SUMMABILITY CONDITION AND RIGIDITY FOR FINITE TYPE MAPS

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Abstract. We give a bound on the dimension of the Teichmüller space of a finite type map in terms of the number of summable singular values and finite singular orbits. This generalizes rigidity results due to Makienko, Dominguez and Sienra, and Urbanski and Zdunik. We also recover a shorter proof of a transversality theorem due to Levin. Our methods are based on the deformation theory introduced by Epstein.

1. Introduction

Finite type maps are a class of analytic maps $f : W \to X$ of complex 1-manifolds introduced and studied by A. Epstein in [7]. A holomorphic map $f : W \to X$ is a finite type map if X is a compact Riemann surface, and f only has finitely many singular values (see Definition 1 for more details).

Many well-studied families of holomorphic maps are finite type maps: for instance, rational self-maps of $\mathbb{P}^1$, or entire functions of the complex plane with a finite set of singular values (called the Speiser class, and introduced in [9]). The Speiser class includes for instance the exponential family $f_\lambda(z) = \lambda e^z$. The class of finite type maps also contains the so-called horn maps appearing in the theory of parabolic implosion, see [5], or the Weierstrass $\wp$-function.

When $W \subset X$, one can study the dynamics of the map f, that is, the orbits $z, f(z), f^2(z), \dots$ for as long as $f^n(z) \in W$. If $z \in W$ is such that for all $n \in \mathbb{N}$, $f^n(z) \in W$, then we say that z is non-escaping. We define the Fatou set $\mathcal{F}(f)$ of f as the set of points $z \in W$ such that there exists a neighborhood U of z in W such that either all points in U escape, or the family $\{f^n_{|U} : W \to X, n \in \mathbb{N}\}$ is well-defined and normal. The Julia set is $\mathcal{J}(f) = X - \mathcal{F}(f)$.

Epstein established in his thesis [7] the basic results of the study of finite type map dynamics: they do not possess wandering domains, repelling cycles are dense in the Julia set, and they cannot have Baker domains (as in the case of rational maps, the only possible periodic Fatou components are Siegel disks, Herman rings, parabolic domains and attracting or super-attracting basins).

Moreover, finite type maps are endowed with natural finite dimensional parameter spaces. They were first constructed in [9] in the setting of the Eremenko-Lyubich class. Epstein generalized these constructions in [7], introducing so-called deformation spaces $\mathrm{Def}_A^B(f)$, which are abstractly defined complex manifolds but may be thought of as natural parameter spaces for f. Here, A and B are dynamically marked finite subsets of X, see Definition 9 for the precise requirements. Recently, it has been proved by Hironaka-Koch ([13]) and Firsova-Kahn-Selinger ([10]) independantly that these deformation spaces need not be connected in general.

Another important complex manifold parametrizing deformations of a finite type map $f : W \to X$ is the Teichmüller space of f, denoted by Teich(f). This space was first introduced by McMullen and Sullivan ([24]) in the context of rational maps on $\mathbb{P}^1$; it is so named by analogy with the Teichmüller space of a surface, and it is an entry in the Sullivan dictionary between the dynamics of Kleinian groups and the iteration of holomorphic maps. Epstein extended in [7] that notion to the case of finite type maps.

Informally, the Teichmüller space of f describes the space of holomorphic maps that are quasiconformally conjugated to f. The larger its dimension, the more parameters of quasiconformal deformations exist for f. If Teich(f) is reduced to a point, then f is said to be rigid: this means that if $g : W' \to X'$ is another finite type map that is quasiconformally conjugate to f, then it is in fact holomorphically conjugate to f. Hence, the dynamics of f have no non-trivial deformation.

On the other hand, if Teich(f) has maximal dimension (that is, equal to the dimension of an ambient parameter space), then every nearby parameter is quasiconformally conjugate to f; we say that f is structurally stable.

The question of describing when a rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ is structurally stable, or oppositely when it is rigid, is equivalent to a central conjecture in holomorphic dynamics, which state that with the exception of one well-understood family (flexible Lattès maps), no rational map may have an invariant line field supported on its Julia set.

Certain progress has been made towards that conjecture in the last few decades. In the series of papers [20], [21], [6], [19], [28] and [22], Makienko, Dominguez and Sienra, and Urbanski and Zdunik proved that some sufficient expansion along at least one critical orbit was an obstruction to structural stability, and that expansion along *all* critical orbits implied in fact rigidity. Let us give a slightly more precise historical account: this was proved first for rational maps in [20] under certain assumptions, then for unicritical polynomials in [15] under only the summability condition. Ideas from [15] were used in [21] to remove some unnecessary assumptions. Avila ([3]) then gave a different proof of [15], [21]. Similar results were obtained for the exponential family in [19] and [28], and at last for some subset of the class of entire functions with only finitely many singular values in [6]. In particular, all of these maps are of finite type. We will present in this paper a proof extending the aforementionned result to the setting of finite type maps.

1.1. Definitions and statements of the main results.

Definition 1. *An analytic map $f : W \to X$ of complex 1-manifolds is of finite type if*

- *f is nowhere locally constant,*
- *f has no isolated removable singularities,*
- *X is a compact Riemann surface, and*
- *the set of singular values $S(f)$ is finite.*

The set of singular values $S(f)$ is by definition the smallest closed subset of X such that $f : W - f^{-1}(S(f)) \to X - S(f)$ is a covering map (on each connected component of $W - f^{-1}(S(f))$). Note that we do not require that W is connected; indeed, in the case of horn maps, the natural domain of definition W consists of two disjoint simply connected domains in $X := \mathbb{P}^1$. Singular values are either critical values (images of critical points) or asymptotic values.

A finite type map $f : W \to X$ is called exceptionnal if either of the following holds:

(1) f is an automorphism of X
(2) f is an endomorphism of a torus
(3) $W = X = \mathbb{P}^1$, and f is a flexible Lattès map.

Definition 2. *Let $f : W \to X$ be a finite type map, with $W \subset X$, and let $z \in W$ be a non-escaping point. We say that z is* summable *if there is a Hermitian metric on X such that the series*

$$\sum_{n \geq 0} \|Df^n\left(f^N(z)\right)\|^{-1}$$

is convergent for some $N \in \mathbb{N}$.

Note that by compacity of X, the choice of the metric does not matter. Our definition differs slightly from the one usually found in the literature, to allow for the orbit of z to meet some finite number of critical points before enjoying sufficient derivative growth.

To our knowledge, this type of condition was first introduced by Tsujii ([27]) for real quadratic polynomials. Stronger expansivity conditions (the so-called Collet-Eckmann condition) had previously been known to imply rigidity (see [25]).

The following terminology is due to Levin.

Definition 3. *A compact set $K \subset X$ is called an A-compact if it satisfies the following property: any continuous function on K that is analytic in the interior of K can be uniformly approximated by restrictions of functions that are holomorphic on a neighborhood of K.*

If moreover $\mathring{K} = \emptyset$, we say that K is a C-compact.

This condition is relatively mild; it is in particular always satisfied by Julia sets of polynomials (see [16] or Proposition 3 for general sufficient conditions).

Definition 4. *Let $f : W \to X$ be a finite type map, with $W \subset X$.*

- *We say that the orbit of a singular value v has* good geometry *if*
  (1) *the ω-limit set $\omega(v)$ of v is a C-compact, and*
  (2) *if v belongs to the boundary of a Herman ring, then the closure of the union of all Herman rings meeting $\omega(v)$ is an A-compact.*
- *Let $p(f)$ denote the number of singular values with a finite orbit.*
- *Let $s(f)$ denote the number of summable singular values with infinite forward orbit and with good geometry.*

Let us make a few remarks on condition (2). First, it is vacuously satisfied if no summable singular value belongs to the boundary of a Herman ring (even if the singular value accumulates on the boundary of a Herman ring). It is also obviously satisfied in the absence of Herman rings, as is the case for instance for polynomials or maps in the Speiser class, or for Collet-Eckmann rational maps. It seems likely that the closure of two different Herman rings must be disjoint, but to our knowledge this is not known. If it were indeed the case, the condition could be reformulated as: the closure of the Herman ring containing v must be A-compact.

Finally, if the boundary of a Herman ring consists of simple closed curves, then its closure is an A-compact (since in this case the complement of the boundary has only finitely many connected components, see Proposition 3), and so in that case condition (2) would be satisfied.

The reason why Herman rings require special treatment is that there exists an integrable invariant quadratic differentials q_A supported in any Herman ring A; and we will need a delicate argument to prove that some other quadratic differentials constructed in the paper cannot coincide with q_A in A.

The following is the main result, and generalizes the aforementienned results of Avila, Levin, Makienko, Dominguez, Sienra, Urbanski and Zdunik:

Theorem A. *Let $f : W \to X$ be a non-exceptional finite type analytic map, with $W \subset X$. We have:*

$$\dim \operatorname{Teich}(f) \leq \operatorname{card} S(f) - p(f) - s(f).$$

In particular, if at least one singular value is summable with an ω-limit set that has good geometry, then f is not structurally stable, and if all singular values either are summable with good geometry or have finite orbits, then f is rigid and therefore does not have any invariant line field.

Our second result is a simplified proof of a theorem of Levin ([16]). Before we state it, let us introduce some notations. Let Rat_d be the space of degree d rational maps, and let rat_d be the quotient of Rat_d by the group of Möbius transformation acting by conjugacy. We will call Rat_d *the parameter space* of degree d rational maps, and rat_d the *moduli space* of degree d rational maps. We denote by $\mathcal{O}(f)$ the orbit of f under the action by conjugacy of the group of Möbius transformations.

The parameter space Rat_d is a $2d + 1$ dimensional complex manifold, and rat_d is a $2d - 2$ complex orbifold. Denote by $\operatorname{Crit}(f)$ the critical set of f, i.e. the set of points z where $Df(z) = 0$. For a rational map $f \in \operatorname{Rat}_d$, the set of singular values $S(f)$ coincides with the set of *critical values*, $f(\operatorname{Crit}(f))$.

It will be convenient to represent the tangent space $T_f\operatorname{Rat}_d$ to Rat_d at some f in the following way: given a complex curve $\lambda \mapsto f_\lambda$ passing through f at $\lambda = 0$, let $\dot{f} := \frac{d}{d\lambda}_{|\lambda=0} f_\lambda$. For $z \in \mathbb{P}^1$, $\dot{f}(z)$ is an element of $T_{f(z)}\mathbb{P}^1$; therefore

$$\eta(z) := (Df(z))^{-1} \cdot \dot{f}(z)$$

defines naturally a meromorphic vector field on $\mathbb{P}^1$; the poles of η are located inside $\operatorname{Crit}(f)$, and have order at most the multiplicities of the critical points of f. Let $T(f)$ denote the space of meromorphic vector fields on $\mathbb{P}^1$ satisfying this condition. Since $T(f)$ and $T_f\operatorname{Rat}_d$ have same dimension, the map $\dot{f} \mapsto \eta$ is a linear isomorphism, through which we identify $T(f)$ and $T_f\operatorname{Rat}_d$ in the rest of the paper.

Definition 5. *Let v be a summable critical value of a rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$. For any $\eta \in T(f)$, let*

$$\xi_\eta(v) := \sum_{k=0}^{\infty} (f^k)^* \eta(v) \in T_v\mathbb{P}^1.$$

Here, $(f^k)^*\eta(z) := (Df^k(z))^{-1} \cdot \left[\eta \circ f^k(z)\right]$ is the pull-back of the vector field η by the map f^k. Explictly, $\xi_\eta(v)$ may be written in local coordinates as

$$\xi_\eta(v) := \sum_{k=0}^{\infty} \frac{\eta \circ f^k(v)}{(f^k)'(v)} = \sum_{k=0}^{\infty} \frac{\dot{f} \circ f^k(v)}{(f^{k+1})'(v)}.$$

Given an analytic submanifold $\Lambda \subset \mathrm{Rat}_d$ containing f, we say that the critical values of f move holomorphically on Λ if for any critical value v_i of f, there are holomorphic maps $\lambda \mapsto v_i(\lambda)$ defined on Λ such that $v_i(\lambda)$ is a critical value of f_λ. In that case, given a tangent vector $\eta \in T_f\Lambda$, we denote by $\dot{v}_i$ the derivative $\frac{d}{d\lambda}_{|\lambda=0} v_i(\lambda)$.

Theorem B (see [16], [18])**.** *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map with s summable critical values v_i, $1 \leq i \leq s$, such that the forward orbits of v_i do not meet $S(f)$. Assume that either f has no invariant line field, or that the $(v_i)_{1\leq i\leq s}$ satisfy the conditions of Definition 4. Then there is a germ of analytic set Λ in Rat_d passing through f and transverse to $\mathcal{O}(f)$ on which the critical values of f move holomorphically, and such that the linear map*

$$\mathcal{V} : T_f\Lambda \subset T(f) \to \bigoplus_{1\leq i\leq s} T_{v_i}\mathbb{P}^1$$
$$\eta \mapsto (\dot{v}_i + \xi_\eta(v_i))_{1\leq i\leq s}$$

has maximal rank, i.e. equal to s.

As a consequence of Theorem B, one can recover a result due to van Strien ([29]), see [[16], Corollary 1.1]:

Corollary 1. *Assume that $f : \mathbb{P}^1 \to \mathbb{P}^1$ has s critical values v_i that each belong to a compact hyperbolic set K. Let $(\lambda, z) \mapsto h_\lambda(z)$ denote the holomorphic motion of K over Λ. Then the hypersurfaces of Λ defined as $H_i := \{\lambda \in \Lambda : v_i(\lambda) = h_\lambda(v_i)\}$ are smooth and transverse at $f_{\lambda_0} := f$.*

We refer to [16] for the details of the proof. The key point is that the vectors $\xi_\eta(v_i)$ coincide exactly with $-\frac{d}{d\lambda}_{|\lambda=\lambda_0} h_\lambda(v_i)$. In the general summable case, such a holomorphic motion needs not exist, but the vectors $\xi_\eta(v_i)$ remain well-defined.

Theorem B found another important application: it was used crucially in [2] to prove that the support of the bifurcation measure has positive Lebesgue volume in the moduli space of rational maps. More precisely, it was applied to some Collet-Eckmann rational maps (for which all singular values are summable and have dense orbits, but are known to have no invariant line fields) in order to prove that they belong to the support of the bifurcation measure.

Strategy of the proof. The idea of of the proof of Theorem A goes as follows: there is a natural immersion $\Psi : \mathrm{Teich}(f) \to \mathrm{Def}_A^B(f)$, which means that the codifferential $D\Psi^*([0]) : T^*_{[0]}\mathrm{Def}_A^B(f) \to T^*_{[0]}\mathrm{Teich}(f)$ is surjective. The dimension of $\mathrm{Def}_A^B(f)$ is known; therefore computing the dimension of $\mathrm{Teich}(f)$ reduces to computing the kernel of the codifferential $D\Psi^*$. We then construct one explicit element of $\ker D\Psi^*$ for each summable singular values, and prove that they are linearly independant, which gives the desired bound on the dimension of $\mathrm{Teich}(f)$.

The proof of Theorem B consists in proving that those elements of $\ker D\Psi^*$ that we constructed represent precisely the linear forms $\eta \mapsto \dot{v}_i + \xi_\eta(v_i)$. Theorem B then follows from their linear independance.

Outline. In section 2, we will recall some facts about the Teichmüller space and the deformation space of a finite type map, as well as a description of their cotangent bundle and the immersion of the Teichmüller space into the deformation space. In sections 3 and 4, we will prove some technical lemmas on quadratic differentials that will be useful to the proof of the main theorems. Finally, sections 5 and 6 are devoted to the proofs of Theorem A and Theorem B respectively.

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2. Teichmüller spaces and deformation spaces

We start by recalling a few objects related to Teichmüller spaces, and their dynamical counterparts. Although the definitions may be a little technical, the main goal of this section is only to introduce the complex vector spaces representing the cotangent space to the Teichmüller space and deformation space respectively, and explain how one surjects into the other. The reader unfamiliar with Teichmüller spaces may find some background in [14] and [12]; we refer the reader to [24] for a detailed discussion of Teichmüller spaces of dynamical systems. We assume the reader to be familiar with the concepts of Beltrami forms, quasiconformal maps and the Measurable Riemann Mapping Theorem, and we will give more detailed definitions of less standard objects.

2.1. **Beltrami forms, Beltrami differentials and Teichmüller spaces.** In what follows, we will call *Beltrami differentials* the objects obtained by relaxing the condition $\|\mu\|_\infty < 1$ from the definition of Beltrami forms. If X is a Riemann surface, we will denote by $\mathrm{Bel}(X)$ the space of Beltrami forms, and by $\mathrm{bel}(X)$ the space of Beltrami differentials. As the terminology indicates, we will think of Beltrami differentials as tangent vectors to the space of Beltrami forms. If $f : W \to X$ is a finite type map, and $\mu \in \mathrm{bel}(X)$ (resp. $\mathrm{Bel}(X)$), the pullback $f^*\mu$ is by definition the Beltrami differential (resp. form) on X defined by

$$f^*\mu(z) = \begin{cases} \mu(f(z))\frac{\overline{f'(z)}}{f'(z)} & \text{if } z \in W \\ 0 & \text{else.} \end{cases}$$

Note that when W has full measure in X, as is the case e.g. for rational maps or maps in the Eremenko-Lyubich class, this is the usual definition of the pullback. We denote by $\mathrm{Bel}(f)$ (resp. $\mathrm{bel}(f)$) the space of pullback-invariant Beltrami forms (resp. differentials) on X.

We denote by $\mathrm{QC}(f)$ the group of quasiconformal homeomorphisms $h : X \to X$ commuting with f, and by $\mathrm{QC}_0(f)$ the subgroup of quasiconformal homeomorphisms commuting with f and *uniformly quasiconformally isotopic to the identity.* This means that $h \in \mathrm{QC}_0(X)$ if and only if there is $0 < k < 1$ and a continuous map $H : [0,1] \times X \to X$ such that

1. for all $t \in [0,1]$, $h_t := H(t, \cdot) \in \mathrm{QC}(f)$
2. h_0 is the identity on X, and $h_1 = h$
3. for all $t \in [0,1]$, the quasiconformal dilatation of h_t is less than k.

The group $\mathrm{QC}_0(f)$ acts on $\mathrm{Bel}(f)$ by pullback.

Definition 6. *The Teichmüller space of f, denoted by* $\mathrm{Teich}(f)$*, is defined as the quotient* $\mathrm{Bel}(f)/\mathrm{QC}_0(f)$.

Although it is not obvious from that definition, $\mathrm{Teich}(f)$ is a finite dimensional, connected complex manifold. We refer the reader to [24] or [1] for two different proofs in the case of rational maps, or to [7] (Corollary 9 p. 137) in the setting of arbitrary finite type maps.

These definitions are modelled on the more classical definition of the Teichmüller space of a Riemann surface (with a marked finite set), which we recall here for reference. Let $A \subset X$ be a finite set, and let $\mathrm{QC}(A)$ denote the group of quasiconformal homeomorphisms fixing A pointwise. Let $\mathrm{QC}_0(A)$ denote the subgroup of $\mathrm{QC}(A)$ of elements that are uniformly quasiconformally isotopic to the identity relatively to A (it is the same definition as before, only we are now asking that $h_t \in \mathrm{QC}(A)$ instead of $\mathrm{QC}(f)$). Then the Teichmüller space of X with marked set A is $\mathrm{Teich}(X, A) := \mathrm{Bel}(X)/\mathrm{QC}_0(A)$. Note that the zero Beltrami form induces a basepoint in $\mathrm{Teich}(f)$ (resp. $\mathrm{Teich}(X, A)$), that we shall denote by $[0]$.

2.2. **Quasiconformal vector fields.**

Definition 7. *A continuous vector field on X is called quasiconformal if* $\overline{\partial}\xi \in \mathrm{Bel}(X)$ *(in the sense of distributions).*

More explictly, if in local coordinates $\xi = \xi(z)\frac{d}{dz}$, then $\overline{\partial}\xi(z) = \frac{\partial \xi}{\partial \bar{z}}\frac{\overline{dz}}{dz}$, and ξ is quasiconformal if and only if $\frac{\partial \xi}{\partial \bar{z}} \in L^\infty$.

Given a finite type map $f : W \to X$, we let Λ_f denote the closure of the grand orbit of the singular values, and $\Omega_f := X - \Lambda_f$. One can check that

(1) Λ_f contains the Julia set, and
(2) $W \cap \Omega_f$ is a subset of the Fatou set of f.

We have the following canonical identifications of the tangent space to the Teichmüller space at the basepoint:

$$T_{[0]}\mathrm{Teich}(f) \simeq \mathrm{bel}(f)/\{\overline{\partial}\xi : \xi = 0 \text{ on } \Lambda_f\} \tag{1}$$

$$T_{[0]}\mathrm{Teich}(X, A) \simeq \mathrm{bel}(X)/\{\overline{\partial}\xi : \xi = 0 \text{ on } A\} \tag{2}$$

where ξ are quasiconformal vector fields on X.

2.3. **Quadratic differentials.** We now introduce quadratic differentials, which are dual (or in fact, pre-dual) objects to Beltrami differentials. A quadratic differential on a Riemann surface X is a section of $T^*X \otimes T^*X$, where T^*X is the cotangent bundle of X and $\otimes$ denotes the tensor product. In other terms, if q is a quadratic differential on X and $x \in X$, then $q(x)$ defines a complex quadratic form on T_xX. This is often expressed by writing q in local coordinates as $q = q(z)dz^2$. Most of the time, we will work with meromorphic or holomorphic quadratic differentials, but in this paper it shall be also sometimes convenient to consider quadratic differentials which are only integrable in the following sense: if q is a quadratic differential on X, then $|q|$ defines in a natural way a volume form on X. We say that q is integrable if $\int_X |q| < \infty$, and we set $\|q\| := \int_X |q|$.

Definition 8. *Let W be a complex 1-manifold, and let $A \subset W$ be a finite set. We denote by $Q(W)$ the space of integrable holomorphic quadratic differentials on W, and by $Q(W, A)$ the space of integrable meromorphic quadratic differentials on W, with at worst simple poles, whose poles are in A.*

If $f : X_1 \to X_2$ is any holomorphic map between Riemann surfaces, and q is a holomorphic quadratic differential on X_2, then the pullback f^*q defined by

$$f^*q(x; v) := q(f(x); Df(x) \cdot v)$$

for $(x, v) \in TX_1$. Then f^*q is a holomorphic quadratic differential on X_1. If now $f : X_1 \to X_2$ is a holomorphic ramified cover, and q is a holomorphic quadratic differential on X_1, then the pushforward f_*q is defined as

$$(f_*q)(x) = \sum_i (g_i^*q)(x)$$

where $x \in X_2 \backslash S(f)$ and the g_i are the inverse branches of f. If f has finite degree, the sum is always well-defined and f_*q is a meromorphic quadratic differential on X_2, which may have poles at $S(f)$.

One can check that if q is integrable, then f_*q must also be integrable and well-defined even if f has infinite degree, and that integrable holomorphic quadratic differentials on a finite type Riemann surface $X \backslash A$ (with X compact and $A \subset X$ finite) are exactly the meromorphic quadratic differentials on X with at worst simple poles, all of which are in A. Therefore, if $f : W \to X$ is a finite type map with $W \subset X$, the pushforward operator f_* maps $Q(X, A)$ to $Q(X, f(A) \cup S(f))$.

Finally, let us mention that there is a natural duality pairing between quadratic differentials and Beltrami forms. Indeed, if μ is a Beltrami form and q is an integrable quadratic differential, then $q \cdot \mu$ may be canonically identified with a $(1, 1)$ form, and can thus be integrated over X. One case of particular interest, which we will use repeatedly, is the following: $q \in Q(\mathbb{P}^1, A)$, and $\mu = \overline{\partial}\xi$ for some quasiconformal vector field ξ on $\mathbb{P}^1$. Then, an application of Stokes' theorem gives

$$\int_{\mathbb{P}^1} q \cdot \mu = 2i\pi \sum_{a \in A} \operatorname{Res}(q \cdot \xi, a). \tag{3}$$

Here, $\operatorname{Res}(q \cdot \xi, a)$ denotes the residue of the 1-form $q \cdot \xi$ at $z = a$. Explictly, if in local coordinates $q = \alpha \frac{dz^2}{z-a} + \text{h.o.t.}$, then $\operatorname{Res}(q \cdot \xi, a) = \alpha\xi(a)$. Note that this number does not depend on the choice of local coordinates.

All these operations on quadratic differentials, quasiconformal vector fields and Beltrami differentials are described in greater details in e.g. [1].

2.4. **Deformation spaces.** We now introduce the key ingredient, Epstein's deformation spaces. The reader is referred to the manuscript [8] or to [13] for a more complete exposition. As ever, $f : W \to X$ is a finite type map with $W \subset X$.

Definition 9. *Let $A \subset W$ and $B \subset X$ be two finite sets. We say that (A, B) is admissible for f if all of the following conditions hold:*

(1) $A \subset B$
(2) $f(A) \subset B$

(3) $S(f) \subset B$
(4) *if* X *has genus* 0*, then* $\operatorname{card} A \geq 3$*, and if* X *has genus* 1*, then* $\operatorname{card} A \geq 1$.

If (A, B) is admissible for f, then $A \subset B$, so $\mathrm{QC}_0(B)$ is a subgroup of $\mathrm{QC}_0(A)$. Therefore, the identity map on $\mathrm{Bel}(X)$ induces a natural "forgetful map" $\varpi : \mathrm{Teich}(X, B) \to \mathrm{Teich}(X, A)$.

Similarly, the pullback operator $f^* : \mathrm{Bel}(X) \to \mathrm{Bel}(X)$ descends to a holomorphic map $\sigma_f : \mathrm{Teich}(X, B) \to \mathrm{Teich}(X, A)$.

Definition 10 ([7])**.** *We define* $\mathrm{Def}_A^B(f)$ *by:*

$$\mathrm{Def}_A^B(f) = \{\tau \in \mathrm{Teich}(X, B), \varpi(\tau) = \sigma_f(\tau)\}.$$

Let us explain briefly what this definition means. By standard arguments, if ϕ and ψ are quasiconformal homeomorphisms associated to τ and $\sigma_f(\tau)$ respectively, then $\phi \circ f \circ \psi^{-1} : \psi(W) \to \phi(X)$ is holomorphic, hence a finite type map. Informally speaking, the condition $\varpi(\tau) = \sigma_f(\tau)$ ensures that $\phi_{|A} = \psi_{|A}$. Therefore, points in $\mathrm{Def}_A^B(f)$ can be represented by triples (ϕ, ψ, g), where $g = \phi \circ f \circ \psi^{-1}$ is a finite type map, and ϕ and ψ are quasiconformal homeomorphisms agreeing on the marked set A. In practice, A will consist in unions of cycles and pieces of orbits of singular values, and so is dynamically meaningful. Although knowing this is helpful for gaining some intuition about the proof of Theorem A, in this paper we will really only need information about the cotangent space of $\mathrm{Def}_A^B(f)$ at the basepoint.

From now on, we will always assume that (A, B) is admissible. From its definition, $\mathrm{Def}_A^B(f)$ is clearly an analytic subset of the finite dimensional complex manifold $\mathrm{Teich}(X, B)$. In fact, Epstein proved more:

Theorem 1 (Epstein)**.** *If* f *is non-exceptionnal, then* $\mathrm{Def}_A^B(f)$ *is a smooth complex manifold, of dimension* $\operatorname{card}(B - A)$.

If $B \subset \Lambda_f$, the identity map $\mathrm{Bel}(X) \to \mathrm{Bel}(X)$ descends to a holomorphic map $\Psi : \mathrm{Teich}(f) \to \mathrm{Def}_A^B(f)$, mapping basepoint to basepoint.

The next theorem has been proved in [1] in the case of rational maps, through a new construction of the complex structure of $\mathrm{Teich}(f)$ that bypasses the use of certain sophisticated tools from Teichmüller theory. Epstein has a different (unpublished) proof.

Theorem 2. *Let* (A, B) *be admissible, such that* $B \subset \Lambda_f$*. If* f *is non-exceptionnal, then* Ψ *is an immersion (its differential is injective).*

We finish this subsection with the following description of the cotangent spaces of $\mathrm{Teich}(f)$ and $\mathrm{Def}_A^B(f)$. Let Q_f denote the space of quadratic differentials integrable on X and holomorphic on Ω_f, endowed with $\|\cdot\|$.

From the description (1), one obtains by duality (see [1]):

$$T^*_{[0]}\mathrm{Teich}(f) \simeq Q_f/\mathrm{closure}(\nabla_f Q_f). \tag{4}$$

Here, the operator ∇_f is defined as $\mathrm{Id} - f_*$, where f_* is the pushforward operator acting on quadratic differentials. The closure is taken with respect to the L^1 norm on quadratic differentials. From the proof of Theorem 1, one also obtains the description

$$T^*_{[0]}\mathrm{Def}_A^B(f) \simeq Q(X, B)/\nabla_f Q(X, A). \tag{5}$$

Both are finite dimensional complex vector spaces; and the codifferential $D\Psi([0])^*$ is the linear map induced by by the injection $Q(X,B) \hookrightarrow Q_f$. This linear map is surjective by Theorem 2. By elementary linear algebra, its kernel is the quotient vector space

$$\ker D\Psi([0])^* = (Q(X,B) \cap \text{closure}(\nabla_f Q_f))/\nabla_f Q(X,A). \tag{6}$$

A large part of the proof of Theorem A essentially boils down to estimating the dimension of (6).

3. Action of quadratic differentials on vector fields

In this section, X will denote a compact Riemann surface of genus g. If we chose an arbitrary Hermitian metric on X, we get a topology on the space $\Gamma(TX)$ of continuous vector fields on X, induced by the norm

$$\|\xi\|_\infty = \sup_{s\in X} \|\xi(s)\|.$$

This norm depends on the particular choice of the Hermitian metric, but not the topology it induces (since X is compact). We will refer to it as the *uniform topology* for continuous vector fields on X.

Definition 11. *Denote by $\Gamma(TX)^*$ the (topological) dual of the topological vector space of continuous vector fields on X, equipped the topology dual to the uniform topology.*

Again, the choice of a Hermitian metric on X gives by duality a norm generating the topology on $\Gamma(TX)^*$, but that topology is independant from the choice of the norm. Depending on the genus g of X, it will be convenient to use different choices of metrics in the following proofs.

Definition 12. *Let q be an integrable quadratic differential on X. Then q induces a linear form on the space of smooth vector fields in the following way:*

$$\xi \mapsto \int_X q \cdot \overline{\partial}\xi.$$

If that linear form extends continuously to an element of $\Gamma(TX)^*$, we denote that extension by $\overline{\partial}q$ and we say that q is regular.

Note that if q is written in local coordinates as $q = q(z)dz^2$, then q is regular if and only if $\frac{\partial q}{\partial \overline{z}}$ (in the sense of distributions) is a complex Radon measure. It is in particular the case when q is meromorphic with at worst simple poles, in which case $\overline{\partial}q$ has finite support.

An immediate consequence of Weyl's lemma is that if q is a regular quadratic differential such that $\overline{\partial}q$ is supported in a compact K, then q is holomorphic outside of K.

Proposition 1. *Let $\mathcal{M}$ be the space of Radon measures on X, and $A = C^0(X,\mathbb{C})$. Let $\Omega^{1,0}(X)$ denote the space of complex-valued continuous forms of bidegree $(1,0)$ on X. The map*

$$\mathcal{M} \otimes_A \Omega^{1,0}(X) \to \Gamma(TX)^*$$
$$\mu \otimes \alpha \mapsto \left(\xi \mapsto \int_X \alpha(\xi) d\mu\right)$$

is an isomorphism of A-modules (and therefore of $\mathbb{C}$-vector spaces).

Remark 1. *Since $\Omega^{1,0}(X)$ is an A-module of rank 1, every element of $\Omega^{1,0}(X) \otimes_A \mathcal{M}$ can be written as $\alpha \otimes_A \mu$, where $\alpha \in \Omega^{1,0}(X)$ and $\mu \in \mathcal{M}$.*

Proof. The considered map is clearly an injective morphism of A-modules. It is therefore enough to prove that it is surjective. Let $u \in \Gamma(TX)^*$. If the support of u in included in a local coordinate domain (U, z), then it is a consequence of Riesz's representation theorem that u can be written as $u = dz \otimes_A \mu$, where μ is a Radon measure of support included in U. We conclude easily using a partition of unity. □

We will therefore identify from now on $\Gamma(TX)^*$ with $\Omega^{1,0}(\mathbb{P}^1) \otimes_A \mathcal{M}$.

Definition 13. *Let $f : W \to X$ be a holomorphic map that is not constant on any connected component of W and let $u = \alpha \otimes \mu \in \Gamma(TX)^*$ be such that $\frac{\|\alpha\|}{\|Df\|} \in L^1(|\mu|)$ (for any continuous Hermitian metric on X). We define the pushforward of u, denoted by f_*u, by :*

$$\langle f_*u, \xi \rangle := \langle u, f^*\xi \rangle = \int_{\mathbb{P}^1} \alpha(f^*\xi) d\mu.$$

Note that $f_*u \in \Gamma(TX)^*$. In particular, if $u \in \Gamma(TX)^*$ has support K that does not meet $S(f)$, then f_*u is well-defined and has a support included in $f(K)$.

Lemma 1. *Let $Z \subset X$ be a subset of cardinal $|3g - 3|$. Let $u \in \Gamma(TX)^*$ be supported in $\{y\}$, where $y \in X$. Then there is a unique meromorphic quadratic differential q on X with at worst simple poles such that:*

- *if $g = 0$, then $\overline{\partial} q - u$ is supported in Z*
- *if $g \geq 1$, $\overline{\partial} q = u$ and for all $z \in Z$, $q(z) = 0$.*

Moreover, for any choice of Hermitian metric on X, there is a constant $C > 0$ depending only on that metric and on Z such that $\|q\|_{L^1} \leq C\|u\|_\infty$.

Proof. We will treat separately the three following cases: $g = 0$, $g = 1$ and $g \geq 2$.

The case of genus 0. If X has genus 0, then X is isomorphic to the Riemann sphere $\mathbb{P}^1$. Note that a meromorphic quadratic differential q with at worst simple poles on X will satisfy the property that $\overline{\partial} q - u$ if and only if q has at worst four simple poles, located in $Z \cup \{y\}$, and for all smooth vector fields ξ vanishing on Z,

$$\int_X q \cdot \overline{\partial} \xi = \langle u, \xi \rangle.$$

If we work in affine coordinates in which $Z = \{0, 1, \infty\}$, then q has the form:

$$q(z) = \alpha \frac{y(y-1)}{z(z-1)(z-y)} dz^2$$

where α is such that $u = \delta_y \otimes (\alpha dz)$, δ_y being the Dirac mass at y. Up to permuting the order of points in Z, we may assume that $|y| < 1$. Then it is easy to see that $\|q\| \leq C|\alpha|$ for some constant $C > 0$ (depending on the coordinates z and therefore on Z) and that $\|u\| = |\alpha| \sup_{|y|<1} \|dz\|$. Therefore there is a constant $C_2 > 0$ depending only on the metric and on Z such that $\|q\|_{L^1} \leq C_2\|u\|$. This concludes the case of genus 0.

The case of genus 1. In this case, Z is empty, so we need to prove that there is a unique quadratic differential with at worst simple poles such that $\overline{\partial} q = u$. Any such quadratic differential must have at worst one simple pole, located at y. According to the Riemann-Roch formula, such quadratic differentials form a complex vector space of dimension one. Moreover, Stokes' theorem implies that for all smooth vector fields ξ on X,

$$\langle \overline{\partial} q, \xi \rangle = \int_X q \cdot \overline{\partial} \xi = 2i\pi \mathrm{Res}(q \cdot \xi, y).$$

Therefore, there is exactly one choice of polar part at y (and therefore exactly one choice of q) such that $\overline{\partial} q = u$. Let us now prove the inequality. If $g = 1$, then X is a complex torus $\mathbb{C}/\Lambda$, and for any $y \in X$, there is a translation descending to an automorphism T_y of X mapping the basepoint $[0]$ to y. The pullback map T_y^* induces an isometry for the L^1 norm of integrable quadratic differentials, as well as for linear forms in $\Gamma(TX)^*$ (endowed with the norm induced by the flat Hermitian metric on X). In other words, we lose no generality in assuming that $y = [0]$. Then the desired inequality is trivial, since the map $u \mapsto q$ is a complex linear map between finite dimensional normed vector spaces (in fact one-dimensional), therefore is continuous.

The case of genus at least 2. The proof of existence and unicity is similar to the previous case: notice that if $\overline{\partial} q = u$ and q has at worst simple poles, then q must have at worst a simple pole at y and must vanish on Z. According to the Riemann-Roch formula, such quadratic differentials form a vector space of complex dimension one, and once again, the choice of the right polar part at y uniquely determines q.

Now let us prove the desired inequality. Since $g \geq 2$, X is hyperbolic, so we may pick its hyperbolic metric as a choice of Hermitian metric inducing a norm on $\Gamma(TX)^*$. We will work by duality. According to Theorem A in [1], for any quasiconformal vector field ξ on X, we have $\|\xi\| \leq 4\|\overline{\partial}\xi\|_\infty$ (here $\|\xi\|$ is the supremum of the length of ξ is the hyperbolic metric on X, which is finite since X is compact). Therefore $\|q\|_{L^1} \leq 4\|u\|$. □

Theorem 3. *Let $Z \subset X$ be a subset of cardinal $|3g-3|$. Let $u \in \Gamma(TX)^*$. Then there is a unique regular quadratic differential q on X such that:*

- *if $g = 0$, then $\overline{\partial} q - u$ is supported in Z*
- *if $g \geq 1$, $\overline{\partial} q = u$ and for all $z \in Z$, $q(z) = 0$.*

Moreover, for any choice of Hermitian metric on X, there is a constant $C > 0$ depending only on that metric such that $\|q\|_{L^1} \leq C\|u\|_\infty$.

We will say that the quadratic differential q given by the above theorem is *the* Z-normalized quadratic differential corresponding to u.

Proof. According to Proposition 1, we can write $u = \mu \otimes \alpha$. For any $y \in X$, let $u_y = \delta_y \otimes \alpha$, where δ_y is the Dirac mass at y, and q_y be the corresponding quadratic differential given by the preceding lemma. Let $r_y = \overline{\partial} q_y - u_y$: by definition, r_y is supported in Z. The second part of that lemma implies that there is a constant $C > 0$ depending only on the choice of Hermitian metric and on α such that for all $y \in X$, $\|q_y\|_{L^1} \leq C$. Let $q = \int_X q_y \, \mu(y)$. Note that q is integrable and $\|q\|_{L^1} \leq C$. We will prove that q satisfies the desired property. Let ξ be a smooth vector field on X. We have:

$$\begin{aligned}\int_X q \cdot \overline{\partial}\xi &= \int_X \left(\int q_y \, \mu(y)\right) \cdot \overline{\partial}\xi \\ &= \int \left(\int_X q_y \cdot \overline{\partial}\xi\right) \mu(y) \\ &= \int \langle u_y, \overline{\partial}\xi\rangle \mu(y) \\ &= \int \alpha_y(\xi)\, \mu(y) + \int \langle r_y, \xi\rangle \mu(y) \\ &= \langle u, \xi\rangle + \langle r, \xi\rangle\end{aligned}$$

where $\langle r, \xi\rangle := \int \langle r_y, \xi\rangle \mu(y)$ is supported in Z if $g = 0$, and is identically zero otherwise. Thus the theorem is proved. □

Remark 2. *In terms of geometric measure theory, the quadratic differential $q = q(z)dz^2$ obtained in this way is simply given by the Cauchy transform of a measure absolutely continuous with respect to μ. This will play a role later on.*

Proposition 2. *Let q be a regular quadratic differential, and $f : W \to X$ a finite type analytic map. Assume that $f_*\overline{\partial}q$ and $\overline{\partial}f_*q$ are well-defined as elements of $\Gamma(TX)^*$. Then*

$$\operatorname{supp}(\overline{\partial}f_*q - f_*\overline{\partial}q) \subset S(f).$$

Proof. Let ξ be a quasiconformal vector field vanishing on a neighborhood of $S(f)$. Then $f^*\xi$ is also quasiconformal (and it vanishes in the neighborhood of $\mathrm{Crit}(f)$), and $\overline{\partial}f^*\xi = f^*\overline{\partial}\xi$. Therefore:

$$\langle \overline{\partial}f_*q, \xi\rangle = \langle f_*\overline{\partial}q, \xi\rangle,$$

and so $\langle f_*\overline{\partial}q - \overline{\partial}f_*q, \xi\rangle = 0$. This exactly means that $\operatorname{supp}(\overline{\partial}f_*q - f_*\overline{\partial}q) \subset S(f)$. □

4. Extended infinitesimal Thurston rigidity

Recall that a compact $K \subset X$ is called a C-compact if any continuous function on K can be uniformly approximated by restrictions to K of functions that are holomorphic on a neighborhood of K. Note that a C-compact must have empty interior. The following proposition gives sufficient conditions for a compact to be a C-compact. The proof is adapted from [22] to the case of a general Riemann surface.

Remark 3. *In fact, it can be proved (see* [4, Th. 2]*) that being a C-compact is a local property, in the sense that K is a C-compact if and only if for every point $p \in K$, there is a basis of neighborhoods $(U_n)_{n\in\mathbb{N}}$ such that $K \cap \overline{U_n}$ is a C-compact. Therefore we can replace functions by vector fields (or sections of any holomorphic line bundle) without changing the definition of C-compact.*

The proposition below is classical, especially in the case where $M = \mathbb{P}^1$. We give below some references for the general case.

Proposition 3. *Let K be a compact subset of X. Each of the following properties imply that K is a C-compact :*

i) K has zero Lebesgue measure, or

ii) *K is the boundary of some open connected subset U of X.*

Proof. Those conditions have been observed to imply that K is a C-compact in [21] and [16] in the case where $X = \mathbb{P}^1$. The following are immediate adaptations to the general case of a compact Riemann surface X.

i) This follows from the local nature of being a C-compact (remark 3) and Vitushkin's theorem (see e.g. [11]).

ii) This follows from [26], by taking M to be X with a closed disk removed from U. □

The proposition below is a reformulation of [[17], Lemma 2.1(b)].

Proposition 4. *Let K be a proper compact subset of X, and let q be a regular quadratic differential supported in K. If for some open set $W \subset X$ we have that $K \cap \overline{W}$ is a C-compact, then $q = 0$ on W.*

Proof. Since q is regular, there exists a constant $C > 0$ such that for any test vector field ξ on X, $\left|\int_X q\overline{\partial}\xi\right| \leq C\|\xi\|$. This applies in particular for vector fields supported in W, so q restricted to W is also regular. Since $\overline{W} \cap K$ is a C-compact, any test vector field can be uniformly approximated on $K \cap \overline{W}$ by vector fields that are analytic near $K \cap \overline{W}$; but for such a vector field, $\int_W q\overline{\partial}\xi = 0$. So $\overline{\partial}q = 0$ on W, and so by Weyl's lemma q is holomorphic on W. But since $q = 0$ on $W - K$, we have $q = 0$ on W. □

Recall the following fundamental fact:

Proposition 5 (see [7], Corollary 8 p. 124)**.** *Let $f : W \to X$ be a non-exceptional finite type analytic map, with $W \subset X$. Let $A \subset X$ be a finite set. Then if $q \in Q(X, A)$ and $q = f_*q$, then $q = 0$.*

We will now investigate what happens if we relax the assumption that q is meromorphic. The following result will be needed:

Theorem 4 ([7])**.** *Let U be a non-escaping Fatou component for f. Then U is eventually mapped to a periodic component which is is either an attracting basin, a parabolic basin, a Herman ring or a Siegel disk.*

Definition 14. *A rotation annulus for f is a connected component of Ω_f which is an annulus of finite modulus and on which the dynamics of f is conjugate to an irrational rotation.*

A cycle of rotation annuli for f of period p is a family of components $(\mathcal{A}, \dots, f^{p-1}(\mathcal{A}))$ of Ω_f which are all rotation annuli for f^p.

To each rotation annulus $\mathcal{A}$, we may canonically associate a quadratic differential $q_{\mathcal{A}}$ in the following way: let $\phi : \mathcal{A} \to \mathbb{C}$ be a linearizing coordinate for f on $\mathcal{A}$, mapping $\mathcal{A}$ to a straight annulus $A(R) = \{1 < |z| < R\}$. Let

$$q_{\mathcal{A}} = \phi^* \left(\frac{dz^2}{z^2}\right). \tag{7}$$

One can easily check using Laurent series that $\frac{dz^2}{z^2}$ is up to scalar multiplication the only holomorphic quadratic differential on $Q(A(R))$ that is rotation-invariant: in particular, there are no rotation invariant quadratic differential that are integrable near 0. Therefore $q_{\mathcal{A}}$ is the only integrable holomorphic quadratic differential on $Q(\mathcal{A})$ that is forward-invariant under f. We can extend it by zero outside of $\mathcal{A}$ to obtain a forward invariant quadratic differential in $Q(\Omega_f)$, that we still denote by $q_{\mathcal{A}}$.

Similarly, if $(A, \dots, f^{p-1}(A))$ is a cycle of rotation rings for f, then we get a quadratic differential $\tilde{q}_{\mathcal{A}} \in Q(\mathcal{A})$ that is invariant under f^p. It is then easy to check that $q_{\mathcal{A}} := \sum_{k=0}^{p-1} f_*^k \tilde{q}_{\mathcal{A}}$ is forward invariant under f, and it is (up to scalar multiplication) the only one on $Q(\mathcal{A} \cup \ldots \cup f^{p-1}(\mathcal{A}))$.

Proposition 6. *Let $f : W \to X$ be a finite type analytic map, with $W \subset X$. Then the only quadratic differentials on $Q(\Omega_f)$ invariant by f_* are those described above.*

Proof. Let $q \in Q(\Omega_f)$ be an invariant quadratic differential. Then $|q|$ is an invariant measure on Ω_f, that does not charge any escaping Fatou component.

Let U be a component of Ω_f with positive mass for $|q|$, and let V be the non-escaping Fatou component containing U. According to Theorem 4, V is eventually mapped to an attracting basin, a parabolic basin, a Siegel disk or a Herman ring. If V is mapped to an attracting or parabolic basin, then every point in U converges to the same finite cycle of points, so the grand orbit of V cannot support an invariant measure absolutely continuous with respect to the Lebesgue measure. Therefore V must be eventually mapped to either a (periodic) Siegel disk or Herman ring. Such a Fatou component can never be completely invariant, since it maps to itself with degree 1. So V must be in fact in the cycle: indeed, if it were not the case, then the preimages $f^{-n}(V)$ would form a pairwise disjoint family of open sets, each having the same mass as V for $|q|$; but this would contradict the fact that q is integrable.

Therefore U must be in a periodic rotation domain. But by the preceding discussion, the only invariant quadratic differentials on such domains are the quadratic differentials $q_{\mathcal{A}}$ associated to rotation annuli. □

5. Proof of Theorem A

Before proving Theorem A, we state here a recent result of Levin, that is an extension of the classical F. and M. Riesz theorem adapted to the setting of rotation domains.

Theorem 5 ([17], Corollary 2.1)**.** *Let $(A_j)_{j \in J}$ denote a non-empty union of Herman rings of a rational map f, and assume that $F := \overline{\bigcup_j A_j}$ is an A-compact. Let ν be a measure supported in the boundary of F such that the Cauchy transform $\hat{\nu}$ is supported in F. Then for each A_j, $\nu_{|A_j}$ is absolutely continuous with respect to the harmonic measure on A_j.*

Even though Theorem 5 is stated for rational maps, the proof remains valid in our setting. Indeed, the only property of the dynamics of f used in the proof is the fact that f extends continuously to the boundary of each Herman rings (used in the proof of Proposition 2 p. 8), which is guaranteed here by the assumption that each Herman ring is compactly contained in W.

We now restate Theorem A for the reader's convenience:

Theorem A. *Let $f : W \to X$ be a non-exceptional finite type analytic map, with $W \subset X$. We have:*

$$\dim \mathrm{Teich}(f) \leq \mathrm{card}\, S(f) - p(f) - s(f).$$

Proof. Let us start by defining an appropriate marked set A. The main point is to account for the possibility of dynamical relations between the orbits of singular values.

Since repelling periodic points are dense in the Julia set of f, we may chose a finite union of repelling cycles of f of cardinal at least $|3g-3|$, if $g \neq 1$, or of cardinal at least one if $g = 1$. Let Z denote such a set. Let $S_0(f)$ denote the set of singular values of f which are periodic or preperiodic, and let $A_0 := \bigcup_{n\in\mathbb{N}} f^n(S_0(f))$.

Let $S_1(f)$ denote the set of summable singular values with infinite forward orbits whose ω-limit sets are C-compacts. Let N be the smallest integer such that for all $n \geq N$, for all $v \in S_1(f)$, $f'(f^n(v)) \neq 0$ (such a N exists by definition of summability). We let $A_1 := \bigcup_{0\leq n\leq N-1} f^n(S_1(f))$ (with the convention that $A_1 = \emptyset$ if $N = 0$).

Finally, let

$$A := A_0 \cup A_1 \cup Z \quad \text{and} \quad B := S(f) \cup A \cup f(A).$$

Then (A, B) is admissible, and therefore $\mathrm{Def}_A^B(f)$ is a complex manifold of dimension $\mathrm{card}\,(B - A)$. Observe that $B - A = (f(A_1) - A_1) \cup (S(f) - S_0(f))$, so that

$$\mathrm{card}\,(B - A) = \mathrm{card}\, S(f) - p(f) - s(f) + \mathrm{card}\, f^N(S_1(f)). \tag{8}$$

Let $z_i \in f^N(S_1(f))$. Let $u_i \in \Gamma(TX)^*$ be any non-zero linear form with support equal to $\{z_i\}$. Let $v_n := \sum_{k=0}^n f_*^k u$. The pushforwards are well-defined, since by definition of N, the orbit of z_i does not meet critical points. The fact that z_i is summable readily implies that the sequence $(v_n)_{n\in\mathbb{N}}$ converges to some $v_i \in \Gamma(TX)^*$. Let q_i be the Z-normalized quadratic differential on X corresponding to v_s (see Theorem 3). From the definition of v_s, we have that $v_i - f_* v_i = u_i$; moreover, $v_i - \overline{\partial} q_i$ is supported in Z (if $g = 0$, or $v_i = \overline{\partial} q_i$ if $g > 0$). Therefore, in view of Proposition 2, $q_i - f_* q_i$ is supported in $Z \cup S(f)$ (and in fact, if $g > 0$, it is supported only in $S(f)$).

Lemma 2. *The quadratic differentials $(\nabla_f q_i)_{z_i \in f^N(S_1(f))}$ are linearly independant.*

Proof of Lemma 2. First note that the quadratic differentials $(q_i)_{z_i\in f^N(S_1(f))}$ are linearly independant, since the $\overline{\partial} q_i$ are. So let us pick a linear combination $q := \sum_i \lambda_i q_i$, and assume that $\nabla_f q = 0$; we need to prove that $q = 0$.

Since $|q|$ is an invariant measure for f, its gives full measure to non-escaping points. Therefore $q = 0$ outside of $W_\infty := \bigcap_{n\in\mathbb{N}} f^{-n}(W)$. Let $K := \bigcup_i \omega(v_i)$, and let F denote the (possibly empty) union of the closure of all Herman rings whose boundaries meet K. Let us start by proving that $q = 0$ outside of $K \cup F$ (observe that the set F is closed, since a finite type map can only have finitely many Herman rings). Let U be a connected component of $X \backslash (K \cup F)$. By Proposition 6, if U intersects the Fatou set of f, then $q = 0$ on U.

Otherwise, $U \subset J(f)$ and we will prove that if q is not identically zero on U, then f is in fact a Lattès map. Assume that q is not identically zero on U: then also the set of escaping points in U is at most discrete, so that $\tilde{U} := U \cap W_\infty$ is still open. Then by Montel's theorem, $\bigcup_{n\in\mathbb{N}} f^n(\tilde{U})$ cannot be hyperbolic, so that necessarily $W = X = \mathbb{P}^1$ and f is a rational map. Moreover, $\frac{\overline{q}}{|q|}$ is an invariant line field that is holomorphic on

$U \subset J(f)$, so by [23] (lemma 3.16), f is a Lattès map; this contradicts our assumption that f is non-elementary.

We have therefore proved that $q = 0$ outside of $K \cup F$. We now claim that $q = 0$ outside of F. Indeed, if $z \in K \backslash F$, let D be a small disk containing z such that the closure $\overline{D}$ doesn't meet F. Since K is a C-compact, so is $K \cap \overline{D}$, and so by Proposition 4, $q = 0$ on D.

It now only remains to prove that $q = 0$ on F. If v_i is a summable singular value that is not on the boundary of a Herman ring, then we have proved that $q = 0$ near v_i; therefore $\lambda_i = 0$ and so $q = 0$ on any Herman ring whose boundary does not contain a summable singular value. In particular, there can be only finitely many Herman rings in F; and their closure must be pairwise disjoint,

We therefore now assume that F is a union of Herman rings each containing a summable singular value. Observe that by Remark 2, q is the Cauchy transform of a discrete measure ν (namely, a converging series of Dirac masses at points of the orbits of the v_i). Moreover, we proved that q is supported in F. But by Theorem 5, this implies that ν is also absolutely continuous with respect to the harmonic measures of boundary components of the Herman rings; therefore $\nu = 0$ and $q = 0$. □

Let us now return to the proof of Theorem A. Recall that by construction, for any $z_i \in f^N(S_1(f))$, $\nabla_f q_i \in Q(X, B)$. It is a consequence of the previous lemma that the classes $([\nabla_f q_i])_{z_i \in f^N(S_1(f))}$ are linearly independent in $Q(X, B)/\nabla_f Q(X, A)$: indeed, ∇_f is injective on the vector space spanned by the $q_i, z_i \in f^N(S_1(f))$ and by $Q(X, A)$. Therefore no non-trivial linear combination of the $\nabla_f q_i$ can be in $\nabla_f Q(X, A)$, since none of the q_i are in $Q(X, A)$. This means that $\dim \left(Q(X, B) \cap \overline{\nabla_f Q_f}\right) / \nabla_f Q(X, A) \geq s(f)$. But by Theorem 2 and (6):

$$\dim T^*_{[0]} \mathrm{Teich}(f) = \dim T^*_{[0]} \mathrm{Def}^B_A(f) - \dim \left(Q(X, B) \cap \overline{\nabla_f Q_f}\right) / \nabla_f Q(X, A)$$

so that

$$\dim \mathrm{Teich}(f) \leq \mathrm{card}\,(B - A) - s(f) = \mathrm{card}\, S(f) - p(f) - s(f),$$

which is the desired inequality. □

6. Proof of Theorem B

In this section, we will focus on the case where $W = X = \mathbb{P}^1$, so that $f : W \to X$ is a rational map. We will recover from the work done in the previous sections a simpler proof of a result due to Levin (see [16]). First let us introduce some notations.

We assume A to be some finite union of repelling cycles, and $B := A \cup S(f)$. Recall that elements of $\mathrm{Def}^B_A(f)$ may be represented as a triple (ϕ, ψ, g), where $g = \phi \circ f \circ \psi^{-1}$. There is a natural holomorphic map $\Phi : \mathrm{Def}^B_A(f) \to \mathrm{rat}_d$ defined as $[\mu] \mapsto [\phi \circ f \circ \psi^{-1}]$; if $Z \subset A$ is some subset of cardinal 3, Φ admits a holomorphic lift $\Phi_Z : \mathrm{Def}^B_A(f) \to \mathrm{Rat}_d$ obtained by requiring that ϕ and ψ fix Z pointwise.

If $\lambda \mapsto [\mu_\lambda]$ is a holomorphic curve in $\mathrm{Def}^B_A(f)$ passing through the basepoint [0] at $\lambda = 0$, let $(\phi_\lambda, \psi_\lambda, f_\lambda)$ be a corresponding representation. Then, differentiating with respect to λ, we have

$$\eta = Df^{-1} \cdot \left(\frac{d}{d\lambda}_{|\lambda=0} f_\lambda \right) = f^*\dot{\phi} - \dot{\psi}, \tag{9}$$

and η is a meromorphic vector field in $T(f)$.

With the identification $T(f) \simeq T_f \mathrm{Rat}_d$, this means that $D\Phi_Z([0]) \cdot [\dot{\mu}] = \eta$.

We will need the following result, proved by Epstein in [8] for more general choices of A and B, using elaborate cohomology arguments. We give here a very elementary and simple proof for the convenience of the reader.

Proposition 7. *The map Φ_Z is an immersion, that is, the linear map $D\Phi_Z$ is injective.*

Proof. It is enough to prove this statement at the basepoint $[0]$. Given the discussion above, and using the same notations, we have to prove the following: if $\eta = 0$, then $[\dot{\mu}] = 0$ as an element of $T_{[0]}\mathrm{Teich}(\mathbb{P}^1, B)$, i.e. $\dot{\phi} = 0$ on B. Recall that $\eta = f^*\dot{\phi} - \dot{\psi}$, and that $\dot{\phi} = \dot{\psi}$ on A. Therefore, for all $z \in A$, $\dot{\phi}(z) = f^*\dot{\phi}(z)$. Since A was chosen to be a finite union of repelling cycles, for every $z \in A$, there is some k such that $f^k(z) = z$ and $\rho := (f^k)'(z)$ satisfies $|\rho| > 1$. Therefore $\dot{\phi}(z) = \rho\dot{\phi}(z)$, and so $\dot{\phi}(z) = 0$.

By definition of B, it only remains to prove that $\dot{\phi}(v) = 0$ for all $v \in S(f)$. Let $c \in \mathrm{Crit}(f)$ be a critical point such that $f(c) = v$. From the equation $\eta = f^*\dot{\phi} - \dot{\psi}$ and $\eta = 0$, we get $\dot{\phi} \circ f = Df \cdot \dot{\psi}$; evaluating at c, we have

$$\dot{\phi}(v) = Df(c) \cdot \dot{\psi}(c) = 0.$$

□

In fact, using the same arguments, it is not difficult to extend Proposition 7 to the case where A is some finite union of cycles with multipliers different from 1, and of pieces of (forward) critical orbits, and $B := A \cup f(A) \cup S(f)$.

We now come to our second result, that we state again here for the reader's convenience:

Theorem B. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map with s summable critical values v_i, $1 \leq i \leq s$, such that the orbits of each v_i do not meet $S(f)$. Assume that either f has no invariant line field, or that the ω-limit set of those s summable critical values are C-compacts. Then there is a germ of analytic set Λ in Rat_d passing through f and transverse to $\mathcal{O}(f)$ on which the critical values of f move holomorphically, and such that the linear map*

$$\mathcal{V} : T_f\Lambda \subset T(f) \to \bigoplus_{1 \leq i \leq s} T_{v_i}\mathbb{P}^1$$
$$\eta \mapsto (\dot{v}_i + \xi_\eta(v_i))_{1 \leq i \leq s}$$

has maximal rank, i.e. equal to s.

Proof. Let A be a repelling cycle of cardinal 3 that does not contain any critical value, and let $Z := A$. Let $B = A \cup S(f)$: then $\mathrm{Def}_A^B(f)$ is a complex manifold of dimension $\mathrm{card}\, S(f)$. Let $\Phi : \mathrm{Def}_A^B(f) \to \mathrm{rat}_d$ be the natural map from the deformation space to the moduli space, and let $\Phi_Z : \mathrm{Def}_A^B(f) \to \mathrm{Rat}_d$ be its lift to the parameter space obtained by chosing quasiconformal homeomorphisms fixing Z pointwise. Then $D\Phi_Z([0])$ takes values in $T_Z(f)$, the subspace of $T(f)$ of vector fields vanishing on Z. For each summable critical value v_i, let u_i be a non-zero element of $\Gamma(TX)^*$ supported in $\{v_i\}$, and let q_i be the quadratic differentials constructed in the proof of Theorem A (recall that q_i is the Z-normalized quadratic differential associated to $\sum_{n=0}^{\infty} f_*^n u_i$).

If $\lambda \mapsto [\mu_\lambda]$ is a holomorphic curve in $\mathrm{Def}_A^B(f)$ tangent to $[\mu] \in T_{[0]}\mathrm{Def}_A^B(f)$ at the basepoint, we can lift $\lambda \mapsto [\mu_\lambda]$ to a holomorphic curve of representatives $\lambda \mapsto \mu_\lambda$, and the normalized solutions ϕ_λ of the associated Beltrami equation will satisfy the property that $\overline{\partial}\dot{\phi} = \overline{\partial}\frac{d}{d\lambda}_{\lambda=0}\phi_\lambda$ is a representative of $[\mu]$.

We will prove that that the linear map:

$$\mathcal{U} : T_{[0]}\mathrm{Def}_A^B(f) \to \mathbb{C}^s$$
$$[\mu] \mapsto (\langle u_i, \dot{\phi}(v_i) + \xi_\eta(v_i)\rangle)_{1\leq i\leq s}$$

has rank s. This will imply Theorem B. Indeed, since $D\Phi_Z$ is injective, Φ_Z is a local biholomorphism onto its image, and therefore defines a germ of analytic submanifold Λ passing through f. It is transverse to $\mathcal{O}(f)$, and the critical values of f move holomorphically on Λ since they are given by $v_i(\lambda) = \phi_\lambda(v_i)$. Additionnally, since the linear map

$$\bigoplus_{i\leq s} T_{v_i}\mathbb{P}^1 \to \mathbb{C}^s$$
$$(\xi_i)_{i\leq s} \mapsto (\langle u_i, \xi_i\rangle)_{i\leq s}$$

is an isomorphism, it is clear that if $\mathcal{U}$ has rank s then $\mathcal{V}$ also has rank s.

Let $\mathcal{U}_i : [\mu] \mapsto \langle u_i, \dot{\psi}(v_i) + \xi_\eta(v_i)\rangle$, so that $\mathcal{U} = (\mathcal{U}_i)_{i\leq s}$. Let $[\mu] \in T_{[0]}\mathrm{Def}_A^B(f)$ and $\eta = D\Phi_Z([0]) \cdot [\mu] \in T_Z(f)$.

For $n \in \mathbb{N}$, denote by $q_{i,n}$ the Z-normalized quadratic differential associated to $f_*^n u_i$: then $q_i = \sum_{n=0}^{\infty} q_{i,n}$ and $q_{i,n}$ is a quadratic differential with exactly four poles, all simple, which are in $Z \cup \{f^n(v_i)\}$.

Note that

$$2i\pi \sum_{n\in\mathbb{N}} \mathrm{Res}(q_{i,n} \cdot \eta, f^n(v_i)) = \mathcal{U}_i([\mu]).$$

Indeed, by definition:

$$\langle f_*^n u_i, \eta\rangle = 2i\pi \mathrm{Res}(q_{i,n} \cdot (f^n)^*\eta, f^n(v_i))$$

and $\xi(v_i) = \sum_{n\geq 0}((f^n)^*\eta)(v_i)$.

The next lemma essentially says that the linear form $\mathcal{U}_i$ is represented by $\frac{1}{2i\pi}\nabla_f q_i$ in $T_{[0]}^*\mathrm{Def}_A^B(f)$:

Lemma 3. *Let $\mu = \overline{\partial}\dot{\phi}$ be a representative of $[\mu] \in T_0\mathrm{Def}_A^B(f)$. Then :*

$$\int_{\mathbb{P}^1} \nabla_f q_i \cdot \mu = \langle u_i, \dot{\phi}(v_i) + \xi_\eta(v_i)\rangle.$$

Proof of lemma 3. For all $n \in \mathbb{N}$, the differential form $q_{i,n} \cdot \eta$ is meromorphic on $\mathbb{P}^1$, therefore the sum of its residues is null. The quadratic differential $q_{i,n}$ has poles at

$Z \cup \{f^n(v_i)\}$, and the vector field η has poles at $\mathrm{Crit}(f)$. So:

$$\mathrm{Res}(q_{i,n} \cdot \eta, f^n(v_i)) = -\sum_{z \in Z \cup \mathrm{Crit}(f)} \mathrm{Res}(q_{i,n} \cdot \eta, z).$$

Therefore:

$$\sum_{n \in \mathbb{N}} \mathrm{Res}(q_{i,n} \cdot \eta, f^n(v_i)) = -\sum_{n \in \mathbb{N}} \sum_{z \in Z \cup \mathrm{Crit}(f)} \mathrm{Res}(q_{i,n} \cdot \eta, z)$$

According to equation (9), we have: $\eta = f^* \dot{\phi} - \dot{\psi}$, and $\dot{\phi} = \dot{\psi}$ on $Z \subset A$. Therefore for all $z \in A$ and for all $n \in \mathbb{N}$:

$$\mathrm{Res}(q_{i,n} \cdot \eta, z) = \mathrm{Res}(q_{i,n} \cdot (f^* \dot{\phi} - \dot{\phi}), z) = \mathrm{Res}(f_* q_{i,n} \cdot \dot{\phi}, f(z)) - \mathrm{Res}(q_{i,n} \cdot \dot{\phi}, z).$$

Therefore:

$$-\sum_{n \in \mathbb{N}} \sum_{z \in Z} \mathrm{Res}(q_{i,n} \cdot \eta, z) = \sum_{z \in Z} \mathrm{Res}(\nabla_f q_i \cdot \dot{\phi}, z).$$

Moreover:

$$\begin{aligned} -\sum_{c \in \mathrm{Crit}(f)} \mathrm{Res}(q_{i,n} \cdot \eta, c) &= -\sum_{c \in \mathrm{Crit}(f)} \mathrm{Res}(q_{i,n} \cdot (f^* \dot{\phi} - \dot{\psi}), c) \\ &= -\sum_{c \in \mathrm{Crit}(f)} \mathrm{Res}(q_{i,n} \cdot f^* \dot{\phi}, c) \\ &= -\sum_{v \in S(f)} \mathrm{Res}(f_* q_{i,n} \cdot \dot{\phi}, v). \end{aligned}$$

For all $n \geq 1$, since $q_{i,n}$ has no pole at any $v \in S(f)$, we therefore have

$$-\sum_{c \in \mathrm{Crit}(f)} \mathrm{Res}(q_{i,n} \cdot \eta, c) = \sum_{v \in S(f)} \mathrm{Res}(\nabla_f q_{i,n} \cdot \dot{\phi}, v).$$

On the other hand, for $n = 0$, $q_{i,0}$ has a pole at v_i, and so:

$$-\sum_{c \in \mathrm{Crit}(f)} \mathrm{Res}(q_{i,0} \cdot \eta, c) = \sum_{v \in S(f)} \mathrm{Res}(\nabla_f q_{i,0} \cdot \dot{\phi}, v) - \mathrm{Res}(q_{i,0} \cdot \dot{\phi}, v_i)$$

Therefore:

$$-\sum_{n \in \mathbb{N}} \sum_{c \in \mathrm{Crit}(f)} \mathrm{Res}(q_{i,n} \cdot \eta, c) = -\mathrm{Res}(q_{i,0} \cdot \dot{\phi}, v_i) + \sum_{v \in S(f)} \mathrm{Res}(\nabla_f q_i \cdot \dot{\phi}, v).$$

To sum things up, we have:

$$\sum_{n \in \mathbb{N}} \mathrm{Res}(q_{i,n} \cdot \eta, f^n(v_i)) = -\mathrm{Res}(q_{i,0} \cdot \dot{\phi}, v_i) + \sum_{z \in B} \mathrm{Res}(\nabla_f q_i \cdot \dot{\phi}, z).$$

Finally, by Stokes' theorem,

$$2i\pi \sum_{z \in B} \mathrm{Res}(\nabla_f q_i \cdot \dot{\phi}, z) = \int_{\mathbb{P}^1} \nabla_f q_i \cdot \overline{\partial} \dot{\phi},$$

and by definition of $q_{i,0}$ (and the fact that $\dot{\phi} = 0$ on Z),

$$2i\pi \mathrm{Res}(q_{i,0} \cdot \dot{\phi}, v_i) = \int_{\mathbb{P}^1} q_{i,0} \cdot \overline{\partial} \dot{\phi} = \langle u_i, \dot{\phi}(v_i) \rangle.$$

To conclude the proof of lemma 3, just observe that

$$2i\pi \sum_{n\in\mathbb{N}} \operatorname{Res}(q_{i,n}\cdot\eta, f^n(v_i)) = \sum_{n=0}^{\infty}\langle f_*^n u_i, \eta\rangle = \langle u_i, \xi_\eta(v_i)\rangle.$$

□

Let us now return to the proof of Theorem B. According to the preceding lemma, we have:

$$\mathcal{U}_i^Z([\mu]) = \frac{1}{2i\pi}\int_{\mathbb{P}^1} \nabla_f q_i \cdot [\mu]$$

for all $[\mu] \in T_{[0]}\mathrm{Def}_A^B(f)$. In other words, the class of the quadratic differential $\frac{1}{2i\pi}\nabla_f q_i$ in $Q(\mathbb{P}^1, B)/\nabla_f Q(\mathbb{P}^1, A)$ represents the linear form $\mathcal{U}_i$ in $T^*_{[0]}\mathrm{Def}_A^B(f)$. Moreover, according to lemma 2, the quadratic differentials $(\nabla_f q_i)_{i\leq s}$ are linearly independent, and as in the proof of Theorem A, this implies that the classes $([\nabla_f q_i])_{i\leq s}$ are linearly independent in $Q(\mathbb{P}^1, B)/\nabla_f Q(\mathbb{P}^1, A)$. Therefore, the $(\mathcal{U}_i)_{i\leq s}$ are linearly independent, which proves that $\mathcal{U}$ has rank s. □

References

[1] Astorg, M. The Teichmüller space of a rational map immerses into moduli space. *Adv. Math. 313* (2017), 991–1023.

[2] Astorg, M., Gauthier, T., Mihalache, N., and Vigny, G. Collet, Eckmann and the bifurcation measure. *Invent. Math. 217*, 3 (2019), 749–797.

[3] Avila, A. Infinitesimal perturbations of rational maps. *Nonlinearity 15*, 3 (2002), 695–704.

[4] Boivin, A., and Jiang, B. Uniform approximation by meromorphic functions on Riemann surfaces. *J. Anal. Math. 93* (2004), 199–214.

[5] Buff, X., Écalle, J., and Epstein, A. Limits of degenerate parabolic quadratic rational maps. *Geom. Funct. Anal. 23*, 1 (2013), 42–95.

[6] Domínguez, P., Makienko, P. M., and Sienra, G. Ruelle operator and transcendental entire maps. *Discrete Contin. Dyn. Syst. 12*, 4 (2005), 773–789.

[7] Epstein, A. L. *Towers of finite type complex analytic maps.* PhD thesis, 1993. Thesis (Ph.D.)–City University of New York.

[8] Epstein, A. L. Transversality in holomorphic dynamics. `http://www.warwick.ac.uk/mases`.

[9] Erëmenko, A. E., and Lyubich, M. Y. Dynamical properties of some classes of entire functions. *Ann. Inst. Fourier (Grenoble) 42*, 4 (1992), 989–1020.

[10] Firsova, T., Kahn, J., and Selinger, N. On deformation spaces of quadratic rational maps. *ArXiv preprint:1606.00723* (2016).

[11] Gamelin, T. W. *Uniform algebras.* Prentice-Hall, Inc., Englewood Cliffs, N. J., 1969.

[12] Gardiner, F. P., and Lakic, N. *Quasiconformal Teichmüller theory*, vol. 76 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2000.

[13] Hironaka, E., and Koch, S. A disconnected deformation space of rational maps. *J. Mod. Dyn. 11* (2017), 409–423.

[14] Hubbard, J. H. *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1.* Matrix Editions, Ithaca, NY, 2006. Teichmüller theory, With contributions by Adrien Douady, William Dunbar, Roland Roeder, Sylvain Bonnot, David Brown, Allen Hatcher, Chris Hruska and Sudeb Mitra, With forewords by William Thurston and Clifford Earle.

[15] Levin, G. On an analytic approach to the Fatou conjecture. *Fund. Math. 171*, 2 (2002), 177–196.

[16] Levin, G. Perturbations of weakly expanding critical orbits. 163–196.

[17] Levin, G. On the Cauchy transform vanishing outside a compact. *ArXiv preprint:1911.05336* (2019).

[18] Levin, G. Fixed points of the Ruelle-Thurston operator and the Cauchy transform. *Fund. Math. 254*, 1 (2021), 49–67.

[19] Makienko, P., and Sienra, G. Poincaré series and instability of exponential maps. *Bol. Soc. Mat. Mexicana (3) 12*, 2 (2006), 213–228.

[20] Makienko, P. M. Remarks on Ruelle operator and invariant line field problem. *ArXiv preprint: 0110093* (2001).

[21] Makienko, P. M. Remarks on the Ruelle operator and the invariant line fields problem. II. *Ergodic Theory Dynam. Systems 25*, 5 (2005), 1561–1581.

[22] Makienko, P. M. Remarks on the dynamic of the Ruelle operator and invariant differentials. *Dal nevost. Mat. Zh. 8*, 2 (2008), 180–205.

[23] McMullen, C. T. *Complex dynamics and renormalization*, vol. 135 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1994.

[24] McMullen, C. T., and Sullivan, D. P. Quasiconformal homeomorphisms and dynamics. III. The Teichmüller space of a holomorphic dynamical system. *Adv. Math. 135*, 2 (1998), 351–395.

[25] Przytycki, F., and Rohde, S. Rigidity of holomorphic Collet-Eckmann repellers. *Ark. Mat. 37*, 2 (1999), 357–371.

[26] Scheinberg, S. Uniform approximation by functions analytic on a Riemann surface. *Ann. of Math. (2) 108*, 2 (1978), 257–298.

[27] Tsujii, M. A simple proof for monotonicity of entropy in the quadratic family. *Ergodic Theory Dynam. Systems 20*, 3 (2000), 925–933.

[28] Urbański, M., and Zdunik, A. Instability of exponential Collet-Eckmann maps. *Israel J. Math. 161* (2007), 347–371.

[29] van Strien, S. Misiurewicz maps unfold generically (even if they are critically non-finite). *Fund. Math. 163*, 1 (2000), 39–54.

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