# HORN MAPS OF SEMI-PARABOLIC HÉNON MAPS

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Abstract. We prove that horn maps associated to quadratic semi-parabolic fixed points of Hénon maps, first introduced by Bedford, Smillie, and Ueda, satisfy a weak form of the Ahlfors island property. As a consequence, two natural definitions of their Julia set (the non-normality locus of the family of iterates and the closure of the set of the repelling periodic points) coincide. As another consequence, we also prove that there exist small perturbations of semi-parabolic Hénon maps for which the Hausdorff dimension of the forward Julia set $J^+$ is arbitrarily close to 4.

## 1. Introduction

Following the seminal work of Douady-Hubbard [DH84, DH85a] and Lavaurs [Lav89], the study of perturbations of maps with a parabolic fixed point, often referred to as *parabolic implosion*, has been a major theme in one-variable complex dynamics. A first consequence of this theory is the discontinuity of the Julia sets at parameters where a parabolic bifurcation occurs [Dou94]. Further notable applications include the celebrated result by Shishikura [Shi98] that parabolic maps can be approximated by hyperbolic maps with arbitrarily large hyperbolic dimension and, as a consequence, the proof that the boundary of the Mandelbrot set has maximal Hausdorff dimension, see also [DSZ97, McM00, Tan98, Zin98] for refinements of this result and further consequences and [Shi00, PV20] for an overview of the theory.

Shishikura's proof involves the so-called *horn maps*. These maps describe the limit behaviour of the return maps of large iterates of the perturbed maps near the parabolic point. Another important aspect of horn maps (also called *Écalle-Voronin invariants*) is that they classify parabolic germs up to analytic conjugacy, i.e., they form a *complete invariant* for this notion of equivalence [Éca85, Vor81]. Techniques from parabolic implosion have been extended by Inou and Shishikura [IS06], who introduced the *near-parabolic renormalization*, a powerful tool which was used, for instance, to construct quadratic Julia sets with positive area [BC12], and to make significant steps towards the settling of the Fatou hyperbolicity conjecture [CS15].

In recent years, techniques of parabolic implosion have started to be developed and successfully applied also in higher dimensions. In [BSU17], Bedford, Smillie, and Ueda extended Lavaurs' results to diffeomorphisms of $(\mathbb{C}^2, 0)$ with a semi-parabolic fixed point, that is, with one multiplier equal to 1 and one in the unit disk. In the important particular case of dissipative Hénon map, the authors introduced a horn map analogous to the one-dimensional case, which they used to show the discontinuity of various dynamically meaningful sets at parameters with a semi-parabolic fixed point. In [DL15], Dujardin and Lyubich adapted and improved the results from [BSU17] to construct homoclinic tangencies in some regions of the parameter space of complex Hénon maps, as part of their characterization of stability and bifurcation in families of such maps. Parabolic implosion techniques in two complex variables were also used by Buff, Dujardin, Peter, Raissy, and the first author to give the first example of an endomorphism of $\mathbb{P}^2(\mathbb{C})$ with a wandering domain [ABD+16], which solved a long-standing open question in the domain, see also [ABTP23]. Adaptations of these techniques from Boc Thaler and the first author have lead to a precise description of the local dynamics near a parabolic point of a significant class of maps tangent to the identity in $\mathbb{C}^2$ [Aba01], in particular solving a long-standing open question by Abate [Aba05] on the topological classification of such maps. The first result on the parabolic implosion of a two-dimensional map tangent to the identity was also established by the second author in [Bia19].

Coming back to the original work by Bedford, Smillie, and Ueda, very little is known about the dynamics of the horn maps of semi-parabolic Hénon maps; for instance, it was not even known until now whether they always had periodic cycles (besides the two "trivial" fixed points 0 and $\infty$). In this paper, we prove that they satisfy a weak version of the so-called Ahlfors island property. As a consequence, we can show the density of the repelling periodic points in their Julia sets, and an analogous of Shishikura's result for the forward Julia set $J^+$ of dissipative Hénon maps.

The class of *Ahlfors island maps* was implicitly present in the work of Epstein [Eps93], see also [RR12, MR12]. Roughly speaking, given an open set $W \subset \mathbb{P}^1$, a holomorphic map $f : W \to \mathbb{P}^1$ has the $N$ islands property if, given any $N$ Jordan domains with pairwise disjoint closures, one can find univalent inverse branches of $f$ on at least one of these Jordan domains, whose image is close to any given point in the boundary of $W$ (see Definition 2.3 for a precise formulation). A celebrated theorem by Ahlfors [Ahl35] states that every entire or meromorphic map has the 5 islands property (in this case, we have $W = \mathbb{C}$ and the only boundary point is $\infty$), see also [Ber98]. This remarkable result can be used to give a simple proof of the density of repelling cycles in the Julia set of any transcendental entire or meromorphic map, see for instance [Ber93].

Another class of maps extensively studied by Epstein [Eps93] is the class of *finite type maps*, that is, holomorphic maps with finitely many singular values (see Definition 2.2). Finite type maps with $N$ singular values have the $(N + 1)$ islands property; however, finite type maps form a much smaller class than Ahlfors island maps. For instance, Epstein proved that finite type maps have no wandering or Baker domains and admit only finitely many non-repelling cycles; none of these statements is true in general for Ahlfors island maps. While horn maps of one-variable rational maps are always finite type maps [Eps93], it seems unlikely that this holds true for horn maps of dissipative semi-parabolic Hénon maps in general (indeed, this would be equivalent to proving that only finitely many stable manifolds have tangencies with a certain entire curve $\Sigma$, defined below).

In this paper, we introduce the following slightly weaker version of the island property. Observe that in the usual versions, the property below is true for every positive real number instead of just those smaller than $r(z_0)$, see Definition 2.3. On the other hand, here we can specify which points $z_0$ should be excluded, see Remark 2.4.

**Definition 1.1.** *Let $W \subset \mathbb{P}^1$ be an open set and $h : W \to \mathbb{P}^1$ a holomorphic map. We say that $h$ has the* small island property *if, for every $z_0 \in \mathbb{C}^*$, there exists $r(z_0) > 0$*

*such that, for every domain $U$ intersecting $\partial W$, there exists $\Omega \Subset U \cap W$ such that $h : \Omega \to \mathbb{D}(z_0, r(z_0))$ is a conformal isomorphism.*

Let us emphasize that $r(z_0)$ does not depend on the choice of $U$, but only on $f$ and $z_0$. We will show in Theorem 1.2 that the small island property as in Definition 1.1 is enough to prove the density of the repelling periodic points in the Julia set.

Let now $f$ be a dissipative Hénon map with a semi-parabolic fixed point $p$ of order 2, see Section 3 for the precise definitions. Let $\mathcal{B}$ denote the parabolic basin of the semi-parabolic point, which is a two-dimensional open set with $p$ on its boundary, and $\Sigma$ the parabolic curve, which can thought of as the one-dimensional unstable curve of $p$. By [BSU17], there exist two maps $\phi^\iota \colon \mathcal{B} \to \mathbb{C}$ (usually called the *incoming Fatou coordinate*) and $\psi^o : \mathbb{C} \to \Sigma$ (usually called the *outgoing Fatou parametrization*) which semi-conjugate the map $f$ on $\mathcal{B}$ and on $\Sigma$ to the translation by 1 on $\mathbb{C}$, respectively. The *Hénon-Lavaurs map* (or *transition map*) associated to $f$ is the composition $\mathcal{L}_0 := \psi^o \circ \phi^\iota$. Observe that $\mathcal{L}_0$ commutes with $f$. By [BSU17], $\mathcal{L}_0$ can be seen as a limit of large iterates of suitable perturbations of $f$ near the semi-parabolic point. We refer to Section 3.4 for more details on this.

By the definition of $\mathcal{L}_0$, we immediately see that this map is semi-conjugated to the map $H_f := \phi^\iota \circ \psi^o \colon (\psi^o)^{-1}(\mathcal{B}) \to \mathbb{C}$. This is the *lifted horn map* associated to $f$. As $\mathcal{L}_0$ commutes with $f$, we see that $H_f$ commutes with the translation by 1 on $\mathbb{C}$. We can then quotient its action by this translation, and obtain a map from a subset of the cylinder $\mathbb{C}^*$, containing pointed neighbourhoods of 0 and $\infty$, to the cylinder itself. By [BSU17], $h_f$ extends to 0 and $\infty$. The following is our first main result.

**Theorem 1.2.** *Let $f$ be a dissipative semi-parabolic Hénon map as above. The horn map $h_f$ has the small island property as in Definition 1.1.*

The following is then a consequence of Theorem 1.2.

**Corollary 1.3.** *Let $f$ be a dissipative semi-parabolic Hénon map as above. The repelling periodic points of the horn map $h_f$ are dense in its Julia set.*

The proof of Theorem 1.2 is based on techniques from Pesin theory, which were first adapted to this context in [BLS93b], and on local equidistribution results towards the Green currents $T^+$ and $T^-$ of the Hénon map $f$, which follow from the local approch to these problems developed in [HOV95, Duj04, DS06, DNS08].

As mentioned above, as an application of Theorem 1.2, we will also deduce an analogous of Shishikura's result for Hénon maps.

**Theorem 1.4.** *Let $f$ be a dissipative semi-parabolic Hénon map as above. Then, there exists a sequence $f_n \to f$ of dissipative Hénon maps of the same algebraic degree such that*

$$\dim_H J^+(f_n) \to 4 \quad \text{as } n \to \infty.$$

In order to prove Theorem 1.4, we show that every holomorphic map with the small island property can be suitably modified (i.e., can be multiplied by a suitable constant) so that it has arbitrarily large (i.e., close to 2) hyperbolic dimension. Assume for the sake of simplicity that the multiplication is not necessary (this is just a minor technical point). We can apply this to the horn map $h_f$, and recall that the Hénon-Lavaurs map $\mathcal{L}_0$ is the limit of suitable large iterates $f_{\epsilon_n}^n$ of perturbations $f_{\epsilon_n}$ of $f$. By the conjugacy, large hyperbolic sets for $h_f$ give a large limit set for suitable iterates of $\mathcal{L}_0$. As these large hyperbolic sets persist under the perturbation $f \mapsto f_{\epsilon_n}$ for $n$ sufficiently large, this leads to hyperbolic limit sets $\mathcal{H}_n$ for suitable iterates of $f_{\epsilon_n}^n$ with large (i.e., close to 2) unstable dimension. As the forward Julia set $J^+(f_\epsilon)$ contains the union of the stables manifolds of the points of $\mathcal{H}_n$, this leads to the desired result.

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## 2. Island properties and consequences

### 2.1. Ahlfors island maps and finite type maps.

**Definition 2.1.** *Let $X$ and $W$ be a Riemann surface, with $X$ connected, and $f : W \to X$ a holomorphic map. The* singular value set *$S(f)$ of $f$ is the smallest subset of $X$ such that $f : W_0 \setminus f^{-1}(S(f)) \to X \setminus S(f)$ is a covering map for every connected component $W_0$ of $W$.*

As covering maps are surjective, it follows from the above definition that we have $X \setminus f(W) \subset S(f)$.

**Definition 2.2.** [Eps93] *Let $W \subset \mathbb{P}^1$ be a non-empty open set and $f : W \to \mathbb{P}^1$ a holomorphic map. We say that $f$ is a* finite type map *on $\mathbb{P}^1$ if*

*(1) $f$ is non-constant on every connected component of $W$;*
*(2) $f$ has no removable singularities;*
*(3) $S(f)$ is finite.*

**Definition 2.3.** [RR12, MR12] *Let $W \subset \mathbb{P}^1$ be a non-empty open set and $f : W \to \mathbb{P}^1$ a holomorphic map. We say that $f$ has the $N$* islands property *if, given any $N$ Jordan domains $D_1, \dots, D_N \subset \mathbb{P}^1$ with pairwise disjoint closures and any open set $U$ intersecting $\partial W$, there exists $1 \leq i_0 \leq N$ and an open set $\Omega \Subset U \cap W$ such that $f : \Omega \to D_{i_0}$ is a conformal isomorphism. If there exists $N \geq 1$ such that $f$ has the $N$ islands property, we say that $f$ is an* Ahlfors island map*.*

**Remark 2.4.** *In Definition 1.1, $z_0$ must be chosen different from $0$ and $\infty$. Hence, the small island property as in that definition can be seen as a weaker version of the 3 islands property above. We could give a more general definition of the small $N$ islands property, admitting $N-1$ exceptions as in the Definition 2.3. The proofs in this section would be the same. We just define the precise version of the property that we will prove for the horn maps of semi-parabolic Hénon maps for simplicity.*

Also note that the 1 island property is vacuously satisfied if $\partial W = \emptyset$, that is, if $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational map. This case is however very special, and our arguments will often use the fact that $\partial W \neq \emptyset$.

2.2. **Julia sets.** We fix in this section an open set $W \subset \mathbb{P}^1$ and a holomorphic map $f : W \to \mathbb{P}^1$. We give here two natural definitions of the Julia set of $f$. The first is related to the notion of non-normality. Since $W \neq \mathbb{P}^1$, the definition needs to take into account orbits leaving the domain $W$.

**Definition 2.5** (Definition of the Julia set as non-normality locus)**.** *The Fatou set $F(f)$ of $f$ is the union of all open sets $U \subset \mathbb{P}^1$ such that either*

*(1) $f^n(U) \subset W$ for all $n \in \mathbb{N}$, and $\{f^n : U \to W\}$ is normal; or*

*(2) there exists $n \in \mathbb{N}$ such that $f^n(U) \subset \mathbb{P}^1 \setminus \overline{W}$, where $\overline{W}$ denotes the closure of $W$ in $\mathbb{P}^1$.*

*The set $J_F(f)$ is the complement of $F(f)$ in $\mathbb{P}^1$.*

Observe that according to this definition, we always have $\partial W \subset J_F(f)$ and $\mathbb{P}^1 \setminus \overline{W} \subset F(f)$. Given a point in the Fatou set, all its orbit is in the Fatou set; conversely, given a point in $J_F(f)$, all preimages are in $J_F(f)$. Moreover, it is clear that repelling periodic points are always in $J_F(f)$. This leads to the second definition, and to the inclusion $J_R(f) \subset J_F(f)$.

**Definition 2.6** (Definition of the Julia set by means of repelling periodic points)**.** *The set $J_R(f)$ is the closure of the set of all repelling periodic points of $f$.*

2.3. **A consequence of the small island property.** We fix again an open set $W \subset \mathbb{P}^1$ and a holomorphic map $f : W \to \mathbb{P}^1$. By [Eps93], we have $J_F(f) = J_R(f)$ if $f$ is an Ahlfors island map. In this section we show that the small island property as in Definition 1.1 is still enough to ensure this property. For simplicity, we will denote by $W_\infty$ the interior of $\bigcap_{n\geq 0} f^{-n}(W)$ in the rest of this section.

**Lemma 2.7.** *Assume that either $W_\infty = \emptyset$ or all connected components of $W_\infty$ are hyperbolic. Then we have $J_F(f) = \overline{\bigcup_{n\geq 0} f^{-n}(\partial W)}$.*

*Proof.* The inclusion $\overline{\bigcup_{n\geq 0} f^{-n}(\partial W)} \subset J_F(f)$ is always true by the Definition 2.5 of $J_F(f)$.

Conversely, if $W_\infty \neq \emptyset$, $W_\infty$ is completely invariant and $f : W_\infty \to W_\infty$ is non-increasing for the hyperbolic metric. Therefore, we have $W_\infty \subset F(f)$. Clearly, this inclusion still holds if $W_\infty = \emptyset$.

Let $U$ be a connected open set intersecting $J_F(f)$. By the inclusion proved above, $U$ cannot be contained in $W_\infty$. Therefore, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap (\mathbb{P}^1 \setminus W) \neq \emptyset$. Moreover, we must have $f^n(U) \cap \partial W \neq \emptyset$, for otherwise we would have $f^n(U) \subset \mathbb{P}^1 \setminus W$. By definition, this would give $U \subset F(f)$, contradicting the assumption $U \cap J_F(f) \neq \emptyset$.

The above proves that, for any open domain $U$ intersecting $J_F(f)$, there exists $n \in \mathbb{N}$ such that $f^{-n}(\partial W) \cap U \neq \emptyset$. Again by definition, we have $f^{-n}(\partial W) \subset J_F(f)$. This shows that $\bigcup_{n\geq 0} f^{-n}(\partial W)$ is indeed dense in $J_F(f)$, and concludes the proof. $\square$

**Theorem 2.8.** *Assume that $f$ has the small island property. Then $J_F(f) = J_R(f)$.*

*Proof.* Recall that the inclusion $J_R(f) \subset J_F(f)$ always holds by definition. Hence, we only have to prove the reversed inclusion.

If $W_\infty$ is non-empty and has a non-hyperbolic component, it is isomorphic to $\mathbb{P}^1$, $\mathbb{C}^*$, or $\mathbb{C}$. Then, $f$ is either a rational map, a transcendental self-map of $\mathbb{C}^*$ or a transcendental entire map. In all these cases, the result is classical. Therefore, we only need to deal with the case where $W_\infty$ is either empty or has a hyperbolic connected component. In particular, Lemma 2.7 applies.

Fix $z_0 \in J_F(f)$. Since $J_F(f)$ has no isolated points, we may assume without loss of generality that $z_0 \in \mathbb{C}^*$. By Lemma 2.7, we may also assume that $f^n(z_0) \in \partial W$ for some $n \in \mathbb{N}$. Let $D$ be a small disk centered at $z_0$ small enough so that $f^n : D \to f^n(D)$ is a branched cover, ramified only possibly at $f^n(z_0)$. We can also assume that the radius of $D$ is smaller than the quantity $r(z_0)$ as in the Definition 1.1 of the small island property. It is enough to show that there exists a repelling point in $D$.

Let $\Omega \Subset f^n(D)$ be the simply connected domain given by the small island property applied with $U = f^n(D)$. Then, the map $f^{n+1} : f^n(D) \Supset \Omega \to f^n(D)$ is a branched cover with at most one critical value $f^n(z_0)$, which lies outside of $\overline{\Omega}$, and in particular, it is polynomial-like. It follows from, e.g., Douady-Hubbard's Straightening theorem [DH85b] that $f^{n+1} : \Omega \to f^n(D)$ has a repelling periodic point[1].

Therefore, $D$ contains a repelling periodic point for $f$. This proves the inclusion $J_F(f) \subset J_R(f)$ and completes the proof. □

**Remark 2.9.** *A holomorphic family of maps $f_\lambda : W_\lambda \to \mathbb{P}^1$ is* natural [EL92, ABF21] *if it is of the form $f_\lambda = \phi_\lambda \circ f \circ \psi_\lambda^{-1}$, where $W \subset \mathbb{P}^1$ is an open set, $f : W \to \mathbb{P}^1$ a holomorphic map, $W_\lambda := \psi_\lambda(W)$ and $\phi_\lambda, \psi_\lambda : \mathbb{P}^1 \to \mathbb{P}^1$ are homeomorphisms depending holomorphically on $\lambda \in M$. It is straightforward to check that if $f$ has the small island property, then each map $f_\lambda$ in a natural family as above also has the small island property. In the proof of Theorem 1.4, we will be interested in the case of a family of maps of the form $h_\lambda = \lambda h$, where $h$ has the small island property and $\lambda \in \mathbb{C}^*$. It is in particular a natural family, with $\phi_\lambda(z) := \lambda z$ and $\psi_\lambda :=$* Id.

## 3. Preliminaries on Hénon and horizontal-like maps

We will consider in this section an automorphism $f$ of $\mathbb{C}^2$ of the form

$$f(z, w) = (p(z) + aw, z), \tag{3.1}$$

where $p$ is a monic polynomial of degree $d \geq 2$ and $a$ is some constant in $\mathbb{C}^*$. Any $f$ as above is usually referred to as a *(generalized) Hénon map*. Observe that the Jacobian of $f$ is constant, and equal to $|a|$. We say that $f$ is *dissipative* if $|a| < 1$.

By results of Jung [Jun42] and Friedland-Milnor [FM89], every polynomial automorphism $f$ of $\mathbb{C}^2$ is conjugated (in the group of polynomial automorphisms) to either an *elementary automorphism*, i.e., a map of the form $(z, w) \mapsto (az + p(w), bw + c)$ for some $a, b \in \mathbb{C}^*$, $c \in \mathbb{C}$, and polynomial $p$ of degree $d \geq 0$, or to a *Hénon-type map*, i.e., a finite composition of generalized Hénon maps as in (3.1). As the dynamics of elementary automorphisms is simple to describe, we will just consider in the following maps of the

[1] Actually, since this polynomial-like map has only one critical value which is escaping, it can also be proved by elementary means that it has a repelling fixed point, see for instance [Eps93].

second type. For simplicity, we will just consider Hénon maps, but the picture is the same when considering finite compositions as above.

3.1. **Filtration property and induced horizontal-like map.** Let $f$ be as in (3.1). Friedland and Milnor [FM89] showed that it is possible to decompose $\mathbb{C}^2$ in a dynamically meaningul way, as follows. Let $\mathbb{D}(0,R) \subset \mathbb{C}$ be the disc of center 0 and radius $R$. Set

$$\begin{aligned} D_R &:= \mathbb{D}(0,R)^2, \\ V_R^+ &:= \{(z,w) \in \mathbb{C}^2 \colon |z| > \max(|w|, R)\}, \text{ and} \\ V_R^- &:= \{(z,w) \in \mathbb{C}^2 \colon |w| > \max(|z|, R)\}. \end{aligned}$$

We also denote by $K^+$ (resp. $K^-$) the set of points whose orbit under $f$ (resp. $f^{-1}$) is bounded.

**Lemma 3.1.** *The following assertions hold for every $R$ sufficiently large.*

*(1) $f(V_R^+) \subset V_R^+$ and $V_R^+ \cap K^+ = \emptyset$;*
*(2) $f(D_R \cup V_R^+) \subset D_R \cup V_R^+$;*

Similar assertions hold replacing $f$, $V_R^+$, and $K^+$ with $f^{-1}$, $V_R^-$, and $K^-$, respectively.

Although our maps will always be globally defined, in the following we will sometimes need to work in a semi-local setting, that we now describe. A *vertical subset* of $D_R$ is a subset of $D_R$ whose closure in $\mathbb{C}^2$ is disjoint from the *vertical boundary* $\partial\mathbb{D}(0,R) \times \mathbb{D}(0,R)$ of $D_R$. Similarly, a *horizontal subset* of $D_R$ is a subset of $D_R$ whose closure is disjoint from the *horizontal boundary* $\mathbb{D}(0,R) \times \partial\mathbb{D}(0,R)$.

We fix $R$ sufficiently large for Lemma 3.1 to hold. The map $f$ can be seen as a map from the set $D_R \cap f^{-1}(D_R)$ to the set $D_R \cap f(D_R)$. In particular, since $f^{-1}(D_R) \Subset D_R \cup V^-$ and $f(D_R) \Subset D_R \cup V^+$, $f$ is a holomorphic map from a vertical subset of $D_R$ to a horizontal one and it is a so-called *horizontal-like map*. We refer to [HOV95, Duj04, DS06, DNS08] for the precise definition and their properties. We will use the notation $\tilde{f}$ for the horizontal-like map associated to $f$ as above, when we will need to emphasize it.

3.2. **The Green currents $T^+$ and $T^-$.** Let $f$ be a Hénon map. Denote by $G^\pm$ the functions $G^\pm(z,w) := \lim_{n\to\infty} d^{-n} \log^+ \|f^{\pm 1}(z,w)\|$, where $\log^+(\cdot) = \max(0, \log(\cdot))$. Such functions, usually called the *Green functions* of $f$ and $f^{-1}$ respectively, are well defined (as the convergences are uniform of every compact subset of $\mathbb{C}^2$). They are Hölder continuous and plurisubharmonic [FS92, Hub86]. Hence, the $(1,1)$-currents given by $T^\pm := dd^c G^\pm$ are positive closed. They are the *Green currents* of $f$ and $f^{-1}$ and they describe – in a quantified sense – the distribution of the iterate of curves under forward and backward iteration of $f$ [BS91a, FS92]. Their support is equal to $J^\pm := \partial K^\pm$, respectively. More precisely, they are the unique positive $dd^c$-closed currents supported on $K^\pm$, respectively [DS14]. By [BS91b], the convergence

$$d^{-n}[f^{-n}(M)] \to c_M T^+ \tag{3.2}$$

holds for every locally closed submanifold $M \subset \mathbb{C}^2$ satisying $M \subset J^+$ or $M \subset X$, where $X$ is algebraic. Here $c_M$ is a constant depending on $M$, and we have $c_M > 0$ if, for instance, one has $T^-|_M > 0$. A similar property holds for $T^-$.

In the semi-local setting described in Section 3.1, the convergence above can improved. For a given large $R$, recall that we denote by $\tilde{f}$ the associated horizontal-like map on $D_R$. By [Duj04], we can associate to $\tilde{f}$ its Green current $\tilde{T}^+$, which is vertical, and the Green current of $\tilde{f}^{-1}$, $\tilde{T}^-$, which is horizontal in $D_R$. By [Duj04, DNS08], we have

$$d^{-n}\tilde{f}^*R \to c_R\tilde{T}^+ \tag{3.3}$$

for every positive closed vertical current on $D_R$. Here $c_R > 0$ is a constant depending on $R$ (and it is equal to its *vertical mass*, see [Duj04, DS06]; when $R$ is smooth, it is equal to the mass of the restriction of $R$ to any horizontal line in $D_R$). It follows from (3.2) and (3.3) that $\tilde{T}^\pm = T^\pm_{|D_R}$.

3.3. **The equilibrium measure and Pesin boxes.** As the Green functions $G^\pm$ are continuous, the intersection $\mu := T^+ \wedge T^-$ is a well-defined probability measure, and is the unique measure of maximal entropy of $f$ [BLS93b, Sib99]. It satisfies remarkable ergodic properties [BLS93b, BLS93a, Din05, BD24]. We recall here those that we will need in the sequel. We denote by $\lambda^+$ and $\lambda^-$ the two Lyapunov exponents of $\mu$. Recall [BS98] that we have $\lambda^+ > 0$ and $\lambda^- < 0$, hence $\mu$ is hyperbolic in the sense of Pesin theory.

By Oseledec's ergodic theorem, there exist a full measure subset $\mathscr{R}$ of the support of $\mu$ and two measurable distributions of 1-dimensional subspaces $E^s, E^u \colon \mathscr{R} \to T\mathbb{C}^2$ with $E^s(x), E^u(x) \in T_x\mathbb{C}^2$ with the property that for every $x \in \mathscr{R}$, we have $E^s(x) \neq E^u(x)$, $Df(E^{s/u}(x)) = E^{s/u}(f(x))$ and

$$\lim_{n\to\infty} \|Df^n(v)\| = \lambda^+ \quad \forall v \notin E^s(x) \quad \text{and} \quad \lim_{n\to\infty} \|Df^{-n}(v)\| = \lambda^- \quad \forall v \notin E^u(x).$$

The angle between $E^s$ and $E^u$ along an orbit is also controlled.

Given $r > 0$ and $x \in \mathscr{R}$, we denote by $B^s_r(x)$ and $B^u_r(x)$ the 1-dimensional affine discs in $\mathbb{C}^2$ centred at $x$, of radius $r$, and whose tangent at $x$ is given by $E^s(x)$ and $E^u(x)$, respectively. By Pesin theory, for every $x$ there is an $r(x)$ such that the stable and unstable manifolds $W^s(x)$ and $W^u(x)$ of $x$ are locally graphs over $B^s_r(x)$ and $B^u_r(x)$, respectively. We denote by $W^s_r(x)$ and $W^u_r(x)$ these local stable and unstable manifolds, respectively.

Fix $r > 0$ and denote by $\mathscr{R}_r$ the set of points $x \in \mathscr{R}$ such that $r(x) \geq r$ [2]. Let $F$ be a compact subset of $\mathscr{R}_r$, and assume that the diameter of $F$ is $\ll r$. We denote by $W^s_r(F)$ and $W^u_r(F)$ the union of the sets $W^s_r(x)$ and $W^u_r(x)$ for $x \in F$. We call the set $P := W^s_r(F) \cap W^u_r(F)$ the *Pesin box* [Pes77] generated by $F$. By [BLS93b] (see also [Duj04]), there exist a compact set $P^s$ which is homeomorphic to $W^u_r(x) \cap W^s_r(F)$ for all $x \in F$ and a compact set $P^u$ which is homeomorphic to $W^s_r(x) \cap W^u_r(F)$ for all $x \in F$. Then, $P$ is homeomorphic to $P^s \times P^u$. Moreover, there exists a neighbourhood $N = N(P)$ of $P$, biholomorphic to a bidisk, such that (the image of) every $W^s_N(x) := W^s_r(x) \cap N$ (resp. $W^u_N(x) := W^u_r(x) \cap N$) is a vertical (resp. horizontal) graph, and for every $x, y \in P$ the unique intersection point between $W^s_N(x)$ and $W^u_N(x)$ is in $P$.

It follows from Pesin theory [Pes77] that, up to a negligible set, the support of $\mu$ can be covered by means of just countably many Pesin boxes.

[2]Further conditions should be imposed on $\mathscr{R}_r$, see [BLS93b, (4.2), (4.3), and (4.4)]. Since we will not need them and they are always true up to a zero-measure subset, we will not focus on this issue here.

3.4. **Semi-parabolic dynamics and horn maps.** Let us assume from now on that a Hénon map $f$ has a semi-parabolic point of order 2 at the origin $\mathbb{O}$ of $\mathbb{C}^2$. A description of the local dynamics of $f$ near the semi-parabolic point is given in [Ued86, Ued91, BSU17]. Following [BSU17], we recall here the definitions and results that we will need in the sequel. We refer to [Lav89, Shi00] for the earlier one-dimensional counterparts of these definitions and results.

Up to suitable changes of coordinates, we can assume that the local form of $f$ near $\mathbb{O}$ is given by

$$f(z,w) = (z + z^2 + O(z^3), bw + O(zw))$$

for some $0 < |b| < 1$ (a more precise development is given in [DL15], but we will not need it here).

We denote by $\mathcal{B}$ the *parabolic basin* of $\mathbb{O}$, i.e., the open set of points $x$ such that $f^n(x) \to \mathbb{O}$ as $n \to \infty$. Observe that, as $f$ is invertible, $\mathcal{B}$ is connected. There exists a holomorphic submersion $\phi^\iota \colon \mathcal{B} \to \mathbb{C}$, called the (one dimensional) *incoming Fatou coordinate*, that semi-conjugates the dynamics of $f$ on $\mathcal{B}$ to a translation by 1 on $\mathbb{C}$; i.e., we have

$$\phi^\iota(f(x)) = \phi^\iota(x) + 1 \quad \forall x \in \mathcal{B}. \tag{3.4}$$

There also exists a second (open) holomorphic map $\phi_2 \colon \mathcal{B} \to \mathbb{C}$ such that the map $\Phi = (\phi^\iota, \phi_2) \colon \mathcal{B} \to \mathbb{C}^2$ is a biholomorphism which satisfies $\Phi(f(x)) = \Phi(x) + (1,0)$ for every $x \in \mathcal{B}$. Given any $p \in \mathcal{B}$, the fiber $\{q \in \mathcal{B} : \phi^\iota(p) = \phi^\iota(q)\}$ is called the *strongly stable manifold* of $p$, denoted by $W^{ss}(p)$. It is an injectively immersed entire curve in $\mathbb{C}^2$, and it is characterized by the following property:

$$q \in W^{ss}(p) \Longleftrightarrow \lim_{n\to\infty} \frac{1}{n} \log d(f^n(p), f^n(q)) = \log |b|.$$

Let us now consider the set of points converging to $\mathbb{O}$ under the iteration of $f^{-1}$. This set is an $f$-invariant complex curve $\Sigma \subset \mathbb{C}^2$, with $\mathbb{O}$ on its boundary. The *Fatou parametrization* of $\Sigma$ is a holomorphic map $\phi^o \colon \mathbb{C} \to \mathbb{C}^2$ satisfying

$$f(\psi^o(y)) = \psi^o(y+1) \quad \forall y \in \mathbb{C}. \tag{3.5}$$

**Definition 3.2.** *The map $\mathcal{L}_0 := \psi^o \circ \phi^\iota \colon \mathcal{B} \cap \Sigma \to \Sigma$ is the* Hénon-Lavaurs*, or* transition map *of $f$ (associated to the semi-parabolic point $\mathbb{O}$). The map $H_f := \phi^\iota \circ \psi^o \colon (\psi^o)^{-1}(\mathcal{B} \cap \Sigma) \to \mathbb{C}$ is the* lifted horn map *of $f$ (associated to the semi-parabolic point $\mathbb{O}$).*

The following properties of $H_f$ directly follow from its definition and the local description of $\mathcal{B}$ and $\Sigma$, see [BSU17, DL15]. The last item is a key point in the characterization of bifurcations in [DL15] by means of homoclinic tangencies.

**Proposition 3.3.** *The following properties hold.*

1. *The domain $\psi^o(\mathcal{B} \cap \Sigma)$ contains the set $\{|\Im z| > R\}$ for every $R$ large enough.*
2. *For every $z \in (\psi^o)^{-1}(\mathcal{B})$ and every $w \in \mathbb{C}$, we have $H_f(z) = w$ if and only if $\psi^o(z)$ lies in the intersection of $\Sigma$ and the strongly stable manifold $\{(x,y) \in \mathcal{B} : \phi^\iota(x,y) = w\}$.*
3. *Given $z \in \mathbb{C}$, we have $H_f'(z) = 0$ if and only if the strongly stable manifold $\{(x,y) \in \mathcal{B} : \phi^\iota(x,y) = w\}$ is tangent to $\Sigma$ at $\psi^o(x)$.*

The maps $\mathcal{L}_0$ and $H_f$ are conjugated to each other, so their dynamics are very similar. In particular, one can use the map $H_f : \mathbb{C} \to \mathbb{C}$ as a model for the map $\mathcal{L}_0$, whose domain and image are in $\mathbb{C}^2$. It follows from (3.4) and (3.5) that we have

$$f \circ \mathcal{L}_0 = \mathcal{L}_0 \circ f \quad \text{ and } \quad H_f \circ \tau_1 = \tau_1 \circ H_f,$$

where we denote by $\tau_1$ the translation by 1 in $\mathbb{C}$. In particular, $H_f$ induces a map $h_f$ on (a subset of) the cylinder $\mathcal{C}$ obtained by taking the quotient of $\mathbb{C}$ by the $\mathbb{Z}$-action of $\tau_1$. By Proposition 3.3 (1), the domain of $h_f$ contains the two extremities of $\mathcal{C}$. By [BSU17], $h_f$ extends to such extremities, that we can identify with 0 and $\infty$ in $\mathbb{P}^1$. In particular, we can see $h_f$ as a map from an open subset of $\mathbb{P}^1$ (containing two neighbourhoods of 0 and $\infty$) to $\mathbb{P}^1$.

**Definition 3.4.** *The map $h_f$ is the* horn map *of $f$ (associated to the semi-parabolic point $\mathbb{O}$).*

The maps $\mathcal{L}_0$, $H_f$, and $h_f$ are deeply related to the so-called (semi-)parabolic implosion phenomenon for the perturbations of the map $f$, see [BSU17] and [Lav89, Shi00] for their counterparts in one-dimensional parabolic dynamics. While we will not need results in this direction in the proof of Theorem 1.2, we recall here what we will need in the proof of Theorem 1.4.

Observe that the map $\mathcal{L}_0$ is actually defined on $\mathcal{B}$. For every $\alpha \in \mathbb{C}$, define the *Hénon-Lavaurs maps of phase $\alpha$* $\mathcal{L}_\alpha \colon \mathcal{B} \to \mathbb{C}^2$ as

$$\mathcal{L}_\alpha := \psi^o \circ \tau_\alpha \circ \phi^\iota,$$

where $\tau_\alpha$ denotes the translation by $\alpha \in \mathbb{C}$ in $\mathbb{C}$. Observe that the image of $\mathcal{L}_\alpha$ is contained in $\Sigma$, and that the definition of $\mathcal{L}_0$ is coherent with its previous definition above.

For small $\epsilon$, we consider holomorphic perturbations $f_\epsilon$ of $f$ of the form

$$f_\epsilon(z, w) = (z + z^2 + \epsilon^2 + O(z^2), b_\epsilon w + O(zw)),$$

where in particular $b_\epsilon$ depends holomorphically in $\epsilon$. Following [BSU17], we say that a sequence $(n_j, \epsilon_j) \subset (\mathbb{N}, \mathbb{R}^+)^{\mathbb{N}}$ is an *$\alpha$-sequence* if $\epsilon_j \to 0$ and $n_j - \pi/\epsilon_j \to \alpha$. Observe that this condition implies that $n_j \to \infty$, and prescribes that the convergence $\epsilon_j \to 0$ happens tangentially to the positive real axis.

**Theorem 3.5** (Bedford-Smillie-Ueda [BSU17])**.** *Let $(n_j, \epsilon_j)$ be an* $\alpha$-sequence. *Then,*

$$f_{\epsilon_j}^{n_j} \to \mathcal{L}_\alpha$$

*locally uniformly in $\mathcal{B}$.*

## 4. Proof of Theorem 1.2

We continue to use the notation of the previous sections. Let $R > 0$ be large enough so that the filtration property in Section 3.1 holds for $f$ and $R$. We will only be concerned with the dynamics of $f$ inside $D_R$, where we recall that $f$ can be seen as an invertible horizontal-like map. Fix $z_0 \in \mathbb{C}$ and denote by $S(z_0)$ a connected component of $D_R \cap \{\phi^\iota(x, y) = z_0\}$. We also let $D$ be a small disk in $\Sigma$ intersecting the boundary of $\mathcal{B}$. As we will see, the proof of Theorem 1.2 will essentially consist in finding suitable intersections between preimages of $S(z_0)$ and images of $D$.

In order to find such intersections, let us fix an arbitrary Pesin box $P$ for $\mu$, see Section 3.3. Recall that we denote by $N$ a given neighbourhood of $P$, where the stable/unstable manifolds associated to $P$ can be thought as vertical/horizontal, see Section 3.3.

We first consider the preimages of $S(z_0)$. Here is where the semi-local setting of horizontal-like maps will turn out to be useful, as $S(z_0)$ does not a priori satisfy the conditions for the convergence 3.2.

**Lemma 4.1.** *We have $d^{-n}(f^n)^*[S(z_0)] \to T^+$ on $D_R$.*

*Proof.* Observe that, up to choosing $R > 0$ large enough, we have $S(z_0) \subset K^+$, which implies that $S(z_0) \cap V_R^+ = \emptyset$. Hence, $S(z_0)$ is a vertical analytic set in $D_R$. Moreover, we have $[S(z_0)] \wedge [L] = 1$ for every horizontal line $L$ in $D_R$. The assertion is then a consequence of (3.3). □

**Lemma 4.2.** *For $\mu$-a.e. $x_1 \in P$, there exists $n_1 \in \mathbb{N}$ such that $f^{-n_1}(S(z_0))$ intersects $W^u(x_1)$ transversally at some $y_1 \in P$.*

*Proof.* In the case where $S(z_0)$ is replaced by a disc $\Delta$ in the stable manifold of a saddle point, the assertion follows from [BLS93b, Lemma 9.1]. Since the proof only uses that $\Delta$ satisfies $d^{-n}[f^{-n}\Delta] \to T^+$, the same proof applies here thanks to Lemma 4.1. □

For any $r > 0$, let $\mathcal{S}(r)$ denote the connected component of $\{(x, y) \in D_R : \phi^\iota(x, y) \in \mathbb{D}(z_0, r)\}$ which contains $S(z_0)$. For any $z \in \mathbb{D}(z_0, r)$, we let $S(z)$ denote the connected component of $\{\phi^\iota(x, y) = z\}$ contained in $\mathcal{S}(r)$.

By Lemma 4.2, there exist $n_1 \in \mathbb{N}$ and $x_1 \in P$ such that $f^{-n_1}(S(z_0))$ intersects transversally $W^u_N(x_1)$ in $P$. By the description of the Pesin boxes in Section 3.3 and continuity, we get the following result.

**Lemma 4.3.** *For every $z_0 \in \mathbb{C}$ there exists $r(z_0) > 0$ such that*

*(1) the connected component $U$ of $\mathcal{S}(r(z_0)) \cap W^u_N(x_1)$ containing $y_1$ is simply connected and relatively compact in $N$; and*

*(2) for all $z \in \mathbb{D}(z_0, r(z_0))$, $f^{-n_1}(S(z))$ intersects $W^u_N(x_1)$ transversally at a point in $U$.*

We now prove a result analogous to Lemma 4.2 for a disk in $\Sigma$ intersecting the boundary of $\mathcal{B}$. We first need the following analogous of Lemma 4.1.

**Lemma 4.4.** *Let $D \subset \Sigma$ be a small disk which intersects $\partial\mathcal{B}$. Then*

*(1) ${T^+}_{|D} > 0$, and*

*(2) $d^{-n}(f^n)_*[D] \to c_D T^-$, for some positive constant $c_D$.*

*Proof.* By assumption, there exists $x \in \partial\mathcal{B} \cap D$, so that $G^+(x) = 0$. Let us prove that there exists $x_1 \in D$ such that $G^+(x_1) > 0$. Assume, by contradiction, that $G^+ \equiv 0$ on $D$. Then, we have $D \subset K^+ \cap K^-$. In particular, we have $f^n(D) \subset D_R$ for all $n \in \mathbb{N}$. Therefore, $\{f^n : D \to \mathbb{C}^2\}$ is a normal family. As there is an open subset of $D$ contained in $\mathcal{B}$, any limit of any subsequence $f^{n_k}_{|D}$ must be constantly equal to the semi-parabolic point. This implies that $D \subset \mathcal{B}$ and gives a contradiction with the assumption that $D \cap \partial\mathcal{B} \neq \emptyset$.

Therefore, $G^+ : D \to \mathbb{R}$ is not constantly equal to 0. On the other hand, it is a non-constant subharmonic function on $D$ which admits a local minimum at $x \in D$. It follows by the maximum principle that $G^+$ cannot be harmonic on $D$, which means that $dd^cG^+ > 0$ on $D$. This proves the first assertion.

By assumption, we have $D \subset \Sigma \subset J^-$. Applying (3.2) (with $f^{-1}$ instead of $f$), by the first item we have $d^{-n}(f^n)_*[D] \to c_D T^-$, for some $c_D > 0$. The assertion follows. □

**Lemma 4.5.** *For $\mu$-a.e. $x_2 \in P$, there exists $n_2 \in \mathbb{N}$ such that $f^{n_2}(D)$ intersects transversally $W^s_N(x_2)$ at some $y_2$ in $N$.*

*Proof.* As in Lemma 4.2, thanks to Lemma 4.4 the proof is the same as that of [BLS93b, Lemma 9.1]. □

The following lemma gives the desired tranverse intersections between preimages of disks in stable manifolds in $\mathcal{B}$ and images of disks in $\Sigma$ intersecting the boundary of $\mathcal{B}$.

**Lemma 4.6.** *There are two sequences of integers $N_j, M_j \to +\infty$ such that the following properties hold.*

1. *$f^{-N_j}(U) \subset N$ for all $j$;*
2. *$\operatorname{diam}(f^{-N_j}(U)) \to 0$ as $j \to +\infty$;*
3. *for all $z \in \mathbb{D}(z_0, r(z_0))$, the connected component of $f^{-n_1-N_j}(S(z)) \cap N$ intersecting $f^{-N_j}(U)$ is a vertical graph in $N$;*
4. *$f^{M_j}(y_2) \in N$ for all $j$;*
5. *the connected component of $f^{n_2+M_j}(D) \cap N$ containing $f^{M_j}(y_2)$ is a horizontal graph in $N$.*

*In particular, for all $z \in \mathbb{D}(z_0, r(z_0))$, $f^{-n_1-N_j}(S(z))$ intersects $f^{n_2+M_j}(D)$ transversally at a point in $N$.*

*Proof.* We fix points $x_1, x_2$ and integers $n_1, n_2$ as in Lemmas 4.2 and 4.5.

By Poincaré's recurrence theorem, there exists $N_j \to \infty$ such that $f^{-N_j}(x_1) \in P$. Since $U \subset W^u_N(x_1)$, we have $f^{-N_j}(U) \subset N$. Up to extracting a subsequence, we may assume that $\lim_{j\to+\infty} f^{-N_j}(x_1) \to \hat{x}_1 \in P$, and that we have $f^{-N_j}_{|U} \to \hat{x}_1$ uniformly. This proves (1) and (2).

Let $A_j$ denote the connected component of $f^{-N_j-n_1}(\mathcal{S}(r(z_0))) \cap N$ containing $U$; by definition, $A_j$ is a union of pieces of strongly stable manifolds $\{\phi^t = z - n_1 - N_j\}$. Let $\tilde{S}$ be one such piece. By the inclination lemma, each $\tilde{S}$ converges in the $C^1$ topology to $W^s_N(\hat{x}_1)$, which by Pesin theory (see the discussion in Section 3.3) is a horizontal graph in $N$. This proves (3).

On the other hand, by a symmetric argument, there exists $M_j \to +\infty$ such that $f^{M_j}(x_2) \in P$, and $f^{M_j}(x_2) \to \hat{x}_2 \in P$. Since $f^{n_2}(D)$ intersects $W^s(x_2)$ transversally in $P$, the inclination lemma implies that the connected component of $f^{M_j+n_2}(D) \cap N$ containing $f^{M_j}(y_2)$ converges in the $C^1$ topology to $W^u_N(\hat{x}_2)$, which is a horizontal graph in $N$. This proves (4) and (5). □

*End of the proof of Theorem 1.2.* Set $W := (\psi^o)^{-1}(\mathcal{B})$. By the definition of $h_f$ and $H_f$, and in particular by the fact that $H_f$ commutes with the translation $\tau_1$, it is enough to prove the following property.

($\star$) for every $z_0 \in \mathbb{C}$ there exists $r(z_0) > 0$ such that for any open set $\tilde{W}$ intersecting $\partial W$ there exist $n \in \mathbb{Z}$ and $V \Subset W \cap (\tilde{W} + n)$ such that $H_f : V \to \mathbb{D}(z_0, r(z_0))$ is a conformal isomorphism.

Fix $z_0 \in \mathbb{C}$ and, without loss of generality, let $\tilde{W}$ be an open ball intersecting $\partial W$. Set $D := \psi^o(\tilde{W}) \subset \Sigma$, and observe that $D$ satisfies the assumption of Lemma 4.4. Let also $r(z_0)$ and $U$ be as in Lemma 4.3, $n_1, n_2$ be as in Lemmas 4.2 and 4.5, and let the sequences $\{N_j\}, \{M_j\}$ be as in Lemma 4.6. By Lemma 4.6, for every $z \in \mathbb{D}(z_0, r(z_0))$ and every $j$, the set $f^{-n_1-N_j-n_2-M_j}(S(z))$ intersects $D$ transversally at a point in $f^{-n_2-M_j}(U)$. By the second item in Lemma 4.6, we can fix $j_0$ such that the diameter of $f^{-n_1-N_j}(U)$ is much smaller than the diameter of $N$. For convenience, we also set $\tilde{n} := n_1 + n_2 + N_{j_0} + M_{j_0}$.

Set $V := \tilde{n} + (\psi^o)^{-1}(f^{-n_2-M_{j_0}-N_{j_0}}(U)) \subset (\psi^o)^{-1}(D) = \tilde{n} + \tilde{W}$. By the choice of $j_0$ and Lemma 4.6 (4), we have $V \Subset \tilde{n} + \tilde{W}$. For every $v \in V$, let $u = u(v)$ be the point in $U$ given by $u = f^{n_2+M_{j_0}+N_{j_0}}(\psi^o(v - \tilde{n}))$. The map $v \mapsto u(v)$ is a biholomorphism from $V$ to $U$. Recall also that, by construction, the map $\phi^\iota \circ f^{n_1}$ is a biholomorphism between $U$ and $\mathbb{D}(z_0, r(z_0))$. Hence, the map $\tilde{H}(\cdot) := \phi^\iota \circ f^{\tilde{n}} \circ \psi^o(\cdot - \tilde{n})$ is a biholomorphism from $V \Subset \tilde{n} + \tilde{W}$ to $\mathbb{D}(z_0, r(z_0))$. Recalling that $f \circ \psi^o(\cdot) = \psi^o(\cdot + 1)$, we see that $\tilde{H} = H_f$. Hence, $H_f$ is a biholomorphism from $\tilde{n} + V \Subset \tilde{n} + \tilde{W}$ to $\mathbb{D}(z, r(z_0))$. This proves ($\star$), and the assertion follows. ☐

**Remark 4.7.** *Observe that our proof of Theorem 1.2 does not really need that $f$ is a globally defined Hénon map, but it is enough that it is an invertible horizontal-like map. In particular, the map $f$ does not need to be algebraic.*

## 5. Almost maximal dimension for $J^+$ and Theorem 1.4

5.1. **Preliminaries and McMullen's result.** In order to prove Theorem 1.4, the idea will be to first construct a hyperbolic (repelling) set of large Hausdorff dimension for some horn map, and then to transfer it to some suitable perturbations of the initial semi-parabolic Hénon map. While we could follow more closely Shishikura's arguments [Shi98], we will use a result by McMullen [McM00], see Theorem 5.4 below, to prove that there are small quasiconformal copies of the Mandelbrot set in the parameter space of horn maps. This will allow us to bypass a part of the proof. In this preliminary section, we recall some terminology and McMullen's result.

**Definition 5.1.** *Let $W \subset \mathbb{P}^1$ be an open set and $f : W \to \mathbb{P}^1$ a holomorphic map. We say that $z \in \mathbb{P}^1$ is* unramified *(for $f$) if the closure of the set $\{x \in W : \exists n \in \mathbb{N}, f^n(x) = z \text{ and } (f^n)'(x) \neq 0\}$ is dense in $J_F(f)$.*

For rational maps, some points may be completely ramified. For instance, it may happen that every preimage of some point is a critical point. However, we will see in Lemma 5.8 that this is not possible for maps with the small island property.

**Definition 5.2.** *Let $M$ be a complex manifold and $(f_\lambda)_{\lambda\in M} : W \to \mathbb{P}^1$ a holomorphic family of holomorphic maps. We say that the family $(f_\lambda)_{\lambda\in M}$ has a* local Misiurewicz bifurcation of degree $d \geq 2$ at $\lambda_0 \in M$ *if:*

*(1) there exists a marked critical point $c_\lambda$ of $f_\lambda$ and $n \in \mathbb{N}$ such that $x_{\lambda_0} := f^n_{\lambda_0}(c_{\lambda_0})$ is a repelling periodic point;*
*(2) the map $\lambda \mapsto f^n_\lambda(c_\lambda)$ is not constantly equal to $x_\lambda$ in a neighborhood of $\lambda_0$;*
*(3) $c_{\lambda_0}$ is unramified;*
*(4) the local degree of $f^n_\lambda$ at $c_\lambda$ is constantly equal to $d$ in a neighborhood of $\lambda_0$.*

Recall that a critical point $c_\lambda$ is *marked* if it can be followed holomorphically as a function $\lambda \mapsto c_\lambda = c(\lambda)$ of the parameter $\lambda$. Such a critical point is *active* at $\lambda_0$ if the sequence $f^n_\lambda(c_\lambda)$ is not normal on any neighbourhood of $\lambda_0$. As in [McM00], we remark that the subtle point in the above definition is the fourth request.

We can now recall the main technical results in [McM00]. We will not use the following proposition, but we quote it since proving a version of it in our context will be a main point of our construction.

**Proposition 5.3.** *Let $M$ be a complex manifold and $(f_\lambda)_{\lambda\in M} : \mathbb{P}^1 \to \mathbb{P}^1$ a holomorphic family of rational maps. Assume that a marked critical point $c_\lambda$ is active and unramified at $\lambda_0$. Then, there exists two sequences $M \ni \lambda_n \to \lambda_0$ and $\mathbb{N} \ni m_n \to \infty$ such that, for every $n \in \mathbb{N}$, the family $f^{m_n}$ has a local Misiurewicz bifurcation at $\lambda_n$.*

We denote by $\mathbf{M_d}$ the *degree $d$ Mandelbrot set*, i.e., the connectedness locus of the family $z^d + c$, for $c \in \mathbb{C}$.

The theorem below was proved by McMullen in the context of holomorphic families of rational maps. However, the arguments are purely local (indeed, the proof consists in finding suitable polynomial-like restrictions of the maps $f_\lambda$) and the proof carries through verbatim in our more general setting.

**Theorem 5.4** (McMullen [McM00])**.** *If a holomorphic family $(f_\lambda)_{\lambda\in M} \colon W \to \mathbb{P}^1$ has a local Misiurewicz bifurcation of degree $d$ at some parameter $\lambda_0$, then there is a sequence of embeddings $\phi_n \colon \mathbf{M_d} \to M$ such that for every $c \in \mathbf{M_d}$, we have $\lambda_n := \phi_n(c) \to \lambda_0$ and, for every $n \in \mathbb{N}$, $f_{\lambda_n}$ has a polynomial-like restriction which is hybrid-equivalent to $z^d + c$. Moreover, the quasiconformal distortion of these copies tends to $0$ as $n \to +\infty$.*

In practice, McMullen's results give that, whenever – in a family of rational maps – a bifurcation occurs, one can create local Misiurewicz bifurcations, and thanks to them, by means of a renormalization process, a large set of bifurcation parameters. As a consequence, by [Shi98], one can then find maps with large hyperbolic dimension close to any bifurcation parameter. In the next section, we will adapt the above ideas in the setting of maps with the small island property.

5.2. **Large hyperbolic sets from the small island property.** In this section, we fix an open set $W \subset \mathbb{P}^1$ and a holomorphic map $h \colon W \to \mathbb{P}^1$ satisfying the small island property. We also assume that $\partial W \neq \emptyset$ and that $h$ has at least one critical point (both of these properties hold for horn maps of dissipative semi-parabolic Hénon maps; in particular, the existence of a critical point was proved in [DL15]). For every $\lambda \in \mathbb{C}^*$, we denote $h_\lambda := \lambda h$. The following is the main result of this section.

**Proposition 5.5.** *For every $\epsilon > 0$ there exists $N \in \mathbb{N}$, $\lambda_1 \in \mathbb{C}^*$, simply connected domains $U, U_1, \ldots, U_N \subset W \cap \mathbb{C}^*$, and integers $m_1, \ldots, m_N$ such that:*

*(1) for all $1 \leq i < j \leq N$, we have $U_i, U_j \Subset U$ and $\overline{U_i} \cap \overline{U_j} = \emptyset$;*
*(2) for all $1 \leq i \leq N$, the map $h_{\lambda_1}^{m_i} : U_i \to U$ is a conformal isomorphism;*
*(3) if $c := \max_{1 \leq i \leq N} \sup_{z \in U_i} |(h_{\lambda_1}^{m_i})'(z)|$, then*

$$2 - \epsilon \leq \frac{\log N}{\log c}.$$

In particular, it follows from the Bowen formula [Bow08] that the limit set of the Conformal Iterated Function System (CIFS) given by the holomorphic maps $h_{\lambda_1}^{-m_i} : U \to U_i$ as in the above statement has Hausdorff dimension at least $2 - \epsilon$.

We begin with the following preliminary result, which we will use to replace the non-normality of the critical orbit near a bifurcation parameter in a family of rational maps.

**Lemma 5.6** (Shooting Lemma). *Let $(h_\lambda)_{\lambda \in \mathbb{C}^*}$ be the holomorphic family defined by $h_\lambda = \lambda h$ for every $\lambda \in \mathbb{C}^*$. Fix $\lambda_0 \in \mathbb{C}^*$ and let $\gamma_1, \gamma_2$ be two holomorphic maps defined in a neighborhood of $\lambda_0$ and such that $h_{\lambda_0}^n \circ \gamma_1(\lambda_0) \in \partial W$ for some $n \in \mathbb{N}$. Then there exists $\lambda'$ arbitrarily close to $\lambda_0$ such that*

$$h_{\lambda'}^{n+1}(\gamma_1(\lambda')) = \gamma_2(\lambda').$$

In the proof of Lemma 5.6 we will need the following consequence of the argument principle.

**Lemma 5.7.** *Let $V$ be a Jordan domain, and let $f, g$ be holomorphic functions in a neighborhood of $\overline{V}$. Suppose that $g(\overline{V}) \subset f(V)$ and $g(\partial V) \cap f(\partial V) = \emptyset$. Then there exists $\lambda \in V$ such that $f(\lambda) = g(\lambda)$.*

*Proof of Proposition 5.6.* For every $\lambda \in \mathbb{C}^*$, set $G(\lambda) := h_\lambda^n(\gamma_1(\lambda))$ and $g(\lambda) := \lambda^{-1}\gamma_2(\lambda)$. Then, the equation $h_\lambda^{n+1}(\gamma_1(\lambda)) = \gamma_2(\lambda)$ may be rewritten as

$$h_\lambda \circ G(\lambda) = g(\lambda).$$

We will apply Lemma 5.7 to the functions $f(\lambda) := h_\lambda \circ G(\lambda)$ and $g(\lambda)$.

Let $D$ be a disk centered at $g(\lambda_0)$ small enough for the small island property of $h$ to apply (i.e., whose radius is smaller than $r(g(\lambda_0))$). Fix also an $\epsilon > 0$ very small compared to the diameter of $D$, and also small enough so that $G : \mathbb{D}(\lambda_0, \epsilon) \to G(\mathbb{D}(\lambda_0, \epsilon))$ is a branched cover without critical points besides possibly $\lambda_0$. By assumption, $G(\mathbb{D}(\lambda_0, \epsilon))$ intersects $\partial W$.

By the small island property of $h$ (which implies the small island property of $h_\lambda$ for every $\lambda \in \mathbb{C}^*$, see Remark 2.9), there exists $U \Subset G(\mathbb{D}(\lambda_0, \epsilon))$ such that $f_{\lambda_0}(U) = D$. Let $V$ denote a connected component of $G^{-1}(U)$ inside $\mathbb{D}(\lambda_0, \delta)$. By the choice of $\epsilon$, $V$ is a Jordan domain. By construction, we have $f \circ G(U) = D$. On the other hand, $g(U)$ is contained in the disk $\mathbb{D}(g(\lambda_0), C\epsilon^{1/d})$, where $d$ is the local degree of $G$ at $\lambda_0$ and $C$ is a positive constant independent of $\epsilon$. Therefore, for $\epsilon$ small enough, we have $g(U) \Subset f \circ G(U)$ and $g(\partial U) \cap f \circ G(\partial U) = \emptyset$. It follows from Lemma 5.7 that there exists $\lambda' \in U$ such that $f \circ G(\lambda') = g(\lambda')$. The proof is complete. □

Proposition 5.3 has an assumption on the non-ramification of the critical point. In our setting, we will be able to get rid of that assumption thanks to the following lemma.

**Lemma 5.8.** *Every $z \in \mathbb{C}^*$ is unramified for $h$.*

*Proof.* Fix $z \in \mathbb{C}^*$ and let $U \Subset W$ be an open set intersecting $J_F(h)$. Define

$$B^-(z) := \{x \in W : \exists n \in \mathbb{N}, h^n(x) = z \text{ and } (h^n)'(x) \neq 0\}.$$

It is enough to show that $U \cap B^-(z) \neq \emptyset$.

As $U \cap J_F(h) \neq \emptyset$, by Montel's lemma there exists $n \in \mathbb{N}^*$ such that $h^n(U) \cap \partial W \neq \emptyset$. We choose the smallest such $n$. The map $h^n : U \to h^n(U)$ has only finitely many critical points. On the other hand, by the small island property, there are infinitely many $y \in h^n(U)$ such that $h(y) = z$ and $h'(y) \neq 0$. In particular, we can find one such $y$ such that there exists $x \in U$ with $h^n(x) = y$ and $(h^n)'(x) \neq 0$. As such $x$ belongs to $U \cap B^-(z)$, this concludes the proof. □

We can now prove the following version of Proposition 5.3.

**Proposition 5.9.** *Let $W \subset \mathbb{P}^1$ be an open set with $\partial W \neq \emptyset$ and $h : W \to \mathbb{P}^1$ a holomorphic map satisfying the small island property and with at least one critical point. There exists $\lambda_0 \in \mathbb{C}^*$ such that the family $(h_\lambda := \lambda h)_{\lambda \in \mathbb{C}^*}$ has a local Misiurewicz bifurcation at $\lambda_0$.*

*Proof.* Take $\lambda_0 \in \mathbb{C}^*$. We allow ourselves to modify $\lambda_0$ in what follows. We let $\gamma_1(\lambda)$ be the motion of a critical point near $\lambda_0$, and $\gamma_2(\lambda)$ the motion of a repelling periodic cycle (which exists by Theorem 1.2 and the implicit function theorem). Observe that, as the maps $h_\lambda$ have the form $h_\lambda = \lambda h$, the map $\gamma_1$ is in fact constant; we will therefore simply write $\gamma_1$ instead of $\gamma_1(\lambda)$. As we are allowed to modify $\lambda_0$, we can also assume that $\lambda_0 h(\gamma_1) \in \partial W$. Hence, we are in the assumptions of Lemma 5.6. That lemma shows that, up to slighly modifying $\lambda_0$, we have a Misiurewicz relation at $\lambda_0$. As in [McM00], since we are allowed to slightly perturb the starting parameter (and the critical point is automatically unramified by Lemma 5.8), the first three conditions in Definition 5.2 can be achieved from the above construction. Hence, we only have to show that (up to a further small perturbation) we can get the fourth condition in Definition 5.2. Let us recall that, as in [McM00], this property may fail if the local degree of the critical point at $\lambda_0$ is larger than at nearby parameters $\lambda' \neq \lambda_0$.

We assume for simplicity that the integer $n$ in Definition 5.2 is equal to 1. Following [McM00], we denote by $a_\lambda$ the holomorphic motion of the repelling point giving the Misiurewicz relation at $\lambda_0$. We denote by $U$ a small linearization domain for $a_\lambda$, for all $\lambda$ in given small neighbourhood of $\lambda_0$. We will always work with $\lambda$ in this neighbourhood. We can also assume that the local degree of $c_\lambda$ is constant outside of $\lambda_0$. We let $b_\lambda$ be the motion of a second repelling point, which stays in $U$ for all $\lambda$ in consideration. By the small island property, there exist preimages of $b_\lambda$ accumulating on $a_\lambda$. We denote by $b'_\lambda$ one of these preimages. Applying again Lemma 5.6, we can find $\lambda'$ close to $\lambda_0$ such that $f(c_{\lambda'}) = b'_{\lambda'}$. The local degree of $f$ is constant near $\lambda'$, and since there are no critical points in $U$, we see that the same is true for the iterate of $f$ mapping $c_{\lambda'}$ to $b'_{\lambda'}$. The proof is complete. □

*Proof of Proposition 5.5.* Fix $\epsilon > 0$ and let $d \geq 2$ be the local degree at a given critical point of $h$. For any $c \in \mathbb{C}$, denote $g_c(z) := z^d + c$. By [Shi98], there exists $c \in \mathbf{M_d}$, $N \in \mathbb{N}$, simply connected domains $V, V_1, \ldots, V_N \subset \mathbb{C}$, and integers $m_1, \ldots, m_N$ such that:

(1) for all $1 \leq i < j \leq N$, we have $V_i, V_j \Subset V$ and $\overline{V_i} \cap \overline{V_j} = \emptyset$;
(2) for all $1 \leq i \leq N$, $g_c^{m_i} : V_i \to V$ is a conformal isomorphism;
(3) if $A := \max_{1 \leq i \leq N} \sup_{z \in V_i} |(g_c^{m_i})'(z)|$, then

$$2 - \frac{\epsilon}{2} \leq \frac{\log N}{\log A}.$$

By Proposition 5.9, there exists a local Misiurewicz bifurcation of degree $d \geq 2$ at some $\lambda_0 \in \mathbb{C}^*$ in the family $(h_\lambda := \lambda h)_{\lambda \in \mathbb{C}^*}$. By Theorem 5.4, there exists a sequence $\lambda_n \in \mathbb{C}^*$ and, for every $n$, a $K_n$-quasiconformal homeomorphisms $\phi_n : V \to \phi_n(V)$ with $K_n \to 1$ as $n \to \infty$ which conjugates $g_c$ to a polynomial-like restriction of $h_{\lambda_n}$. Then, for $n$ large enough, the open sets $U := \phi_n(V)$ and $U_i := \phi_n(V_i)$ satisfy the conditions in the statement. □

5.3. **Proof of Theorem 1.4.** By Theorem 1.2, we can apply Proposition 5.5 to the horn map $h_f$ associated to the semi-parabolic fixed point of $f$ as in Definition 3.4. The following proposition is a rewriting of that statement in terms of the Hénon-Lavaurs maps $\mathcal{L}_\alpha$. Observe that the multiplicative constant in Proposition 5.5 translates to the phase $\alpha$ in the statement below.

**Proposition 5.10.** *For every $\epsilon > 0$ there exists $\alpha \in \mathbb{C}$, disjoint open sets $U, U_1, \ldots, U_N \subset \Sigma$, and integers $m_1, \ldots, m_N$ such that*

*(1) for every $1 \leq i < j \leq N$, we have $U_i, U_j \Subset U$ (in the topology induced by $\Sigma$) and $\overline{U_i} \cap \overline{U_j} = \emptyset$;*
*(2) for every $1 \leq i \leq N$, the map $\mathcal{L}_\alpha^{m_i} : U_i \to U$ is a conformal isomorphism;*
*(3) if $c := \max_{1 \leq i \leq N} \sup_{z \in U_i} |(\mathcal{L}_\alpha^{m_i})'(z)|$, then*

$$2 - \epsilon \leq \frac{\log N}{\log c}. \tag{5.1}$$

In the proposition above, up to taking preimages, we may assume that $U$ and all the $U_i$'s have small diameter, and that they are transverse to the strong stable foliation in $\mathcal{B}$. In particular, we may choose a small open set $V \subset \mathbb{C}^2$ with $V \cap \Sigma = U$ and a coordinate system $(x, y) \in \mathbb{D}^2$ for $V$ in which $U = \mathbb{D} \times \{0\}$ and the strongly stable foliation is the vertical foliation given by $x = c$, for $c \in \mathbb{D}$. We allow ourselves to reduce the size of $V$ (in the transversal direction to $\Sigma$) in the following, as well as $U$ and the $U_i$'s. We only work on $V$ (resp. $\mathbb{D}^2$) in the following. By a slight abuse of notation, we will still denote by $U_i \times \{0\}$ the images of the $U_i$ in the chart $(x, y)$. Moreover, given two vertical subsets $A, B \subset \mathbb{D} \times \mathbb{D}$, we will write $A \Subset B$ whenever $A \cap \{y = y_0\} \Subset B \cap \{y = y_0\}$ for every $y_0 \in \mathbb{D}$. For every $1 \leq i \leq N$, we also denote $V_i := U_i \times \mathbb{D}$ and by $L_i$ the reading of $\mathcal{L}_\alpha^{m_i}$ in the coordinates $(x, y) \in \mathbb{D}^2$. We also denote by $L$ the holomorphic map on $\cup V_i$ which is equal to $L_i$ on $V_i$. Observe that, in particular, with the above notations we have

$$L_i^{-1}(V_j) \Subset V_i \text{ for all } i, j \quad \text{and} \quad L^{-1}(\cup V_i) \Subset \cup V_i. \tag{5.2}$$

By Theorem 3.5, given an $\alpha$-sequence $(\epsilon_j, n_j)$, we have $f_{\epsilon_j}^{n_j} \to \mathcal{L}_\alpha$ locally uniformly in $\mathcal{B}$. For simplicity of notation, we will fix an $\alpha$-sequence of the form $(\epsilon_n, n)$, so that the convergence takes the form

$$f_{\epsilon_n}^n \to \mathcal{L}_\alpha. \tag{5.3}$$

For every $n$ and $i$, we denote by $g_{n,i}$ the reading in the coordinates $(x, y) \in \mathbb{D}^2$ of the restriction of $f_{\epsilon_n}^{nm_i}$ to $V_i$. We also denote by $g_n$ the holomorphic map which is equal to $g_{n_i}$ on $V_i$. Then, in the coordinates $(x, y)$, for every $n$ sufficiently large, $g_n$ can be seen as a horizontal-like map [DNS08] from a vertical subset to a horizontal subset of $\mathbb{D} \times \mathbb{D}$. Up to slightly reducing the $U_i$ and $V$, this vertical subset is actually a subset of $\cup V_i$. The horizontal subset consists of $N$ small horizontal sets, contained in a small neighbourhood of $U = \{0\} \times \mathbb{D}$.

It follows from (5.2) and (5.3) that, for every $n$ sufficiently large and up to slightly reducing the $U_i$'s, we have

$$g_{n,i}^{-1}(V_j) \Subset V_i \text{ for all } i, j \quad \text{and} \quad g_n^{-1}(\cup V_i) \Subset \cup V_i. \tag{5.4}$$

Let us denote by $K_n^+$ the set of points in $\mathbb{D}^2$ whose orbit under $g_n$ never leaves $\mathbb{D}^2$ (which corresponds to point never leaving $V$ under appropriate iterates of $f_{\epsilon_n}^n$). It follows from (5.4) that, for all $n$ sufficiently large, $K_n^+$ is a collection of vertical graphs in $\mathbb{D} \times \mathbb{D}$, which are stable manifolds in $\mathbb{D}^2$. Similarly, we can also consider the horizontal set $K_n^-$ which is a union of unstable manifolds in $\mathbb{D}^2$. The intersection $C_n := K_n^+ \cap K_n^-$ is a Cantor set, on which the action of $f_{\epsilon_n}^n$ is uniformly hyperbolic. We denote by $\mu_n$ the measure of maximal entropy $\log N$ on $C_n$. This measure admits two Lyapunov exponents $\chi_n^- < 0 < \chi_n^+$. The exponent $\chi_n^+$ can be estimated by means of the transversal contraction of the vertical sets $V_i$ under the inverse iteration of $g_n$. In particular, we deduce the following upper bound from (5.3) and the definition of $c$ in Proposition 5.10:

**Lemma 5.11.** *Fix $\delta > 1$ and let $c$ be as in Proposition 5.10. Then, for all $n \in \mathbb{N}$ large enough, we have*

$$\chi_n^+ < \delta \log c.$$

Let $W^u(p)$ denote a generic unstable manifold of some $p \in C_n$, and let $\nu_n$ denote the conditional measure of $\mu_n$ on $W^u(p)$. It follows from [LY85] that the Hausdorff dimension of $\nu_n$ is equal to $(\log N)/\chi_n^+$. As a consequence, we deduce from Lemma 5.11 that, for every $\delta > 1$, the Hausdorff dimension of $\nu_n$ is larger than $\delta^{-1} \log N/c$ for every $n$ sufficiently large.

Recall that through every $p \in C_n$ there is a stable manifold $W^s(p)$ which is a vertical graph in $\mathbb{D} \times \mathbb{D}$ and an unstable manifold $W^u(p)$ which is a horizontal graph in $\mathbb{D} \times \mathbb{D}$; in particular, $W^s(p)$ intersects every horizontal graph at exactly one point. In particular, for every $y_0 \in \mathbb{D}$ there is a well-defined holonomy map $\phi_{y_0} : \operatorname{supp} \nu_n \to \mathbb{D}$, associated to the stable foliation of $g_n$ in $\mathbb{D}^2$, between the transversals $W^u(p)$ and the horizontal disk $\{y = y_0\}$. By [Lyu99] (see also [LP21]), the map $\phi_{y_0}$ is Lipschitz continuous. It follows that $K_n^+ = \bigcup_{y \in \mathbb{D}} \phi_y(\operatorname{supp} \nu_n)$ has Hausdorff dimension at least $2 + \delta^{-1}(2 - \epsilon)$. By the relation (5.1) between $N$ and $c$, this implies that we have

$$\dim_H(K_n^+) \geq 2 + \delta^{-1}(2 - \epsilon)$$

for every $n$ sufficiently large. Up to taking $\delta$ sufficiently close to 1 and $n$ sufficiently large, we deduce the lower bound $\dim_H(K_n^+) \geq 4 - 2\epsilon$. As the local coordinates $(x, y)$ are conformal and the image of $K_n^+$ is contained in $J^+(f_{\epsilon_n}^n) = J^+(f_{\epsilon_n})$, this concludes the proof of Theorem 1.4.

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