$ such that for all $1 \leq j \leq k$, there exists $n_j \in \mathbb{N}^*$ such that $f^{n_j}(c_j) \in K$.*

We say that f is a Thurston map or a PCF map if the orbit of each of its critical points are finite, i.e. all its critical points are either periodic or strictly preperiodic.

Finally, we say that a f is a Misiurewicz-Thurston map if all of its critical points are strictly preperiodic and if their orbit eventually lands on repelling cycles.

THEOREM 1.6 ([**BE09**], [**DF08**]). *For $M = \mathrm{Poly}_d$ or $\mathcal{M}_d$, the support of μ_{bif} is exactly the closure of the set of Misiurewicz-Thurston maps.*

In fact, the statements in [**BE09**] and [**DF08**] involve Misiurewicz-Thurston maps which are not flexible Lattès maps, but it was proved later by other means in [**BG13**] that flexible Lattès maps also belong to the support of μ_{bif} in $\mathcal{M}_d$.

1.1.4. *Equidistribution results.* The current T_{bif} and its self-intersections T_{bif}^k provide a more quantitative description of the bifurcations than simply through their support. A number of results provide *equidistribution theorems* of some dynamically defined parameters to T_{bif} or μ_{bif}. We will only mention a few for illustration purposes, rather than try to present a complete account of the literature on this topic.

DEFINITION 1.3. *Let $n \in \mathbb{N}^*$, $\eta \in \mathbb{C}$. We define*

$$\mathrm{Per}_n(\eta) := \{f \in \mathrm{Poly}_d : f \text{ has a cycle of exact period } n \text{ and multiplier } \eta\}.$$

The first parametric equidistribution result is the following, obtained by Levin:

THEOREM 1.7. [**Lev90**] *Let $f_\lambda(z) := z^2 + \lambda$ be the quadratic family. Then*

$$2^{-n} \sum_{\lambda_j \in \mathrm{Per}_n(0)} \delta_{\lambda_j} \to \mu_{\mathrm{bif}}$$

in the sense of measures, where μ_{bif} is the bifurcation measure of the quadratic family $(f_\lambda)_{\lambda\in\mathbb{C}}$ (which also coincides with the harmonic measure of the Mandelbrot set).

This result can be recoverd in a quantitative way by arithmetic methods (see [**FRL06**])

In higher degree, the set $\mathrm{Per}_n(\eta)$ is an algebraic hypersurface of Poly_d; it therefore supports an integration current $[\mathrm{Per}_n(\eta)]$, which is closed and of bidegree $(1,1)$.

Bassanelli and Berteloot first proved in [**BB11**] the equidistribution of the $\mathrm{Per}_n(\eta)$ towards the bifurcation current, either for $\eta \in \mathbb{D}$ or for generic $\eta \in \mathbb{C}$ (outside of a polar set). Gauthier strengthened these results by proving the equidistribution for any $\eta \in \mathbb{C}$:

THEOREM 1.8. [**Gau16**] *For every $\eta \in \mathbb{C}$ and $d \geq 2$, we have*

$$d^{-n}[\mathrm{Per}_n(\eta)] \to T_{\mathrm{bif}}$$

in the sense of currents.

Gauthier and his coauthors later improved Theorem 1.8 in various significant ways: for instance in [**GV17**], Gauthier and Vigny obtain a speed of convergence of the measures supported by $\bigcap_{j=1}^{d} \mathrm{Per}_n(\eta_j)$ (for $\eta_j \in \mathbb{D}$) to μ_{bif} in Poly_d, which has applications to some counting problems in parameter spaces. Similar results were also obtained in the moduli space of rational maps.

Another type of equidistribution results address the repartition of parameters whose critical points are preperiodic, with prescribed combinatorics.

DEFINITION 1.4. *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of degree $d \geq 2$ rational maps, with a marked critical point $c(\lambda)$. Let $0 \leq k < n$. We define* $\mathrm{Preper}(n,k)$ *as the set of $\lambda \in M$ for which $f_\lambda^{n+k}(c(\lambda)) = f_\lambda^n(c(\lambda))$, that is, $c(\lambda)$ is preperiodic with preperiod k and period n.*

THEOREM 1.9 ([**DF08**]). *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of degree $d \geq 2$ rational maps, with $M = \mathrm{Poly}_d$ or $\mathcal{M}_d$; assume that there is a marked critical point $c(\lambda)$. Let $k(n) \to +\infty$ be a sequence of integers. Then*

$$\frac{1}{d^n + d^{(1-e)k(n)}}[\mathrm{Preper}(k(n), n)] \to T_{\mathrm{bif},c},$$

where e is the generic cardinal of the exceptional locus of f_λ, and $T_{\mathrm{bif},c}$ is the bifurcation current of c.

We will not define here the bifurcation current of a marked critical point, but just mention that it is analoguous to the bifurcation current, and that its support is exactly the activity locus of c in the family.

1.2. The support of μ_{bif} has positive volume. We present in this section the main result of [**AGMV19**] (with T. Gauthier, N. Mihalache and G. Vigny), which is the fact that the support of μ_{bif} in $\mathcal{M}_d$ has positive Lebesgue volume. Since $\mathrm{supp}\ \mu_{\mathrm{bif}} \subsetneq \mathrm{Bif}$ and since Bif is a closed set of empty interior (Corollary 1.1), this is the strongest possible result on the "size" of the set $\mathrm{supp}\ \mu_{\mathrm{bif}}$. In particular, it is strictly stronger that a previous theorem of Gauthier ([**Gau12**]) which states that the *bifurcation locus* (and not the support of μ_{bif}) has maximal Hausdorff dimension (which is weaker than positive volume).

First, we need to recall the definition of a Collet-Eckmann rational map:

DEFINITION 1.5. *A rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ is Collet-Eckmann if there exists $\gamma, \gamma_0 > 0$ such that for all critical value $v \in \mathrm{CV}(f)$,*

$$|(f^n)'(v)| \geq e^{n\gamma - \gamma_0}.$$

In particular, this definition implies that all critical points are in the Julia set (some authors use a different definition and allow some critical points to be captured by attracting cycles). Rational maps which are $(2d-2)$-Misiurewicz as defined in Definition 1.2 are Collet-Eckmann, but the converse is not true, as critical points may be recurrent for a Collet-Eckmann rational map. The trade-off is that Collet-Eckmann rational maps are more abundant than Misiurewicz maps in parameter space.

THEOREM 1.10. [**AGMV19**] *Let $d \geq 2$ and let $[f] \in \mathrm{supp}\,(\mu_{\mathrm{bif}})$. Then for any open neighborhood $\Omega \subset \mathcal{M}_d$ of $[f]$, there is a set $CE_\Omega \subset \Omega \cap \mathrm{supp}\,(\mu_{\mathrm{bif}})$ of Collet-Eckmann maps with:*

$$\mathrm{Vol}_{\mathcal{M}_d}(CE_\Omega)) > 0.$$

The strategy of the proof is as follows: we first give a general sufficient condition for a conjugacy class of rational maps to belong to the support of the bifurcation measure (Theorem 1.11). Then we exhibit a large set of rational maps fulfilling this condition. To this end, we invoke a result of Aspenberg [**Asp13**] which gives a set of positive volume of Collet-Eckmann (conjugacy classes of) rational maps satisfying good properties (see Theorem 1.12). Then we prove that we can choose this set so that its elements satisfy our criterion to be in the support of μ_{bif} (Theorem 1.13).

The first step of the proof uses the so-called large scale condition as a sufficient condition to be in the support of the bifurcation measure.

Let us first introduce some notations: given marked critical points $c_1, \dots, c_m$ and a multiindex $\underline{n} \in \mathbb{N}^m$, we let $\xi^j_{n_j}(\lambda) := f^{n_j}_\lambda(c_j(\lambda))$ for $1 \leq j \leq m$, and $\xi_{\underline{n}} := (\xi^1_{n_1}, \dots, \xi^m_{n_m})$. We let $\mathcal{V}_{\underline{n}} \subset \Lambda \times (\mathbb{P}^1)^m$ denote the graph of $\xi_{\underline{n}}$.

DEFINITION 1.6 (Large scale condition). *We say that a parameter $\lambda_0 \in \Lambda$ satisfies the* large scale condition *at f_{λ_0} for the m-tuple $(c_1, \dots, c_m)$ in Λ if there exist a sequence $\underline{n}_k = (n_{k,1}, \dots, n_{k,m}) \in \mathbb{N}^m$ with $\lim_{k\to+\infty} n_{k,j} = +\infty$ for all $1 \leq j \leq m$, and a basis of neighborhood $\{\Omega_k\}_{k\geq 1}$ of λ_0 in Λ and $\delta > 0$ such that for any $k \in \mathbb{N}$, the connected component of $\mathcal{V}_{\underline{n}_k} \cap \Omega_k \times \Big(\mathbb{D}(\xi^1_{n_{k,1}}(\lambda_0), \delta) \times \cdots \times \mathbb{D}(\xi^m_{n_{k,m}}(\lambda_0), \delta)\Big)$ containing $(\lambda_0, \xi_{\underline{n}_k}(\lambda_0))$ is contained in $\Omega'_k \times \Big(\mathbb{D}(\xi^1_{n_{k,1}}(\lambda_0), \delta) \times \cdots \times \mathbb{D}(\xi^m_{n_{k,m}}(\lambda_0), \delta)\Big)$ for some $\Omega'_k \Subset \Omega_k$.*

This means that a rational map f satisfies the large scale condition if, for appropriate sequences of iterates, the map $\xi_{\underline{n}_k}$ sends a small neighborhood of $[f]$ in the moduli space $\mathcal{M}_d$ to a polydisk of fixed size in $(\mathbb{P}^1)^{2d-2}$ and its graph is vertical-like near $[f]$. This condition is quite flexible, and is for instance satisfied by $(2d-2)$-Misiurewicz maps (and a fortiori by Misiurewicz-Thurston maps), compare Theorem 1.6.

THEOREM 1.11. [**AGMV19**] *Let $f \in \mathrm{Rat}_d$. Assume that $\omega(c) \subset J(f)$ for all $c \in \mathrm{Crit}(f)$, that f has simple critical points and that f satisfies the large scale condition. Then $[f] \in \mathrm{supp}\,(\mu_{\text{bif}})$.*

Following a fruiful idea in complex dynamics, the proof of Theorem 1.11 is based on a phase-parameter transfer (which may be seen as a sort of measurable version of Tan Lei's work [**Tan90**]). These arguments still work in higher dimension, and were used in a very similar way e.g. in [**AB22**].

As mentionned above, the critical orbits of a Collet-Eckmann rational map are in general recurrent. In many cases, it is crucial to control quantitatively this recurrence; this is why many additional conditions on the critical orbits have been considered by different authors. Let us now define more precisely some of these conditions appearing in the papers [**Tsu93**] and [**Asp13**]:

DEFINITION 1.7. *Let $f \in \mathrm{Rat}_d$.*

(1) We say that f satisfies the Topological Collet-Eckmann property (TCE) if there exists $0 < \lambda_0 < 1$ and $r_0 > 0$ such that for $z \in J(f)$, $n \in \mathbb{N}^*$ and for every component W of $f^{-n}(\mathbb{D}(z, r_0))$, we have $\mathrm{diam}(W) \leq \lambda_0^n$.

(2) We say that f satisfies the Backward Collet-Eckamnn property (CE2) if for all $n \geq 0$ and $x \in f^{-n}(\mathrm{Crit}(f))$

$$|(f^n)'(x)| > e^{n\mu - \mu_0}.$$

(3) *We say that f satisfies the Basic Assumption (BA) if there exists $\alpha > 0$ such that for all $v \in f(\mathrm{Crit}(f))$ and $n \geq 0$*

$$\ln |f'(f^n(v))| > -n\alpha.$$

(4) *We say that f satisfies the Free Assumption (FA) if there exist $\eta, \iota > 0$ such that for all $v \in f(\mathrm{Crit}(f))$ and $n > 0$*

$$\sum_{\substack{j=0 \\ \mathrm{dist}(f^j(v),\mathrm{Crit}(f))\leq\eta}}^{n-1} \ln |f'(f^j(v))| > -n\iota.$$

(5) *We say that f satisfies the summability condition (SC) or is summable if for every critical point c, $\sum_{n\geq 1} \|df(f^n(z))\|^{-1} < +\infty$ (in the spherical metric).*

REMARK 1.1. *Let us make a few comments:*

(1) *Any Collet-Eckmann rational map is also summable.*
(2) *There are many equivalent definitions of (TCE), see* [**PRLS03**]*; here we only gave one.*
(3) *It was proved in* [**PRLS03**] *that (CE) implies (TCE), but the converse does not hold in general.*

We will use the notation BA(α), CE(γ, γ_0), FA(η, ι), etc. to refer to those conditions with the specified constants $\alpha, \gamma, \eta, \iota$, etc.

Reformulating the main result of Aspenberg in terms of the conditions CE(γ, γ_0), CE2(μ, μ_0), BA(α) and FA(η, ι) gives the following:

THEOREM 1.12. [**Asp13**] *Assume that $f \in \mathrm{Rat}_d$ is Misiurewicz-Thurston, has simple critical points and is not a flexible Lattès map. Then, there exist $\mu, \mu_0, \gamma, \gamma_0 > 0$ and $\hat{\alpha} > 0$ such that for all $\alpha < \min(\frac{\gamma}{200}, \hat{\alpha})$, there exist $\hat{\eta} > 0$ and $\hat{\iota} > 0$ such that for all $\eta < \hat{\eta}$ and for all $\iota < \hat{\iota}$, the map f is a Lebesgue density point of rational maps satisfying CE(γ, γ_0), CE2(μ, μ_0), BA(α) and FA(η, ι).*

The last and main ingredient of the proof of Theorem 1.10 can be formulated as follows.

THEOREM 1.13. [**AGMV19**] *Let $\gamma, \gamma_0, \mu, \mu_0, \eta, \kappa > 0$ and $\alpha < \gamma/200$. There exists $\iota > 0$ such that any $f \in \mathrm{Rat}_d$ with simple critical points and satisfying CE(γ, γ_0), CE2(μ, μ_0), BA(α), FA(η, ι) satisfies the large scale condition.*

The proof of Theorem 1.13 is long and technical and we will not reproduce it here. Instead, we only give a very brief account of the main ideas:

(1) Using the properties CE(γ, γ_0), CE2(μ, μ_0), BA(α), FA(η, ι), we find for each critical value v of f a large set of integers n (of density close to 1) such that $f^n : \mathbb{D}(v, \delta_n) \to \mathbb{C}$ has small distortion and large derivative $(f^n)'(v)$ (here $\delta_n \asymp \frac{1}{|(f^n)'(v)|}$). This is the main part of the proof, and it follows Tsujii's generalization [**Tsu93**] of Benedicks and Carleson construction [**BC85**], adapted to the complex setting.
(2) We now consider a one-parameter family $(f_\lambda)_{\lambda\in\mathbb{D}}$ such that $f_0 := f$ satisfies the assumptions of Theorem 1.12. Using Step (1) and a phase-parameter transfer argument, we prove that $\xi_n^j : \mathbb{D}(0, \delta_n) \to \mathbb{P}^1$ defined by $\xi_n(\lambda) := f_\lambda^n(v_j(\lambda))$ has small distortion and large derivative at $\lambda = 0$ (comparable to $\frac{1}{\delta_n}$).
(3) Finally, a theorem of Sibony-Wong and a transversality result obtained in [**Ast22**] allow us to get good control of $\xi_n = (\xi_n^j)_{1\leq j\leq n}$ from the estimates obtained at Step (2) for each ξ_n^j separately, allowing us to get the large scale condition for $f = f_0$.

Finally, we note that these results were recently much improved by Lefèvre in his PhD thesis ([**Lef23**]), who proved that Lebesgue a.e. $[f] \in \mathcal{M}_d$ satisfying (SC) and (TCE) are Collet-Eckmann with even better recurrence regularity than (BA) or (FA), hence also satisfy the large scale condition and are in the support of the bifurcation measure.

2. Bifurcations for skew-products

We now move on to the topic of bifurcations in higher dimension, in the general setting of endomorphisms of $\mathbb{P}^k$, of which rational maps on $\mathbb{P}^1$ are a particular case.

The theory of bifurcations for families of endomorphisms of $\mathbb{P}^k$ has only recently been developped by Berteloot-Bianchi-Dupont ([**BBD18**], see Theorem 2.3 below), following a parallel theory for polynomial automorphisms of $\mathbb{C}^2$ developped by Dujardin and Lyubich ([**DL15**]).

The theory is still far from being as developped as in dimension one, and although many important and striking results have been obtained (e.g. the presence of *robust bifurcations*, see [**Duj17**] and [**Taf21**]), it is useful to study and understand special, simpler families to build intuition before tackling the study of bifurcations in the full generality of endomorphisms of $\mathbb{P}^k$, in the same way that the quadratic family serves as a model for one-dimensional rational dynamics.

To this end, and motivated in part by the results of [**ABD$^+$16**], together with F. Bianchi, we started studying bifurcations in families of *regular polynomial skew-products* in the papers [**AB23**] and [**AB22**].

Polynomial skew-products are maps of the form

$$f(z,w) = (p(z), q(z,w)) \tag{2}$$

where p is a polynomial in one complex variable, and q is a polynomial in 2 complex variables. Unless otherwise stated, we will only consider *regular* skew-products, that is, maps of the form (2) which extend to endomorphisms of $\mathbb{P}^2$.

If we write $p(z) = \sum_{k=0}^{d_1} a_k z^k$ and $q(z,w) = \sum_{0 \leq k+\ell \leq d_2} b_{k,\ell} z^k w^\ell$ with $a_{d_1} \neq 0$, then f is regular if and only if $d_1 = d_2$ and $b_{0,d_2} \neq 0$.

The dynamics of these maps was studied in detail in [**Jon99**]. Despite their seemingly simple form, they have already provided examples of dynamical phenomena not displayed by one-dimensional polynomials, see for instance [**ABD$^+$16, Duj16, Duj17, Taf21**].

2.1. General theory of bifurcations for endomorphisms of projective spaces.

Before exposing our results on bifurcations in families of skew-products, we begin with a short description of the ergodic properties of endomorphisms of $\mathbb{P}^k$ and their link with the bifurcation theory recently developped by Berteloot, Bianchi and Dupont. We refer the reader to [**DS10**] for full details.

2.1.1. *Equilibrium measure for endomorphisms of* $\mathbb{P}^k$. An endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ is given in homogeneous coordinates by $k+1$ homogeneous polynomials of degree d:

$$f([x_0 : \ldots : x_k]) = [P_0(x_0, \ldots, x_k) : \ldots : P_k(x_0, \ldots, x_k)]$$

such that $Z(P_0) \cap \ldots \cap Z(P_k) = \{0\}$, where $Z(P_j) = P_j^{-1}(\{0\})$. The integer d is called the algebraic degree of f. The topological degree of f (generic number of preimages) is d^k.

THEOREM 2.1 ([**HP94**],[**BD01**], [**DS03**]). *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be an endomorphism of algebraic degree $d \geq 2$. There exists a unique invariant measure μ_f of maximal entropy $k \log d$. It also satisfies $f^*\mu_f = d^k \mu_f$, is ergodic and mixing.*

The support of μ_f is denoted by $J_k(f)$ and called the small Julia set (to be distinguished from the big Julia set $J_1(f)$ which is the complement of the Fatou set, see [**DS10**]). Repelling cycles are dense in $J_k(f)$, but in contrast to the case of dimension one, there may be some repelling cycles outside of $J_k(f)$. We will refer to repelling cycles contained in $J_k(f)$ as J_k-repelling cycles.

As in the case of rational maps on $\mathbb{P}^1$, a holomorphic family of endomorphisms of (algebraic) degree $d \geq 2$ is a holomorphic map $F : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ of the form

$$F(\lambda, z) = (\lambda, f_\lambda(z)),$$

such that for all $\lambda \in M$, $f_\lambda = F(\lambda, \cdot) : \mathbb{P}^k \to \mathbb{P}^k$ is an endomorphism of algebraic degree d. As before, we will use the notation $(f_\lambda)_{\lambda \in M}$. We let $\mathcal{H}_d(\mathbb{P}^k)$ denote the space of all endomorphisms of $\mathbb{P}^k$ of algebraic degree d.

The theory developped by Mañé-Sad-Sullivan, Lyubich and DeMarco for self-maps of $\mathbb{P}^1$ was extended to families of endomorphisms of $\mathbb{P}^k$ in any dimension by Berteloot, Bianchi and Dupont in [**BBD18, Bia19a**] (see also [**DL15, BD17**] for a parallel theory in the setting of polynomial diffeomorphisms of $\mathbb{C}^2$). Before we state their result, we need to introduce some further notations and definitions.

DEFINITION 2.1. *Let $L : \mathcal{H}_d(\mathbb{P}^k) \to \mathbb{R}$ be the sum of the Lyapunov exponents of the measure of maximal entropy:*

$$L(f) := \int_{J_k(f)} \log |\mathrm{Jac} f(z)| d\mu_f(z)$$

THEOREM 2.2 ([**BB09**], [**Pha05**]). *The function L is continuous and plurisubharmonic.*

In particular, we may define as in dimension 1 the *bifurcation current* $T_{\mathrm{bif}} := dd^c L$. Let $F : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ be a holomorphic family of endomorphisms of $\mathbb{P}^k$ of degree $d \geq 2$ (i.e. $F(\lambda, z) = (\lambda, f_\lambda(z))$).

2.1.2. *Berteloot-Bianchi-Dupont's theorem.* We let

$$\mathcal{J} := \{\gamma : M \to \mathbb{P}^k, \gamma \text{ is holomorphic and } \gamma(\lambda) \in J_k(f_\lambda)\ \} \tag{3}$$

and we let $\mathcal{F} : \mathcal{J} \to \mathcal{J}$ denote the map induced by F, that is,

$$\mathcal{F}(\gamma) = (\lambda \mapsto f_\lambda \circ \gamma(\lambda))$$

DEFINITION 2.2. *An equilibrium web for F is a measure $\mathcal{M}$ on $\mathcal{J}$ such that*

(1) $\mathcal{M}$ is $\mathcal{F}$-invariant and its support is a compact subset of $\mathcal{J}$
(2) for every $\lambda \in M$, $\mu_\lambda = \int_{\mathcal{J}} \delta_{\gamma(\lambda)} d\mathcal{M}(\gamma)$.

DEFINITION 2.3. *Given $\gamma \in \mathcal{J}$, we let $\Gamma_\gamma \subset M \times \mathbb{P}^k$ denote its graph. An equilibrium lamination for F is a relatively compact subset $\mathcal{L}$ of $\mathcal{J}$ such that*

(1) $\Gamma_\gamma \cap \Gamma_{\gamma'} = \emptyset$ for every $\gamma, \gamma' \in \mathcal{L}$
(2) $\mu_\lambda(\{\gamma(\lambda), \gamma \in \mathcal{L}\}) = 1$ for every $\lambda \in M$
(3) for every $\gamma \in \mathcal{L}$, Γ_γ does not meet the grand orbit of the critical set of F
(4) the map $\mathcal{F} : \mathcal{L} \to \mathcal{L}$ is d^k to 1.

DEFINITION 2.4. *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of endomorphisms, and let $\lambda_0 \in M$. We say that λ_0 is a Misiurewicz parameter if there exists a holomorphic map $\gamma : U \to \mathbb{P}^k$ (where U is an open neighborhood of λ_0) such that*

(1) $\gamma(\lambda) \in J_k(f_\lambda)$ and is a repelling periodic point for all $\lambda \in U$
(2) $\gamma(\lambda_0) \in f^{n_0}(\mathrm{Crit}(f_{\lambda_0}))$ for some $n_0 \in \mathbb{N}^*$
(3) the graph Γ_γ is not contained in $f^{n_0}(\mathrm{Crit}(f))$.

THEOREM 2.3 ([**BBD18**]). *Let M be a simply connected complex manifold and let $F = (f_\lambda)_{\lambda\in M}$ be a holomorphic family of endomorphisms of degree $d \geq 2$ on $\mathbb{P}^k$. The following are equivalent:*

(1) the repelling J_k-cycles move holomorphically
(2) the function L is pluriharmonic on M
(3) F admits an equilibrium web
(4) F admits an equilibrium lamination
(5) any $\lambda_0 \in M$ admits a neighborhood U such that:

$$\liminf \frac{1}{d^{kn}} \|(F^n)_*[\mathrm{Crit}(F)]_{U\times\mathbb{P}^k}\| = 0$$

(6) there are no Misiurewicz parameters in M.

Here, $\|\cdot\|$ denotes the mass of a current.

2.1.3. *A parametric equidistribution result in higher dimension.* We now state an equidistribution result obtained in [**AB23**], the first in higher dimension. Contrary to the other results from [**AB23**], this parametric equidistribution theorem applies not just to skew-products but in the full family of degree d endomorphisms of $\mathbb{P}^k$.

THEOREM 2.4. [**AB23**] *Let $(f_\lambda)_{\lambda\in\mathcal{H}_d(\mathbb{P}^k)}$ be the family of all endomorphisms of $\mathbb{P}^k$ of degree $d \geq 2$. For all $\eta \in \mathbb{C}$ outside of a polar subset, we have*

$$\frac{1}{d^{kn}}[\mathrm{Per}_n(\eta)] \to T_{\mathrm{bif}},$$

where $\mathrm{Per}_n(\eta) := \{\lambda\colon \exists z \in J_k(f_\lambda)$*of exact period n for f_λ and such that* $\mathrm{Jac}_z f_\lambda = \eta\}$.

We note here than contrary to similar results in dimension one (where the case $\eta \in \mathbb{D}$ is special), we are not able to give any explicit value of η for which the equidistribution holds.

Since the proof of Theorem 2.4 is not very long, we have chosen to include it here. The general strategy follows the main line of the one dimensional case and is based on techniques and tools from pluripotential theory. However, one of the difficulties we have to face is the possible presence of infinitely many non-repelling cycles for an endomorphism of $\mathbb{P}^k$ – something which is excluded for $k = 1$ by a Theorem due to Fatou. We thus need more quantitative estimates on the number of repelling cycles with small multiplier, which are related to the approximation formula for the Lyapunov exponent valid in any dimension established in [**BDM08**]

Let M be a connected complex manifold, and let $F : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ be a holomorphic map, defining a holomorphic family $F(\lambda, z) = (\lambda, f_\lambda(z))$ of endomorphisms of $\mathbb{P}^k$. We assume here the following:

$$\forall n \in \mathbb{N}^* \quad \exists \lambda \in M \text{ such that for all periodic points of exact period } n \text{ for } f_\lambda :$$

$$\det(Df_\lambda^n(z) - \mathrm{id}) \neq 0. \tag{4}$$

Observe that the above condition is for instance satisfied if the family contains the map $[z_0 : z_1 : \dots z_k] \mapsto [z_0^d : z_1^d : \dots : z_k^d]$. Denote by Jac the determinant of the Jacobian matrix and set

$$\widetilde{\mathrm{Per}_n^J} = \{(\lambda, \eta) \in M \times \mathbb{C} : \exists z \in \mathbb{P}^k \text{ of exact period } n \text{ for } f_\lambda \text{ and such that } \mathrm{Jac} f_\lambda^n(z) = \eta\}.$$

Let Per_n^J be the closure of $\widetilde{\mathrm{Per}_n^J}$ in $M \times \mathbb{C}$. The following result in particular implies that Per_n^J is an analytic hypersurface in $M \times \mathbb{C}$.

PROPOSITION 2.1. *Let $(f_\lambda)_{\lambda \in M}$ be a holomorphic family of endomorphisms of $\mathbb{P}^k$ satisfying* (4). *There exists a sequence of holomorphic maps $P_n : M \times \mathbb{C} \to \mathbb{C}$ such that*

(1) for all $\lambda \in M$, $P_n(\lambda, \cdot)$ is a monic polynomial of degree $\delta_n \sim \frac{d^{nk}}{n}$;
(2) $P_n(\lambda, \eta) = 0$ if and only $(\lambda, \eta) \in \mathrm{Per}_n^J$.

Moreover, if $(\lambda, \eta) \in \mathrm{Per}_n^J \backslash \widetilde{\mathrm{Per}_n^J}$, there exists $z \in \mathbb{P}^k$ and $m < n$ dividing n such that $f_\lambda^m(z) = z$, $\mathrm{Jac}(f_\lambda^n)(z) = \eta$, and 1 is an eigenvalue of $Df_\lambda^n(z)$.

PROOF. Let Ω_n denote the set of $\lambda \in M$ such that periodic cycles of period less than or equal to n do not have 1 as an eigenvalue. By Assumption (4), Ω_n is Zariski open in M. Let $p_n \colon \Omega_n \times \mathbb{C} \to \mathbb{C}$ be defined by

$$p_n(\lambda, \eta) := \prod_{z \in E_n(\lambda)} (\eta - \mathrm{Jac}\, f_\lambda^n(z))$$

where $E_n(\lambda)$ denotes the set of periodic points of exact period n for f_λ.

By the implicit function theorem and the definition of Ω_n, p_n is holomorphic on $\Omega_n \times \mathbb{C}$. Since it is locally bounded, Riemann's extension theorem implies that it extends holomorphically to all of $M \times \mathbb{C}$.

Now notice that for all $\lambda \in \Omega_n$, n divides the multiplicity of every root w of the polynomial $p_n(\lambda, \cdot)$. Indeed, if $z \in E_n(\lambda)$ is such that $w = \mathrm{Jac}\, f_\lambda^n(z)$, then it is also the case for the other points of the cycle, namely the $f^m(z)$, $0 \leq m \leq n-1$. So, for every $\lambda \in \Omega_n$, there is a unique monic polynomial map $P_n(\lambda, \cdot)$ such that $P_n(\lambda, \cdot)^n = p_n(\lambda, \cdot)$. Its degree δ_n satisfies $\delta_n \sim \frac{\mathrm{card}\, E_n(\lambda)}{n}$, and by classical computations $\mathrm{card}\, E_n(\lambda) \sim d^{nk}$. The map $\lambda \mapsto P_n(\lambda, \cdot)$ is holomorphic on Ω_n and locally bounded, hence extends holomorphically to M.

Finally, for all $(\lambda, \eta) \in \Omega_n \times \mathbb{C}$, $P_n(\lambda, \eta) = 0$ if and only if $\lambda \in \widetilde{\mathrm{Per}_n^J}$. If $\lambda \notin \Omega_n$, by considering a sequence $(\lambda_i, \eta_i) \in \Omega_n \times \mathbb{C}$ converging to (λ, η), we find that $P_n(\lambda, \eta) = 0$ if and only if f_λ has a cycle with Jacobian η whose period divides n. The drop in period may occur if two points of the cycle collide, creating an eigenvalue 1. □

DEFINITION 2.5. *For $\eta \in \mathbb{C}$, we denote by $\mathrm{Per}_n^J(\eta)$ the analytic hypersurface of M defined by $\mathrm{Per}_n^J(\eta) := \{\lambda \in M : (\lambda, \eta) \in \mathrm{Per}_n^J\}$ and by $L_n \colon M \times \mathbb{C} \to \mathbb{C}$ the function $L_n(\lambda, \eta) := d^{-nk} \log |P_n(\lambda, \eta)|$.*

By the Lelong-Poincaré equation, we have that $dd^c_{\lambda,\eta} L_n = d^{-nk}[\mathrm{Per}_n^J]$, where $[\mathrm{Per}_n^J]$ is the current of integration on Per_n^J. Likewise, we have $dd^c_\lambda L_n(\cdot, \eta) = d^{-nk}[\mathrm{Per}_n^J(\eta)]$.

THEOREM 2.5. **[AB23]** *Let $(f_\lambda)_{\lambda \in M}$ be a holomorphic family of endomorphisms of $\mathbb{P}^k$ containing the map $[z_0, z_1, \dots z_k] \mapsto [z_0^d, z_1^d, \dots, z_k^d]$. Then $L_n \to L$, the convergence taking place in $L^1_{\mathrm{loc}}(M \times \mathbb{C})$. In particular, for any $\eta \in \mathbb{C}$ outside of a polar set, we have $d^{-nk}[\mathrm{Per}_n^J(\eta)] \to T_{\mathrm{bif}}$.*

Recall that we denote by $L : M \to \mathbb{R}^+$ the sum of the Lyapunov exponents of f_λ with respect to its equilibrium measure μ_λ. The assumptions of Theorem 2.5 are clearly satisfied when we consider the family of all endomorphisms of $\mathbb{P}^k$ of a given algebraic degree. Thus Theorem 2.5 implies Theorem 2.4. In order to prove the convergence in Theorem 2.5, in the spirit of [**BB09**] we first study the convergence of the following modifications of L_n:

(1) $L_n^+(\lambda, \eta) = (nd^{nk})^{-1} \sum_{z\in E_n(\lambda)} \log^+ |\eta - \eta_n(z, \lambda)|$ where $\eta_n(z, \lambda) := \mathrm{Jac} f_\lambda^n(z)$
(2) $L_n^r(\lambda) = (2\pi d^{nk})^{-1} \int_0^{2\pi} \log |P_n(\lambda, re^{it})|dt$.

We will need the following quantitative approximation of L by Berteloot-Dupont-Molino.

LEMMA 2.1 ([**BDM08**], Lemma 4.5). *Let f be an endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$. Let $\epsilon > 0$ and let $R_n^\epsilon(f)$ be the set of repelling periodic points z of exact period n for f, such that $\left|\frac{1}{n}\log|\mathrm{Jac} f^n(z)| - L(f)\right| \leq 2\epsilon$. Then for n large enough, $\mathrm{card}\, R_n^\epsilon(f) \geq d^{nk}(1-\epsilon)^3$.*

LEMMA 2.2. *For all $\eta \in \mathbb{C}$, $L_n^+(\cdot, \eta) \to L$ pointwise and in $L^1_{\mathrm{loc}}(M)$.*

PROOF. In what follows, the notation $O(\cdot)$ denotes quantities that are bounded by constants depending only on f_λ and η, and not on n or ϵ. Fix $\eta \in \mathbb{C}$ and $\epsilon > 0$. We have, for all $n \in \mathbb{N}^*$:

$$|L_n^+(\lambda, \eta)| \leq \frac{\mathrm{card}\, E_n(\lambda)}{d^{nk}} \sup_{z\in\mathbb{P}^k} \|Df_\lambda(z)\|$$

which is locally bounded from above. Moreover,

$$\begin{aligned}
L_n^+(\lambda, \eta) =& \frac{1}{nd^{nk}} \left(\sum_{z\in R_n^\epsilon(\lambda)} \log|\eta - \eta_n(z, \lambda)| + \sum_{z\in E_n(\lambda)\setminus R_n^\epsilon(\lambda)} \log^+ |\eta - \eta_n(z, \lambda)| \right) \\
=& \frac{1}{nd^{nk}} (\sum_{z\in R_n^\epsilon(\lambda)} \log|\eta_n(z, \lambda)| + O\left((L(\lambda) - 2\epsilon)^{-n}\right)) \\
&+ O\left(\frac{\mathrm{card}\, E_n(\lambda) \setminus R_n^\epsilon(\lambda)}{nd^{nk}} \log(|\eta| + (L(\lambda) + 2\epsilon)^n) \right)
\end{aligned}$$

For any $\epsilon > 0$ small enough, $\lim_{n\to\infty}(L(\lambda) - 2\epsilon)^{-n} = 0$. By Lemma 2.1, for n large enough, $\frac{\mathrm{card}\, E_n(\lambda)\setminus R_n^\epsilon(\lambda)}{d^{nk}} = O(\epsilon)$. Hence, for n large enough,

$$L_n^+(\lambda, \eta) = \frac{1}{nd^{nk}} \sum_{z\in R_n^\epsilon(\lambda)} \log|\eta_n(z, \lambda)| + O(\epsilon) = L(\lambda) + O(\epsilon).$$

Therefore the sequence of maps L_n^+ converges pointwise to $(\lambda, \eta) \mapsto L(\lambda)$ on $M \times \mathbb{C}$. Since the L_n's are psh and locally uniformly bounded from above, by Hartogs lemma, the convergence also happens in L^1_{loc}. □

LEMMA 2.3. *For all $r > 0$, $L_n^r \to L$ pointwise and in $L^1_{\mathrm{loc}}(M)$.*

The proof is a straightforward adaptation of that of [**BB09**, Theorem 3.4 (2)].

PROOF OF THEOREM 2.5. First, note that the sequence L_n does not converge to $-\infty$. Indeed, denote by λ_0 the parameter corresponding to the map $[z_0, z_1, \ldots z_k] \mapsto [z_0^d, z_1^d, \ldots, z_k^d]$. For all n-periodic cycles at f_{λ_0}, the eigenvalues at those cycles are either d or 0; in particular, the modulus of the Jacobian takes values in the set $\{0\} \cup \{d^{kn}, n \in \mathbb{N}\}$. Let us choose $\eta_0 := i$: we claim that $(\lambda_0, \eta_0) \notin \overline{\bigcup_{n\in\mathbb{N}^*} \mathrm{Per}_n}$. Indeed, for any $\epsilon > 0$, there exists a neighborhood

V of the map $[z_0, z_1, \dots z_k] \mapsto [z_0^d, z_1^d, \dots, z_k^d]$ such that for all $\lambda \in V$, any eigenvalue ρ of any m-periodic cycle of f_λ satisfies either $|\rho| < \epsilon^m$ or $(d-\epsilon)^m < |\rho| < (d+\epsilon)^m$. Therefore, there exists $N = N(\epsilon) \in \mathbb{N}$ such that the modulus of the Jacobian at any cycle of f_λ of period larger than N avoids the annulus $\{\frac{3}{4} < |z| < \frac{4}{3}\}$. Moreover, as mentioned above, the map $[z_0, z_1, \dots z_k] \mapsto [z_0^d, z_1^d, \dots, z_k^d]$ has no cycles with Jacobian in the larger annulus $\{\frac{1}{2} < |z| < 2\}$. Since there are only finitely many cycles of period at most N, we conclude by continuity that up to restricting V, no f_λ with $\lambda \in V$ has a cycle with Jacobian in $\{\frac{3}{4} < |z| < \frac{4}{3}\}$. In particular, the sequence $L_n(\lambda_0, \eta_0)$ does not converge to $-\infty$, as desired.

Let now $\phi : M \times \mathbb{C} \to \mathbb{R}$ be a psh function such that a subsequence L_{n_j} converges L^1_{loc} to ϕ. Let $(\lambda_0, \eta_0) \in M \times \mathbb{C}$. We have to prove that $\phi(\lambda_0, \eta_0) = L(\lambda_0)$.

First, let us prove that $\phi(\lambda_0, \eta_0) \le L(\lambda_0)$. Take $\epsilon > 0$ and let B_ϵ be the ball of radius ϵ centered at (λ_0, η_0) in $M \times \mathbb{C}$. Using the submean inequality and the L^1_{loc} convergence of L_n^+, we have

$$\phi(\lambda_0, \eta_0) \le \frac{1}{|B_\epsilon|} \int_{B_\epsilon} \phi \le \frac{1}{|B_\epsilon|} \lim_j \int_{B_\epsilon} L_{n_j} \le \frac{1}{|B_\epsilon|} \lim_j \int_{B_\epsilon} L_{n_j}^+ \le \frac{1}{|B_\epsilon|} \int_{B_\epsilon} L.$$

Then letting $\epsilon \to 0$, we have that $\phi(\lambda_0, \eta_0) \le L(\lambda_0)$, which gives the desired inequality.

Now let us prove the opposite inequality. Assume for now that $\eta_0 \neq 0$. Let $r_0 = |\eta_0|$, and let us first notice that

$$\text{for almost every } t \in S^1, \quad \limsup_j L_{n_j}(\lambda_0, r_0 e^{it}) = L(\lambda_0). \tag{5}$$

Indeed, for any $t \in S^1$ we have

$$\lim_j L_{n_j}(\lambda_0, r_0 e^{it}) \le \limsup_j L_{n_j}^+(\lambda_0, r_0 e^{it}) = L(\lambda_0) \tag{6}$$

and by Fatou's lemma (applied to the functions $t \mapsto -L_{n_j}(\lambda_0, r_0 e^{it})$, which are bounded from below by a constant) and the pointwise convergence of $L_n^{r_0}$ we get:

$$\begin{aligned} L(\lambda_0) = \lim_n L_n^{r_0}(\lambda_0) &= \limsup_j \frac{1}{2\pi} \int_0^{2\pi} L_{n_j}(\lambda_0, r_0 e^{it}) dt \\ &\le \frac{1}{2\pi} \int_0^{2\pi} \limsup_j L_{n_j}(\lambda_0, r_0 e^{it}) dt \end{aligned}$$

which, together with (6), concludes the proof of (5).

Suppose now to obtain a contradiction that $\phi(\lambda_0, \eta_0) < L(\lambda_0)$. Since L is continuous and ϕ is upper semi-continuous, there is $\epsilon > 0$ and a neighbourhood V_0 of (λ_0, η_0) such that $\phi(\lambda, \eta) - L(\lambda) < -\epsilon$ for all $(\lambda, \eta) \in V_0$, We may assume without loss of generality that $V_0 = B_0 \times \mathbb{D}(\eta_0, \gamma)$, where B_0 is a ball containing λ_0. Hartogs' Lemma then gives

$$\limsup_j \sup_{V_0} L_{n_j} - L \le \sup_{V_0} \phi - L \le -\epsilon.$$

But this contradicts (5).

Therefore, we have proved that any convergent subsequence of L_n in the L^1_{loc} topology of $M \times \mathbb{C}$ must agree with L on $M \times \mathbb{C}^*$. Since $M \times \{0\}$ is negligible, this proves that L_n converges in L^1_{loc} to L on $M \times \mathbb{C}$. The proof is complete. $\square$

2.2. The family of quadratic skew-products. We now turn our attention to a very specific, low-dimensional family of endomorphisms of $\mathbb{P}^2$: the family of *quadratic skew-products*, i.e., polynomial skew-products of (algebraic) degree 2, that are in this context the analogue of the family $z^2 + c$. By means of an affine change of coordinates, the dynamical study of this family can be reduced to that of the family

$$(*) \qquad f_\lambda : (z, w) \mapsto (z^2 + d, w^2 + az^2 + bz + c)$$

with d and $\lambda := (a, b, c)$ as (complex) parameters. Since bifurcations due to the parameter d are of one-dimensional nature, we fix here $p(z) := z^2 + d$ and consider the parameter space $\mathbb{C}^3$ of the family $\mathbf{Sk}(p, 2) := \{f_\lambda : (a, b, c) \in \mathbb{C}^3\}$. Our goal here is to understand the geometry of the bifurcation locus in this specific family.

2.2.1. *Degeneration of T_{bif} near infinity.* We are especially interested in parameters near the boundary of this space, i.e., near the hyperplane at infinity, that we denote by $\mathbb{P}^2_\infty$. The following is our first main result, giving a complete description of the bifurcation locus near $\mathbb{P}^2_\infty$ from both a topological and measure-theoretical point of view. We denote by J_p the Julia set of p. Given $z \in \mathbb{C}$, we set $E_z := \{[a, b, c]\colon az^2 + bz + c = 0\} \subset \mathbb{P}^2_\infty$ and $E := \cup_{z \in J_p} E_z$. An analogous result for quadratic rational maps is proved in [**BG15**].

THEOREM 2.6. [**AB23**] *The accumulation on $\mathbb{P}^2_\infty$ of the bifurcation locus of the family* (*) *coincides with E. Moreover, the bifurcation current T_{bif} on $\mathbb{C}^3$ extends as a positive closed current $\tilde{T_{\text{bif}}}$ to $\mathbb{P}^3 = \mathbb{C}^3 \cup \mathbb{P}^2_\infty$ and*

$$\tilde{T_{\text{bif}}} \wedge [\mathbb{P}^2_\infty] = \int_{J_p} [E_z] \mu_p(z).$$

The proof of this result relies on several ingredients. The first is a decomposition for the bifurcation current (and locus), see Theorem 2.8. We then prove that special dynamically defined hypersurfaces $\text{Per}_n^v(\eta)$ in $\mathbb{C}^3$ equidistribute towards the bifurcation current T_{bif} and $\tilde{T_{\text{bif}}}$ (Theorem 2.7 below). This last theorem is a straightforward adaptation to the skew-product setting of the more general equidistribution theorem (Theorem 2.4) proved above.

Moreover, we can precisely control the intersections of these hypersurfaces with $\mathbb{P}^2_\infty$. We thus obtain the convergences

$$\frac{1}{d^{2n}} [\text{Per}_n^v(\eta)] \to \tilde{T_{\text{bif}}} \quad \text{and} \quad \frac{1}{d^{2n}} [\text{Per}_n^v(\eta)] \wedge [\mathbb{P}^2_\infty] \to \int_{z \in J_p} [E_z] \mu_p.$$

Theorem 2.6 then reduces to proving that the convergences above imply that

$$\frac{1}{d^{2n}} [\text{Per}_n^v(\eta)] \wedge [\mathbb{P}^2_\infty] \to \tilde{T_{\text{bif}}} \wedge [\mathbb{P}^2_\infty],$$

which is a problem of intersection of currents. To do this, we exploit the theory of *horizontal positive closed currents* as developed by Dujardin [**Duj04**], see also [**DS06**]. This requires proving some uniform estimates on the directions at which the bifurcation locus approaches $\mathbb{P}^2_\infty$.

DEFINITION 2.6. *For any $\eta \in \mathbb{C}$, we set* $\text{Per}_n^v(\eta) := \{\lambda \in M : P_n^v(\lambda, \eta) = 0\}$.

Let $f(z, w) = (p(z), q(z, w))$ be a polynomial skew-product, and let $Q_z^n(w) := \pi_2 \circ f^n(z, w)$. We will say that f is *vertically expanding* if there exists constants $C > 0$ and $A > 1$ such that for all $n \in \mathbb{N}$ and $(z, w) \in J_2(f)$, $|Q_z^n(w)| \geq CA^n$. This condition is a convenient adaptation of the classical notion of hyperbolicity in the context of a skew-product.

THEOREM 2.7. [**AB23**] *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of polynomial skew-products of $\mathbb{P}^2$ of degree $d \geq 2$ over a fixed base p. Assume that the family contains a vertical expanding map. For all $\eta \in \mathbb{C}$ outside of a polar subset, we have $d^{-kn}[\mathrm{Per}_n^v(\eta)] \to T_{\mathrm{bif}}$.*

Finally, the last ingredient is a description of the bifurcation current as an average of bifurcations associated to each fibers. The set Bif_z denotes the bifurcation locus of the fiber $z \in J_p$, which may be defined as the set of parameters around which the vertical Julia set J_z (i.e. the non-normality locus of $\{Q_z^n : n \in \mathbb{N}\}$) does not move locally holomorphically. The current $T_{\mathrm{bif},z}$ is an associated $(1,1)$ positive closed bifurcation current, whose support is exactly Bif_z.

THEOREM 2.8. [**AB23**] *Let $f_\lambda(z,w) = (p(z), q_\lambda(z,w))$, $\lambda \in M$, be a holomorphic family of polynomial skew-products of degree d over a fixed base. Then*

$$T_{\mathrm{bif}} = \int_{z\in J_p} T_{\mathrm{bif},z}\mu_p \quad \textit{and} \quad \mathrm{Bif}\left((f_\lambda)_{\lambda\in M}\right) = \overline{\bigcup_{z\in J_p} \mathrm{Bif}_z}.$$

We note that Theorem 2.8 has since been extended in [**DT21**] to the slightly more general setting of an endomorphism of $\mathbb{P}^k$ preserving a linear fibration.

2.2.2. *Hyperbolic components.* Equipped with the description above of the geometry of the bifurcation locus near infinity, we turn our attention to its complement, and in particular to the characterization of the *hyperbolic components*. Notice that, in order for those to exist, p must be hyperbolic; however, in $\mathbf{Sk}(p,d)$ the more natural notion is to ask for vertical expansion rather than full hyperbolicity.

DEFINITION 2.7 ([**Jon99**]). *Let $f(z,w) = (p(z), q(z,w))$ be a polynomial skew product and $Z \subset \mathbb{C}$ be invariant for p. We say that f is* vertically expanding over Z *if there exist constants $c > 0$ and $K > 1$ such that $|(Q_z^n)'(w)| \geq cK^n$ for every $z \in Z$, $w \in J_z$ and $n \geq 1$.*

For polynomials on $\mathbb{C}$, hyperbolicity is equivalent to the fact that the closure of the post-critical set is disjoint from the Julia set. In our situation, we have the following analogous characterization.

THEOREM 2.9 ([**Jon99**]). *Let $f(z,w) = (p(z), q(z,w))$ be a polynomial skew product. Then f is vertically expanding over Z if and only if $D_Z \cap J_Z = \emptyset$, and the following conditions are equivalent:*

(1) f is hyperbolic on its Julia set;
(2) $D_{J_p} \cap J = \emptyset$;
(3) p is hyperbolic, and f is vertically expanding over J_p.

The stability of a polynomial skew product as in (*) is hence determined by the behaviour of the critical points of the form $(z,0)$ with $z \in J_p$. As is the case for polynomials, when all these points escape to infinity by iteration, the map is hyperbolic. It is however not clear a priori that the presence of a hyperbolic map in a component forces all the other maps in the same stability component to be hyperbolic.

In our next result we solve this general problem in the setting of polynomial skew-products, thus giving meaning to the expression *hyperbolic components* here).

THEOREM 2.10. [**AB23**] *Let $(f_\lambda)_{\lambda\in M}$ be a stable family of polynomial skew-products, and let λ_0 be a parameter.*

(1) *If $(f_\lambda)_{\lambda\in M}$ has constant base p and f_{λ_0} is vertically expanding over J_p, then for all $\lambda \in M$, f_λ is vertically expanding over J_p.*
(2) *If f_{λ_0} is hyperbolic, then for all $\lambda \in M$, f_λ is hyperbolic.*

We note that the statement of Theorem 2.10 is still open in the full generality of endomorphisms of $\mathbb{P}^k$. Since the proof of Theorem 2.10 is also not very long, we choose to include it here.

LEMMA 2.4. *Let f be a polynomial skew product with base p. Assume $(z,w) \in J_p \times \mathbb{C}$ is accumulated by D_{J_p}. Then there exists a sequence $(z_m, w_m) \in J_p \times \mathbb{C}$ of iterates of critical points such that $(z_m, w_m) \to (z,w)$ and z_m is a repelling periodic point for p.*

PROOF OF LEMMA 2.4. By assumption, there is a sequence $(y_m, c_m) \in J_p \times \mathbb{C}$, such that $q'_{y_m}(c_m) = 0$ and $f^{n_m}(y_m, c_m) \to (z,w)$. Given any $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that $\|f^{n_m}(y_m, c_m) - (z,w)\| \le \epsilon$ for all $m \ge M$. Since f^{n_m} is continuous, there exists $\delta_m > 0$ such that if $\|(z_m, c'_m) - (y_m, c_m)\| \le \delta_m$, then $\|f^{n_m}(z_m, c'_m) - f^{n_m}(y_m, c_m)\| \le \epsilon$. This implies that $\|f^{n_m}(z_m, c'_m) - (z,w)\| \le 2\epsilon$. Since repelling periodic points are dense in J_p, we can find z'_m periodic and repelling arbitrarily close to y_m. We can then take c'_m such that $(z'_m, c'_m) \in C_{J_p}$ is δ_m-close to (y_m, c_m). The point $(z_m, w_m) := f^{n_m}(z_m, c'_m)$ is then 2ϵ-close to (z,w). Since z'_m is periodic and repelling for p, the same holds for $z_m = p^{n_m}(z'_m)$. Since $\epsilon > 0$ was arbitrary, the lemma is proved. $\square$

PROOF OF THEOREM 2.10. Assume by contradiction that there exists λ_1 such that f_{λ_1} is not vertically expanding. We can replace our parameter space with any relatively compact connected open subset containing λ_0 and λ_1. By Theorem 2.9, there exists $(z,w) \in J_{f(\lambda_1)}$ such that (z,w) is accumulated by the post-critical set of f_{λ_1} over J_p. By Lemma 2.4, there is a sequence (z_m, w_m) of iterates of critical points such that z_m is periodic for p and $(z_m, w_m) \to (z,w)$.

We first treat the case where it is possible to follow holomorphically all critical points over J_p as holomorphic functions of the parameter λ. Notice that this is the case in particular for the polynomial skew-products of degree 2, whose critical points are of the form $(z, 0)$ (and so independent from λ). Set

$$h_m(\lambda) := f_\lambda^{n_m}(y_m, c_m(\lambda))$$

where $f_{\lambda_1}^{n_m}(y_m, c_m(\lambda_1)) = (z_m, w_m)$, and $(y_m, c_m(\lambda))$ is a critical point of f_λ. By definition we have $(\lambda, h_m(\lambda))$ is in the postcritical set. We write $h_m(\lambda) =: (z_m, w_m(\lambda))$.

Since $(z,w) \in J_{f(\lambda_1)}$, there exists a sequence of repelling cycles of the form $(z_m, \gamma_m(\lambda_1)$ converging to (z,w) (by the lower semi-continuity of $z \mapsto J_z$ and the density of repelling cycles). Since f_λ is stable, repelling cycles can be followed holomorphically. We denote by $(z_m, \gamma_m(\lambda))$ the motion of $(z_m, \gamma(\lambda_1))$. Again by the stability of the family, since there are no Misiurewicz parameters, we must have $\gamma_m(\lambda) \ne w_m(\lambda)$ for all m and for all λ. Since the sequence γ_m is uniformly bounded, it is normal and we can assume that γ_m converges to some holomorphic map γ with $\gamma(\lambda_1) = w$.

CLAIM 1. *The sequence $(w_m(\lambda))$ is also normal.*

Assuming this claim, we can get the desired contradiction by taking a limit $w(\lambda)$ for the sequence $w_m(\lambda)$. Indeed, recall that $\gamma(\lambda) \ne w_m(\lambda)$ for all λ and m. Since $\gamma(\lambda_1) = w(\lambda_1)$, by Hurwitz's Theorem the only possibility if that $\gamma(\lambda) \equiv w(\lambda)$ for all λ. Since $\gamma(\lambda_0) \ne w(\lambda_0)$ by assumption, this gives the desired contradiction.

PROOF OF CLAIM 1. The argument is classical, see for instance [**MSS83**]. Since the family is stable, $w_m(\lambda)$ avoids the repelling cycles for all m and λ. Let $a_m(\lambda), b_m(\lambda)$ be two sequences of (holomorphic motions of) repelling periodic points in the fibre z_m. Up to passing to subsequences, we can assume that $a_m(\lambda) \to a(\lambda)$ and $b_m(\lambda) \to b(\lambda)$ (as holomorphic functions in λ). We can also assume that $|a_m(\lambda) - b_m(\lambda)| \geq \epsilon_0 > 0$ for all m and λ. Then, for all λ, we have $w_m(\lambda) \notin \{a_m(\lambda), b_m(\lambda), \infty\}$. It follows that the family $g_m(\lambda) := \frac{w_m(\lambda) - a_m(\lambda)}{b_m(\lambda) - a_m(\lambda)}$ avoids $0, 1, \infty$, hence is normal by Montel Theorem and converges, up to extraction, to some $g(\lambda)$. Since $|a_m(\lambda) - b_m(\lambda)| \geq \epsilon_0$, the sequence $w_m(\lambda) = a_m(\lambda) + g_m(\lambda) \cdot (b_m(\lambda) - a_m(\lambda))$ converges to $w(\lambda) := a(\lambda) + g(\lambda) \cdot (b(\lambda) - a(\lambda))$. The claim is proved. □

We now explain how to adapt the above arguments in the case where it is not possible to follow all critical points as holomorphic functions of λ. As before, we start with sequences of integers n_m and points $y_m \in J_p$ such that $f_{\lambda_1}^{n_m}(y_m, c_m)$ accumulates to some point in $J(f_{\lambda_1})$, and c_m is a critical point of q_{y_m,λ_1}. The accumulation point in $J(f_{\lambda_1})$ can also be accumulated by repelling periodic points $(z_m, \gamma_m(\lambda_1))$.

We now define the function
$$h_m(\lambda) := \prod_{c_i} (f^{n_m}(y_m, c_i) - \gamma_m(\lambda)),$$
where the product is taken over the set of critical points c_i of $q_{y_m,\lambda}$ whose orbits are bounded. Observe now that the function h_m is holomorphic. Indeed, *for a fixed* m, it is always possible to mark the critical points of $q_{y_m,t}$ as holomorphic functions $c_i(t)$, up to passing to a reparametrization $\phi(t) = \lambda$, where ϕ is a finite branched cover.

Since the family is stable, each critical point $c_i(t)$ either has bounded orbit for all t or unbounded orbit for all t. Therefore, $t \mapsto h_m \circ \phi(t)$ is holomorphic, and since $\lambda \mapsto h_m(\lambda)$ is continuous and holomorphic outside the branch locus of ϕ, it is also holomorphic on the whole family. Moreover, the sequence (h_m) is locally uniformly bounded in λ, hence normal; and for all m and λ we must have $h_m(\lambda) \neq 0$ since otherwise this would create a Misiurewicz parameter, contradicting the stability of the family. From there, the proof works as in the previous case. □

For families of polynomial skew-products, it thus makes sense to talk about *hyperbolic components* (respectively *vertically expanding components*), i.e., stable components whose elements are (all) hyperbolic (respectively, vertically expanding). We now give a classification of hyperbolic components that are analoguous to the so-called *shift locus* from dimension 1.

More precisely, let $\mathcal{D}$ be the set of parameters for which all critical points in $J_p \times \mathbb{C}$ escape, and let $\mathcal{D}' \subset \mathcal{D}$ be the subset of parameters λ for which there is an arc joining λ to $\mathbb{P}^2_\infty \backslash E$ inside $\mathcal{D}$. We note that these are not the only unbounded hyperbolic components in $\mathbf{Sk}(p, 2)$, see [**AB23**] for more details.

Set
$$\mathcal{S}_p := \Big\{ s : \pi_0(\mathring{K}_p) \to \{0, 1, 2\} : \sum_{U \in \pi_0(\mathring{K}_p)} s(U) \leq 2 \Big\},$$
where $\pi_0(\mathring{K}_p)$ denotes the set of bounded Fatou components of p.

THEOREM 2.11. [**AB23**] *Let $(f_\lambda)_{\lambda \in \mathbf{Sk}(p,2)}$ be the holomorphic family of polynomial skew-products above the base polynomial p. All connected components of $\mathcal{D}'$ are hyperbolic components, and there is a natural bijection between $\mathcal{S}_p$ and the connected components of $\mathcal{D}'$.*

The proof of the first item of Theorem 2.11 is based on Jonsson's characterization of hyperbolicity (and vertical expansion) (Theorem 2.9). The proof of the second item is topological in nature. Our main task is to exclude that a given hyperbolic component can accumulate two distinct components of $\mathbb{P}^2_\infty \backslash E$. To prove this, we show that the combinatorial invariants $s \in \mathcal{S}_p$ encode the isotopy class of the Julia set in $J_p \times \mathbb{C}$, and that this isotopy class remains the same within a hyperbolic component.

2.3. Higher order bifurcations. We present in this section the results of [**AB22**]. We now consider the family $\mathbf{Sk}(p,d)$ of skew-products of degree $d \geq 2$ above a given polynomial base map p, and investigate the higher bifurcation currents T^k_{bif} within this family.

Recall that for families of rational maps, the current T^k_{bif} is known to equidistribute many kinds of dynamically defined parameters, such as maps possessing k cycles of prescribed multipliers and periods tending to infinity (see, e.g., [**BB07, Gau16**]). This gives rise to a natural stratification of the bifurcation locus as

$$\operatorname{supp} T_{\mathrm{bif}} \supseteq \operatorname{supp} T^2_{\mathrm{bif}} f \supseteq \cdots \supseteq \operatorname{supp} T^{k_{\max}}_{\mathrm{bif}}$$

where $k_{\max}$ is the dimension of the parameter space. The inclusions above are not equalities in general, and are for instance strict when considering the family of all polynomial or rational maps of a given degree (where $k_{\max}$ is equal to $d-1$ and $2d-2$, respectively). Indeed, it is enough to prove the existence of polynomials or rational maps with a prescribed number of critical points captured by attracting cycles, and the other ones being strictly preperiodic to repelling cycles. It is worth pointing out that this stratification is often compared with an analogous stratification for the Julia sets of endomorphisms of $\mathbb{P}^k$. We refer to [**Duj11**] for a more detailed exposition.

The situation in families of higher dimensional dynamical systems is completely different from the one-dimensional counterpart. Namely, we establish the following result.

THEOREM 2.12. [**AB22**] *Let p be a polynomial with Julia set not totally disconnected, which is neither conjugated to $z \mapsto z^d$ nor to a Chebyshev polynomial. Let $\mathbf{Sk}(p,d)$ denote the family of polynomial skew-products of degree $d \geq 2$ over the base polynomial p, up to affine conjugacy, and let D_d be its dimension. Then the associated bifurcation current T_{bif} satisfies*

$$\operatorname{supp} T_{\mathrm{bif}} \equiv \operatorname{supp} T^2_{\mathrm{bif}} \equiv \cdots \equiv \operatorname{supp} T^{D_d}_{\mathrm{bif}}.$$

Theorem 2.12 is stated for the full family $\mathbf{Sk}(p,d)$ of all polynomial skew-products of degree d over p (up to affine conjugacy). One could ask whether such a result holds for algebraic subfamilies of $\mathbf{Sk}(p,d)$: clearly, some special subfamilies have to be ruled out, such as the family of trivial product maps of the form $(p,q) : (z,w) \mapsto (p(z), q(w))$. A less obvious example in degree 3 is given by the subfamily of polynomial skew-products over the base polynomial $z \mapsto z^3$ of the form

$$f_{a,b} : (z,w) \mapsto (z^3, w^3 + awz^2 + bz^3), \qquad (a,b) \in \mathbb{C}^2.$$

One can check that $f_{a,b}$ is semi-conjugated to the product map

$$g_{a,b} : (z,u) \mapsto (z^3, u^3 + au + b)$$

via the blow-up $\pi : (z,w) \mapsto (z, zw)$, that is, that $f_{a,b} \circ \pi = \pi \circ g_{a,b}$. It follows that $\operatorname{supp} T^2_{\mathrm{bif}}(\Lambda) \subsetneq \operatorname{supp} T_{\mathrm{bif}}(\Lambda)$, where $\Lambda := \{f_{a,b}, (a,b) \in \mathbb{C}^2\}$.

The proof of Theorem 2.12 indeed uses the fact that the family $\mathbf{Sk}(p,d)$ is general enough so that it is possible to perturb a bifurcation parameter to change the dynamical behaviour of a critical point in a vertical fibre without affecting all other fibres. It would be interesting to classify algebraic subfamilies of $\mathbf{Sk}(p,d)$ that, like Λ, are degenerate in the sense that a bifurcation in one fibre implies a bifurcation in all other fibres; for such families, the conclusion of Theorem 2.12 will not hold.

The proof of Theorem 2.12 essentially consists of two ingredients, respectively of analytical and geometrical flavours.

The first is an analytical sufficient condition for a parameter to be in the support of $T^k_{\rm bif}$. This is inspired by analogous results by Buff-Epstein [**BE09**] and Gauthier [**Gau12**] in the context of rational maps, and is based on the notion of *large scale condition* (Definition 1.6).

The second ingredient is a procedure to build these multiple independent bifurcations at a common parameter starting from a simple one. The idea is to start with a parameter with a *Misiurewicz* bifurcation, i.e., a non-persistent collision between a critical orbit and a repelling point, and to construct a new parameter nearby where two – and actually, D_d – independent Misiurewicz bifurcations occur simultaneously. Here we say that k Misiurewicz relations are *independent* at a parameter λ if the intersection of the k hypersurfaces given by the Misiurewicz relations has codimension k in $\mathbf{Sk}(p,d)$, and we denote by ${\rm Bif}^k$ the closure of such parameters.

This geometrical construction is our main technical result. Together with the analytic arguments mentioned above (which give ${\rm Bif}^k \subseteq {\rm supp}\, T^k_{\rm bif}$ for all $1 \le k \le D_d$) and the trivial inclusion ${\rm supp}\, T^{D_d}_{\rm bif} \subseteq {\rm supp}\, T_{\rm bif}$, it implies Theorem 2.12.

In order to construct the desired Misiurewicz parameter, we will consider the motion of a sufficiently large hyperbolic subset of the Julia set near a parameter in the bifurcation locus. This hyperbolic set needs to satisfy some precise properties, and this is where the assumptions on p come into play. The construction uses tools from the thermodynamic formalism of rational maps, and more generally of endomorphisms of $\mathbb{P}^k$. Once the hyperbolic set is constructed, the proof proceed by induction. We show that, given a Misiurewicz relations satisfying a given list of further properties, it is possible to construct, one by one, the extra Misiurewicz relations happening simultaneously.

Our main theorems and the method developed for their proof have a number of consequences and corollaries. We list here a few of them.

COROLLARY 2.1. [**AB22**] *Let p be a polynomial with Julia set not totally disconnected, which is neither conjugated to $z \mapsto z^d$ nor to a Chebyshev polynomial. Near any bifurcation parameter in $\mathbf{Sk}(p,d)$ there exist algebraic subfamilies M^k of $\mathbf{Sk}(p,d)$ of any dimension $k < D_d$ such that the support of the bifurcation measure of M^k has non-empty interior in M^k.*

These families are given by the maps satisfying a given critical relation. Notice that d (and thus D_d) can be taken arbitrarily large. This result is for instance an improvement of the main result in [**BT17b**], where 1-parameter families with the same property are constructed.

More strikingly, in [**Duj17, Taf21**], Dujardin and Taflin construct open sets in the bifurcation locus in the family $\mathcal{H}_d(\mathbb{P}^k)$ of all endomorphisms of $\mathbb{P}^k$, $k \ge 1$, of a given degree $d \ge 2$ (see also [**Bie19**] for further examples). Their strategy also works when considering the subfamily of polynomial skew-products (and actually these open sets are built close to this family). Combining Theorem 2.12 with their result we thus get the following consequence.

COROLLARY 2.2. [**AB22**] *Let p be a polynomial with Julia set not totally disconnected, which is neither conjugated to $z \mapsto z^d$ nor to a Chebyshev polynomial. The support of the bifurcation measure in $\mathbf{Sk}(p,d)$ has non empty interior.*

Notice that it is not known whether the bifurcation locus is the closure of its interior (see the last paragraph in [**Duj17**]). Hence, a priori, the open sets as above could exist only in some regions of the parameter space. The last consequence of our main theorems is a uniform and optimal bound for the Hausdorff dimension of the support of the bifurcation measure, which is a generalization to this setting of the main result in [**Gau12**].

COROLLARY 2.3. [**AB22**] *Let p be a polynomial with Julia set not totally disconnected, which is neither conjugated to $z \mapsto z^d$ nor to a Chebyshev polynomial. The Hausdorff dimension of the support of the bifurcation measure in $\mathbf{Sk}(p,d)$ is maximal at all points of its support.*

Notice that, in the family of all endomorphisms of a given degree, such a uniform estimate is not known even for the bifurcation locus, see [**BB18**] for some local estimates. A stronger question, asked by Dujardin, is whether robust bifurcations are dense in the bifurcation locus, that is $\overline{\mathring{\mathrm{Bif}}} = \mathrm{Bif}$. This seems plausible.

REMARK 2.1. *In the recent article* [**GTV23**]*, Gauthier, Taflin and Vigny construct an open subset $U \subset \mathcal{H}_d(\mathbb{P}^k)$ such that $U \subset \operatorname{supp} \mu_{\mathrm{bif}}$ (and such that U does not contain any PCF endomorphism). There are some similarities between this construction and the one from* [**AB22**]*, with two main differences: first, the construction from* [**GTV23**] *is clearly taking place in a larger family of maps with less structure (not skew-products), which makes things more difficult. But on the other hand, the authors of* [**GTV23**] *just need the existence of one such open set anywhere in parameter space, whereas in* [**AB22**] *we need to construct parameters in $\operatorname{supp} \mu_{\mathrm{bif}}$ near any $\lambda_0 \in \operatorname{supp} T_{\mathrm{bif}}$. Indeed, this open set is constructed as a neighborhood of a product map.*

3. Bifurcations for finite type meromorphic maps

In this last section devoted to bifurcations, we finally come back to the setting of complex dimension one, but we exit the algebraic world to consider families of meromorphic maps $f_\lambda : \mathbb{C} \to \mathbb{P}^1$ with an essential singularity at ∞. Our goal is to present here the results of [**ABF21**], which extend to this setting the theory of Mañé-Sad-Sullivan-Lyubich (Theorem 1.1). Note however that none of the potential-theoretic aspect of the theory carries over, as transcendental maps always have infinite entropy and there is no obvious equivalent of the Green function, the measure of maximal entropy or its Lyapunov exponent.

When trying to extend Mañé-Sad-Sullivan-Lyubich's theorem to families of maps defined on non-compact Riemann surfaces (such as the complex plane) one must deal with a new possibility for the failure of periodic orbits being analytically continued, namely the possibility of periodic orbits *exiting the domain* at a certain parameter value. As an example, one can observe this new type of bifurcation occurring at $\lambda_0 = 0$ for the family $f_\lambda(z) = z + \lambda + e^z$, where the fixed points *disappear to infinity* (the essential singularity) when considering curves of parameters converging to 0. Eremenko and Lyubich [**EL92**] showed that this phenomenon does *not* occur for entire maps of the complex plane with a finite number of *singular values* (points where some branch of the inverse fails to be well-defined), namely entire maps that are of finite type as defined in Chapter 1. Consequently, they were able to conclude that $J-$stability is also dense in this class of functions.

By Picard's Theorem, the remaining cases of 1D-holomorphic maps in non-compact manifolds require not only essential singularities but also the presence of poles (i.e. preimages of the essential singularities). In this context, simple examples (e.g. $z \mapsto \lambda \tan z$ for $\lambda_0 = \pi/2$) show that this new type of bifurcations of cycles disappearing to infinity *do* occur and hence obstruct most of the arguments used for the Stability Theorem in [**MSS83, Lyu83**].

The results presented will prove that J-stability is dense also in this setting. This result will be obtained by performing a detailed analysis of the new type of bifurcation. In particular we will see how these bifurcation parameters relate to the stability of singular orbits (Theorems 3.2 and 3.3) and to parabolic parameters (Theorem 3.4), resulting in a Stability Theorem (Theorem 3.5), from which we conclude that the bifurcation locus has empty interior (Corollary 3.2).

One may wonder whether these results might extend beyond finite type (or in other directions). In this respect, we show that density of J-stability may in general fail without the finiteness assumption on singular orbits by constructing a natural family of entire maps (not of finite type) which is not $J-$stable in any open subset of the complex plane. In the same spirit, an unpublished result of Epstein and Rempe provides an example of a family of maps with an infinite (but bounded) set of singular values (hence not of finite type) and infinitely many attractors, in the spirit of the Newhouse pheonomenon [**Buz97**].

We start by giving some necessary definitions, starting by the holomorphic families which are the object of our study.

DEFINITION 3.1 (Natural family). [**ABF21**] *Let M be a complex connected manifold. A* natural family *of finite type meromorphic maps is a family $(f_\lambda)_{\lambda\in M}$ of the form*

$$f_\lambda = \phi_\lambda \circ f_{\lambda_0} \circ \psi_\lambda^{-1},$$

where $f := f_{\lambda_0}$ is a finite type meromorphic map, and $\phi_\lambda, \psi_\lambda$ are quasiconformal homeomorphisms depending holomorphically on $\lambda \in M$, and such that $\psi_\lambda(\infty) = \infty$.

Under these conditions, one can check that f_λ depends holomorphically on λ, (i.e. $\lambda \mapsto f_\lambda(z)$ is holomorphic for every fixed $z \in \mathbb{C}$).

Let $S(f)$ denote the set of singular values (critical or asymptotic, see Definition 1.1) of a meromorphic map f. If f is of finite type, then it can be embedded in a finite dimensional complex manifold $\mathrm{Def}_A^B(f)$ (defined in Section 3, for instance with A a 3-cycle and $B = S(f) \cup A$). Hence natural families can be viewed (locally) as subfamilies in this natural parameter space.

Many explicit, well-studied families are natural, like for example the exponential $E_\lambda(z) = \lambda e^z$, the tangent family $T_\lambda(z) = \lambda \tan(z)$ or the quadratic family $Q_\lambda(z) = z^2 + \lambda$. In these three examples, the map ϕ_λ is conformal, and $\psi_\lambda = \mathrm{id}$.

Notice that the singular values of f_λ are marked points that can be followed holomorphically in $\lambda \in M$, hence their number and their nature (critical or asymptotic) remain constant throughout the entire family. The same is true for their preimages: both critical points and asymptotic tracts can be followed holomorphically in λ and their multiplicity remains constant. One may naturally ask how restrictive is the concept of a natural family. As we show in Theorem 3.1, the answer is that the properties described above are necessary and sufficient conditions for an arbitrary holomorphic family of maps to be locally natural. Hence, since all of our main results are local in parameter space, they still apply if we replace the assumption that $(f_\lambda)_{\lambda\in M}$ is natural by the assumption that $S(f_\lambda)$ and $f_\lambda^{-1}(S(f_\lambda))$ both move holomorphically over M.

THEOREM 3.1 (Characterization of natural families). [**ABF21**] *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of finite type meromorphic maps, on which $S(f_\lambda)$ and $f_\lambda^{-1}(S(f_\lambda))$ both move holomorphically. Then for every $\lambda \in M$, there is a neighborhood V of λ such that $(f_\lambda)_{\lambda\in V}$ is a natural family.*

Next we define the concept of a cycle disappearing to infinity or exiting the domain (both terms will be used indistinctively).

DEFINITION 3.2 (Cycle disappearing to infinity). *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of meromorphic maps. We say that a cycle of period $m \geq 1$* disappears to infinity *at $\lambda_0 \in M$ (or* exits de domain *at λ_0) if there exist two continuous curves $t \mapsto \lambda(t)$ and $t \mapsto z(t)$ such that:*

(1) for all $t > 0$, $\lambda(t) \in M$ and $z(t) \in \mathbb{C}$, with $f^m_{\lambda(t)}(z(t)) = z(t)$
(2) $\lim_{t\to+\infty} \lambda(t) = \lambda_0 \in M$ and $\lim_{t\to+\infty} z(t) = \infty$.

As mentioned above, Eremenko and Lyubich [**EL92**] showed that cycles cannot exit the domain for holomorphic families of entire functions.

The phenomenon of cycles disappearing to infinity was previously observed in several particular slices of meromorphic functions starting with the early studies of the tangent family $T_\lambda(z) = \lambda \tan(z)$ by Devaney, Keen and Kotus [**KK97, DK89, DK88**], followed by several other families with two asymptotic values [**CK19, CJK22**] and generalized to some dynamically defined one-dimensional families in [**FK21**]. Following the terminology in the literature we define virtual cycles which, as we will see, describe limits of actual cycles after they disappear at infinity.

DEFINITION 3.3 (Virtual cycle). *Let $f : \mathbb{C} \to \mathbb{P}^1$ be a meromorphic map. A* virtual cycle *of length n is a finite, cyclically ordered set $z_0, z_1, \dots, z_{n-1}$ such that for all i, either $z_i \in \mathbb{C}$ and $z_{i+1} = f(z_i)$, or $z_i = \infty$ and z_{i+1} is an asymptotic value for f, and at least one of the z_i is equal to ∞. If $z_i = \infty$ only for one value of i then we say that the virtual cycle has* minimal length *n.*

If a virtual cycle remains after perturbation within the family, then it is called *persistent*.

DEFINITION 3.4 (Persistent virtual cycle). *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of meromorphic maps, let $\lambda_0 \in M$ and assume that f_{λ_0} has a virtual cycle $z_0, \dots, z_{n-1}$. If there exist holomorphic germs $\lambda \mapsto z_i(\lambda)$ defined near λ_0 in M such that*

(1) $z_i(\lambda_0) = z_i$
(2) $z_i(\lambda) \equiv \infty$ if $z_i = \infty$
(3) and $z_0(\lambda), \dots, z_{n-1}(\lambda)$ is a virtual cycle for f_λ,

then we say that $z_0, \dots, z_{n-1}$ is a persistent *virtual cycle.*

In particular in a holomorphic family, if $v(\lambda_0)$ is an asymptotic value such that

$$f^n_{\lambda_0}(v(\lambda_0)) = \infty$$

for some $n \geq 0$, then $(v(\lambda_0), f_{\lambda_0}(v(\lambda_0)), \dots, \infty)$ is a virtual cycle of minimal length $n+1$. (The case $n = 0$ corresponds to the situation where ∞ itself is an asymptotic value). This virtual cycle is persistent if and only if the singular relation $f^n_\lambda(v(\lambda)) = \infty$ is satisfied in all of M. If this is not the case, i.e. if a virtual cycle for λ_0 is non-persistent, we will say that λ_0 is a *virtual cycle parameter*.

Our next and last definition concerns the concept of activity or passivity of a singular value. Compared to the definition already introduced for rational maps, we need to account for the fact that the orbit of a point may land after finitely many iterations on the essential singularity. In such a situation, we will also say that the singular value in question is active.

DEFINITION 3.5 (Passive (active) singular value). *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of finite type rational, entire or meromorphic maps. Let $v(\lambda)$ be a singular value (or a critical point) of f_λ depending holomorphically on λ near some $\lambda_0 \in M$. We say that $v(\lambda)$ is* passive *at λ_0 if there exists a neighborhood V of λ_0 in M such that:*

(1) either $f_\lambda^n(v(\lambda)) = \infty$ for all $\lambda \in V$; or

(2) the family $\{\lambda \mapsto f_\lambda^n(v(\lambda))\}_{n\in\mathbb{N}}$ is well-defined and normal on V.

We say that $v(\lambda)$ is active *at λ_0 if it is not passive.*

We are now ready to state our first result, which connects the three concepts defined above: cycles disappearing to infinity, virtual cycles and the activity of a singular value.

THEOREM 3.2 (Activity Theorem). [**ABF21**] *Let $(f_\lambda)_{\lambda\in M}$ be a natural family of finite type meromorphic maps, and assume that a cycle of period n disappears to infinity at $\lambda_0 \in M$. Then, this cycle converges to a virtual cycle for f_{λ_0}, which contains (at least) either an active* asymptotic value*, or an active* critical point*.*

Note that by definition, activity means here that there exists parameters arbitrarily close to λ_0 for which one of the critical points or asymptotic values in the virtual cycle does not remain in the backward orbit of ∞.

Let us observe how Theorem 3.2 implies that cycles cannot disappear at infinity in the finite type entire setting, hence recovering the main theorem [**EL92**, Theorem 2]. Indeed, because of the lack of poles, it is easy to see that if a cycle of period n disappears at infinity, then every point of the cycle must converge to infinity (and not just one). This means that the limit virtual cycle is of the form $\infty, \dots, \infty$. In particular, it does not contain any critical points; Theorem 3.2 then asserts that ∞ itself must be an active asymptotic value for f_{λ_0}. But this is impossible, since for families of finite type entire maps ∞ is always a *passive* asymptotic value.

Finally, we remark that if the virtual cycle contains an active asymptotic value, then this virtual cycle is non-persistent (see Definition 3.3). It would be interesting to know if there are examples of such limit virtual cycles in which all asymptotic values are passive.

We do not include here the full proof of Theorem 3.2, but the following lemma shows easily the link between cycles disappearing and virtual cycles:

LEMMA 3.1 (Cycle exiting the domain implies virtual cycle for f_{λ_0}). *Let $(f_\lambda)_{\lambda\in M}$ be a natural family of meromorphic maps, and*

$$P_n := \{(\lambda, z) \in M \times \mathbb{C} : f_\lambda^n(z) = z\}.$$

Let $t \mapsto (\lambda(t), z(t))$ be a continuous real curve in P_n with $\lim_{t\to+\infty} \lambda(t) = \lambda_0 \in M$ and $\lim_{t\to+\infty} z(t) = \infty$. Then there exists a cyclically ordered set $\infty = a_0, \dots, a_{n-1} \in \mathbb{P}^1$ such that:

(1) for all $0 \le m \le n-1$, $a_m = \lim_{t\to+\infty} f_{\lambda(t)}^m(z(t))$;

(2) if $a_m \in \mathbb{C}$, then $a_{m+1} = f_{\lambda_0}(a_m)$;

(3) if $a_m = \infty$, then a_{m+1} is an asymptotic value of f_{λ_0} (possibly equal to ∞) and a_{m-1} is either ∞ or a pole of f_{λ_0}.

In other words, the set $a_0, \dots, a_{n-1}$ is a *virtual cycle* for f_{λ_0}. Notice that the lemma implies that, as $t \to \infty$ (and hence $\lambda(t) \to \lambda_0$), either the whole cycle corresponding to $z(t)$ tends to infinity (in which case ∞ must be an asymptotic value for f_{λ_0}), or there exists at least one finite asymptotic value and one pole in the virtual cycle (possibly more, if there is more than one a_i which equals infinity). We also observe that the finite type assumption is not needed for this lemma.

PROOF. To simplify notation, let us denote $x_m(t) := f^m_{\lambda(t)}(z(t))$, and $f = f_{\lambda_0}$. By assumption $\lim_{t\to+\infty} f^{n-m}_{\lambda(t)}(x_m(t)) = \lim_{t\to+\infty} z(t) = \infty$, so any finite accumulation point of the curve $t \mapsto x_m(t)$ must be a pre-pole of f of order at most $n-m$. In particular, the set of finite accumulation points of this curve is discrete, and so $\lim_{t\to+\infty} x_m(t)$ exists (and is possibly ∞). Let $a_m := \lim_{t\to\infty} x_m(t) \in \mathbb{P}^1$. Item (2) follows easily.

Next, assume that $a_m = \infty$ for some $0 \leq m \leq n-1$. Since $\{f_\lambda\}_{\lambda\in M}$ is a natural family, we have

$$x_{m+1}(t) = f_{\lambda(t)}(x_m(t)) = \phi_{\lambda(t)} \circ f \circ \psi^{-1}_{\lambda(t)}(x_m(t)),$$

where $f := f_{\lambda_0}$, $\phi_\lambda, \psi_\lambda : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ are quasiconformal homeomorphisms depending holomorphically on λ, and $\phi_{\lambda_0} = \psi_{\lambda_0} = \mathrm{id}$. Therefore, we have

$$f \circ \psi^{-1}_{\lambda(t)}(x_m(t)) = \phi^{-1}_{\lambda(t)}(x_{m+1}(t)),$$

and $\lim_{t\to+\infty} \psi^{-1}_{\lambda(t)}(x_m(t)) = a_m = \infty$, whereas $\lim_{t\to+\infty} \phi^{-1}_{\lambda(t)}(x_{m+1}(t)) = a_{m+1}$ since $\phi^{-1}_{\lambda(t)}$ tends to the identity. Therefore a_{m+1} is indeed an asymptotic value of f.

Finally, still under the assumption that $a_m = \infty$, it follows from item (2) that if a_{m-1} is finite then it is a pole. □

It is therefore quite easy to see that a disappearing cycle must converge to a virtual cycle. We emphasize here that the key content of Theorem 3.2 is the claim that at least one asymptotic value or one critical point in the limiting virtual cycle is active, or in other words, that it is possible to destroy the virtual cycle by perturbation in M. The proof, inspired from [**EL92**], is a proof by contradiction involving careful estimates of the respective speed of convergence of the curves $t \mapsto \psi^{-1}_{\lambda_t} \circ f^k_{\lambda_t}(z_t)$.

We now state our second result, which in a sense is a converse to Theorem 3.2.

THEOREM 3.3 (Accessibility Theorem). [**ABF21**] *Let $(f_\lambda)_{\lambda\in M}$ be a natural family of finite type meromorphic maps, and $\lambda_0 \in M$ be such that f_{λ_0} has a non-persistent virtual cycle of minimal length $n+1$*

$$v(\lambda_0), f_{\lambda_0}(v(\lambda_0)), \ldots, f^n(v(\lambda_0)) = \infty.$$

Then there is a cycle of period $n+1$ exiting the domain at λ_0. Moreover, this cycle can be chosen so that its multiplier goes to zero as it disappears to infinity.

In particular, by the definition of a virtual cycle, $v(\lambda_0)$ is an asymptotic value (finite or infinite) and hence λ_0 is a virtual cycle parameter. In the terminology of [**FK21**], the Accessibility Theorem states that every virtual cycle parameter is also a *virtual center* (since the multiplier of the disappearing cycle is tending to 0 at the limit parameter), and it is accessible from the interior of a component in parameter space for which the analytic continuation of this cycle remains attracting. This proves the main conjecture in [**FK21**] in much greater generality than originally stated.

Putting together Theorems 3.2 and 3.3, we obtain the following immediate corollary.

COROLLARY 3.1. [**ABF21**] *Let $(f_\lambda)_{\lambda\in M}$ be a natural family of finite type meromorphic maps, and assume that this family does not have any persistent virtual cycle. Let $\lambda_0 \in M$. Then a cycle disappears to infinity at λ_0 if and only if $f^n_{\lambda_0}(v(\lambda_0)) = \infty$ for some asymptotic value $v(\lambda_0)$.*

Up to this point we have described the new type of bifurcation that occurs in the presence of poles and asymptotic values. Observe that this phenomenon is *a priori* unrelated to the collision of periodic orbits forming parabolic cycles, in contrast to what occurs for rational or entire maps for which this is the only possible obstruction for the holomorphic motion of the Julia set. Our next result shows that, nevertheless, when an attracting cycle disappears at some parameter value, this can be approximated by parabolic parameters.

THEOREM 3.4 (Approximation by parabolic parameters). [**ABF21**] *Let $(f_\lambda)_{\lambda\in M}$ be a natural family of finite type meromorphic maps, and assume that an attracting cycle of period n disappears to infinity at $\lambda_0 \in M$. Then there exists a sequence $\lambda_k \to \lambda_0$ such that f_{λ_k} has a non-persistent parabolic cycle of period at most n.*

In particular, by Theorem 3.3, this happens at every virtual cycle parameter. With some extra work, it follows from Theorem 3.4 that parabolic parameters are dense in the bifurcation locus.

We now state the last main result of this section, which is an extension of Mañe-Sad-Sullivan's and Lyubich's bifurcation theory in the setting of finite type meromorphic maps. We stress here the fact that the proof relies in a crucial way on both Theorems 3.2 and 3.3.

THEOREM 3.5 (Characterizations of J-stability). [**ABF21**] *Let $(f_\lambda)_{\lambda\in M}$ be a natural family of finite type meromorphic maps. Let $U \subset M$ be a simply connected domain of parameters. The following are equivalent:*

(1) The Julia set moves holomorphically over U (i.e. f_λ is J-stable for all $\lambda \in U$)
(2) Every singular value is passive on U.
(3) The maximal period of attracting cycles is bounded on U.
(4) The number of attracting cycles is constant in U.
(5) For all $\lambda \in U$, f_λ has no non-persistent parabolic cycles.

In view of Theorem 3.5 it makes sense to define the *bifurcation locus* of the natural family as

$$\mathrm{Bif} = \{\lambda \in M \mid f_\lambda \text{ is not } J\text{-stable}\},$$

or equivalently as the set of parameters for which some of the conditions in Theorem 3.5 is not satisfied. Since $J - stable$ parameters form an open set by definition, following the arguments in [**MSS83**] we obtain the following statement, well-known for rational maps:

COROLLARY 3.2 (J- stable parameters form an open and dense set in M). [**ABF21**] *If $(f_\lambda)_{\lambda\in M}$ is a natural family of finite type meromorphic maps, then* $\mathrm{Bif}(M)$ *has no interior or, equivalently, $J-$stable parameters are open and dense in M.*

Our last Corollary gives meaning to the notion of hyperbolic components in our setting.

COROLLARY 3.3. [**ABF21**] *If $(f_\lambda)_{\lambda\in M}$ is a natural family of finite type meromorphic maps, and U is a connected component of $M \setminus \mathrm{Bif}(M)$ containing a parameter λ_0 such that all singular values of f_{λ_0} are captured by attracting cycles, then the same holds for every $\lambda \in U$.*

4. Perspectives

4.1. Higher order bifurcations for polynomials and rational maps. Although bifurcations of rational maps in one complex variable have already been the focus of a lot of research activity, there remains many interesting questions.

In particular, despite the results of e.g. [**BE09**], [**DF08**] and [**AGMV19**], we are still lacking a good understanding of the support of μ_{bif} in $\mathcal{M}_d$ or Poly_d, and in particular of the difference between the intersection of the activity loci of the critical points and the support of μ_{bif}.

For instance, the following example is due to Douady ([**DF08**], Example 6.13): let $f(z) := z + \frac{1}{2}z^2 + z^3 \in \text{Poly}_3$. The polynomial map f has one parabolic fixed point which attracts both critical points, so they are both active at f in the family Poly_3. However, f is *parabolic attracting*, which means that any small perturbation of f has as either a parabolic or attracting fixed point. It follows that $f \notin \text{supp}\ \mu_{\text{bif}}$. A more subtle example is given in [**IM20**] by Inou and Mukherjee. They construct a real-analytic interval $(f_t)_{t\in I}$ of *parabolic repelling* cubic polynomials, for which both critical points are also captured by the parabolic fixed point, such that none of the f_t are in the support of μ_{bif}. To my knowledge, these are the only examples of rational maps that are in the intersection of the activity loci of the critical points but not in the support of μ_{bif}.

We may therefore ask the more general question:

QUESTION 1. *Let P_3 (resp. R_2) denote the set of cubic polynomials (resp. quadratic rational maps) with one parabolic fixed point capturing both critical points. Which $f \in P_3$ are in* $\text{supp}\ \mu_{\text{bif}}$*? Same question for R_2.*

We can expect this question to be related to the bifurcations of the family of horn maps $(e^{2i\pi\sigma}h_\lambda)_{(\sigma,\lambda)\in\mathbb{C}\times M}$, where h_λ is the normalized horn map of $f_\lambda \in M$, $M = P_3$ or R_2 (see Section 1.2 next chapter for a definition of horn maps).

We also observe that both Douady's and Inou-Mukherjee's examples have a complex parameter of deformations (their Teichmüller space has positive dimension). It is therefore also natural to ask:

QUESTION 2. *Let $M = \text{Poly}_d$ or $\mathcal{M}_d$, and $\lambda_0 \in M$. Assume that all critical points are active at λ_0, and that f_{λ_0} is* rigid*, in the sense that* $\dim \text{Teich}(f_{\lambda_0}) = 0$*. Must λ_0 lie in the support of μ_{bif}?*

In the case where all critical points are in the Julia set, this is probably a very difficult question, as it is a lot more general than Theorem 1.13 (indeed, it is known that Collet-Eckmann rational maps, or even summable rational maps, are rigid, see [**Ast22**]).

4.2. Higher order bifurcations for endomorphisms of $\mathbb{P}^k$, $k \geq 2$. A natural goal is to extend the results of [**AB22**] to the total family $\mathcal{H}_d(\mathbb{P}^k)$ of degree $d \geq 2$ endomorphisms of $\mathbb{P}^k$, or at least to $\mathcal{H}_d(\mathbb{P}^2)$.

QUESTION 3. *Is it true that* $\text{supp}\, T_{\text{bif}} = \text{supp}\, \mu_{\text{bif}}$ *in $\mathcal{H}_d(\mathbb{P}^k)$?*

Let us mention a few of the difficulties involved when going from skew-products to general endomorphisms.

In the arguments from [**AB22**], we used crucially the following fact: if (z, w) is a repelling periodic point for a skew-product f for which the most repelling eigenvalue is the vertical, then $\{z\} \times \mathbb{C}$ intersects $J_2(f)$. This follows easily from the fibered structure of $J_2(f)$ described in [**Jon99**].

For the general case of an endomorphism of say $\mathbb{P}^2$, we are led to the following question. Let $f \in \mathcal{H}_d(\mathbb{P}^2)$, and let $x \in \mathbb{P}^2$ be a repelling periodic point in $J_2(f)$, with eigenvalues of different modulus. Let $v \in T_x\mathbb{P}^2$ be an eigenvector associated to the strongest repelling eigenvalue, and let $W^{uu}_{\mathrm{loc}}(x)$ denote the local strongly unstable manifold, tangent to v at x. Must we have $W^{uu}_{\mathrm{loc}}(x) \cap J_2(f)$? It would be enough to construct hyperbolic sets with this property, such as the blenders used in [**GTV23**]. However, while in [**GTV23**] the authors only need to construct one open set with good properties anywhere in parameter space, we would need to do it near any parameter in the bifurcation locus.

Another delicate issue appearing both in [**AB22**] and [**GTV23**] is that contrary to the case of dimension one, Misiurewicz loci need not intersect transversally in higher dimension (indeed, in dimension $k \geq 2$ there are infinitely many critical points, so a single map $f \in \mathcal{H}_d(\mathbb{P}^k)$ may have infinitely many different Misiurewicz relations; but $\mathcal{H}_d(\mathbb{P}^k)$ is still finite-dimensional). But some kind of transversality is required for an m-Misiurewicz map to be in the support of T^m_{bif} (i.e. in order to satisfy the large scale condition or a similar criterion, see definition 1.6). In order to construct parameters in $\operatorname{supp} \mu_{\mathrm{bif}}$, it is then necessary to not only construct parameters satisfying a large number of Misiurewicz relations, but also to ensure that the loci where these Misiurewicz relations are preserved intersect with the right codimension in parameter space. In [**AB22**], this is done using in part a description of the intersection at infinity of certain dynamically defined algebraic hypersurfaces in parameter space, and in part due to a result of Gorbovickis ([**Gor16**]) on the independance of multipliers of one-variable polynomials. The approach of [**GTV23**], which relies on Berteloot-Bianchi-Dupont's theorem (Theorem 2.3) and on a weak version of McMullen's theorem on the generic finiteness of the multiplier map, seems to be a useful variation of these ideas.

4.3. Bifurcations for finite type maps. Another natural goal is to extend the results of [**ABF21**] to families of finite type maps (not necessarily meromorphic), and in particular to families of horn maps or generalized horn maps appearing in [**AT22**]. This is an ongoing project with A.M. Benini and N. Fagella.

The most serious difficulty seems to be the following. For a natural family of finite type maps $(f_\lambda)_{\lambda \in M}$, the correct notion of a periodic point disappearing (generalizing definition 3.2) would be this:

DEFINITION 4.1. *A periodic point of period n disappears at $\lambda_0 \in M$ if there exists a real, continuous curve $t \mapsto (\lambda_t, z_t) \in M \times \mathbb{P}^1$ such that $f^n_{\lambda_t}(z_t) = z_t$, $\lambda_t \to \lambda_0$ as $t \to 0$ and $z_t \to \partial W(f_{\lambda_0})$ as $t \to 0$.*

In other words, a periodic point of period n disappears at λ_0 if and only if λ_0 is an asymptotic value of the projection $\pi_1 : \mathrm{Per}_n \to M$, where

$$\mathrm{Per}_n := \{(\lambda, z) \in M \times \mathbb{P}^1 : f^n_\lambda(z) = z\}.$$

The difficulty is that this allows the curve $t \mapsto z_t$ to accumulate on a continuum in $\partial W(f_{\lambda_0})$. If the maps f_λ are meromorphic, then $\partial W(f_\lambda) = \{\infty\}$ and so this situation does not occur. Most of our arguments in [**ABF21**] would break down in the event of such behaviour of the curve $t \mapsto z_t$.

On the other hand, Theorem 4.3 still indicates a strong link between J-stability and stability of singular orbits.

CHAPTER 3

Parabolic implosion and wandering domains in dimension 2

1. Fatou coordinates, Lavaurs maps, horn maps

We recall in this section the definition of Fatou coordinates of one-variable holomorphic maps, as well as some classical facts about their respective domains of definitions, asymptotic expansion near the parabolic fixed point, and covering properties. We will always restrict ourself to the case of a non-degenerate parabolic germ or map, i.e. of the form $f(z) = z + a_2 z^2 + O(z^3)$, with $a_2 \neq 0$.

1.1. Fatou coordinates.

1.1.1. *Local properties.* We start by recalling local properties of Fatou coordinates; here, f may be just a germ. Unless otherwise stated, we refer the reader to [[**ABD$^+$16**], Appendix] for the proofs of the statements appearing in this Subsection.

Consider a holomorphic germ $f(z) = z + a_2 z^2 + a_3 z^3 + O(z^4)$ where $a_2 \neq 0$. For $r > 0$ small enough we define incoming and outgoing petals by

$$\mathcal{P}_f^\iota = \{|a_2 z + r| < r\} \text{ and } \mathcal{P}_f^o = \{|a_2 z - r| < r\}.$$

The incoming petal $\mathcal{P}_f^\iota$ is forward invariant, and all orbits in $\mathcal{P}_f^\iota$ converge to 0. The outgoing petal $\mathcal{P}_f^o$ is backwards invariant, with backwards orbits converging to 0.

On $\mathcal{P}_f^\iota$ and $\mathcal{P}_f^o$ one can define incoming and outgoing univalent Fatou coordinates $\phi_f^\iota : \mathcal{P}_f^\iota \to \mathbb{C}$ and $\phi_f^o : \mathcal{P}_f^o \to \mathbb{C}$, solving the functional equations

$$\phi_f^\iota \circ f(z) = \phi_f^\iota(z) + 1 \text{ and } \phi_f^o \circ f(z) = \phi_f^o(z) + 1.$$

Moreover, the set $\phi_f^\iota(\mathcal{P}_f^\iota)$ contains a right half plane and $\phi_f^o(\mathcal{P}_f^o)$ contains a left half plane.

Neither the incoming nor the outgoing Fatou coordinates may be extended to a meromorphic function in a neighborhood of the origin in general; however they do satisfy the following asymptotic expansion as $z \to 0$ inside $\mathcal{P}_f^{\iota/o}$ respectively:

$$\phi_f^\iota(z) = -\frac{1}{a_2 z} - \mathfrak{b} \log\left(-\frac{1}{a_2 z}\right) + o(1) \tag{7}$$

and

$$\phi_f^o(z) = -\frac{1}{a_2 z} - \mathfrak{b} \log\left(\frac{1}{a_2 z}\right) + o(1) \tag{8}$$

where $\mathfrak{b} := 1 - \frac{a_3}{a_2^2}$.

Fatou coordinates are only unique up to an additive constant; from now on, we will work with the unique normalized Fatou coordinates for which the asymptotic expansions above hold, with no constant terms.

From the estimate (7), we first deduce that $\left(\phi_f^\iota\right)^{-1}(Z) \sim -\frac{1}{a_2 Z}$ as $\operatorname{Re} Z \to +\infty$. Then, substituting $\left(\phi_f^\iota\right)^{-1}(Z) = -\frac{1}{a_2 Z} + o\left(\frac{1}{Z}\right)$ in 7 again, we obtain:

$$\left(\phi_f^\iota\right)^{-1}(Z) = -a_2^{-1}\left(Z + \mathfrak{b}\log\left(-\frac{1}{a_2 Z}\right) + o(1)\right)^{-1}. \tag{9}$$

1.1.2. *Global properties.* We now recall global properties of Fatou coordinates. From now on, we will assume that f is a globally defined map (entire or rational).

Any orbit which converges to 0 but never lands at 0 must eventually be contained in $\mathcal{P}_f^\iota$. Therefore, we have the following description of the parabolic basin:

$$\mathcal{B}_f = \bigcup_{n\geq 0} f^{-n}(\mathcal{P}_f^\iota).$$

Using the relation $\phi_f^\iota \circ f^n = \phi_f^\iota + n$, the incoming Fatou coordinates can be uniquely extended to the parabolic basin $\mathcal{B}_f$. On the other hand, the inverse of ϕ_f^o can be extended to an entire map denoted by ψ_f^o, which satisfies the functional equation

$$f \circ \psi_f^o(Z) = \psi_f^o(Z+1).$$

This entire function is then called an *outgoing Fatou parametrization*.

1.1.3. *Covering properties of Fatou coordinates.* We first record here the covering properties of ϕ_f^ι and ψ_f^o, in the next two propositions :

PROPOSITION 1.1 ([**BE02**], Proposition 2). *The set of critical points of the map $\phi_f^\iota : \mathcal{B}_f \to \mathbb{C}$ is exactly*

$$\operatorname{crit}(\phi_f^\iota) = \bigcup_{n\in\mathbb{N}} f^{-n}\left(\operatorname{Crit} \cap \mathcal{B}_f\right).$$

Moreover, $\phi_f^\iota : \mathcal{B}_f \to \mathbb{C}$ is a branched cover.

PROPOSITION 1.2 ([**BE02**], Proposition 3). *A point $Z \in \mathbb{C}$ is a critical point of ψ_f^o if and only if there exists $n \in \mathbb{N}^*$ such that $\psi_f^o(Z-n) \in \operatorname{Crit}(f)$. Moreover, the map $\psi_f^o : \mathbb{C} \setminus (\psi_f^o)^{-1}(P_f) \to \mathbb{C} \setminus P_f$ is a covering, where $P_f := \overline{\bigcup_{n\geq 1} f^n(\operatorname{Crit})}$ is the post-critical set of f.*

1.2. Lifted horn maps, horn maps and Lavaurs maps.

DEFINITION 1.1. *The Lavaurs map of phase $\sigma \in \mathbb{C}$ is the map $\mathcal{L}_{f,\sigma} : \mathcal{B}_f \to \mathbb{C}$ defined by $\mathcal{L}_{f,\sigma}(w) := \psi_f^o\left(\phi_f^\iota(w) + \sigma\right)$.*

In order to better study the dynamics of $\mathcal{L}_{f,\sigma}$, it is convenient to introduce the following map which is semi-conjugated to it:

DEFINITION 1.2. *The lifted horn map of phase $\sigma \in \mathbb{C}$ is the map defined on $\mathcal{U}_f := (\psi_f^o)^{-1}(\mathcal{B}_f)$ by $\mathcal{E}_{f,\sigma}(W) := \phi_f^\iota \circ \psi_f^o(W) + \sigma$. We will simply denote by $\mathcal{E}_f$ the lifted horn map of phase 0.*

The open set $\mathcal{U}_f$ has at least two connected components, one containing an upper half-plane and the other containing a lower half-plane. We record here the following property of the lifted horn maps:

PROPOSITION 1.3 ([**BE02**], Proposition 4). *The set of critical values of $\mathcal{E}_f$ is*

$$\operatorname{CV}(\mathcal{E}_f) = \{\phi_f^\iota(c) + n, \quad c \in \operatorname{Crit}(f) \cap \mathcal{B}_f \text{ and } n \in \mathbb{Z}.\}$$

It is not difficult to check that $\mathcal{E}_f(W+1) = \mathcal{E}_f(W)+1$, so that $\mathcal{E}_f$ (and $\mathcal{E}_{f,\sigma}$, for any $\sigma \in \mathbb{C}$) descends to a well-defined map on the cylinder $\mathbb{C}/\mathbb{Z}$. Then, identifying $\mathbb{C}/\mathbb{Z}$ with $\mathbb{C}^*$, we obtain a unique map $h : U \to \mathbb{C}^*$ such that

$$h(e^{2i\pi W}) = \exp(2i\pi \mathcal{E}_f(W)),$$

where U is the image of $\mathcal{U}_f = (\psi_f^o)^{-1}(\mathcal{B}_f)$ under $W \mapsto e^{2i\pi W}$. The map h is called the horn map of f, and the horn map of phase σ is $h_\sigma := e^{2i\pi\sigma} h$. It can be proved that it extends holomorphically at 0 and ∞, and that this extension fixes both points (see [[**ABD$^+$16**], Appendix] and references therein).

Moreover, by [[**BÉE13**], Prop. 7.3], it is a finite type map as defined in Chapter 1, with $\mathrm{A}(h) = \{0, \infty\}$ and

$$\mathrm{CV}(h) = \{e^{2i\pi W} : W \in \mathrm{CV}(\mathcal{E}_f)\} = \{e^{2i\pi \phi^\iota_f(z)} : z \in \mathrm{Crit}(f) \cap \mathcal{B}_f\}.$$

In particular, it has exactly 2 asymptotic values which are fixed, and at least one critical value.

1.2.1. *Lavaurs' theorem.* We now give the statement of Lavaurs' theorem, which has found important applications in one-dimensional complex dynamics (see e.g. [**Shi98**]):

THEOREM 1.1. [**Lav89**] *Let $\epsilon_j \to 0$, $n_j \in \mathbb{N}$ and $\sigma \in \mathbb{C}$ satisfy $n_j - \frac{\pi}{\epsilon_j} \to \sigma \quad \text{as} \quad j \to \infty$. Then*

$$f_{\epsilon_j}^{n_j} \to \mathcal{L}_{f,\sigma}$$

locally uniformly on $\mathcal{B}_f$.

There are more general versions of Lavaurs' theorem applying to maps of the form $f(z) = e^{2i\pi p/q} z + O(z^2)$, see e.g. [**Shi98**], [**Shi00**], [**Oud99**], [**Kap22**].

Some proofs of (strong versions of) Lavaurs' theorem use the Measurable Riemann mapping theorem, and are not suitable to the higher dimensional generalization that we will expose in the following. We refer the reader to [**BSU17**] for a more self-contained proof of Theorem 1.1. It is also explained in [**ABD$^+$16**] how the proof of the main technical result may be adapted to recover a proof of Theorem 1.1.

1.2.2. *Topological and analytic classification in dimension 1.* In dimension one, the topological classification of germs tangent to the identity is simply given by the parabolic multiplicity ([**Cam78**], [**Shc82**]). In other words, all germs of the form

$$f(z) = z + a_k z^k + O(z^{k+1})$$

(with $a_k \neq 0$) are topologically conjugated in a neighborhood of 0.

However, the *analytic* classification of germs tangent to the identity is considerably more complicated: by a result proved independantly by Écalle and Voronin ([**Vor81**], [**É85**]) the germs of the horn maps (defined above) at 0 and ∞ are complete invariants for analytic conjugacy in the family of germs of the form

$$f(z) = z + a_2 z^2 + O(z^3), \quad a_2 \neq 0 \tag{10}$$

These invariants may also be described by a pair of power series, and are also called Écalle-Voronin invariants. Since horn maps fix 0 and ∞, the constant terms in these two power series must be 0; and the two degree 1 coefficients $\rho^\pm$ are linked by the equation $\rho^+\rho^- = e^{2\pi^2(1-a_3/a_2^2)}$. Aside from these relations, any pair of power series at 0 and ∞ is realizable as the horn map of a parabolic germ. In particular, there are uncountably many analytic conjugacy class among maps of the form (10), and they do not depend on any finite k-jet.

These results extend to more parabolic germs of higher multiplicity, although the exposition becomes more technical.

2. Characteristic directions, parabolic curves, parabolic domains

Let P be a holomorphic germ of $(\mathbb{C}^2, 0)$ which is tangent to the identity of order $k \geq 2$, i.e. a map with a homogeneous expansion

$$P = \mathrm{Id} + P_k + P_{k+1} + \dots$$

where the P_j are homogeneous degree j polynomial maps from $\mathbb{C}^2$ to $\mathbb{C}^2$, and $P_k \not\equiv 0$. We say that $v \in \mathbb{C}^2 \setminus \{(0,0)\}$ is a *characteristic direction* for P if there exists a $\lambda \in \mathbb{C}$ so that $P_k(v) = \lambda v$. If $\lambda \neq 0$ then v is said to be *non-degenerate*; otherwise it is degenerate. We shall denote by $v \mapsto [v]$ the canonical projection of $\mathbb{C}^2 \setminus \{(0,0)\}$ onto $\mathbb{P}^1$. The *director* of a characteristic direction v is the eigenvalue of the linear operator

$$d(P_k)_{[v]} - \mathrm{id} : T_{[v]}\mathbb{P}^1 \to T_{[v]}\mathbb{P}^1.$$

A *parabolic curve* for P is an injective holomorphic map $\varphi : \Delta \to \mathbb{C}^2$, satisfying the following properties:

(1) Δ is simply connected domain in $\mathbb{C}$ with $0 \in \partial\Delta$
(2) φ is continuous at the origin and $\varphi(0) = (0,0)$,
(3) $\varphi(\Delta)$ is invariant under P and $P^n|_{\varphi(\Delta)} \to (0,0)$ uniformly on compact subsets.

It is important to keep in mind that the map φ can typically not be extended holomorphically at 0, just like Fatou coordinates cannot be extended meromorphically near 0, even though both admit a formal, in general divergent asymptotic expansion.

We say that a parabolic curve is tangent to $[v] \in \mathbb{P}^1$ if $[\varphi(\xi)] \to [v]$ as $\xi \to 0$ in Δ. This implies that for any point given point z in the parabolic curve the orbit $(P^n(z))$ converges to the origin *tangentially* to v, i.e. $[P^n(z)] \to [v]$ in $\mathbb{P}^1$. We now recall the following classical result due to Écalle ([**É85**]) and Hakim ([**Hak94, Hak98**]):

THEOREM 2.1. *Let $P : \mathbb{C}^2 \to \mathbb{C}^2$ be a holomorphic germ fixing the origin which is tangent to the identity of order $k+1 \geq 2$. Then for any non-degenerate characteristic direction v there exist (at least) k parabolic curves for P tangent to $[v]$. Moreover, if the real part of the director of a non-degenerate characteristic direction v is strictly positive, then there exists an invariant parabolic domain in which every point is attracted to the origin along a trajectory tangent to v.*

Additionnally, by [[**Hak98**], Section 3], when the director of a non-degenerate parabolic curve is not a natural number, then the corresponding parabolic curve is asymptotic to a unique (in general divergent) invariant formal power series.

EXAMPLE 1. *Let $f(z,w) := (p(z), q(z,w))$ be a polynomial skew-product of the form $p(z) = z - z^2 + O(z^3)$ and $q(z,w) = w + w^2 + bz^2 + O(\|(z,w)\|^3)$, with $b \in \mathbb{C}^*$ (such maps will be investigated extensively in Section 4). Its characteristic directions $[z:w]$ are given by the equations*

$$\begin{cases} -z^2 & = \lambda z \\ w^2 + bz^2 & = \lambda w. \end{cases}$$

It follows that aside from the trivial non-degenerate characteristic direction $(0,1)$, there are two other non-degenerate characteristic directions $(1, c^{\pm})$, where $c^{\pm}$ are the roots of

$$u^2 + u + b = 0. \tag{11}$$

Note that $c^\pm = -\frac{1}{2} \pm ic$ where c is the solution of $c^2 = b - \frac{1}{4}$ satisfying $\mathrm{Re}(c) \geq 0$. Clearly, $\mathrm{Im}(c) = 0$ if and only if $b \geq \frac{1}{4}$. Moreover, for $b = \frac{1}{4}$ we have $c^+ = c^- = -\frac{1}{2}$. The directors of the characteristic directions $(1, c^\pm)$ are $\mp 2ic$; in particular, when $b \in]\frac{1}{4}, +\infty[$, neither of them are natural numbers.

It follows from Theorem 2.1 that aside from the trivial parabolic curve contained in the invariant line $z = 0$, there are two parabolic curves which are tangent to the non-degenerate characteristic directions $(1, c^\pm)$ respectively. By Hakim's construction, these parabolic curves may be written as holomorphic graphs $z \mapsto (z, \zeta^\pm(z))$ over a small petal $\mathcal{P}_p^\iota = \mathbb{D}(r, r)$. Since parabolic curves are invariant under f, the functions $\zeta^\pm$ satisfy the functional equations

$$q_z(\zeta^\pm(z)) = \zeta^\pm(p(z)).$$

When the characteristic direction is degenerate, the question of the existence of parabolic curves is more delicate. In [**LHR20**], Lopez-Hernanz and Rosas give a more complete description of the picture in dimension 2:

THEOREM 2.2. [**LHR20**] *Let $F : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a germ tangent to the identity of order $k+1$, and let $[v] \in \mathbb{P}^1$ be a characteristic direction of F. Then at least one of the following possibilities hold:*

(1) There exists an analytic curve pointwise fixed by F and tangent to $[v]$.
(2) There exists at least k invariant sets $\Omega_1, \dots, \Omega_k$ where each Ω_i is either a parabolic curve tangent to $[v]$ or a parabolic domain along $[v]$ and such that all the orbits in $\Omega_1 \cup \ldots \cup \Omega_k$ are mutually asymptotic. Moreover, at least one of the Ω_j is a parabolic curve.
(3) There exists at least k parabolic domains $\Omega_1, \dots, \Omega_k$ along $[v]$, where each Ω_i is foliated by parabolic curves and such that all the orbits in $\Omega_1 \cup \ldots \cup \Omega_k$ are mutually asymptotic.

In particular, if F does not have a curve of fixed points, then F has at least one parabolic curve along each of its characteristic directions. This is for instance the case for skew-products of the form

$$f(z, w) = (z - z^3 + O(z^4), w + w^2 + bz^2 + O(w^3)) \tag{12}$$

which appear in [**ABD$^+$16**] and [**ABTP23**], for which $(0, 1)$ is a non-degenerate characteristic direction (with a "trivial" parabolic curve $\{0\} \times \mathbb{D}(-r, r)$), and two degenerate characteristic directions $(1, \pm\sqrt{-b})$.

The methods from [**LHR20**] are involved: they use a comparison of F with the formal time 1 flow of a formal vector field, and Camacho-Sad's theorem on the existence of separatrices in dimension 2. In [**ABTP23**], we give (independantly) an elementary but delicate construction of the parabolic curves for maps of the form (12), based on a graph transform argument.

3. Wandering domains and Lavaurs map with Siegel disks

We present in this section the results of [**ABTP23**], with H. Peters and L. Boc Thaler. To this end, we must first recall here the main theorem of [**ABD$^+$16**]:

THEOREM 3.1. [**ABD$^+$16**] *Let $p(z) = z - z^2 + O(z^3)$ and $q(w) = w + w^2 + O(w^3)$ be polynomial maps, such that the Lavaurs map $\mathcal{L}_{q,0}$ has an attracting fixed point. Then the map*

$$f : (z, w) \mapsto \left(p(z), q(w) + \frac{\pi^2}{4} z\right)$$

has a wandering Fatou component.

The main technical tool is the following non-autonomous version of Lavaurs' theorem (compare Theorem 1.1):

THEOREM 3.2. [**ABD$^+$16**] *Let* $p(z) = z - z^2 + O(z^3)$ *and* $q(w) = w + w^2 + O(w^3)$ *be polynomial maps, and* $f : (z, w) \mapsto \left(p(z), q(w) + \frac{\pi^2}{4}z\right)$. *Then*

$$f^{2n+1}(p^{n^2}(z), w) = \left(p^{(n+1)^2}(z), \mathcal{L}_{q,0}(w) + o(1)\right).$$

The choice of coefficient $\frac{\pi^2}{4}$ is there to ensure that it takes about $2n$ iterations to "go through the eggbeater" and converge to the Lavaurs map, compare Theorem 1.1. We give here a quick proof of how Theorem 3.1 may be deduced from Theorem 3.2.

PROOF OF THEOREM 3.1. Let $w_0 \in \mathcal{B}(q_0)$ be an attracting fixed point for $\mathcal{L}_{q,0}$, and let $z_0 \in \mathcal{B}(p)$. Choose $r > 0$ small enough that $\mathcal{L}_{q,0}(\mathbb{D}(w_0, r)) \subset \mathbb{D}(w_0, kr)$, for some $0 < k < 1$. Let $n_0 \in \mathbb{N}$ be large enough that for all $n \geq n_0$ and all $(z, w) \in \mathbb{D}(z_0, r) \times \mathbb{D}(w_0, r)$,

$$\pi_2 \circ f^{2n+1}(p^{n^2}(z), w) \in \mathbb{D}(w_0, kr)$$

(here we use Theorem 3.2). Let U be a connected component of $f^{-n_0^2}(p^{n_0^2}(\mathbb{D}(z_0, r) \times \mathbb{D}(w_0, r))$. Then for all $(z, w) \in U$ and all $n \geq n_0$,

$$P^{n^2}(z, w) \in \mathcal{B}(p) \times \mathbb{D}(w_0, r)$$

which implies that U belongs to the Fatou set of f, hence is contained in a Fatou component Ω and that $P^{n^2}_{|\Omega} \to (0, w_0)$. On the other hand, it is not difficult to prove that if V is a preperiodic component with $f^{n+p}(V) = f^n(V)$ and $(n_j)_{j \in \mathbb{N}}$ is a subsequence such that $f^{n_j}_{|V}$ converges to $\xi \in \mathbb{C}^2$, then $f^{n+p}(\xi) = f^p(\xi)$. But since $(0, w_0)$ is not preperiodic for f, Ω_0 cannot be preperiodic either and therefore must be wandering.

$\square$

Now that the existence of wandering Fatou components is known, it is natural to ask questions on their dynamical properties, and in particular the properties of the maps obtained as adherence values of iterates. This leads us to the following definition:

DEFINITION 3.1. *Let* Ω *be a Fatou component of an endomorphism* $f : \mathbb{P}^k \to \mathbb{P}^k$. *We call* Fatou limit map *on* Ω *any adherence value of the sequence of maps* $(f^n_{|\Omega})_{n \in \mathbb{N}}$.

We define the rank of a Fatou component Ω *as the maximal rank of* $\mathrm{d}h_x$, *where* $x \in \Omega$ *and* h *ranges over all Fatou limit maps on* Ω.

Note that for endomorphisms of $\mathbb{P}^2$, any wandering domain either has rank 0 (all Fatou limits are constant) or rank 1. A slightly more careful examination of the proof of Theorem 3.1 above shows that all Fatou limits of the wandering domain Ω are constant, and their values are either $(0, 0)$ or of the form $(0, w_n)$, with $q_0^n(w_n) = w_0$. In particular, they have rank 0. In the paper [**BB23**], which came out around the same time as [**ABTP23**], Berger and Biebler construct by completely different methods Hénon maps with wandering Fatou components; their methods also provide new examples of endomorphisms of $\mathbb{P}^2$ with wandering Fatou components. These new examples also have rank 0, which leads to the question:

QUESTION 4. *Does there exist an endomorphism $f : \mathbb{P}^2 \to \mathbb{P}^2$ with a wandering Fatou component of rank 1?*

It turns out that the answer is yes, but this will only proved in the later work [**AT22**]. Motivated by this question, we investigated in [**ABTP23**] the case of maps of the form

$$f : (z, w) \mapsto \left(p(z), q(w) + \frac{\pi^2}{4} z \right)$$

where $\mathcal{L}_{q,0}$ has a fixed Siegel disk instead of an attracting fixed point.

The idea is that by Theorem 3.2, suitable iterates of f will be compositions of maps that are close to the limit map $\mathcal{L}_{q,0}$ which has a linearizable Siegel fixed point; then one may hope that the composition of these iterates converges to a non-constant map $(z, w) \mapsto (0, h(w))$. More generally, we may ask:

QUESTION 5. *Let $f_1, f_2, \dots$ be a sequence of holomorphic germs, converging locally uniformly to a holomorphic function $\mathcal{L}$ having a Siegel fixed point at 0. Under which conditions does there exist a trapping region?*

By a trapping region we mean the existence of arbitrarily small neighborhoods U, V of 0 and $n_0 \in \mathbb{N}$ such that

$$f_m \circ \dots \circ f_n(z) \in V$$

for all $z \in U$ and $m \geq n \geq n_0$. In other words, any orbit $(z_n)_{n \geq 0}$ that intersects U for sufficiently large n will afterwards be contained in a small neighborhood of the origin. Note that this in particular guarantees normality of the sequence of compositions $f_m \circ \ldots \circ f_0$ in a neighborhood of z_0.

We are particularly interested in the case where the differences $f_n - \mathcal{L}$ are not absolutely summable, i.e. when

$$\sum_{n \geq n_0} \|f_n - \mathcal{L}\|_U = \infty$$

for any n_0 and U. In this situation, the existence of a trapping region is not clear.

We assume that

$$f_n(z) - \mathcal{L}(z) = \frac{h(z)}{n} + O\left(\frac{1}{z^{1+\epsilon}} \right), \tag{13}$$

where h is a holomorphic germ, defined in a neighborhood of the origin.

THEOREM 3.3. [**ABTP23**] *There exists $\kappa \in \mathbb{C}$, a rational expression in the coefficients of f and h, such that the following holds:*

(1) If $\mathrm{Re}(\kappa) = 0$, then there is a trapping region, and all limit maps have rank 1.
(2) If $\mathrm{Re}(\kappa) < 0$, then there is a trapping region, and all orbits converge uniformly to the origin.
(3) If $\mathrm{Re}(\kappa) > 0$, then there is no trapping region. In fact, there can be at most one orbit that remains in a sufficiently small neighborhood of the origin.

REMARK 3.1. *The non-autonomous dynamics of the functions f_n satisfying* (13) *is closely related to the autonomous dynamics of the quasi-parabolic map*

$$F(z, w) = (z - z^2 + O(z^2), f(w) + zh(w) + O(z^2)).$$

The case $\mathrm{Re}(\kappa) < 0$ in Theorem 3.3 corresponds to F being dynamically separating and parabolically attracting, using the terminology of [**BZ13**]*, hence by* [**BZ13**, Corollary 6.3] *the map F*

has a connected basin of attraction at the origin. In particular this implies the existence of a trapping region for the sequence $(f_n)_{n\in\mathbb{N}}$.

In order to apply Theorem 3.3, it is necessary to refine Theorem 3.2 to obtain not just the convergence but also an equivalent of the error term. The computations become considerably more involved, and so we made the simplifying assumption that q has the form $q(w) = w + w^2 + w^3 + O(w^4)$, instead of simply $q(w) = w + w^2 + O(w^3)$. Under this technical assumption, the computation of the error term was achieved with the next result:

THEOREM 3.4. [**ABTP23**] *Let* $f(z,w) = (p(z), q(w) + \frac{\pi^2}{4}z)$*, where* $p(z) = z - z^2 + O(z^3)$ *and* $q(w) = w + w^2 + w^3 + O(w^4)$*. Let* $\mathcal{B}_p, \mathcal{B}_q$ *denote the parabolic basins of* p *and* q *respectively. There exists a holomorphic function* $h : \mathcal{B}_p \times \mathcal{B}_q \to \mathbb{C}$ *such that*

$$f^{2n+1}(p^{n^2}(z), w) = (0, \mathcal{L}_f(w)) + \left(0, \frac{h(z,w)}{n}\right) + O\left(\frac{\log n}{n^2}\right),$$

uniformly on compact subsets of $\mathcal{B}_p \times \mathcal{B}_q$*. More explicitly:*

$$h(z,w) = \frac{\mathcal{L}_q'(w)}{(\phi_q^\iota)'(w)} \cdot \left(C + \phi_q^\iota(w) - \phi_p^\iota(z)\right),$$

where $C \in \mathbb{C}$ *is a constant.*

In particular, we can apply Theorem 3.3 to the non-autonomous compositions of the maps $f_{n,z}(w) := \pi_2 \circ f^{2n+1}(p^{n_0^2}(z), w)$, which all converge to the map $\mathcal{L} := \mathcal{L}_{q,0}$ by Theorem 3.2. We think of the maps $f_{n,z}$ as maps in w depending on a parameter z, so that the index κ from Theorem 3.3 becomes a map $z \mapsto \kappa(z)$ defined on $\mathcal{B}(p)$. Surprisingly, it turns out that when κ is constant, its value must be equal to 1:

PROPOSITION 3.1. [**ABTP23**] *Let* $f(z,w) = (p(z), q(w) + \frac{\pi^2}{4}z)$*, where* $p(z) = z - z^2 + O(z^3)$ *and* $q(w) = w + w^2 + w^3 + O(w^4)$*. Assume that* $\mathcal{L}_{q,0}$ *has a Siegel fixed point* w_0*. There are exactly 2 possibilities:*

(1) either κ *is constant equal to 1 on* $\mathcal{B}_p$*,*
(2) or κ *is a non-constant holomorphic map on* $\mathcal{B}_p$*.*

In view of Theorem 3.3, we deduce:

COROLLARY 3.1. *Let* p, q, f, w_0, κ *be as in Proposition 3.1 above.*

(1) If $\kappa \equiv 1$*, then* f *has no wandering domain whose orbit accumulates* $(0, w_0)$.
(2) Otherwise, there is a rank 0 wandering domain accumulating $(0, w_0)$*. All its limit Fatou fonctions are constants equal to* $(0,0)$ *or* $(0, w_n)$*, where* $q^n(w_n) = w_0$.

In case (2), Theorem 3.3 implies that there exists vertical Fatou disks contained in the boundary of the wandering domains (corresponding to values of z for which $\operatorname{Re}\kappa(z) = 0$), and that these Fatou disks have admit a rank 1 limit of iterates. However, such a disk must always be on the boundary of a Fatou component and cannot lie inside the Fatou set.

4. Generic polynomial skew-products of $\mathbb{C}^2$ tangent to the identity

We now present the results of [**AT22**]. After [**ABD$^+$16**] and [**ABTP23**] which focused on skew-products with very specific forms, the purpose of this work is to investigate the generic case of skew-products P which are tangent to the identity.

By this we mean holomorphic maps $P : \mathbb{C}^2 \to \mathbb{C}^2$ of the form

$$P(z,w) = \left(z + \sum_{i\geq 2} a_i z^i, w + \sum_{i+j\geq 2} b_{i,j} z^i w^j \right)$$

with $a_2 \neq 0$, $b_{2,0} \neq 0$ and $b_{0,2} \neq 0$. Since the results will be mostly local, we do not necessarily assume that P is a polynomial map, or that P is regular (i.e. extends holomorphically to $\mathbb{P}^2$), although that will be the main case we have in mind.

Up to conjugacy by a linear automorphism of $\mathbb{C}^2$, such maps may be reduced to a map of the form

$$P : (z,w) \mapsto \left(z - z^2 + O(z^3), w + w^2 + bz^2 + O(\|(z,w)\|^3) \right),$$

and after a second conjugacy by an automorphism of $\mathbb{C}^2$ of the form

$$(z,w) \mapsto (z, e^{Az} w + Bz^2),$$

we may finally assume that P is of the form $P(z,w) = (p(z), q(z,w))$ with

$$\left\{ \begin{array}{l} p(z) := z - z^2 + az^3 + O(z^4) \\ q(z,w) := w + w^2 + bz^2 + b_{0,3} w^3 + b_{3,0} z^3 + O(\|(z,w)\|^4) \end{array} \right. \tag{14}$$

where $a, b, b_{0,3}, b_{3,0} \in \mathbb{C}$.

A study of the local dynamics of skew-products in the case $b = 0$ in (14) has been undertaken in [**Viv20**], where a full description of the dynamics on a neighborhood of a parabolic fixed point at the origin was achieved. However, most of the difficulty and richness of the dynamics (including the phenomenon of parabolic implosion and the existence of wandering domains) comes precisely from this term bz^2.

In fact, although maps of the form (14) are generic among polynomial skew-products which are tangent to the identity (after analytic conjugacy), we will see that they have considerably complicated local dynamics. We see the investigation of those maps (14) and the results presenter here as a first step (generic case) towards the systematic analysis of the local dynamics of all polynomial skew-products which are tangent to the identity.

We begin by discussing the existence of parabolic domains for maps of the form (14), which depends only on b:

THEOREM 4.1. [**AT22**] *Let P be a map of the form* (14). *Then*

(1) If $b \in (\frac{1}{4}, +\infty)$, the map P has an invariant parabolic domain which is not tangent to any directions.

(2) If $b \in \mathbb{C}\backslash(\frac{1}{4}, +\infty)$, the map P has an invariant parabolic domain which is tangent to one of its non-degenerate characteristic directions.

The main novelty here lies in the first statement of this theorem, while the second statement can be deduced from results of Hakim and Vivas. Invariant parabolic domains which are not tangent to any direction are also sometimes called *spiral domains*. Such domains were first constructed by Rivi in her thesis [**Riv98**, Proposition 4.4.4]. In [**Ron14**], Rong gave sufficient conditions for the existence of spiral domains for some class of maps tangent to the identity (see [**Ron14**, Theorem 1.4]). However, his result does not apply to maps of the form (14).

From now on we will assume that $b > \frac{1}{4}$, and we introduce the following notations:

$$c := \frac{\sqrt{4b-1}}{2}, \quad \alpha_0 := e^{\pi/c}, \quad \beta_0 := (b_{0,3} - a)(\alpha_0 - 1). \tag{15}$$

Observe that since $b > \frac{1}{4}$, we have $c > 0$ and $\alpha_0 > 1$.

In what follows we will see that in the case $b > \frac{1}{4}$ and $\beta_0 \in \mathbb{R}$, there is parabolic implosion, which has many interesting dynamical consequences.

DEFINITION 4.1. *Let P be of the form* (14)*, and $\alpha, \sigma \in \mathbb{C}$. Its generalized Lavaurs map of phase σ and parameter α is defined as*

$$\mathcal{L}(\alpha, \sigma; z, w) := \psi^o_{q_0}\left(\alpha\phi^\iota_{q_0}(w) + (1-\alpha)\phi^\iota_p(z) + \sigma\right), \tag{16}$$

where ϕ^ι_p is the incoming Fatou coordinate of p, $\phi^\iota_{q_0}$ the incoming Fatou coordinates of q_0 and $\psi^o_{q_0}$ the outgoing Fatou parametrization of q_0.

The generalized Lavaurs map is defined for $(z, w) \in \mathcal{B}_p \times \mathcal{B}_{q_0}$, where $\mathcal{B}_p$ and $\mathcal{B}_{q_0}$ are basins of a parabolic fixed point at the origin for p and q_0 respectively, and takes values in $\mathbb{C}$. If $\alpha = 1$, then the map $w \mapsto \mathcal{L}(\alpha, \sigma; z, w)$ does not depend on z and coincides with the classical Lavaurs map of phase σ of the one-variable polynomial q_0. Moreover, generalized Lavaurs maps satisfy the following functional relation:

$$\mathcal{L}(\alpha, \sigma; p(z), q_0(w)) = q_0 \circ \mathcal{L}(\alpha, \sigma; z, w) = \mathcal{L}(\alpha, \sigma + 1; z, w) \tag{17}$$

for all $(z, w) \in \mathcal{B}_p \times \mathcal{B}_{q_0}$.

DEFINITION 4.2. *Given real numbers $\alpha > 1$ and $\beta \in \mathbb{R}$, we say that a strictly increasing sequence of positive integers $(n_k)_{k\geq 0}$ is (α, β)-*admissible *if and only if its* phase sequence $(\sigma_k)_{k\geq 0}$*, defined by $\sigma_k := n_{k+1} - \alpha n_k - \beta \ln n_k$, is bounded. In the case where $\beta = 0$, we will simply call such a sequence α-*admissible.

Observe that for any $\alpha > 1$ and $\beta \in \mathbb{R}$, there always exists (α, β)-admissible sequences: it suffices to define inductively $n_{k+1} := \lfloor \alpha n_k + \beta \ln n_k \rfloor$ and take $n_0 \in \mathbb{N}$ large enough, where $\lfloor \cdot \rfloor$ denotes the floor function. For this particular type of (α, β)-admissible sequence, we have $\sigma_k \in (-1, 0]$ for all $k \in \mathbb{N}$. However, describing the phase sequence is in general a difficult problem; for instance, even in the particular case of the $\frac{3}{2}$-admissible sequences of the form $n_{k+1} = \lfloor \frac{3}{2} n_k \rfloor$, the phase sequence is not fully understood (see [**Dub09**]). An interesting question is the existence of (α, β)-admissible sequences with *converging* phase sequence, which will be discussed in detail below.

The following is the main technical result of [**AT22**]:

THEOREM 4.2. [**AT22**] *Let P be a map of the form* (14)*. Let α_0, β_0 be as in* (15)*, and assume that $b > \frac{1}{4}$ and $\beta_0 \in \mathbb{R}$. Let $(n_k)_{k\geq 0}$ be an (α_0, β_0)-admissible sequence and let $(\sigma_k)_{k\geq 0}$ denote its phase sequence. Then*

$$P^{n_{k+1}-n_k}(p^{n_k}(z), w) = (0, \mathcal{L}(\alpha_0, \Gamma + \sigma_k; z, w)) + o(1) \qquad \textit{as } k \to +\infty$$

with uniform convergence on compacts in $\mathcal{B}_p \times \mathcal{B}_{q_0}$, and where Γ is a constant depending only on $a, b, b_{0,3}, b_{3,0}$ (see (18) *for its explicit expression).*

The constant Γ is given by the explicit expression

$$\Gamma := (e^{\frac{\pi}{c}} - 1)\left(\frac{a - b_{0,3} + b_{3,0}}{2b} + a + \frac{1}{2}(1 - b_{0,3}) + (b_{0,3} - 1)\ln c\right) + (b_{0,3} - a)\frac{\pi}{c} + e^{\frac{\pi}{c}}(1 - b_{0,3})\int_0^{\frac{\pi}{c}} e^{-u}\ln\sin(cu)du. \tag{18}$$

As the expression of Γ may suggest, the computations involved in the proof of Theorem 4.2 are quite heavy, and we will not try to give details on the proof beyond the following heuristic: let $z_j := p^{n_k+j}(z)$. The orbit of $(p^{n_k}(z), w)$ "rotates" in $\mathbb{C}^2$ around the two parabolic curves $\zeta^{\pm}$ given by Hakim's theorem (see Theorem 2.1 and Example 1), with an angle of about cz_j per iteration. Using the classical estimate $z_j \sim \frac{1}{n_k+j}$ (which may be deduced from (7)), the number N of iterates required to "do a full turn" must satisfy $\sum_{j=0}^{N} \frac{c}{n_k+j} \approx 2\pi$, and since $\sum_{j=0}^{N} \frac{c}{n_k+j} \approx c\int_{n_k}^{n_k+N} \frac{dt}{t} = c\ln\left(1 + \frac{N_k}{n_k}\right)$, we obtain $N_k \approx (e^{2\pi/c} - 1)n_k \approx n_{k+1} - n_k$, as expected.

The usefulness of Theorem 4.2 (and of similar results, such as Theorem 3.2, is that by applying it successively, one can estimate more and more precisely certain high iterates of P in terms of iterates of the maps $\mathcal{L}_z : w \mapsto \mathcal{L}(\alpha_0, \Gamma + \sigma_k; z, w)$. Therefore, one can transfer dynamical properties of $\mathcal{L}_z$ to obtain information on the dynamics of P. These maps $\mathcal{L}_z$ are quite complicated (they are non-explicit, transcendental maps, with infinitely many critical points and in general infinitely many critical values). However, by thinking of them as a one-parameter family of maps $(\mathcal{L}_z)_{z\in\mathcal{B}_p}$, we can use ideas from one-dimensional bifurcation theory to obtain information on the dynamics of $\mathcal{L}_z$ for certain values of z. Moreover, under the additional assumption that $\alpha_0 \in \mathbb{N}_{\geq 2}$, we prove that these maps are semi-conjugated to finite type maps, much in the same way that classical Lavaurs maps are semi-conjugated to horn maps. This allows us to obtain a more precise understanding of their dynamics, and in turn, of the dynamics of P.

We list below some consequences of Theorem 4.2.

4.1. Existence of wandering domains and Pisot numbers. The first examples of polynomial maps with wandering Fatou components were introduced in [**ABD$^+$16**] by Buff, Dujardin, Peters, Raissy and the first author (see also [**ABTP23**]); other examples were constructed by Berger and Biebler in [**BB23**], by completely different methods, for Hénon maps and polynomial endomorphisms of $\mathbb{P}^2$. In the opposite direction, Ji gave in [**Ji23**] and [**Ji20**] sufficient conditions to guarantee the absence of wandering domains near an attracting invariant fiber for a skew-product map.

The examples from [**ABD$^+$16**] are polynomial skew-products of the form

$$(z, w) \mapsto \left(p(z), q(w) + \frac{\pi^2}{4}z\right)$$

with $p(z) = z - z^2 + O(z^3)$ and $q(w) = w + w^2 + O(w^3)$, and are not tangent to the identity at the origin. One can simplify the investigation of these maps by passing to a finite branched cover $y^2 = z$. This brings these maps to a form that is tangent to the identity, but with degenerate second order differential at the origin. In particular, these maps are not of the form (14), which explain the difference in dynamical features.

THEOREM 4.3. **[AT22]** *Let P be a map of the form* (14)*, and assume that there exists an (α_0, β_0)-admissible sequence with converging phase sequence. Then P has a wandering domain of rank 1.*

Recall the definition of the rank of a Fatou component (Definition 3.1). Theorem 4.3 gives the first (and so far only) examples of rank 1 wandering domains in complex dimension 2.

We are therefore led to the question: for which values of α and β does such a sequence exist? Before stating an answer, recall the definition of Pisot numbers:

DEFINITION 4.3. *A real algebraic integer $\alpha > 1$ is called a Pisot number if all of its Galois conjugates are in the open unit disk in $\mathbb{C}$ (in particular, integers ≥ 2 are Pisot numbers).*

The next definition is not standard terminology, but it will be convenient for our purposes:

DEFINITION 4.4. *We say that $\alpha > 1$ has the Pisot property if there exist $\zeta \in \mathbb{R}^*$ such that $\|\zeta\alpha^k\| \to 0$, where $\|\cdot\|$ denotes the distance to the nearest integer.*

We recall here two classical results from number theory that justify the terminology of "*Pisot property*":

Theorem ([Pis46]) *Let $\alpha > 1$ be an algebraic number with the Pisot property. Then α is a Pisot number and ζ lies in the field $\mathbb{Q}(\alpha)$.*

Theorem ([Pis46]) *There are only countably many pairs (ζ, α) of real numbers such that $\zeta \neq 0$, $\alpha > 1$, and the sequence $(\{\zeta\alpha^k\})_{k\geq 0}$ has only finitely many limit points. Moreover if (ζ, α) is such a pair where α is an algebraic number, then α is a Pisot number and ζ lies in the field $\mathbb{Q}(\alpha)$. Here $\{\cdot\}$ denotes the fractional part of the number.*

In particular, an algebraic number has the Pisot property if and only if it is a Pisot number. Moreover, it is a long-standing conjecture known as the *Pisot-Viiayaraghavan problem* that Pisot numbers are the only *real* numbers with the Pisot property.

DEFINITION 4.5. *We say that a sequence $(\sigma_k)_{k\geq 0}$ converges to a cycle of period ℓ if the subsequence $(\sigma_{k\ell+j})_{k\geq 0}$ converges for every $0 \leq j < \ell$.*

We can now state an almost sharp condition on α and β for the existence of an (α, β)-admissible sequence with converging phase:

THEOREM 4.4. **[AT22]** *Let $\alpha > 1$ and $\beta \in \mathbb{R}$. Then*

(1) There exists an α-admissible sequence with phase sequence converging to a cycle if and only if α has the Pisot property. Moreover, in that case there exists an α-admissible sequence with phase sequence converging to 0.

(2) (a) If there exists an (α, β)-admissible sequence with phase sequence converging to a periodic cycle, then α has the Pisot property.

(b) Conversely, if α has the Pisot property and $\beta = \frac{\alpha-1}{\ln\alpha}\frac{k_1}{k_2}$, where k_1 and k_2 are coprime integers with $k_2 \geq 1$, then there exists an (α, β)-admissible sequence whose phase sequence converges to a cycle of period k_2.

Note that if the Pisot-Viijayaraghavan conjecture is true, then there exists an α-admissible sequence with converging phase sequence if and only if α is a Pisot number.

It is natural to ask whether the condition of Theorem 4.3 is necessary or not. In the case that there are no (α, β)-admissible sequences whose phase sequence converge to a periodic cycle, it means that any wandering Fatou component whose orbit remains in $\mathcal{B}_p \times \mathcal{B}_{q_0}$ would have to remain bounded under a sequence of non-autonomous compositions of generalized Lavaurs maps with non-periodic sequences of phases. Proving rigorously whether such a thing is possible or not is likely to be very difficult, but it seems reasonable to expect that for generic values of α it is not the case.

If we now specialize to the case of degree 2, Theorems 4.3 and 4.4 imply that for any Pisot number $\alpha_0 > 1$, the map

$$(z, w) \mapsto \left(z - z^2, w + w^2 + \left(\frac{1}{4} + \frac{\pi^2}{(\ln \alpha_0)^2}\right) z^2\right) \tag{19}$$

has a wandering domain of rank 1. Those are the first completely explicit [1] examples of polynomial maps with wandering domains, as well as the first examples in degree 2 and the first examples of wandering domains with rank 1.

Recall that two Fatou components Ω_1 and Ω_2 are in the same grand orbit (of Fatou components) for P if there exists $n_1, n_2 \in \mathbb{N}$ such that $P^{n_1}(\Omega_1) = P^{n_2}(\Omega_2)$. One may ask whether for polynomial endomorphisms of $\mathbb{P}^2$ there exists a bound on the number of grand orbits of wandering domains that would depend only on the degree. The following theorem gives a negative answer:

THEOREM 4.5. [**AT22**] *Let P be of the form* (19) *and let $\alpha_0 > 1$ be an integer. Then P has countably many distinct grand orbits of rank 1 wandering domains.*

For this result, we need to assume that α_0 is integer rather than Pisot. Note that contrary to e.g. arguments involving the classical Newhouse phenomenon, we do not use perturbative arguments in the proof of Theorem 4.5, and the maps considered are completely explicit. In fact, more precisely, we construct an injective map from the set of hyperbolic components in a specific family of modified horn maps into the set of grand orbits of wandering Fatou components of P.

4.2. Topological invariants and horn maps. We will now investigate a few consequences of Theorem 4.2 on the topological classification of skew-products tangent to the identity (compare Section 1.2.2 for some historical remarks on the topological and analytic classification of parabolic germs in one complex variable).

To our knowledge, no complete topological classification is available for germs tangent to the identity in $\mathbb{C}^2$. Our results imply that such a classification must also be complicated even in the seemingly simple class of skew-products; in fact, it resembles the analytic classification for one-dimensional parabolic germs.

A first remarkable consequence of Theorem 4.2 is that the coefficient b is a topological invariant, among maps of the form (14):

THEOREM 4.6. [**AT22**] *Let P_1 and P_2 be two maps of the form* (14)*, and assume that there exists a homeomorphism $\mathfrak{h}$ defined near the origin, with $\mathfrak{h}(0,0) = (0,0)$, such that*

$$\mathfrak{h} \circ P_1 = P_2 \circ \mathfrak{h}.$$

[1] In [**ABD$^+$16**], there are explicit examples of polynomial maps for which numerical experiments strongly indicates the existence of wandering domains. It is possible that a rigorous argument could be made to prove the existence of wandering domains for these explicit maps as well.

Let b_i, α_i, β_i (with $1 \leq i \leq 2$) be as in (15)*, and assume that $b_i > \frac{1}{4}$ and $\beta_i \in \mathbb{R}$. If both pairs (α_i, β_i) admit an (α_i, β_i)-admissible sequence with a converging phase sequence then $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$, and so in particular $b_1 = b_2$.*

In [**Aba05**] Abate asked whether maps of the form

$$(3_{u,v,1}): \quad f(z,w) = (z + uz^2 + (1-u)zw, w + vw^2 + (1-v)zw), \text{ with } u+v \neq 1 \text{ and } u, v \neq 0$$

are topologically conjugated to each other. Using Theorem 4.6 we can now answer this question negatively. Indeed, observe that for $u = 1$ and $v \neq 0$ this map is conjugate, via a linear automorphism, to the map

$$(z,w) \mapsto \left(z - z^2, w + w^2 + \frac{1-v^2}{4} z^2\right) \tag{20}$$

which is of the form (14). In particular, when $v \in i\mathbb{R}^*$ in (20), such maps satisfy $b > \frac{1}{4}$ and $\beta = 0$. Then Theorem 4.6, together with Theorem 4.4, asserts that all maps of the form $(3_{u,v,1})$ with $u = 1$ and $v = 2\pi i / \ln(\rho)$, where ρ is a Pisot number, belong to different local topological conjugacy classes.

We now turn to a slightly stronger equivalence relation than local topological conjugacy:

DEFINITION 4.6. *We define an equivalence relation $\sim$ on maps of the form* (14) *by :*
$P_1 \sim P_2 \Leftrightarrow$ there exists a homeomorphism $\mathfrak{h}$ defined near the origin, with $\mathfrak{h}(0,0) = (0,0)$, such that $\mathfrak{h} \circ P_1 = P_2 \circ \mathfrak{h}$ and $\mathfrak{h}$ is of the form $\mathfrak{h}(z,w) = (\mathfrak{f}(z), \mathfrak{g}(z,w))$.

We now introduce a two-dimensional analogue of horn maps and lifted horn maps:

DEFINITION 4.7. *Let P be a map of the form* (14). *Let us define the lifted horn map of P of phase σ by*

$$\tilde{H}_\sigma(Z,W) := (Z, \alpha_0 \cdot \mathcal{E}_{q_0}(W) + (1-\alpha_0)Z + \sigma) =: (Z, \tilde{H}_{Z,\sigma}(W)) \tag{21}$$

The map $\tilde{H}_\sigma$ satisfies the functional relation $\tilde{H}_\sigma(Z+1, W+1) = \tilde{H}_\sigma(Z,W) + (1,1)$, so it descends to a map H_σ defined on $\mathbb{C}^2/\langle(1,1)\rangle$, which we call the horn map *of phase σ of P.*

In fact, we have the two following relations:

$$\tilde{H}_\sigma(Z+1, W) = \tilde{H}_\sigma(Z,W) + (1, 1-\alpha_0)$$
$$\tilde{H}_\sigma(Z, W+1) = \tilde{H}_\sigma(Z,W) + (0, \alpha_0).$$

Therefore, the map $\tilde{H}_\sigma$ descends to a map on $\mathbb{C}^2/\mathbb{Z}^2$ if and only if $\alpha_0 \in \mathbb{N}$. However, even when $\alpha_0 \notin \mathbb{N}$, it always descends to the horn map defined above, on $\mathbb{C}^2/\langle(1,1)\rangle$.

THEOREM 4.7. [**AT22**] *Let P_1 and P_2 be of the form* (14)*, with $b_i > \frac{1}{4}$ and $\beta_i \in \mathbb{R}$, and assume that $P_1 \sim P_2$. Let H^i_σ denote their respective horn maps. Then there exists $\sigma_1, \sigma_2 \in \mathbb{C}$ such that $H^1_{\sigma_1}$ and $H^2_{\sigma_2}$ are topologically conjugated on $\mathbb{C}^2/\langle(1,1)\rangle$.*

Finally, using Theorem 4.7, we obtain:

COROLLARY 4.1. *Under the same assumptions as Theorem 4.7, the number of critical points of q_i in $\mathcal{B}_{q_i}$ is the same. In particular, for any $k \in \mathbb{N}$, there exists P_1, P_2 of the form* (14) *such that $P_1(z,w) - P_2(z,w) = O(\|(z,w)\|^k)$, but $P_1 \not\sim P_2$.*

Note that the maps P_1 and P_2 are by assumption globally defined maps on $\mathbb{C}^2$, assumed to be topologically conjugated only on a neighborhood of the origin.

4.3. Fatou components with historic behaviour. In [**BB23**], Berger and Biebler construct wandering Fatou components Ω for some maps f (which are Hénon maps or endomorphisms of $\mathbb{P}^2$) that have *historic behaviour*, meaning that for any $x \in \Omega$, the sequence of empirical measures

$$e_n(x) := \frac{1}{n}\sum_{k=1}^{n} \delta_{f^k(x)}$$

does not converge.

To our knowledge, these are the only known examples so far of Fatou components for endomorphisms of $\mathbb{P}^k$ or for Hénon maps with historic behaviour. Note that in the case of the wandering Fatou components constructed in [**ABD**$^+$**16**] and [**ABTP23**], the sequences $(e_n)_{n\in\mathbb{N}}$ converge to the Dirac mass centered at the parabolic fixed point at the origin. In dimension 1, it follows easily from the Fatou-Sullivan classification that no Fatou component of a rational map on $\mathbb{P}^1$ can have historic behaviour; and for moderately dissipative Hénon maps, it follows from the classification of Lyubich and Peters [**LP14**] that periodic Fatou components cannot have historic behaviour.

Using Theorem 4.2, we give here new, explicit examples of polynomial skew-products (which may be chosen to extend to endomorphisms of $\mathbb{P}^2$) which have a Fatou component with historic behaviour:

THEOREM 4.8. [**AT22**] *Let $P(z,w) = (p(z), q(z,w))$ be a polynomial skew-product satisfying the following properties:*

(1) $p(z) = z - z^2 + O(z^3)$
(2) P has two different fixed points tangent to the identity of the form $(0,w_1)$ and $(0,w_2)$, which both satisfy the conditions that $\alpha_i \in \mathbb{N}^*$ and $\beta_i = 0$, with the same notations as in Theorem 4.2 and in appropriate local coordinates.

Then P has a Fatou component Ω with historic behaviour. More precisely, for any $(z,w) \in \Omega$, the sequences $(e_n(z,w))_{n\in\mathbb{N}}$ accumulate on

$$\mu_1 := \frac{\alpha_1\alpha_2 - \alpha_2}{\alpha_1\alpha_2 - 1}\delta_{(0,w_1)} + \frac{\alpha_2 - 1}{\alpha_1\alpha_2 - 1}\delta_{(0,w_2)}$$

and on

$$\mu_2 := \frac{\alpha_1 - 1}{\alpha_1\alpha_2 - 1}\delta_{(0,w_1)} + \frac{\alpha_1\alpha_2 - \alpha_1}{\alpha_1\alpha_2 - 1}\delta_{(0,w_2)}.$$

More explicitly, these conditions are given by:

(1) $p(z) = z - z^2 + O(z^3)$
(2) P has two different fixed points tangent to the identity of the form $(0,w_1)$ and $(0,w_2)$, with $q_0''(w_i) = 2$
(3) $p'''(0) = q_0'''(w_1) = q_0'''(w_2)$
(4) If $b_i := \frac{1}{2}\frac{\partial^2 q}{\partial z^2}(0,w_i)$, then $b_i > \frac{1}{4}$, and $\alpha_i := e^{\frac{2\pi}{\sqrt{4b_i - 1}}} \in \mathbb{N}^*$.

EXAMPLE 2. *With*

$$p(z) := z - z^2 + O(z^4)$$

and $q(z,w) := q_0(w) + a(z)$ with

$$q_0(w) := w + w^2 - 5w^4 + 6w^5 - 2w^6$$

and

$$a(z) := \left(\frac{1}{4} + \frac{\pi^2}{(\ln 2)^2}\right) z^2(1-z)^2,$$

the map P satisfies the conditions above, with $w_1 = 0$ and $w_2 = 1$, $\alpha_i = 2$ and $\beta_i = 0$.

Note that we could replace p by $z \mapsto z - z^2 + z^6$ in the previous example to obtain an example which extends to an endomorphism of $\mathbb{P}^2$.

Although we believe that the Fatou component constructed in Corollary 4.8 is wandering, we were not able to prove so. Note however that if it is not the case, then this would be the first example of an invariant (for some iterate of P) non-recurrent Fatou component whose limit sets depend on the limit map, which would give an affirmative answer to [**LP14**, Question 30] for the case $X = \mathbb{C}^2$ and $X = \mathbb{P}^2$.

4.4. Proof of Theorem 4.5 and families of finite type maps. Finally, we close this chapter by giving some details on the proof of Theorem 4.5, which ties in with the material exposed in Chapter 1 on finite type maps and their Teichmüller space.

Let $f_b(z,w) := (z + z^2, w + w^2 + bz^2)$, and assume that $b = \frac{1}{4} + \frac{\pi^2}{(\ln \alpha_0)^2}$, where $\alpha_0 \in \mathbb{N}$ and $\alpha \geq 2$. Then $p(z) := z + z^2$ and $q_0(w) := w + w^2$.

We let $\hat{h}$ denote the classical horn map of q_0 (see Section 1.2), and observe that

$$e^{2i\pi(1-\alpha_0)Z + 2i\pi\sigma}\hat{h}(e^{2i\pi W})^{\alpha_0} = e^{2i\pi \tilde{H}_{Z,\sigma}(W)} \tag{22}$$

where $\tilde{H}_{Z,\sigma}$ is the lifted horn map of f_b as defined in Definition 4.7. We let $h := \hat{h}^{\alpha_0}$, and consider the family $(h_\lambda)_{\lambda \in \mathbb{C}^*}$, defined by $h_\lambda := \lambda h$. Observe that by the choice of q_0, the maps h_λ have exactly 3 singular values:

(1) 0 and ∞, which are asymptotic values that are also superattracting fixed points
(2) one free critical value $v_\lambda := \lambda v$, where $v := e^{2i\pi\phi^\iota_{q_0}(-\frac{1}{2})}$.

In particular, if h_λ has an attracting cycle different from 0 and ∞, then it must capture v_λ. Moreover, the family $(h_\lambda)_{\lambda \in \mathbb{C}^*}$ is natural in the sense of Definition 4.5 (with $\phi_\lambda(z) := \lambda z$ and $\psi_\lambda := \mathrm{id}$).

DEFINITION 4.8. *A hyperbolic component of period m in the family $(h_\lambda)_{\lambda\in\mathbb{C}^*}$ is a connected component of the set of $\lambda \in \mathbb{C}^*$ such that h_λ has an attracting cycle of period m different from 0 and ∞.*

In order to prove that the Fatou components that we construct are indeed wandering, we will use the following result, which also has intrinsic interest:

THEOREM 4.9. [**AT22**] *Hyperbolic components in the family $(h_\lambda)_{\lambda\in\mathbb{C}^*}$ are simply connected.*

Before proving Theorem 4.9, we introduce some further notations:

DEFINITION 4.9. *We let $P_m := \{(\lambda, z) \in \mathbb{C}^* \times \mathbb{C}^* : z = h_\lambda^m(z)\}$, and $\tilde{\rho} : P_m \to \mathbb{C}$ be the map defined by $\tilde{\rho}(\lambda, z) = (h_\lambda^m)'(z)$.*

Let U be a hyperbolic component of period m and $\mathbb{D} \subset \mathbb{C}$ the unit disk. Then $U = \mathrm{proj}_1(\Pi)$, where Π is a connected component of $\tilde{\rho}^{-1}(\mathbb{D})$ and $\mathrm{proj}_1 : \mathbb{C}^2 \to \mathbb{C}$ is the projection on the first coordinate. Since for every $\lambda \in \mathbb{C}^*$, h_λ has only one free singular value, it may have at most one attracting cycle different from 0 and ∞; therefore if (λ, z_1) and (λ, z_2) are in a same fiber of the map $\mathrm{proj}_1 : \Pi \to U$, then z_1 and z_2 must be periodic points of the same attracting cycle. This means that the function $\tilde{\rho} : \Pi \to \mathbb{D}$ descends to a well-defined holomorphic function $\rho : U \to \mathbb{D}$ satisfying $\tilde{\rho} = \rho \circ \mathrm{proj}_1$.

LEMMA 4.1. *Let $U_0 := U \backslash \rho^{-1}(\{0\})$. The map $\rho : U_0 \to \mathbb{D}^*$ is a covering map.*

PROOF. We will prove this using a classical surgery argument, originally due to Douady-Hubbard for the case of the quadratic family ([**DH84**]). Let $\lambda_0 \in U_0$, and let V be a simply connected open subset of $\mathbb{D}^*$ containing $\rho(\lambda_0)$. Using a standard surgery procedure, we construct for any $t \in V$ a quasiconformal homeomorphism g_t such that $g_t \circ h_{\lambda_0} \circ g_t^{-1}$ is holomorphic, and $g_t(z_0)$ is a periodic point of period m and multiplier t. We refer to [**ABTP23**, Proposition 6.7], for the details (see also e.g. [[**FK21**], Theorem 6.4]).

We let $\phi : V \to \mathrm{Teich}(h_{\lambda_0})$ be the holomorphic map induced by $t \mapsto \mu_t$, where μ_t is the Beltrami form associated to g_t and $\mathrm{Teich}(h_{\lambda_0})$ is the dynamical Teichmüller space of h_{λ_0}.

Let $\hat{V} \subset U_0$ be a simply connected domain containing λ_0. We let $\hat{\phi} : \hat{V} \to \mathrm{Teich}(h_{\lambda_0})$ denote the map given by Theorem 4.3, which applies here since the only singular relations are $h_\lambda(0) = 0$ and $h_\lambda(\infty) = \infty$, which are persistent. Let $\xi := \frac{d}{d\lambda}_{|\lambda=\lambda_0} \hat{g}_\lambda$, and observe that since $\hat{g}_\lambda(v_{\lambda_0}) = v_\lambda = \lambda v$, we have $\xi(v_{\lambda_0}) \neq 0$. By [**Ast17**, Proposition 5], the derivative $\hat{\phi}'(\lambda_0)$ is therefore non-zero. Therefore, up to restricting V, we may assume that $\phi(V) \subset \hat{\phi}(\hat{V})$ and that there exists a well-defined inverse branch $\hat{\phi}^{-1} : \phi(V) \to \hat{V}$. Let $c : V \to \hat{V}$ be the map defined by $c := \hat{\phi}^{-1} \circ \phi$. Then c is a holomorphic local inverse of ρ, which maps $\rho(\lambda_0)$ to λ_0; since this construction is valid for any simply connected domain $V \subset \mathbb{D}^*$, the Lemma is proved. □

PROOF OF THEOREM 4.9. By the lemma above, $\rho : U_0 \to \mathbb{D}^*$ is a covering map. There are two cases: either it is a finite cyclic cover, or an infinite degree universal cover.

In the first case, there exists $\lambda_0 \in U$ such that $U_0 = U \backslash \{\lambda_0\}$, and U_0 is isomorphic to a punctured disk and U to a disk; then we are done.

In the second case, $U_0 = U$ is isomorphic to a disk and we are also done. □

We state here a slightly more precise statement of Theorem 4.5:

THEOREM 4.10. [**AT22**] *To each hyperbolic component U of the family $(h_\lambda)_{\lambda \in \mathbb{C}^*}$, we can associate a wandering Fatou component Ω_U of f_b. Moreover, if $U_1 \neq U_2$, then Ω_{U_1} and Ω_{U_2} are in different grand orbits of f_b.*

We sketch the proof of Theorem 4.10 below.

Since α_0 is an integer, we may choose an α_0-admissible sequence (n_k) to be simply $n_k = \alpha_0^k$, which has zero phase sequence, and where $n_0 \in \mathbb{N}^*$. By Theorem 4.2, we have that $f_b^{n_{k+1}-n_k}(p^{n_k}(z), w) \to (0, \mathcal{L}(\alpha_0, \sigma; z, w))$ uniformly on compacts in $\mathcal{B}_p \times \mathcal{B}_{q_0}$. Here σ is simply the constant Γ from Theorem 4.2, since the phase sequence of $(n_k)_{k\in\mathbb{N}}$ is zero. Let $(\lambda_0, x_0) \in \mathbb{C}^* \times \mathbb{C}^*$ be such that x_0 is an attracting periodic point of exact period ℓ for h_{λ_0}. Let $(z_0, w_0) \in \mathcal{B}_p \times \mathcal{B}_{q_0}$ be such that $e^{2i\pi(1-\alpha_0)\phi_p^\iota(z_0)+2i\pi\sigma} = \lambda_0$ and $e^{2i\pi\phi_{q_0}^o(w_0)} = x_0$: then w_0 is an attracting fixed point of $\mathcal{L}(\alpha_0, \sigma; z_0, \cdot)$ with same multiplier.

With an argument similar to that of the proof of Theorem 3.1, we may prove that the point (z_0, w_0) is contained in a Fatou component of f_b, which we will denote by Ω. By construction, there is a Fatou limit map on Ω of the form $(z, w) \mapsto (0, \eta(z))$, where $\eta(z)$ is an attracting periodic point of period ℓ of the map $\mathcal{L}(\alpha_0, \sigma; z_0, \cdot)$, and such that $\eta(z_0) = w_0$.

Let us first justify that Ω is wandering. Indeed, if it were say fixed, then there would be a continuous curve contained in Ω and joining (z_0, w_0) to $f_b(z_0, w_0)$. Composing with $\eta \circ \pi_1$, we obtain a curve $t \mapsto \eta(z_t)$ (with $z_1 = p(z_0)$) such that $\eta(z_t)$ is an attracting periodic point of period ℓ for $\mathcal{L}(\alpha_0, \sigma; z_0, \cdot)$. Transporting this to the family $(h_\lambda)_{\lambda\in\mathbb{C}^*}$ via the semi-conjugating map

$$e : (z, w) \mapsto (e^{2i\pi(1-\alpha_0)\phi^\iota_p(z)+2i\pi\sigma}, e^{2i\pi\phi^\iota_{q_0}(w)}), \tag{23}$$

we obtain an essential loop in $\mathbb{C}^*$, contained in a hyperbolic component. But this contradicts Theorem 4.9.

The proof of the fact that two different hyperbolic components in the family $(h_\lambda)_{\lambda\in\mathbb{C}^*}$ give rise to two different grand orbits of wandering domains is similar: if two Fatou components Ω_1 and Ω_2 given by the construction above are in the same grand orbit of Fatou components, then there exists $m_i \in \mathbb{N}^*$, $(z_i, w_i) \in \Omega_i$ and a continuous curve joining $f_b^{m_i}(z_i, w_i)$ inside a third Fatou component Ω_3 (with $1 \leq i \leq 2$). Lifting this curve to the family $(h_\lambda)_{\lambda\in C^*}$ using the maps η and e as above, we obtain a curve $t \mapsto \lambda_t$ in $\mathbb{C}^*$ such that h_{λ_t} is hyperbolic for all $t > 0$. Therefore, it must be contained in a single hyperbolic component.

5. Perspectives

5.1. Local dynamics of parabolic skew-products. We have treated in [**AT22**] the case of a skew-product tangent to identity

$$f(z, w) = \left(z + \sum_{i\geq 2} a_i z^2, w + \sum_{i+j\geq 2} b_{i,j} z^i w^j \right) \tag{24}$$

under the *generic* assumption

$$a_2 \neq 0, \quad b_{0,2} \neq 0 \quad \text{and} \quad b_{2,0} \neq 0. \tag{25}$$

Another natural question is to investigate the general case of maps of the form (24), without these assumptions.

QUESTION 6. *Which maps or germs of the form* (24) *exhibit some form of parabolic implosion?*

Here, by parabolic implosion we mean a renormalization property such as Theorem 4.2 (which does not involve any actual perturbation of the map f). When (25) holds, it follows from the results of [**AT22**] that the condition is $b_{2,0} \in (\frac{1}{4}, +\infty)$; this is related to the directors of the two non-trivial parabolic curves of f, see Theorem 2.1 and the discussion in Example 1.

In the general case of maps of the form (24), the discussion will involve the number of parabolic curves (which depends on the order of vanishing of $f - \mathrm{id}$ at $(0, 0)$), their directors, and whether or not they are degenerate. It is analoguous to parabolic implosion for perturbations of general one-dimensional maps or germs $f(z) = z + a_k z^k + O(z^{k+1})$, $k \geq 2$ and $a_k \neq 0$, which involves a discussion of the nature of the small cycles created near 0 by perturbation. Even in dimension one, these questions are far from easy (see [**Oud99**]); this complexity is also cited by Bedford, Smillie and Ueda in [**BSU17**] as the reason why their paper only deals with the non-degenerate parabolic case.

The following question remains as a motivation for Question 6:

QUESTION 7. *Which maps or germs of the form* (24) *have wandering domains?*

This could be a potential topic for a future PhD student.

5.2. Perturbations of a parabolic map in dimension 2. F. Bianchi has studied in [**Bia19b**] actual parabolic implosion of a map tangent to the identity in $\mathbb{C}^2$. More precisely, he considers a family of maps of the form

$$F_\epsilon(x,y) = \big(x + (x^2+\epsilon^2)\alpha_\epsilon(x,y), y(1+\rho x + \beta_\epsilon(x,y))\big) \tag{26}$$

where $\rho > 1$, and $(\epsilon, x, y) \mapsto \alpha_\epsilon(x,y)$ and $(\epsilon, x, y) \mapsto \beta_\epsilon(x,y)$ are holomorphic.

In particular, F_0 is tangent to the identity and leaves invariant both axes $x = 0$ and $y = 0$. Moreover, the condition $\rho > 1$ implies, by Hakim's work, that $[1, 0]$ is a non-degenerate characteristic direction contained in a parabolic basin $\mathcal{B}$ (see Theorem 2.1).

F. Bianchi then proves:

THEOREM 5.1 ([**Bia19b**]). *Let $\epsilon_j \to 0$ and $n_j \to +\infty$ be such that $n_j - \frac{\pi}{\epsilon_j} \to \sigma \in \mathbb{C}$. Then for every compact $K \subset \mathcal{B} \cap (y = 0)$, there is a neighborhood U of K in $\mathbb{C}^2$ and a subsequence $j_k \to +\infty$ such that*

$$F_{\epsilon_{j_k}}^{n_{j_k}} \to \mathcal{L}$$

locally uniformly , where $\mathcal{L}$ is a non-constant holomorphic map.

Compare with Lavaurs' theorem in dimension one (Theorem 1.1). While the limit map $\mathcal{L}$ is a priori not unique in Theorem 5.1, it does satisfy $\phi^o \circ \mathcal{L} = \phi^\iota + \sigma$, where ϕ^o and ϕ^ι are Fatou coordinates for the map F_0, wherever both sides are well-defined.

In an ongoing project with L. Lopez-Hernanz and J. Raissy, we aim to improve Theorem 5.1 in several ways:

(1) We wish to remove the very strong assumption that F_0 fixes any complex line, and only assume a generic non-vanishing condition on the quadratic part of F_0, as well as a natural condition on the real part of a director of a characteristic direction (similar to the condition $\rho > 1$) ensuring that this direction is contained in a parabolic basin.
(2) We wish to obtain a global convergence on $\mathcal{B}$ and not only locally near the characteristic direction $y = 0$.
(3) We wish to obtain an actual convergence to a Lavaurs map of phase σ, without having to take subsequences.

It is our hope that such a general statement on 2-dimensional parabolic implosion could have applications to the study of the bifurcations in e.g. the family $\mathcal{H}_d(\mathbb{P}^2)$.

5.3. Local dynamics of general non-degenerate parabolic germs. Finally, another direction of generalization of the results from [**AT22**] is to remove the assumption that f is a skew-product, and work e.g. with maps of the form

$$f(x,y) = (x + x^2 + O(\|(x,y)\|^3), y + y^2 + bx^2 + O(\|(x,y))\|^3)).$$

This project may seem very close to the one described in Section 5.2, since they would apply to more or less the same class of maps. There are however two main differences: first, here we do not study perturbations of f but the dynamics of f itself. Additionnally, in the previous project, the region where the convergence to the Lavaurs map is expected to take place is the parabolic basin $\mathcal{B}$ containing the characteristic curve tangent to $[1:0]$, where the dynamics of f is trivial. On the other hand, the region considered here is a neighborhood of the characteristic curve $[0:1]$, on which the dynamics is much more complicated, as the results of [**AT22**] attest.

This is certainly a more challenging and long-term project. A motivation for the extension of the results of [**AT22**] to generic maps tangent to the identity is the following (probably even more difficult) question:

QUESTION 8. *Let $\mathcal{W}_d \subset \mathcal{H}_d(\mathbb{P}^2)$ be the set of degree $d \geq 2$ endomorphisms of $\mathbb{P}^2$ with a wandering Fatou component. Is $\mathcal{W}_d$ dense in the bifurcation locus?*

For now, the best result available towards answering this question is obtained in [**BB23**], and is the fact that the closure of $\mathcal{W}_d$ contains an open subset of $\mathcal{H}_d(\mathbb{P}^2)$ for $d \geq 5$.

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Domaines errants et bifurcations en dynamique holomorphe

Résumé :

Ce manuscrit d'HDR regroupe les travaux effectués depuis mon arrivée à Orléans, en 2016. Il comporte trois parties : la première est consacrée à un panorama synthétique sur les applications de type fini, introduites par Epstein. La théorie de Fatou/Julia y est développée dans ce cadre, suivie d'une étude des déformations quasiconformes de la dynamique, reposant sur des outils de théorie de Teichmüller.
La seconde porte sur les bifurcations de familles de systèmes dynamiques holomorphes, dans différents cadres : en une variable complexe (fractions rationnelles, applications méromorphes transcendantes, ou applications de type fini) ou en dimension supérieure (endomorphismes d'espaces projectifs). Dans le cadre algébrique (fractions rationnelles et endomorphismes d'espaces projectifs), on dispose d'outils de théorie ergodique et du pluripotentiel, qui permettent notamment de définir et d'étudier une stratification du lieu de bifurcation.
Enfin, la troisième partie porte sur la construction et l'étude de composantes de Fatou errantes pour certaines classes d'endomorphismes du plan projectif complexe (produits fibrés), ainsi que sur la dynamique semi-locale près d'un point fixe tangent à l'identité.

Mots clés :

Dynamique holomorphe, ensembles de Fatou/Julia, stabilité/bifurcations, dynamique locale de germes tangents à l'identité

Wandering domains and bifurcations in holomorphic dynamics

Abstract :

This HDR manuscript compiles the work done since my arrival in Orléans in 2016. It consists of three parts: the first is devoted to a synthetic overview of finite type maps, which were introduced by Epstein. The theory of Fatou/Julia is developed within this framework, followed by a study of quasiconformal deformations of dynamics, based on tools from Teichmüller theory.
The second part focuses on bifurcations of families of holomorphic dynamical systems, in different settings: in one complex variable (rational maps on the Riemann sphere, transcendental meromorphic maps, or finite type maps) or in higher dimension (endomorphisms of projective spaces). In the algebraic setting (rational maps and endomorphisms of projective spaces), there are tools from ergodic theory and pluripotential theory, which notably allow the definition and study of a stratification of the bifurcation locus.
Finally, the third part deals with the construction and study of wandering Fatou components for certain classes of endomorphisms of the complex projective plane (skew-products), as well as on the semi-local dynamics near a fixed point tangent to the identity.

Keywords :

Holomorphic dynamics, Fatou/Julia sets, stability/bifurcations, local dynamics of germs tangent to the identity.