ON THE DYNAMICAL TEICHMÜLLER SPACE

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Abstract. We prove that the Teichmüller space of a rational map immerses into the moduli space of rational maps of the same degree, answering a question of McMullen and Sullivan. This is achieved through a new description of the tangent and cotangent space of the dynamical Teichmüller space.

1. Notations

The following notations will be used throughout the article :

- $\mathcal{S}$ is a (possibly disconnected) complex 1-manifold. Whenever $\mathcal{S}$ is additionnally referred to as a Riemann surface, it means that $\mathcal{S}$ is assumed to be connected.
- $\mathbb{P}^1$ is the Riemann sphere.
- Ω is a hyperbolic open subset of $\mathbb{P}^1$ (possibly disconnected: then hyperbolic means that all connected components are hyperbolic).
- $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational map of degree $d \geq 2$.
- If $\mathcal{S}$ is hyperbolic, $\rho_{\mathcal{S}}$ is the hyperbolic metric on $\mathcal{S}$. In the case where $\mathcal{S}$ is disconnected, $\rho_{\mathcal{S}}$ refers to the hyperbolic metric of each component of $\mathcal{S}$. More precisely, if $v \in T_s\mathcal{S}$ is a tangent vector attached to a point $s \in \mathcal{S}$, then $\rho_{\mathcal{S}}(s; v)$ is the length of the vector v measured in the hyperbolic metric of the connected component of $\mathcal{S}$ containing s. If ξ is a vector field on $\mathcal{S}$, $\rho_{\mathcal{S}}(\xi)$ denotes the function $s \mapsto \rho_{\mathcal{S}}(s; \xi(s))$.

2. Introduction

Let us denote by Rat_d the space of rational maps of degree d, and by rat_d its quotient under the action by conjugacy of the group of Möbius transformations. For $f \in \mathrm{Rat}_d$, we will denote by $\mathcal{O}(f)$ the orbit of f under the action of the group of Möbius transformations.

In order to study the geometry of the quasiconformal conjugacy class of f in Rat_d and rat_d, McMullen and Sullivan introduced in [MS98] the dynamical Teichmüller space of a rational map f, as a dynamical analogue of the Teichmüller theory of surfaces (see [GL00] and [Hub06] for an introduction to Teichmüller theory). McMullen and Sullivan constructed a natural complex structure on the Teichmüller space of a rational map f of degree d, making it into a complex manifold of dimension at most $2d-2$. They also exhibited a holomorphic map of orbifolds $\Psi : \mathrm{Teich}(f) \to \mathrm{rat}_d$ whose image is exactly the quasiconformal conjugacy class of f : thus one should think of the Teichmüller space of f as a complex manifold parametrizing the conjugacy class of f. In this context, a natural question arises concerning the parametrization Ψ : is it an immersion ? This question was asked by McMullen and Sullivan in their introductory paper. As it turns out, the answer is yes. A. Epstein has an unpublished proof of this result; in [Mak10],

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Makienko also gives a proof in the same spirit. We present here a different approach, using more elementary tools : in particular, we won't need the explicit description of the Teichmüller space of f given in [EM88], and we will give a new method for constructing the complex structure on Teich(f) which does not rely on preexisting Teichmüller theory. A related question, also raised in [MS98], is to know whether or not the image of this map can accumulate on itself. In [Bra92], Branner showed that the answer is yes.

Denote by bel(f) the space of L^∞ Beltrami differentials invariant under f, and by Bel(f) its unit ball. We shall use the term "Beltrami forms" for elements of Bel(f), and "Beltrami differentials" for elements of bel(f), which we will think of as the tangent space to Bel(f). Given a quasiconformal homeomorphism ϕ, we will denote by $K(\phi)$ its dilatation.

Definition 2.1. Let us introduce the following notations:

- Denote by QC(f) the group of quasiconformal homeomorphisms commuting with f.
- Denote by $\mathrm{QC}_0(f)$ the normal subgroup of the elements $\phi \in \mathrm{QC}(f)$ such that there exists an isotopy $\phi_t \in \mathrm{QC}(f)$ with $\phi_0 = \mathrm{Id}$, $\phi_1 = \phi$ and for all $t \in [0,1]$, $K(\phi_t) \leq K$, where $K > 1$ is a constant independant from t.
- The modular group of f is $\mathrm{Mod}(f) = \mathrm{QC}(f)/\mathrm{QC}_0(f)$.
- The Teichmüller space of a rational map f (which we will denote by Teich(f)) is Bel(f) quotiented by the right action of $\mathrm{QC}_0(f)$ by precomposition.

Let $Z \subset \mathbb{P}^1$ be a set of cardinal 3. There is a holomorphic map $\Psi^Z : \mathrm{Bel}(f) \to \mathrm{Rat}_d$ defined by $\Psi^Z(\mu) = \phi_\mu^Z \circ f \circ (\phi_\mu^Z)^{-1}$, where ϕ_μ^Z is the unique solution of the Beltrami equation $\overline{\partial}\phi_\mu^Z = \mu \circ \partial\phi_\mu^Z$ fixing Z. It descends to a holomorphic map of orbifolds $\Psi : \mathrm{Bel}(f) \to \mathrm{rat}_d$ independent from the choice of Z, and to maps $\Upsilon^Z : \mathrm{Teich}(f) \to \mathrm{Rat}_d$ and $\Upsilon : \mathrm{Teich}(f) \to \mathrm{rat}_d$.

The unit ball Bel(f) being an open subset of the Banach space L^∞, it has a natural complex Banach manifold structure, and there exists at most one complex structure on Teich(f) making $\pi : \mathrm{Bel}(f) \to \mathrm{Teich}(f)$ into a split submersion. Using the results of [EM88] on the equivalence between several notions of isotopies (isotopies relative to the ideal boundary, relative to the topological boundary, uniformly quasiconformal isotopies) McMullen and Sullivan constructed such a complex structure on Teich(f) and showed that Teich(f) is isomorphic to the cartesian product of a polydisk and of Teichmüller spaces of some finite type Riemann surfaces associated to the dynamics of f.

Once Teich(f) is endowed with its complex structure, one can verify that Υ^Z and Υ are holomorphic maps between complex manifolds and orbifolds respectively, and McMullen and Sullivan asked whether those maps are immersions. Since rat_d is not a manifold, we have to define what we mean by the statement that Υ is an immersion.

Definition 2.2. We will say that Υ is an immersion if the lift Υ^Z is an immersion whose image is transverse to $\mathcal{O}(f)$. If this is true for one choice of normalization set Z, then it holds for all Z.

It turns out that Υ is indeed an immersion, and Adam Epstein has an unpublished proof of this result. The idea of his proof is a dual approach using quadratic differentials. The key ingredients are the deformation spaces introduced in [Eps09] and a result of Bers concerning the density of rational quadratic differentials (cf [GL00], theorem 9 p.63).

The main result of this article is to give another proof of this result :

Main Theorem. The map $\Upsilon : \mathrm{Teich}(f) \to \mathrm{rat}_d$ is an immersion.

Our proof uses a new and more elementary construction of the complex structure on Teich(f) (we will notably not use the results of [EM88]).

A key tool for this construction is the following analytical result on quasiconformal vector fields (see definition 3.5), which is interesting in its own right.

Theorem A. Let Ω be a hyperbolic open subset of $\mathbb{P}^1$ and ξ be a quasiconformal vector field on Ω. The following properties are equivalent :

i) We have $\rho_\Omega(\xi) \in L^\infty(\Omega)$.
ii) We have $\|\rho_\Omega(\xi)\|_{L^\infty(\Omega)} \leq 4\|\overline{\partial}\xi\|_{L^\infty(\Omega)}$.
iii) There exists a quasiconformal extension $\hat{\xi}$ of ξ on all of $\mathbb{P}^1$ with $\hat{\xi} = 0$ on $\partial\Omega$.
iv) The extension $\hat{\xi}$ defined by $\hat{\xi}(z) = \xi(z)$ if $z \in \Omega$ and 0 else is quasiconformal on $\mathbb{P}^1$, and $\overline{\partial}\hat{\xi}(z) = 0$ almost everywhere in the complement of Ω.

As a consequence of Theorem A and of a lemma used in its proof, we will obtain a shorter and easier proof of the following theorem due to Bers and Lakic:

Corollary 1 (Bers-Lakic density theorem)**.** Let K be a compact subset of $\mathbb{P}^1$ containing at least 3 points, and let Z be a countable dense subset of K. The space of meromorphic quadratic differentials with simple poles in Z is dense (for the L^1 norm) in the space of integrable quadratic differentials which are holomorphic outside of K.

This new proof notably allows us to avoid the use of a delicate mollifying sequence called Ahlfors' Mollifier.

There is in Teichmüller theory a property called infinitesimal triviality (see definition 3.14) which is classically known to be equivalent to item *iv)* of Theorem A for a Beltrami differential on an open hyperbolic set Ω of $\mathbb{P}^1$. We will include a proof of this equivalence, thus proving this new characterization of infinitesimal triviality :

Corollary 2. A Beltrami differential μ on a hyperbolic Riemann surface $\mathcal{S}$ is infinitesimally trival if and only if it is of the form $\mu = \overline{\partial}\xi$, with $\|\rho_{\mathcal{S}}(\xi)\|_{L^\infty(\mathcal{S})} \leq 4\|\overline{\partial}\xi\|_{L^\infty(s)}$.

The proof of the Main theorem will also yield the following description of the tangent and cotangent spaces to the Teichmüller space of f (here, Λ_f is the closure of the grand orbit of the critical points, $Q(\Lambda_f)$ is the space of integrable quadratic differentials holomorphic outside Λ_f, and $\nabla_f = \mathrm{Id} - f_*$) :

Corollary 3. We have the following identification :

$$T_0\mathrm{Teich}(f) = \mathrm{bel}(f)/\{\overline{\partial}\xi, \xi = f^*\xi\}$$

$$T_0^*\mathrm{Teich}(f) = Q(\Lambda_f)/\overline{\nabla_f Q(\Lambda_f)}.$$

In section 3, we will be concerned only with non-dynamical, analytic results on quasiconformal vector fields. The main result of this section is Theorem A. In section 4, we will apply Theorem A to obtain the key fact that $D\Psi^Z$ has constant rank. Lastly, we will prove the Main Theorem in section 5.

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3. Quasiconformal vector fields

3.1. **Generalities.** In this section, we introduce notations and recall important results on several mathematical objects involved in quasiconformal Teichmüller theory : Beltrami forms and differentials, quadratic differentials, and quasiconformal vector fields.

In all of the article, $\mathcal{S}$ will denote a (connected) Riemann surface.

Definition 3.1. A quadratic differential q on $\mathcal{S}$ is a section of the complex line bundle $T^*\mathcal{S} \otimes T^*\mathcal{S}$ (tensor product over $\mathbb{C}$). The norm $|q|$ of a differential quadratic q is the associated hermitian form on $\mathcal{S}$. We say that q is integrable if $\mathcal{S}$ has finite volume for $|q|$, and denote in this case $\int_{\mathcal{S}} |q|$ by $\|q\|$.

In coordinates, a quadratic differential q is written $q = q(z)dz^2$ and $\|q\| = \int_{\mathcal{S}} |q(z)||dz|^2$.

Definition 3.2. If μ is a section of $\overline{\mathrm{Hom}}(T\mathcal{S}, T\mathcal{S})$, i.e. a section of the complex line bundle of anti-$\mathbb{C}$-linear endomorphisms of tangent planes, and $s \in \mathcal{S}$, then let $|\mu|(s)$ denote the norm of the endomorphism $\mu(s)$ of $T_s\mathcal{S}$: $|\mu|$ is a well defined function on $\mathcal{S}$. If μ is such a section verifying $|\mu| \in L^\infty(\mathcal{S})$, μ is called a Beltrami differential. If additionally $\|\mu\|_{L^\infty(\mathcal{S})} < 1$, we say that μ is a Beltrami form.

Here by norm we mean the spectral radius of the endomorphism $\mu(s)$ of $T_s\mathcal{S}$. Since $T_s\mathcal{S}$ has complex dimension one, the spectral radius coincides with the norm of $\mu(s)$ for any hermitian metric on $\mathcal{S}$. Note that in local coordinates, a Beltrami differential μ is written $\mu(z)\frac{d\bar{z}}{dz}$, and $|\mu(z)| = |\mu|(z)$ is independent of the coordinate.

Definition 3.3. Let $k \in \mathbb{Z}$. We denote by $\Gamma(T\mathcal{S}^{\otimes k})$ the space of sections of $(T^*\mathcal{S})^{\otimes k}$ if $k \geq 0$, and the space of sections of $(T\mathcal{S})^{\otimes |k|}$ if $k < 0$ (tensor products over $\mathbb{C}$).

In local coordinates, we write $\phi = \phi(z)dz^k$ if $k \geq 0$ and $\phi = \phi(z)\frac{d}{dz^{|k|}}$ if $k < 0$. Sometimes, by an abuse of notation we will also note $\phi = \phi(z)dz^k$ even if $k < 0$. In new coordinates $w = h^{-1}(z)$, the change of expression is given by: $\phi = \phi \circ h(w)h'(w)^k dw^k$ if $k \geq 0$ and $\phi = \phi \circ h(w)h'(w)^k \frac{d}{dw^k}$ if $k < 0$.

Recall that if ϕ is a section of a holomorphic line bundle E over a Riemann surface $\mathcal{S}$, with sufficient regularity, then $\overline{\partial}\phi$ is naturally a section of the holomorphic line bundle $\Lambda^{0,1}(T\mathcal{S}) \otimes E$.

In particular:

- If $E = T\mathcal{S}$ and $\phi \in \Gamma(E)$ (i.e. ϕ is a vector field), then $\overline{\partial}\phi$ is a Beltrami differential
- If $E = T^*\mathcal{S}$, then $\overline{\partial}\phi$ is a sesquilinear form. After antisymmetrization it becomes a volume form which may be integrated over $\mathcal{S}$.

In local coordinates, if $\phi = \phi(z)dz^k$, then $\overline{\partial}\phi = \frac{\partial\phi}{\partial\bar{z}}(z)d\bar{z} \otimes dz^k$.

Definition 3.4. Let $\phi \in \Gamma(T^*\mathcal{S}^{\otimes m})$ and $\psi \in \Gamma(T^*\mathcal{S}^{\otimes n})$, where $m, n \in \mathbb{Z}$.

- $\phi \cdot \psi$ is naturally a section of $(T^*\mathcal{S})^{\otimes m+n}$, given by either a tensor product or an interior product, depending on the signs of m and n.
- $\phi \cdot \overline{\partial}\psi$ naturally defines a section of $\Lambda^{0,1}(T\mathcal{S}) \otimes (T^*\mathcal{S})^{\otimes m+n}$ in the following way : write locally $\overline{\partial}\psi = \psi_1 \otimes \psi_2$ where $\psi_1 \in \Omega^{0,1}(\mathcal{S})$ and $\psi_2 \in \Gamma((T^*\mathcal{S})^{\otimes n})$, and set $\phi \cdot \overline{\partial}\psi := \psi_1 \otimes (\phi \cdot \psi_2)$.

When $m + n = 1$, $\phi \cdot \overline{\partial}\psi$ is a sesquilinear form on $\mathcal{S}$. Up to antisymmetrization, it canonically yields a volume form on $\mathcal{S}$. Therefore, it makes sense to integrate such an object on $\mathcal{S}$. Note that the operation of antisymmetrization is a canonical isomorphism between sesquilinear forms and volume forms on $\mathcal{S}$, so we may loosely think of $\phi \cdot \overline{\partial}\psi$ before and after antisymmetrization as the same object.

Suppose that ϕ and ψ are written in local coordinates $\phi = \phi(z)dz^m$ and $\psi = \psi(z)dz^n$. Then :

$$\phi \cdot \psi = \phi(z)\psi(z)dz^{m+n}$$
$$\phi \cdot \overline{\partial}\psi = \phi(z)\frac{\partial\psi}{\partial\overline{z}}(z)dz^{m+n}d\overline{z}$$

This makes obvious the following properties, and justify the introduction of these notations :

$$\phi \cdot (\psi \cdot \xi) = (\phi \cdot \psi) \cdot \xi$$
$$\phi \cdot \psi = \psi \cdot \phi$$
$$\overline{\partial}(\phi \cdot \psi) = (\overline{\partial}\phi) \cdot \psi + \phi \cdot \overline{\partial}\psi$$

We shall use these properties in the rest of the article without further justification.

Here are notable instances of these definitions :

- If q is a quadratic differential and ξ is a vector field, then $q \cdot \xi$ is $(1,0)$-differential form and thus my be integrated on curves and has a well-defined notion of residues.
- If q is a quadratic differential and μ is a Beltrami differential, then $q \cdot \mu$ is a sesquilinear form. After antisymmetrization, it becomes a $(1,1)$-differential form and thus may be integrated on $\mathcal{S}$. This includes the case where $\mu = \overline{\partial}\xi$, where ξ is a vector field.
- If q is a quadratic differential and ξ is a vector field, then $\overline{\partial}q \cdot \xi$ is also a sesquilinear form, and after antisymmetrization can be integrated over $\mathcal{S}$.

We will follow the tradition of writing Beltrami differentials in coordinates as $\mu = \mu(z)\frac{\overline{dz}}{dz}$. In view of the above notations and formalism, a slightly more exact notation would be $\mu = \mu(z)d\overline{z} \otimes \frac{d}{dz}$.

Definition 3.5. Let ξ be a vector field on $\mathcal{S}$. We say that ξ is quasiconformal if $\overline{\partial}\xi$ (in the sense of distribution theory) is a Beltrami differential.

More explicitly, this means that a vector field ξ is quasiconformal if and only if there is a L^∞ Beltrami differential μ such that for every smooth, compactly supported quadratic differential ϕ on $\mathcal{S}$,

$$\int_{\mathcal{S}} \overline{\partial}\phi \cdot \xi = -\int_{\mathcal{S}} \phi \cdot \mu$$

Definition 3.6. If $A \subset \mathbb{P}^1$ is closed, we note $Q(A)$ the Banach space of integrable quadratic differentials on $\mathbb{P}^1$ and holomorphic on $\mathbb{P}^1 - A$, equipped with the L^1 norm.

Proposition 1 (Stokes' theorem for quasiconformal vector fields)**.** Let U be an open subset of $\mathbb{P}^1$ with piecewise C^1 boundary, let q be a C^1 quadratic differential continuous on $\overline{U}$ and ξ a quasiconformal vector field on $\mathbb{P}^1$. Then

$$\int_U q \cdot \overline{\partial}\xi + \int_U \xi \cdot \overline{\partial}q = \int_{\partial U} q \cdot \xi$$

Proof. In the case where ξ is a C^1 vector field, this follows from the classical Stokes' theorem applied to $q \cdot \xi$. Indeed, $q \cdot \xi$ is a $(1,0)$-form on a Riemann surface, so $d(q \cdot \xi) = \overline{\partial}(q \cdot \xi)$. Moreover, $\overline{\partial}(q \cdot \xi) = q \cdot \overline{\partial}\xi + \overline{\partial}q \cdot \xi$. We deduce the general case where $\overline{\partial}\xi$ only exists in the sense of distribution with a density argument : let ξ be a quasiconformal vector field and ξ_n a sequence of vector fields which are C^1 in the neighborhood of $\overline{U}$ and converging uniformly to ξ on $\overline{U}$ (such a sequence exists because ξ is continuous).

Then ξ_n converges to ξ as a distribution on U, so $\overline{\partial}\xi_n$ converges to $\overline{\partial}\xi$ in the sense of distributions (by continuity of the $\overline{\partial}$ operator for the topology of distributions). Since we know that $\overline{\partial}\xi$ is in fact a L^∞ Beltrami differential, we deduce from this that for all test quadratic differential ϕ (i.e. smooth and with compact support in U), we have :

$$\lim_{n\to\infty}\int_U \phi\cdot\overline{\partial}\xi_n = \int_U \phi\cdot\overline{\partial}\xi$$

Since test quadratic differentials are dense for the L^1 norm , this still holds for all quadratic differential ϕ integrable on U, and in particular for q.

Therefore $\lim_{n\to\infty}\int_U q\cdot\overline{\partial}\xi_n = \int_U q\cdot\overline{\partial}\xi$ and since ξ_n converges uniformly on $\overline{U}$, we also have

$$\lim_{n\to\infty} -\int_U \xi_n\cdot\overline{\partial}q + \int_{\partial U} q\cdot\xi_n = -\int_U \xi\cdot\overline{\partial}q + \int_{\partial U} q\cdot\xi.$$

□

Definition 3.7. Let $z_0\in\mathcal{S}$ and ξ a vector field on $\mathcal{S}$. If q is a meromorphic quadratic differential with at worst a simple pole at z_0, we define the residue of $q\cdot\xi$ at z_0 (denoted by $\operatorname{Res}(q\cdot\xi,z_0)$) as the residue of $q\cdot\tilde{\xi}$ at z_0, where $\tilde{\xi}$ is a vector field holomorphic in the neighborhood of z_0 with $\xi(z_0)=\tilde{\xi}(z_0)$. This definition does not depend upon the choice of $\tilde{\xi}$ but only on $\xi(z_0)$.

Proposition 2. Let q be a meromorphic quadratic differential on an open domain Ω with smooth boundary, relatively compact in $\mathbb{P}^1$, with simple poles that are included in a finite set P. Let ξ be a quasiconformal vector field on Ω extending continuously to $\overline{\Omega}$. Then :

$$\int_\Omega q\cdot\overline{\partial}\xi = 2i\pi\sum_{z\in P}\operatorname{Res}(q\cdot\xi,z) - \int_{\partial\Omega} q\cdot\xi$$

Proof. As in Proposition 1, let us first prove the result with stronger regularity assumptions on ξ and then conclude by density. Suppose first that ξ is a smooth vector field defined in a neighborhood of $\overline{\Omega}$ and that ξ is holomorphic in the neighborhood of each $z\in P$. Let $U=\Omega-\bigcup_{z_i\in P}D_i$, where D_i denotes a smooth disk centered at $z_i\in P$ small enough to ensure that it is included in Ω and that ξ is holomorphic on each D_i. Then by Proposition 1:

$$\int_\Omega q\cdot\overline{\partial}\xi = \int_U q\cdot\overline{\partial}\xi = -\int_{\partial U} q\cdot\xi$$

Moreover:

$$\int_{\partial U} q\cdot\xi = \int_{\partial\Omega} q\cdot\xi - \sum_{z_i\in P}\int_{\partial D_i} q\cdot\xi.$$

Since ξ is holomorphic in each disk D_i, the residue formula yields :

$$\int_{\partial U} q\cdot\xi = \int_{\partial\Omega} q\cdot\xi - 2i\pi\sum_{z_i\in P}\operatorname{Res}(q\cdot\xi,z_i).$$

This concludes the proof in the case where ξ is smooth and holomorphic in the neighborhood of P.

Notice that such vector fields are dense in the space of continuous vector fields on $\overline{\Omega}$ endowed with uniform convergence. Indeed, by working in local coordinates, the condition of being holomorphic in the neighborhood of P is verified by vector fields that are locally constant in those coordinates, and that condition is clearly dense for uniform convergence among smooth vector fields, which are themselves dense among continuous vector fields. Moreover, the right hand-side $-\int_{\partial\Omega} q\cdot\xi + 2i\pi\sum_{z_i\in P}\operatorname{Res}(q\cdot\xi,z_i)$ of the desired equality is clearly continuous in ξ with respect to uniform convergence. The same density argument used in the proof of Proposition 1 yields the sequential continuity of the left-hand side $\int_\Omega q \cdot \overline{\partial}\xi$ with respect to ξ, which concludes the proof. □

Note that in the particular case $\Omega = \Delta$ and $q = \frac{dz^2}{z}$, we get the usual Cauchy-Pompéiu formula.

Definition 3.8. For a rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$, we will note $\Delta_f = \mathrm{Id} - f^*$ and $\nabla_f = \mathrm{Id} - f_*$, where f^* et f_* are respectively the pullback by f on vector fields (and Beltrami differentials), and f_* is the pushforward by f on quadratic differentials, following the notations of [Eps09].

3.2. **Splitting and hyperbolic metric.**

Definition 3.9. Let $\mathcal{S}$ be a hyperbolic Riemann surface and ξ a vector field on $\mathcal{S}$. We say that ξ is hyperbolically bounded on $\mathcal{S}$ if and only if $\rho_{\mathcal{S}}(\xi) \in L^\infty(\mathcal{S})$, where $\rho_{\mathcal{S}}$ is the hyperbolic metric on $\mathcal{S}$.

Theorem 3.10. *Let ξ be a vector field hyperbolically bounded on an open hyperbolic subset Ω of $\mathbb{P}^1$, quasiconformal on Ω and identically vanishing outside Ω. Then ξ is globally quasiconformal, and $\overline{\partial}\xi = 0$ almost everywhere in the complement of Ω. Moreover, $\|\rho(\xi)\|_{L^\infty(\Omega)} \leq 4\|\overline{\partial}\xi\|_{L^\infty(\Omega)}$.*

Proof. The key point is the following lemma :

Lemma 3.11. *Let q be an integrable quadratic differential of class C^∞ on Ω, and ξ a hyperbolically bounded quasiconformal vector field on Ω. Assume that $\xi \cdot \overline{\partial}q$ is integrable on Ω. Then :*

$$\int_\Omega \overline{\partial}\xi \cdot q = -\int_\Omega \xi \cdot \overline{\partial}q$$

Proof. Let Ω_i be the (at most countable) collection of connected components of Ω. For each of those components, pick an arbitrary base point $z_i \in \Omega_i$. Let $\delta(z) = d_{\Omega_i}(z, z_i)$, for each $z \in \Omega_i$, where d_{Ω_i} is the hyperbolic distance on Ω_i. Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth function such that $\phi(x) = 1$ for $x \in [0, 1]$ and $\phi(x) = 0$ for $x \geq 2$. For all $n \in \mathbb{N}$, let us define $\phi_n : \Omega \to \mathbb{R}^+$ by

$$\phi_n(z) = \phi\left(\frac{\delta(z)}{n}\right).$$

Set $\mu = \overline{\partial}\xi$.

Since $\phi_n q$ is compactly supported in Ω, we have (by definition of $\overline{\partial}$ in the sense of distributions):

$$\int_\Omega \mu \cdot (\phi_n q) = -\int_\Omega \xi \cdot \overline{\partial}(\phi_n q) = \int_\Omega \xi \cdot (\phi_n \overline{\partial} q) + \xi \cdot (q \cdot \overline{\partial}\phi_n)$$

Moreover, $\int_\Omega \xi \cdot (q \cdot \overline{\partial}\phi_n) = \int_\Omega q \cdot (\xi \cdot \overline{\partial}\phi_n)$. Let us now evaluate the L^∞ norm of the Beltrami differential $\overline{\partial}\phi_n \cdot \xi$. Since δ is a locally lipschitz function on Ω, it has locally bounded distributional derivatives. We have :

$$\overline{\partial}\phi_n = \frac{1}{n}\phi'(\delta/n)\overline{\partial}\delta.$$

Let $z \in \Omega$ and $u \in T_z\mathbb{P}^1$. We have $\overline{\partial}\phi_n \cdot \xi(z) : u \mapsto \overline{\partial}\phi_n(u)\xi(z)$, and the norm of this endomorphism for any hermitian metric is $|\overline{\partial}\phi_n \cdot \xi|(z)$. We can therefore work with the hyperbolic metric ρ_Ω. Since δ is 1-lipschitz for the hyperbolic metric in Ω, the derivative

$D\delta$ has hyperbolic norm less than one almost everywhere, and so does the projection $\overline{\partial}\delta$. We have :

$$\overline{\partial}\phi_n \cdot \xi(z;u) = \frac{1}{n}\phi'\left(\frac{\delta(z)}{n}\right)\overline{\partial}\delta(z;u) \times \xi(z)$$
$$\rho_\Omega(\overline{\partial}\phi_n \cdot \xi(z;u)) \leq \frac{\sup_{\mathbb{R}^+}|\phi'|}{n}\|\rho_\Omega(\xi)\|_{L^\infty(\Omega)}\rho_\Omega(u)$$
$$|\overline{\partial}\phi_n \cdot \xi(z)| \leq \frac{\sup_{\mathbb{R}^+}|\phi'|}{n}\|\rho_\Omega(\xi)\|_{L^\infty(\Omega)}.$$

Therefore : $\|\overline{\partial}\phi_n \cdot \xi\|_{L^\infty} = O(1/n)$ and $\left|\int_\Omega q \cdot (\xi \cdot \overline{\partial}\phi_n)\right| \leq \|q\|_{L^1}\|\overline{\partial}\phi_n \cdot \xi\|_{L^\infty} = O(1/n)$.

We then have :

$$\int_\Omega \overline{\partial}\xi \cdot (\phi_n q) = \int_{\mathbb{P}^1}(\overline{\partial}\xi \cdot q)\phi_n = -\int_\Omega \phi_n(\xi \cdot \overline{\partial}q) + O(1/n)$$

so

$$\int_\Omega (\overline{\partial}\xi \cdot q)\phi_n = -\int_\Omega \phi_n(\xi \cdot \overline{\partial}q) + O(1/n)$$

Since we assumed that both $\xi \cdot \overline{\partial}q$ and $|q|$ are integrable, we can apply the dominated convergence theorem to get :

$$\int_\Omega \overline{\partial}\xi \cdot q = -\int_\Omega \xi \cdot \overline{\partial}q.$$

□

Let now q be a C^∞ quadratic differential on $\mathbb{P}^1$: its restriction to Ω verifies the conditions of the lemma, therefore we have :

$$\int_{\mathbb{P}^1} \mu \cdot q = -\int_{\mathbb{P}^1} \xi \cdot \overline{\partial}q,$$

where $\mu = \overline{\partial}\xi$ on Ω and 0 elsewhere. This means precisely that $\overline{\partial}\xi = \mu$ in the sense of distributions on $\mathbb{P}^1$, which proves the first assertion of the theorem.

Let us now prove the second assertion. Denote by $\tilde{\xi} = p^*\xi_{|\Omega}$ where $p : \Delta \to \Omega$ is a universal cover of Ω mapping 0 to an arbitrary point $z_0 \in \Omega$. Proposition 2 applied to $\tilde{\xi}$ and $q = \frac{dz^2}{z}$ on Δ yields :

$$\text{Res}\left(\frac{dz^2}{z} \cdot \tilde{\xi}(0), 0\right) = dz(\tilde{\xi}(0)) = \frac{1}{2i\pi}\int_{\Delta_r}\overline{\partial}\tilde{\xi}(z) \cdot \frac{dz^2}{z} + \frac{1}{2i\pi}\int_{S_r}\tilde{\xi}(z) \cdot \frac{dz^2}{z}$$

where Δ_r and S_r are respectively the disk of radius r and the circle of radius r. Since we assumed that $\|\rho_\Omega(\xi)\|_{L^\infty(\Omega)} = \|\rho_\Delta(\tilde{\xi})\|_{L^\infty(\Delta)}$ is finite, the second term converges to 0 when r tends to 1 (since it is equal to $\frac{1}{2\pi}\int_0^{2\pi}\xi(re^{it})dt$ and $r \mapsto \xi(re^{it})$ is continuous and vanishes when $r = 1$). Therefore, by letting r converging to 1 :

$$dz(\tilde{\xi}(0)) = \frac{1}{2i\pi}\int_\Delta \overline{\partial}\tilde{\xi}(z) \cdot \frac{dz^2}{z}$$

and

$$\rho_\Delta(\tilde{\xi})(0) = 2|\tilde{\xi}(0)| \leq 2\frac{1}{2\pi}\|\frac{dz^2}{z}\|_{L^1(\Delta)}\|\overline{\partial}p^*\xi\|_{L^\infty}.$$

Since $\|\frac{dz^2}{z}\|_{L^1(\Delta)} = 4\pi$ et $\overline{\partial}p^*\xi = p^*\overline{\partial}\xi$, we deduce

$$\rho_\Delta(\tilde{\xi})(0) = \rho_\Omega(\xi)(z_0) \leq 4\|\overline{\partial}\xi\|_{L^\infty(\Omega)}.$$

Since z_0 is arbitrary, this concludes the proof of the second assertion. □

This last theorem states that if we have a hyperbolically bounded quasiconformal vector field on an open set Ω, we can glue it together with the zero vector field outside Ω and still get a globally quasiconformal vector field. The next proposition gives a little more than the converse. We will need the following lemma :

Lemma 3.12. *Let Ω be a hyperbolic open subset of $\mathbb{P}^1$, and X a countable dense subset of $\partial\Omega$. Let (X_n) be an increasing sequence of finite subsets of X with $\cup_n X_n = X$ and* $\mathrm{card} X_n \geq 3$ *for all $n \in \mathbb{N}$, and let $\Omega_n = \mathbb{P}^1 - X_n$. Then the hyperbolic metric ρ_{Ω_n} of Ω_n converges pointwise on Ω to the hyperbolic metric ρ_Ω of Ω.*

Proof. Let $z \in \Omega$ and denote by $\pi : \Delta \to \Omega$ a universal cover of Ω mapping 0 to z. For $n \in \mathbb{N}$, let $\pi_n : \Delta \to \Omega_n$ a universal cover also mapping 0 to z. The inclusion $i : \Omega \subset \Omega_n$ lifts to a map $\psi_n : \Delta \to \Delta$ fixing 0 and such that $\pi_n \circ \psi_n = i \circ \pi = \pi$. According to Schwartz's lemma, $|\psi_n'(0)| \leq 1$ and therefore $|\pi'(0)| \leq |\pi_n'(0)|$.

Moreover, the sequence of maps (π_n) avoids the set X_1 of cardinal 3, and therefore forms a normal family by Montel's theorem. Let ϕ be the limit of a convergent subsequence of (π_n). Since for all $n \in \mathbb{N}$, $|\pi_n'(0)| \geq |\pi'(0)| > 0$, ϕ is not constant, therefore it is an open map. So $\phi(\Delta)$ is an open subset of $\mathbb{P}^1$, connected, containing z but disjoint from X. Therefore $\phi(\Delta)$ is a subset of the connected component of $\mathbb{P}^1 - \overline{X} = \mathbb{P}^1 - \partial\Omega$ containing z, namely Ω. Similarly, Schwarz's lemma applied to a lift of $\phi : \Delta \to \Omega$ shows that $|\phi'(0)| \leq |\pi'(0)|$. In other words, for all $n \in \mathbb{N}$:

$$\limsup_{m\to\infty} |\pi_m'(0)| \leq |\pi'(0)| \leq |\pi_n'(0)|$$

and therefore $\lim_{n\to\infty} |\pi_n'(0)| = |\pi'(0)|$. Thus we have that $\lim_{n\to\infty} \rho_n(z) = \rho(z)$, and $z \in \Omega$ is arbitrary, which concludes the proof. □

Proposition 3. Let ξ be a quasiconformal vector field on $\mathbb{P}^1$ vanishing on the boundary of a hyperbolic open subset Ω of $\mathbb{P}^1$. Then :

$$\|\rho_\Omega(\xi)\|_{L^\infty(\Omega)} \leq 4\|\overline{\partial}\xi\|_{L^\infty(\Omega)}$$

Proof. Denote by K the boundary of Ω. Let $(X_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of $\partial\Omega$ whose union is dense in $\partial\Omega$, with $\mathrm{card} X_n \geq 3$. Then by lemma 3.12, the hyperbolic metric ρ_{Ω_n} of $\Omega_n = \mathbb{P}^1 - X_n$ converges pointwise to the hyperbolic metric ρ_Ω of Ω on Ω. By Theorem 3.10, it then suffices to show that for all $n \in \mathbb{N}$, $\|\rho_{\Omega_n}(\xi)\|_{L^\infty(\Omega)}$ is bounded. This will follow from the following lemma:

Lemma 3.13. *Let ξ be a quasiconformal vector field on a compact Riemann surface $\mathcal{S}$, vanishing on a finite set P (of cardinal at least 3). Then the restriction $\xi_{|\mathcal{S}-P}$ is hyperbolically bounded in $\Omega = \mathcal{S} - P$.*

Proof of lemma 3.13. Since $\rho_\Omega(\xi)$ is a continuous function on $\mathcal{S} - P$, and by compacity of $\mathcal{S}$, it is enough to show that $\rho_\Omega(\xi)$ is bounded in the neighborhood of all $z \in P$. Let $z_0 \in P$, and $r > 0$ such that the punctured disk U of center z_0 and radius r is included in Ω. Then by the Schwarz lemma, the hyperbolic metric of Ω is smaller than that of U, so we have for all $z \in U$:

$$\rho_\Omega(\xi)(z) \leq \rho_U(\xi)(z) \leq C'|\xi(z)| \left(|z - z_0| \log |z - z_0|^{-1}\right)^{-1}.$$

The second inequality is a classical estimate of the hyperbolic metric of the punctured disk in the neighborhood of z_0 (see for example [GL00] or [Hub06]). Furthermore, ξ has a continuity modulus on $-\epsilon \log \epsilon$ by virtue of quasiconformality (cf [GL00], Theorem 7 p. 56), so there exists a constant $C > 0$ (depending only on ξ and on the choice of coordinates) such that in the coordinates z :

$$|\xi(z)| = |\xi(z) - \xi(z_0)| \leq C|z - z_0| \log |z - z_0|^{-1}.$$

We therefore have, for all $z \in D_r(z_0)$:

$$\rho_{\Omega_n}(\xi)(z) \leq C.$$

Thus $\rho_\Omega(\xi)$ is bounded in the neighborhood of P, and therefore $\xi_{|\Omega}$ is hyperbolically bounded. □

Applying lemma 3.13 to $\xi_{|\Omega_n}$, we get that there exists a constant $C_n > 0$ such that $\rho_{\Omega_n}(\xi) \leq C_n$ on Ω_n.

Then Theorem 3.10 applied to ξ on Ω_n allows us to improve that bound to get uniformity with respect to n :

$$\|\rho_{\Omega_n}(\xi)\|_{L^\infty(\Omega_n)} \leq 4\|\overline{\partial}\xi\|_{L^\infty(\Omega_n)} \leq 4\|\overline{\partial}\xi\|_{L^\infty(\mathbb{P}^1)}.$$

By passing to the limit, we get :

$$\|\rho_\Omega(\xi)\|_{L^\infty(\Omega)} \leq 4\|\overline{\partial}\xi\|_{L^\infty(\mathbb{P}^1)},$$

and a second application of the same theorem finally yields :

$$\|\rho_\Omega(\xi)\|_{L^\infty(\Omega)} \leq 4\|\overline{\partial}\xi\|_{L^\infty(\Omega)}.$$

□

By combining the results of Theorem 3.10 and proposition 3, we get :

Theorem A. Let Ω be a hyperbolic open subset of $\mathbb{P}^1$ and ξ be a quasiconformal vector field on Ω. The following properties are equivalent :

i) We have $\rho_\Omega(\xi) \in L^\infty(\Omega)$.

ii) We have $\|\rho_\Omega(\xi)\|_{L^\infty(\Omega)} \leq 4\|\overline{\partial}\xi\|_{L^\infty(\Omega)}$.

iii) There exists a quasiconformal extension $\hat{\xi}$ of ξ on all of $\mathbb{P}^1$ with $\hat{\xi} = 0$ on $\partial\Omega$.

iv) The extension $\hat{\xi}$ defined by $\hat{\xi}(z) = \xi(z)$ if $z \in \Omega$ and 0 else is quasiconformal on $\mathbb{P}^1$, and $\overline{\partial}\hat{\xi}(z) = 0$ almost everywhere in the complement of Ω.

Corollary 4. Let Ω be a hyperbolic open subset of $\mathbb{P}^1$ and ξ be a quasiconformal vector field vanishing on $\mathbb{P}^1 - \Omega$. Let $\Omega = \bigsqcup_i \Omega_i$ a countable partition of Ω into open sets Ω_i. Then

$$\xi = \sum_i \xi_i$$

where ξ_i is a quasiconformal vector field coinciding with ξ on Ω_i and vanishing outside Ω_i.

Proof. By item *iv)* of Theorem A, the vector fields ξ_i are quasiconformal. □

Recall the following notion, which is of importance in Teichmüller theory :

Definition 3.14. A Beltrami differential μ on a Riemann surface $\mathcal{S}$ is infinitesimally trivial if $\int_{\mathcal{S}} q \cdot \mu = 0$ for all quadratic differential q holomorphic on $\mathcal{S}$.

The terminology comes from the fact that the tangent space to the base point $T_0\mathrm{Teich}(\mathcal{S})$ identifies canonically to the quotient of the space of Beltrami differentials on $\mathcal{S}$ by the space of infinitesimally trivial Beltrami differentials (see [GL00] or [Hub06]).

The next result is a theorem due to Bers and Lakic. Its proof classically involves a delicate mollifier introduced by Ahlfors, the so-called Ahlfors Mollifier, see [GL00], theorem 9 p. 63. The mollifier ϕ_n of the proof of theorem 3.10 replaces the Ahlfors Mollifier and yields a simplified proof.

We now prove Corollaries 1 and 2.

Proof of Corollary 1. It is enough to show that any continuous linear form on the space of integrable quadratic differentials holomorphic outside $\mathbb{P}^1$ vanishing against every meromorphic quadratic differentials with only simple poles in A must be trivial. By the Hahn-Banach theorem, any such linear form may be represented by a L^∞ Beltrami differential on $\mathbb{P}^1$. Let μ be such a Beltrami differential and ξ a quasiconformal vector field such that $\mu = \overline{\partial}\xi$, and assume that

$$\int_{\mathbb{P}^1} q \cdot \overline{\partial}\xi = 0$$

for all meromorphic integrable quadratic differential q with simple poles in A. Let $Z \subset A$ a set of cardinal 3 : by adding to ξ a holomorphic vector field, we lose no generality by assuming that ξ vanishes on Z. Then by proposition 2 applied to $\mathbb{P}^1$ and q a quadratic differential with simple poles precisely in Z and at $z \in A\backslash Z$, one sees that ξ must vanish at z. By continuity, ξ vanishes on all of K. So by Theorem A, ξ is hyperbolically bounded on $\Omega = \mathbb{P}^1 - K$. Let q be an integrable quadratic differential that is holomorphic on Ω. In particular, q is C^∞ and integrable on Ω, and $\overline{\partial}q$ vanishes on Ω. Lemma 3.11 yields :

$$\int_\Omega q \cdot \overline{\partial}\xi = -\int_\Omega \overline{\partial}q \cdot \xi = 0.$$

Moreover, by Theorem A, we have $\overline{\partial}\xi = 0$ almost everywhere on K, so :

$$\int_K q \cdot \overline{\partial}\xi = 0,$$

which ends the proof. □

Proof of Corollary 2. Let μ be a Beltrami differential on $\mathcal{S}$ that is infinitesimally trivial. Let $\pi : \Delta \to \mathcal{S}$ a universal cover. Let $q \in Q(\Delta)$ be any holomorphic integrable quadratic differential. Notice that :

$$\int_\Delta q \cdot \pi^*\mu = \int_{\mathcal{S}} \pi_* q \cdot \mu = 0.$$

Therefore $\pi^*\mu \in \mathrm{Bel}(\Delta)$ is infinitesimally trivial. We have proved in the proof of Corollary 1 that this implies that π^μ is of the form $\overline{\partial}\xi$, where ξ is a quasiconformal vector field on $\mathbb{P}^1$ vanishing outside of Δ. Therefore by Theorem A, ξ is hyperbolically bounded on Δ, and $\|\rho_\Delta(\xi)\|_{L^\infty(\Delta)} \leq 4\|\overline{\partial}\xi\|_{L^\infty(\Delta)}$.

Now it remains to prove that ξ descends to vector field on $\mathcal{S}$. Let γ be an element of the group of deck transformations of π. We need to prove that ξ is γ-invariant. Since γ is an isometry for the hyperbolic metric, $\xi - \gamma^*\xi$ is hyperbolically bounded. Moreover, it is holomorphic since $\overline{\partial}\xi - \gamma^*\overline{\partial}\xi = 0$ on Δ. But the only holomorphic and hyperbolically bounded vector field on Δ is the vector field that is identically zero. Therefore ξ descends to a vector field $\hat{\xi}$ on $\mathcal{S}$, such that $\overline{\partial}\hat{\xi} = \mu$ and $\|\rho_{\mathcal{S}}(\hat{\xi})\|_{L^\infty(\mathcal{S})} \leq 4\|\overline{\partial}\hat{\xi}\|_{L^\infty(\mathcal{S})}$.

Conversely, suppose that $\mu = \overline{\partial}\hat{\xi}$ where $\hat{\xi}$ is a quasiconformal vector field on $\mathcal{S}$ such that $\|\rho_{\mathcal{S}}(\hat{\xi})\|_{L^\infty(\mathcal{S})} \leq 4\|\overline{\partial}\hat{\xi}\|_{L^\infty(\mathcal{S})}$. Let $q \in Q(\mathcal{S})$. According to Theorem 4 p. 50 in [GL00], the pushforward operator $\pi_* : Q(\Delta) \to Q(\mathcal{S})$ is surjective, so we may assume that $q = \pi_*\phi$, where $\phi \in Q(\Delta)$. Since $\pi^*\mu$ is of the form $\pi^*\mu = \overline{\partial}\pi^*\hat{\xi}$, where $\pi^*\hat{\xi}$ is a hyperbolically bounded quasiconformal vector field on Δ, we have that $\int_\Delta \phi \cdot \pi^*\mu = 0$. Therefore:

$$\int_{\mathcal{S}} q \cdot \mu = \int_\Delta \phi \cdot \pi^*\mu = 0$$

which proves that μ is infinitesimally trivial on $\mathcal{S}$. □

4. Dynamical Teichmüller space

4.1. **The differential of Ψ.** If $\lambda \mapsto f_\lambda$ is a holomorphic curve in Rat_d passing through $f_0 = f$, then $\dot{f} = \frac{df_\lambda}{d\lambda}_{|\lambda=0}$ is a section of the bundle $f^*T\mathbb{P}^1$: for all $z \in \mathbb{P}^1$, $\dot{f}(z) \in T_z\mathbb{P}^1$. Moreover, $Df^{-1} \circ \dot{f}$ is a meromorphic vector field on $\mathbb{P}^1$, whose poles are included in $\mathrm{Crit}(f)$ and of multiplicity at most that of the critical points of f. Denoting by $T(f)$ the complex vector space of such vector fields, we obtain a canonical identification between $T_f\mathrm{Rat}_d$ and $T(f)$. In the rest of this article, we will implicitly identify $T_f\mathrm{Rat}_d$ with $T(f)$.

Denote as well by $\mathrm{aut}(\mathbb{P}^1)$ the space of holomorphic vector fields on $\mathbb{P}^1$ and by $\mathcal{O}(f)$ the orbit of f by conjuacy via Möbius transformation. By [BE09], proposition 1, $\mathcal{O}(f)$ is a complex submanifold of Rat_d of dimension 3, and $T_f\mathcal{O}(f) = \Delta_f \mathrm{aut}(\mathbb{P}^1) \subset T(f)$.

Proposition 4. Let ξ be a quasiconformal vector field on $\mathbb{P}^1$ such that $\overline{\partial}\xi \in \mathrm{bel}(f)$. Then $\Delta_f\xi \in T(f)$. Moreover, if we assume that ξ vanishes on a set Z of cardinal 3, then :

$$D\Psi^Z(0) \cdot \overline{\partial}\xi = -\Delta_f\xi.$$

Proof. An easy calculation shows that almost everywhere in the complement of $\mathrm{Crit}(f)$, $\overline{\partial} f^*\xi = f^*\overline{\partial}\xi$. Therefore by Weyl's lemma, $\Delta_f\xi = \xi - f^*\xi$ is holomorphic on $\mathbb{P}^1 - \mathrm{Crit}(f)$. Since ξ is continuous, we have $\|\Delta_f\xi\| = O(\|Df\|^{-1})$ in the neighborhood of $\mathrm{Crit}(f)$ (in the spherical metric), so $\Delta_f\xi$ has at every critical point c of f a pole of at most the multiplicity of c cas a critical point of f ; so $\Delta_f\xi \in T(f)$.

Moreover, if $\mu_\lambda \in \mathrm{bel}(f)$ is a holomorphic curve passing through 0, with $\mu_\lambda = \lambda\overline{\partial}\xi + o(\lambda)$, then we have :

$$\phi^Z_{\mu_\lambda} = \mathrm{Id} + \lambda\xi + o(\lambda)$$

where $\phi^Z_{\mu_\lambda}$ is the unique quasiconformal homeomorphism associated to μ_λ fixing Z (see [GL00] or [Hub06]). If we differentiate with respect to λ the equality

$$\phi^Z_{\mu_\lambda} \circ f = f_\lambda \circ \phi^Z_{\mu_\lambda},$$

we get :

$$\xi \circ f = \dot{f} + Df(\xi),$$

où $\dot{f} = \frac{df_\lambda}{d\lambda}_{|\lambda=0}$. This can be rewritten as :

$$\eta := Df^{-1}(\dot{f}) = -\Delta_f\xi.$$

□

With an abuse of notations, we will note $D\Psi(0) : \mathrm{bel}(f) \to T(f)/T_f\mathcal{O}(f)$ the quotient of the linear map $D\Psi^Z(0) : \mathrm{bel}(f) \to T(f)$. This application does not depend on the choice of Z.

Definition 4.1. Let f be a rational map. We will note Λ_f the closure of the grand critical orbit of f, and $\Omega_f = \mathbb{P}^1 - \Lambda_f$.

Proposition 5. Let ξ be a quasiconformal vector field on $\mathbb{P}^1$ such that $\overline{\partial}\xi \in \mathrm{bel}(f)$. The following properties are equivalent :

i) $\overline{\partial}\xi \in \ker D\Psi(0)$
ii) $\Delta_f\xi \in \Delta_f \mathrm{aut}(\mathbb{P}^1)$
iii) There exists $h \in \mathrm{aut}(\mathbb{P}^1)$ such that $\xi - h$ vanishes on $\mathrm{Crit}(f)$ with at least the multiplicity of each critical point of f (in the sense that $\|(\xi - h) \circ f(c)\| = O(\|Df(c)\|)$, where $c \in \mathrm{Crit}(f)$)
iv) There exists $h \in \mathrm{aut}(\mathbb{P}^1)$ such that $\xi - h$ vanishes on Λ_f

Proof. The first two items are equivalent by [BE09], proposition 1.

ii) $\Rightarrow$ *iii)* : if $\Delta_f \xi = \Delta_f h$, $h \in \mathrm{aut}(\mathbb{P}^1)$, then $\xi - h$ is a continuous f-invariant vector field. Hence $\xi - h$ must vanish on $\mathrm{Crit}(f)$ with at least the multiplicity of the critical points of f, since $(\xi - h) \circ f = Df(\xi - h)$.

iii) $\Rightarrow$ *ii)* If $\xi - h$ vanishes on $\mathrm{Crit}(f)$ with at least the multiplicity of the critical points of f, then $f^*(\xi - h)$ is well-defined and continuous at $\mathrm{Crit}(f)$. By the above proposition, $\Delta_f(\xi - h) \in T(f)$, so $\Delta_f(\xi - h) = 0$.

iv) $\Rightarrow$ *ii)* : If $\xi - h$ vanishes on Λ_f, then since Λ_f is invariant $\Delta_f(\xi - h)$ vanishes as well on Λ_f. Therefore $\Delta_f(\xi - h)$ is a meromorphic vector field (by the above proposition) vanishing on Λ_f which is not discrete, so $\Delta_f(\xi - h) = 0$ by the isolated zeros principle.

ii) $\Rightarrow$ *iv)* : If $\Delta_f(\xi - h) = 0$, then we saw that $\xi - h$ must vanish on $\mathrm{Crit}(f)$ (item *iii)*). Therefore $(f^k)^*(\xi - h)(c) = (\xi - h)(c) = 0$ for every $k \geq 0$. Moreover, if $f^p(z) = c \in \mathrm{Crit}(f)$, then $(f^p)^*(\xi - h)(z) = 0 = (\xi - h)(z)$. So $(\xi - h)$ vanishes on the grand critical orbit of f, hence on Λ_f by continuity. □

Note that if we normalize ξ by imposing the condition that it vanishes on on a set Z invariant by f of cardinal 3, then proposition 5 remains true by replacing h by 0 in items *ii)*, *iii)* and *iv)*, and $D\Psi(0)$ by $D\Psi^Z(0)$ in item *i)*.

We will also need to know the differential Ψ^Z in an arbitrary point of $\mathrm{Bel}(f)$. Recall the following fact of Teichmüller theory (see [Hub06]) :

Definition 4.2. Let ψ be a quasiconformal homeomorphism of $\mathbb{P}^1$. For any Beltrami form μ, denote by $\psi^*\mu$ the Beltrami form corresponding to $\phi_\mu \circ \psi$, where ϕ_μ is a quasiconformal homeomorphism associated to μ.

We will also note $\psi_* = (\psi^{-1})^*$.

Proposition 6. For every quasiconformal homeomorphism ψ, the map $\psi^* : \mathrm{Bel}(\mathbb{P}^1) \to \mathrm{Bel}(\mathbb{P}^1)$ is biholomorphic.

We shall need to consider here maps $\Psi_f^Z : \mathrm{Bel}(f) \to \mathrm{Rat}_d$ and $\Psi_g^Z : \mathrm{Bel}(g) \to \mathrm{Rat}_d$ associated to different rational maps f and g. In the rest of the article, there will be no ambiguity and and we will just use the notation Ψ^Z.

Proposition 7. Let $\mu \in \mathrm{Bel}(f)$ and ψ the unique corresponding quasiconformal homeomorphism fixing Z. Let $g = \phi \circ f \circ \phi^{-1}$. Then

$$D\Psi_f^Z(\mu) = D\Psi_g^Z(0) \circ D\phi_*(\mu)$$

In particular, $\mathrm{rank} D\Psi_f^Z(\mu) = \mathrm{rank} D\Psi_g^Z(0)$.

Proof. Let $\mu_\lambda = \mu + O(\lambda)$ be a holomorphic curve in $\mathrm{Bel}(f)$. Let ϕ_λ be the corresponding solution to the Beltrami equation fixing Z pointwise.

Then remark that :

$$\phi_\lambda \circ f \circ \phi_\lambda^{-1} = (\phi_\lambda \circ \phi_0^{-1}) \circ \phi_0 \circ f \circ \phi_0^{-1} \circ (\phi_\lambda \circ \phi_0^{-1})^{-1}$$

which may be rewritten as :

$$\Psi_f^Z(\mu_\lambda) = \Psi_g^Z(\phi_* \mu_\lambda).$$

The result follows by differentiating that last equality. □

4.2. **Constant rank theorem in Banach spaces.** Recall the following version of the constant rank theorem (for a reference, see [Bou07], p. 53).

Theorem 4.3 (Constant rank theorem)**.** *Let $\Psi : U \to F$ be an analytic map, where U is an open subset of a complex Banach space E and F is a complex finite-dimensional manifold. Assume that* $\mathrm{rank} D\Psi = r$ *is constant on U. Then for every $x_0 \in U$, there exists a germ of analytic diffeomorphism $\chi : (F, \Psi(x_0)) \to (T_{\Psi(x_0)}F, 0)$ and a germ of analytic diffeomorphism $\phi : \mathrm{Im} D\Psi(x_0) \oplus \ker D\Psi(x_0) \to E$ such that for all $(u, v) \in \mathrm{Im} D\Psi(x_0) \oplus \ker D\Psi(x_0)$ in the neighborhood of $\phi^{-1}(x_0)$,*

$$\chi \circ \Psi \circ \phi(u, v) = u.$$

Corollary 5. Let $\Psi : U \to F$ verifying the requirements of the above theorem. Then for all $z_0 \in \Psi(U)$, the level set $M = \Psi^{-1}(z_0)$ is a Banach submanifold of E, of codimension r and whose tangent space at $x_0 \in \Psi^{-1}(z_0)$ is $T_{x_0}M = \ker D\Psi(x_0)$.

Proof. With the notations of the constant rank theorem, we have $\Psi(x) = z_0$ if and only if $\chi \circ \Psi(u, v) = \chi(z_0) = u$, where $(u, v) = \psi^{-1}(x)$, which is equivalent to $\psi(\chi(z_0), v)) = x$. Since ψ is a (germ of) diffeomorphism, this gives a local chart at x_0 for M, which is therefore a Banach submanifold modeled on $\ker D\Psi(x_0)$. □

4.3. **Counting dimensions.** The goal of this section is to show that the differential of $\Psi^Z : \mathrm{Bel}(f) \to \mathrm{Rat}_d$ has constant rank.

Definition 4.4. We say that a critical point is acyclic if it is not preperiodic. We say that two acyclic critical points lie in the same foliated acyclic critical class if the closure of their grand orbits are the same.

The key point to apply the constant rank theorem is the following count of dimension :

Theorem 4.5. *Let f be a rational map of degree $d \geq 2$. Then*

$$\mathrm{rank} D\Psi(0) = n_f + n_H + n_J - n_p$$

where n_H is the number of Herman rings of f, n_J is the number of ergodic line fields of f, n_f is the number of foliated acyclic critical classes lying in the Fatou set, and n_p is the number of parabolic cycles.

Definition 4.6. Let $\mathcal{S}$ be a hyperbolic complex 1-manifold. Denote by $M(\mathcal{S})$ the set of Beltrami differentials on the Riemann surface $\mathcal{S}$ and by $N(\mathcal{S})$ the subspace of Beltrami differentials on $\mathcal{S}$ that are of the form $\overline{\partial}\xi$, where ξ is a hyperbolically bounded quasiconformal vector field on $\mathcal{S}$.

Definition 4.7. Let $\mathcal{S}$ be a hyperbolic complex 1-manifold and $f : \mathcal{S} \to \mathcal{S}$ be a holomorphic function. Denote by $M_f(\mathcal{S})$ the space of Beltrami forms that are invariant by f, and by $N_f(\mathcal{S})$ the subspace of $M_f(\mathcal{S})$ of Beltrami differentials of the form $\overline{\partial}\xi$, where ξ is a hyperbolically bounded quasiconformal vector field on $\mathcal{S}$.

Definition 4.8. Let $(\mathcal{S}_i)_{i \in I}$ be a collection (at most countable) of complex 1-manifold $\mathcal{S}_i$. The norm restricted direct product $\Pi^*_{i \in I} M(\mathcal{S}_i)$ is defined as :

$$\{(\mu_i)_{i \in I}, \sup_{i \in I} \|\mu_i\|_{L^\infty(\mathcal{S}_i)} < 1\}$$

Theorem 4.9. *Let f be a rational map, and Ω a hyperbolic open subset of $\mathbb{P}^1$ completely invariant under f. Let $\Omega = \bigsqcup_i \Omega_i$ be a partition of Ω into open subsets Ω_i completely invariant under f. Then*

$$M_f(\Omega)/N_f(\Omega) \simeq \Pi^*_i M(\Omega_i)/N(\Omega_i)$$

Proof. Clearly $M_f(\Omega) = \Pi_i^* M(\Omega_i)$.

Let $\overline{\partial}\xi \in N_f(\Omega)$. By corollary 4, we have :

$$\xi = \sum_i \xi_i$$

where ξ_i is a quasiconformal vector field coinciding with ξ on Ω_i, and such that $\xi_i = 0$ outside of Ω_i. This shows that $N_f(\Omega) = \Pi_i^* N_f(\Omega_i)$.

Hence $M_f(\Omega)/N_f(\Omega) = \Pi_i^* M_f(\Omega_i)/N_f(\Omega_i)$. □

Lastly, we will need the classification of Fatou components, which is a corollary of Sullivan's no wandering domain Theorem. Note that McMullen (see [McM14]) has given a direct and purely infinitesimal proof of Sullivan's theorem, which does notably not rely on the theory of dynamical Teichmüller spaces. His proof is based on quasiconformal vector fields and is in the same spirit as the methods used here.

For the reader's convenience, we include here a version of McMullen's argument, made shorter by the use of Theorem A.

Theorem 4.10 (No Wandering Domain, Sullivan)**.** *Let f be a rational map of degree $d \geq 2$. Then f has no wandering domain.*

As in other proofs of Sullivan's theorem, the argument is greatly simplified by the following lemma due to Baker :

Lemma 4.11 (Baker's lemma)**.** *Suppose $(\Omega_n)_{n\in\mathbb{N}}$ is a wandering domain. Then for n large enough, Ω_n is simply connected and $f_{|\Omega_n}$ is univalent.*

Proof of Theorem 4.10. Assume that f has a wandering domain. According to Baker's lemma, there is a Fatou component U belonging to that wandering domain that is simply connected.

We can construct invariant Beltrami differentials on $\mathbb{P}^1$ by chosing arbitrary Beltrami differentials μ on U and then pulling them back by f and branches of f^{-1} along the wandering domain, and setting it to zero outside (we can do this because according to Baker's lemma, $f_{|f^n(U)}$ is univalent for all $n \in \mathbb{N}^*$). Thus specifying a Beltrami differential on U specifies uniquely an invariant Beltrami differential on $\mathbb{P}^1$. Consider the vector space $V \subset \mathrm{bel}(f)$ of invariant Beltrami differentials obtained by this method through the use of Beltrami differentials on U of the form :

$$\mu_{|U} = \phi^* \left(\sum_{k=1}^{n} a_k k \overline{z}^{k-1} \frac{\overline{dz}}{dz} \right)$$

where the a_k are complex numbers and $\phi : U \to \Delta$ is a uniformizing coordinate.

Lemma 4.12. *Let $\mu \in V$ and let ξ be a quasiconformal vector field on $\mathbb{P}^1$ such that $\mu = \overline{\partial}\xi$. If ξ is hyperbolically bounded on U, then $\mu = 0$.*

Proof. It is enough to prove this for $\tilde{\xi} = \phi_*\xi$, since the property of being hyperbolically bounded is invariant under pullback by the biholomorphism ϕ. We have that

$$\tilde{\xi}(z) = \left(\sum_{k=0}^{n} a_k \overline{z}^k - h(z) \right) \frac{d}{dz}$$

where h is holomorphic on Δ.

If $\tilde{\xi}$ is hyperbolically bounded, then it extends continuously by 0 to the unit circle S^1. This means that h extends continuously to the unit circle by the formula :

$$\hat{h}(z) = \sum_{k=0}^{n} a_k \overline{z}^k = \sum_{k=0}^{n} \frac{a_k}{z^k}$$

and therefore that

$$\hat{h}(z) = \begin{cases} h(z) & \text{if } z \in \Delta \\ \sum_{k=0}^{n} \frac{a_k}{z^k} & \text{else} \end{cases}$$

is holomorphic on all of $\mathbb{P}^1$. Since $\hat{h}$ coincides with $z \mapsto \sum_{k=0}^{n} \frac{a_k}{z^k}$ outside of Δ, this implies that all the a_k must be zero, and therefore that $\mu = 0$ on U. Then by the definition of μ, we have that $\mu = 0$ on $\mathbb{P}^1$. □

In view of Theorem A, we have proved that if any Beltrami differential $\mu \in V$ is infinitesimally trivial on U, then $\mu = 0$. Moreover, according to proposition 5, if $\mu \in \ker D\Upsilon^Z(0)$, then μ is infinitesimally trivial on U. So we have constructed an infinite-dimensional subspace $V \subset \mathrm{bel}(f)$ with $V \cap \ker D\Upsilon^Z(0) = \{0\}$. This contradicts the finite-dimensionality of the rank of $D\Upsilon^Z(0)$. □

Note that this proof is purely functional analytic in nature and does not actually require the information that the linear map $\mathrm{bel}(f) \ni \overline{\partial}\xi \mapsto f^*\xi - \xi \in T(f)$ is in fact the differential $D\Upsilon^Z(0)$, only the description of its kernel and the fact that it has finite rank. In particular, we have not used the Ahlfors-Bers Theorem.

In addition to Sullivan's No Wandering Domain Theorem, we will need a few lemmas.

Lemma 4.13. *Suppose Ω is a hyperbolic open subset of $\mathbb{P}^1$ and $f : \Omega \to \Omega$ is a holomorphic self-cover of Ω, such that Ω/f is a (connected) hyperbolic Riemann surface. Then the projection $\pi_1 : \Omega \to \Omega/f$ induces an identification :*

$$M_f(\Omega)/N_f(\Omega) \simeq M(\Omega/f)/N(\Omega/f)$$

Proof. It is clear that $M_f(\Omega) \simeq M(\Omega/f)$. Let us prove that any element of $N_f(\Omega)$ passes to the quotient to an element of $N(\Omega/f)$: let $\mu = \overline{\partial}\xi \in N_f(\Omega)$, where ξ is hyperbolically bounded. We have to prove that ξ is f-invariant so that it descends to Ω/f. Since μ is f-invariant and $f : \Omega \to \Omega$ is a covering map, $\xi - f^*\xi$ is a holomorphic vector field. It is also hyperbolically bounded since f is an isometry for the hyperbolic metric of Ω. By lifting to the universal cover Δ of Ω, it is easy to see that such a vector field must vanish everywhere. Therefore ξ is f-invariant and μ descends to an element of $N(\Omega/f)$.

Let $\mu = \overline{\partial}\xi \in N(\Omega/f)$. Since the map $\pi_1 : \Omega \to \Omega/f$ is a covering map between hyperbolic Riemann surfaces, it is a local isometry for the hyperbolic metrics, and therefore $\xi_1 = \pi_1^*\xi$ is a hyperbolically bounded quasiconformal vector field, which is invariant by f by construction. By Theorem A, $\hat{\xi_1}$ extended by 0 outside Ω is still quasiconformal (and invariant), and $\mu = \overline{\partial}\hat{\xi_1} \in N_f(\Omega)$. This proves that $N_f(\Omega) \simeq N(\Omega/f)$. □

Lemma 4.14. *Let U be a periodic component of Ω_f, of period $p \in \mathbb{N}^*$. Let Ω be the completely invariant open subset of Ω_f it generates. Then the restriction to U induces an isomorphism $M_f(\Omega) \to M_{f^p}(U)$, mapping $N_f(\Omega)$ onto $N_{f^p}(U)$. In particular,*

$$M_f(\Omega)/N_f(\Omega) \simeq M_{f^p}(U)/N_{f^p}(U)$$

Proof. Every Beltrami differential $\mu \in M_f(\Omega)$ is invariant under f, hence under f^p. Conversely, if μ is a Beltrami differential on U invariant under f^p, then μ extends uniquely to a Beltrami differential $\tilde{\mu}$ invariant on Ω_i in the following way : if V is a component of

Ω_i, then there exists $k \in \mathbb{N}$ (defined up to a multiple of p) such that $f^k_{|V} : V \to U$. We then set $\tilde{\mu}_{|V} = (f^k)^*\mu$, and this definition is valid if V belongs to the same cycle as U since $\mu = (f^p)^*\mu$.

This identification maps $N_f(\Omega)$ onto $N_{f^p}(U)$ since if $\mu = \overline{\partial}\xi \in N_f(\Omega)$, then $\hat{\xi}(z) = \xi(z)$ if $z \in U$ and 0 else is such that $\overline{\partial}\hat{\xi} = \mu_{|U}$ by Theorem A, and therefore $\mu_{|U} \in N_{f^p}(U)$. Conversely, let us prove that the restriction map $N_f(\Omega) \to N_{f^p}(U)$ is surjective: let $\mu \in N_{f^p}(U)$. Since we proved that $M_f(\Omega) \to M_{f^p}(U)$ is surjective, there exists $\tilde{\mu} \in M_f(\Omega)$ which is an extension of μ to Ω, and $\tilde{\mu}$ restricted to each component V of Ω is obtained as a pullback of the form $(f^{-m} \circ f^n)^*\mu$. Moreover, by definition of Ω_f, the branches $(f^{-m} \circ f^n)^*\mu$ are covering maps between components of Ω, and thus are local isometries for the hyperbolic metric of Ω. Since $\mu \in N_{f^p}(U)$, there is a quasiconformal vector field ξ on U such that $\overline{\partial}\xi = \mu$ and ξ is hyperbolically bounded on U. Therefore on any component of Ω, $\tilde{\mu} = \overline{\partial}((f^{-m} \circ f^n)^*\xi)$, and $(f^{-m} \circ f^n)^*\xi$ is hyperbolically bounded on that component. Therefore $\tilde{\mu} \in N_f(\Omega)$. □

Lemma 4.15. *Let μ be a Beltrami differential invariant under a holomorphic function g. In both of the following cases : $g(z) = e^{2i\pi\alpha}z$, $\alpha \notin \mathbb{Q}$, and $g(z) = z^d$, $d \geq 2$, μ is then invariant under all rotations, and we have in local coordinates :*

$$\mu(re^{it}) = c(r)e^{2it}\frac{\overline{dz}}{dz}$$

Proof. The proof is a modification of the usual proof of the ergodicity of rotations of irrational angles.

Let us start with the case of a rotation of irrational angle $g(z) = e^{2i\pi\alpha}z$. Let μ be a Beltrami differential invariant by g. We have, in local coordinates :

$$\mu(z) = g^*\mu(z) = e^{-4i\pi\alpha}\mu(e^{2i\pi\alpha}z) \tag{4.1}$$

By expanding into Fourier series on the circles $|z| = r$, we will see that μ must be of the form

$$\mu(re^{it}) = c(r)e^{2it}\frac{\overline{dz}}{dz}$$

where c is a L^∞ function.

Indeed, if we denote by $c_n(r)$ the n^{th} Fourier coefficient of $t \mapsto \mu(re^{it})$, we deduce from 4.1 that for all $n \in \mathbb{Z}$:

$$c_n(r)e^{4i\pi\alpha} = c_n(r)e^{2in\pi\alpha}$$

Since $\alpha \notin \mathbb{Q}$, this implies that $c_n(r) = 0$ for all $n \neq 2$ and all $r \in (0, 1)$. Therefore if we let $c(r) := c_2(r)$, we get

$$\mu(re^{it}) = c(r)e^{2it}\frac{\overline{dz}}{dz} \tag{4.2}$$

In particular, μ is invariant by rotations, and one easily verifies that all rotation-invariant Beltrami differential must be of this form.

If we now assume that $g(z) = z^2$, $d \geq 2$, and that μ is invariant by g, then μ is invariant by all branches of $g^{-n} \circ g^n$, hence by all rotations of angles $\frac{2k\pi}{d^n}$, for all $k \in \mathbb{N}$:

$$\mu(z) = e^{-4ik\pi/d^n}\mu(e^{2ik\pi/d^n}z).$$

Similarly, by expanding into Fourier series on the circles centered on 0, we obtain :

$$\mu(re^{it}) = c(r)e^{2it}\frac{\overline{dz}}{dz}.$$

□

Lemma 4.16. *Let Ω be a rotation invariant planar open set. Let $M(\Omega)$ be the space of rotation-invariant Beltrami differentials on Ω, and $N(\Omega)$ the subspace of $M(\Omega)$ of elements of the form $\overline{\partial}\xi$, where ξ is a hyperbolically bounded quasiconformal vector field on Ω.*

i) If Ω is the unit disk, $\dim M(\Omega)/N(\Omega) = 0$.
ii) If Ω is a ring of finite modulus, then $\dim M(\Omega)/N(\Omega) = 1$.

Proof. Consider a vector field ξ of the form

$$\xi(re^{it}) = h(r)re^{it}\frac{d}{dz}$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a lipschitz function. One can easily verify that

$$\overline{\partial}\xi(re^{it}) = rh'(r)e^{2it}\frac{\overline{dz}}{dz}. \tag{4.3}$$

Let μ be a rotation-invariant Beltrami differential, hence of the form $\mu(re^{it}) = c(r)e^{2it}\frac{\overline{dz}}{dz}$. Let h be the function defined by:

$$h(r) = \int_1^r \frac{c(u)}{u}du.$$

and let ξ be the continuous vector field defined by

$$\xi(re^{it}) = rh(r)e^{it}\frac{d}{dz}.$$

We have $\overline{\partial}\xi = \mu$ in the sense of distributions, according to (4.3), and ξ is a quasiconformal vector field on all of $\mathbb{P}^1$ vanishing on the unit disk.

Therefore $M(\Delta) = N(\Delta)$. If now Ω denotes a straight ring $\Omega = \{r_0 < |z| < 1\}$, the map

$$\mu = c(r)e^{2it}\frac{\overline{dz}}{dz} \mapsto h(r_0) = \int_1^{r_0} \frac{c(u)}{u}du$$

is a linear form on $M(\Omega)$ whose kernel is exactly $N(\Omega)$. This linear form is not trivial, since if we take $\mu = re^{2it}\frac{\overline{dz}}{dz}$, then $h(r_0) = r_0 - 1 \neq 0$. Therefore $\dim M(\Omega)/N(\Omega) = 1$. □

We can now prove Theorem 4.5.

Proof of Theorem 4.5. Denote by $\mathcal{F}$ the Fatou set of f, and $\mathcal{J}$ its Julia set. We will also denote by $\mathrm{Fix}_{\mathcal{J}}$ the space of invariant line fields. Since $\ker D\Psi(0) = N_f(\Omega_f) = N_f(\mathcal{F})$ by proposition 5, we have :

$$\mathrm{bel}(f)/\ker D\Psi(0) = (\mathrm{Fix}_{\mathcal{J}} \oplus M_f(\mathcal{F}))/N_f(\mathcal{F})$$

If c is a critical point of f, then the closure of its grand orbit is equal to the union of the Julia set $\mathcal{J}$ and of a countable set of points and smooth circles (if the orbit of c is captured by a superattracting cycle, or a cycle of Siegel disks or Herman rings). Therefore Λ_f coincides with $\mathcal{J}$ up to a set of Lebesgues measure zero. Hence $M_f(\mathcal{F})$ canonically identifies with $M_f(\Omega_f)$ through the obvious restriction map. We deduce from this observation that :

$$\mathrm{Fix}(f)/\ker D\Psi(0) = \mathrm{Fix}_{\mathcal{J}} \oplus M_f(\Omega_f)/N_f(\Omega_f)$$

Consider the equivalence relationship on the set of connected components of Ω_f which identifies two components if and only if they have the same grand orbit, and let Ω_i be the union of the elements of a class i of this equivalence relationship. The Ω_i form a partition of Ω_f into completely invariant open subsets. By Theorem 4.9, we have :

$$\text{rank}D\Psi(0) = \dim \text{Fix}_J + \sum_i \dim M_f(\Omega_i)/N_f(\Omega_i).$$

Each component Ω_i is mapped by f^n for n large enough into a periodic Fatou component U. [1] Let us now compute $\dim M_f(\Omega_i)/N_f(\Omega_i)$ depending on the nature of the periodic Fatou component U it meets. There are five cases to consider. Denote by n_i the number of foliated acyclic critical classes meeting the grand orbit of U

a) The case of an attracting cycle

If U is a component of an attractive basin and Ω_i meets U, then Ω_i is the grand orbit of U with the countable set of the critical orbits captured by this cycle (and the cycle itslef) removed. So every component of Ω_i is preperiodic to $U - \Lambda_f$. Thus $f_{|\Omega_i} : \Omega_i \to \Omega_i$ acts discretely, and $X_i = \Omega_i/f$ is a Riemann surface. In a linearizing coordinate for f^k on the immediate basin of attraction (where $k \in \mathbb{N}^*$ is the period of the cycle and ρ is its multiplier), note $A = \{|\rho| \leq z < 1\}$. It is a fundamental domain for the action of f on the cycle of Fatou components V containing U, and $A - \Lambda_f$ is a fundamental domain for the action of f on Ω_i. Therefore X_i is the torus $X = A/f$ with a finite number n_i of points removed, where n_i is the number of points of the post-critical set meeting A, i.e. the number of foliated acyclic critical classes meeting V.

By lemma 4.13, $\dim M_f(\Omega_i)/N_f(\Omega_i) = \dim M(X_i)/N(X_i)$. Since X_i is a finitely punctured torus, any hyperbolically bounded quasiconformal vector field on X_i extends to a quasiconformal vector field on the torus vanishing on the marked points. Then the quotient $M(X_i)/N(X_i)$ is exactly the tangent space to the Teichmüller space of X_i, which has dimension equal to the number n_i of marked points (see for example [Hub06]).

b) The case of a parabolic cycle

If U is a parabolic cycle and Ω_i meets U, then Ω_i is the grand orbit of U minus the grand orbit of the critical points captured by U. In particular, every component of Ω_i is mapped after after finitely many steps into U, and thus is preperiodic. Moreover, $f_{|\Omega_i} : \Omega_i \to \Omega_i$ acts discretely, so $X_i = \Omega_i/f$ is a Riemann surface isomorphic to $X = U/f^p$ minus the grand orbit of critical points captured by U, where p is the period of the parabolic cycle associated to U.

Via a Fatou coordinate, the action of f^p on U is conjugated to that of $z \mapsto z+1$ on an upper half-plane, so X is isomorphic to a cylinder and X_i is isomorphic to a cylinder with n_i points removed, those points corresponding to the n_i grand critical orbits captured by U. So X is isomorphic to the Riemann sphere with two points a_1 and a_2 removed, and X_i is isomorphic to the Riemann sphere with $n_i + 2$ points $a_1, \dots, a_{n_i+2}$ removed, where the a_j, $j \geq 2$ correspond to the grand critical orbit meeting U.

By lemma 4.13, $\dim M_f(\Omega_i)/N_f(\Omega_i) = \dim M(X_i)/N(X_i)$. Since X_i is a finitely punctured sphere, any hyperbolically bounded quasiconformal vector field on X_i extends to a quasiconformal vector field on the sphere vanishing on the marked points. Then the quotient $M(X_i)/N(X_i)$ is exactly the tangent space to the Teichmüller space of X_i, which has dimension equal to the number $n_i + 2 - 3 = n_i - 1$, where n_i is the number of critical grand orbits meeting Ω_i (see for example [Hub06]).

[1] However, in the case of a superattracting cycle, the components Ω_i need not be themselves preperiodic : if there is a critical orbit in a superattracting basin, one gets components Ω_i which are annuli delimited by equipotentials that accumulate on the superattracting cycle.

c) The case of a Siegel disk

If U is a Siegel disk, then the intersection of Λ_f and the cycle of Fatou components containing U consists in a finite union of n_i smooth circles, where n_i is the number of foliated acyclic critical classes captured by the cycle of Siegel disks (it may be that $n_i = 0$). Therefore all components of Ω_i are preperiodic and are mapped in finitely many steps to a periodic ring A_i included in U or a topological disk strictly included in U (if $n_i \neq 0$), or in all of the periodic Siegel disk if $n_i = 0$. In both cases, denote by V the periodic component of Ω_i to which is mapped a given component of Ω_i.

By lemma 4.14, the space $M_f(\Omega_i)$ identifies to the space $M_{f^p}(V)$ of Beltrami differentials on V that are invariant by $f^p_{|V}$, where p is the period of the cycle associated to U, and similarly $N_f(\Omega_i)$ identifies to $N_{f^p}(V)$. A linearizing coordinate ϕ for f^p conjugates $f^p : V \to V$ to $g(z) = e^{2i\pi\alpha} z$ on either the unit disk or an annulus $A(R)$, where α is an irrational rotation number. Therefore, by lemmas 4.15 and 4.16, $\dim M_f(\Omega_i)/N_f(\Omega_i) = 1$ if $n_i \neq 0$ and 0 else. We then obtain $\sum_{j\in J} \dim M_f(\Omega_j)/N_f(\Omega_j) = n_i$.

d) The case of a Herman ring

This case is very similar to the case of a Siegel disk : Ω_i still consists in the grand critical orbit of a periodic annulus. The only difference is that even if there are no critical orbit lying in the Herman ring, the components of Ω_i are still preperiodic to a ring and not a disk, and therefore $\dim M_f(\Omega_i)/N_f(\Omega_i) = 1$. We deduce : $\sum_{j\in J} \dim M_f(\Omega_j)/N_f(\Omega_j) = n_i + 1$ where n_i is the number of foliated acyclic critical classes captured by U.

e) The case of a superattracting cycle

If U is a component of a superattracting cycle, then $\Lambda_f \cap U$ is a countable union of equipotentials (which are smooth circles) and the superattracting cycle itself.

Assume first that there are no critical orbits captured by the superattracting cycle. Then there is a unique Ω_i intersecting U, and it is the whole grand orbit of U. By lemma 4.14, $M_f(\Omega_i)/N_f(\Omega_i) \simeq M_{f^p}(U)/N_{f^p}(U)$, where p is the period of U. Through a Böttcher coordinate, $f^p_{|U} : U \to U$ is conjugated to $g(z) = z^k$, $k \geq 2$. By lemma 4.15, every Beltrami differential $\mu \in M_{f^p}(U)$ is invariant by rotation. By lemma 4.16, we deduce that if there are no critical orbits meeting U, then $\dim M_{f^p}(U)/N_{f^p}(U) = 0$.

Assume now that $n_i > 0$, where n_i is the number of foliated acyclic critical classes meeting U. Let us denote by r_j, $j \leq n_i$, the radii in Böttcher coordinates of the circles corresponding to foliated acyclic critical classes in U. Note $A(r, r')$ the annulus $\{r' < |z| < r\}$. Let $\Omega_j \subset \Omega_f$ meeting U. Then for every component V of Ω_j, there exists a unique branch of $f^{-k} \circ f^l$ mapping V into the annulus $A(r_{j-1}, r_j)$ (with the convention $r_{-1} = 1$). By lemma 4.15, $M_f(\Omega_j)$ identifies to $M(A(r_{j-1,r_j}))$, and $N_f(\Omega_j)$ to $N(A(r_{j-1,r_j}))$. We deduce from this that $\dim M_f(\Omega_i)/N_f(\Omega_i) = 1$. Therefore $\sum_{j\in J} \dim M_f(\Omega_j)/N_f(\Omega_j) = n_i$.

Summing things up, each Fatou component U contributes n_i to the dimension, where n_i is the number of foliated acyclic critical classes meeting U, except for Herman rings which contribute $n_i + 1$ and the parabolic basins which contribute $n_i - 1$.

Moreover, ergodic line fields form a basis of the vector space Fix_J of invariant line fields, therefore $\dim \mathrm{Fix}_J = n_J$. Thus we have :

$$\mathrm{rank} D\Psi(0) = n_H + n_J + n_f - n_p.$$

□

5. Proof of the Main Theorem

The first application of Theorem 4.5 is that Ψ^Z has constant rank :

Corollary 6. Let f be a rational map, Z be an invariant set of cardinal 3 and $\mu \in \mathrm{Bel}(f)$. Then $\mathrm{rank} D\Psi^Z(0) = \mathrm{rank} D\Psi^Z(\mu)$.

Proof. It is clear that n_f, n_p and n_H are invariant under quasiconformal conjugacy. The number n_J is invariant as well since a quasiconformal homeomorphism preserve sets of Lebesgues measure zero (see [GL00]). Therefore if $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is a quasiconformal conjugacy between f and another rational map g, then ϕ^* maps invariant line fields for f to invariant line fields for g. Lemma 7 concludes the proof. □

Recall that $\mathrm{QC}(f)$ is the group of quasiconformal homeomorphisms of $\mathbb{P}^1$ commuting with f.

Definition 5.1. We will denote by $\mathcal{B}\mathrm{QC}(f)$ the space of all Beltrami forms corresponding to elements of $\mathrm{QC}(f)$. Similarly, we will denote by $\mathcal{B}\mathrm{QC}_0(f)$ the space of all Beltrami forms of elements of $\mathrm{QC}_0(f)$.

Note that we have $\mathcal{B}\mathrm{QC}_0(f) \subset \mathcal{B}\mathrm{QC}(f) \subset \mathrm{Bel}(f)$.

Corollary 7. The space $\mathcal{B}\mathrm{QC}(f)$ is a Banach submanifold of $\mathrm{Bel}(f)$, of tangent space at the basepoint equal to the space $N_f(\Omega_f)$ of Beltrami differentials of the form $\overline{\partial}\xi$, where ξ is a quasiconformal vector field invariant by f.

Proof. The space of quasiconformal homeomorphisms commuting with f is exactly the fiber $\Psi^{-1}(f)$. But by the above corollary, Ψ^Z has constant finite rank on $\mathrm{Bel}(f)$, therefore by the constant rank Theorem, $(\Psi^Z)^{-1}(f)$ is a Banach submanifold of finite codimension, whose tangent space to the identity is $\ker D\Psi(0) = N_f(\Omega_f)$. Moreover, $N_f(\Omega_f)$ is also the space of Beltrami differentials of the form $\overline{\partial}\xi$, where ξ is a quasiconformal vector field invariant by f by proposition 5. □

Note that in particular, $\mathcal{B}\mathrm{QC}(f)$ is locally connected at the identity. Therefore any element of $\mathcal{B}\mathrm{QC}(f)$ sufficiently close to the basepoint belongs in fact to $\mathcal{B}\mathrm{QC}_0(f)$.

Proposition 8. The Teichmüller premetric is a metric.

In particular, this implies that $\mathrm{Teich}(f)$ is a Hausdorff space.

Proof. Let $[\mu]$ and $[\nu]$ be points in $\mathrm{Teich}(f)$ such that $d_T([\mu],[\nu]) = 0$. This means that there is a sequence of representatives ϕ_n and ψ_n of $[\mu]$ and $[\nu]$ respectively such that the maximal dilatatation of $\phi_n \circ \psi_n^{-1}$ tends to one.

The rational maps $g = \phi_n \circ f \circ \phi_n^{-1}$ and $h = \psi_n \circ f \circ \psi_n^{-1}$ do not depend on the representatives ϕ_n and ψ_n. Since the Beltrami coefficient of $\phi_n \circ \psi_n^{-1}$ tends to 0 in the L^∞ topology, we have that $\phi_n \circ \psi_n^{-1}$ converges uniformly to the identity, and therefore $g = h$ and all the $\phi_n \circ \psi_n^{-1}$ belong to $\mathrm{QC}(f)$. Since by assumption the Beltrami forms associated to $\phi_n \circ \psi_n^{-1}$ tend to 0, we have that for n large enough it belongs to a path-connected neighborhood of 0 in $\mathcal{B}\mathrm{QC}(f)$, thus in $\mathcal{B}\mathrm{QC}_0(f)$.

This means exactly that $[\mu] = [\nu]$ in $\mathrm{Teich}(f)$. □

Theorem 5.2. *There exists a unique structure of complex manifold on* $\mathrm{Teich}(f)$ *making the projection* $\pi : \mathrm{Bel}(f) \to \mathrm{Teich}(f)$ *holomorphic. For this complex structure,* π *is a split submersion.*

Proof of Theorem 5.2. Let $\mu \in \mathrm{Bel}(f)$. By the constant rank Theorem 6, there exists germs of biholomorphisms $\phi : (\mathrm{Im} D\Psi^Z(\mu) \oplus \ker D\Psi^Z(\mu), 0) \to (\mathrm{Bel}(f), \mu)$ and $\chi : (\mathrm{Rat}_d, g) \to (\mathrm{Rat}_d, g)$ such that $\chi \circ \Psi^Z \circ \phi(u, v) = u$ for all $(u, v) \in \mathrm{Im} D\Psi^Z(\mu) \oplus \ker D\Psi^Z(\mu)$, where $g = \phi_\mu \circ f \circ \phi_\mu^{-1}$.

In particular, $\Psi^Z \circ \phi(u_1, v_1) = \Psi^Z \circ \phi(u_2, v_2)$ if and only if $u_1 = u_2$; moreover, if we note $\mu_i = \phi(u_i, v_i)$, $1 \leq i \leq 2$, then $\Psi^Z(\mu_1) = \Psi^Z(\mu_2)$ if and only if $\phi_1^Z \circ (\phi_2^Z)^{-1} \in \mathrm{QC}(f)$ where ϕ_i^Z is the quasiconformal homeomorphism corresponding to μ_i and fixing Z.

Recall that $\pi : \mathrm{Bel}(f) \to \mathrm{Teich}(f)$ is the quotient map.

Lemma 5.3. *For μ_1 and μ_2 in* $\mathrm{Bel}(f)$ *sufficiently close to the basepoint, we have that* $\pi(\mu_1) = \pi(\mu_2)$ *if and only if* $u_1 = u_2$.

Proof of lemma 5.3. Indeed, if $\pi(\mu_1) = \pi(\mu_2)$, then $\phi_1^Z \circ (\phi_2^Z)^{-1} \in \mathrm{QC}_0(f)$ and in particular $\phi_1^Z \circ (\phi_2^Z)^{-1} \in \mathrm{QC}(f)$, so $u_1 = u_2$. If now we assume that $u_1 = u_2$, then $\psi := \phi_1^Z \circ (\phi_2^Z)^{-1} \in \mathrm{QC}(f)$, and we have to prove that in fact $\psi \in \mathrm{QC}_0(f)$. Let $\phi_i^Z(t)$ be the quasiconformal homeomorphisms corresponding to $\mu_i(t) = \phi(tu_i, tv_i)$, $1 \leq i \leq 2$ and $t \in [0, 1]$, and $\psi_t = \phi_1^Z(t) \circ (\phi_2^Z)(t)^{-1}$. Since for all $t \in [0, 1]$, $\mu_i(t) = \phi^{-1}(tu_i, tvi)$, we have $\psi_t \in \mathrm{QC}(f)$, and $\psi_0 = \mathrm{Id}$. The maps $t \mapsto \mu_i(t)$ are analytic, so by the parametric Ahlfors-Bers Theorem so are the maps $t \mapsto \phi_i^Z(t)$. Therefore, for all $z \in \mathbb{P}^1$, the map $t \mapsto \psi_t(z) = \phi_1^Z(t) \circ (\phi_2^Z)(t)^{-1}(z)$ is continuous and ψ_t is an isotopy to the identity through elements of $\mathrm{QC}(f)$. Moreover, since the $\phi_i^Z(t)$ have uniformly bounded dilatation, so does ψ_t. [2] So $\psi = \psi_1 \in \mathrm{QC}_0(f)$, which ends the proof of this lemma. □

Lemma 5.4. *The parametrization* $\phi : \mathrm{Bel}(f) \to \mathrm{Im} D\Psi^Z(\mu) \oplus \ker D\Psi^Z(\mu)$ *descend to a parametrization* $\tilde{\phi} : \mathrm{Im} D\Psi^Z(\mu) \to \mathrm{Teich}(f)$ *making* $\mathrm{Teich}(f)$ *into a topological manifold. Moreover,* $\pi : \mathrm{Bel}(f) \to \mathrm{Teich}(f)$ *is a fibre bundle.*

Proof of lemma 5.4. Let $\mu \in \mathrm{Bel}(f)$, and $\epsilon > 0$ small enough that the disk $D(\mu, \epsilon)$ is contained into the domain of local coordinates ϕ^{-1} given by the constant rank Theorem. Consider the map $\tilde{\pi} : \mathrm{Im} D\Psi^Z(\mu) \oplus \ker D\Psi^Z(\mu) \to \mathrm{Teich}(f)$ defined by $\tilde{\pi} = \pi \circ \phi$. According to lemma 5.3, there is a map $\tilde{\phi} : \mathrm{Im} D\Psi^Z(\mu) \to \mathrm{Teich}(f)$ such that $\tilde{\pi}(u_1, u_2) = \Phi(u_1)$, and this map is bijective onto its image. Moreover, it is continuous since π and ϕ are continuous. Therefore, up to restricting its domain, it is a homeomorphism onto its image. This gives a local parametrization $\tilde{\phi}$ making the following diagram commute:

$$
\begin{array}{ccc}
\mathrm{Bel}(f) & \xrightarrow{\quad \phi \quad} & \mathrm{Im} D\Psi^Z(\mu) \oplus \ker D\Psi^Z(\mu) \\
\Big\downarrow \pi & & \Big\downarrow \pi_1 \\
\mathrm{Teich}(f) & \xrightarrow{\quad \tilde{\phi} \quad} & \mathrm{Im} D\Psi^Z(\mu)
\end{array}
$$

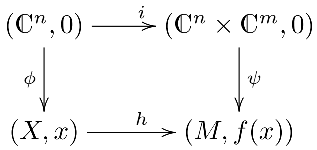
where π_1 is the projection on the first coordinate. Since according to Proposition 8 the space $\mathrm{Teich}(f)$ is Hausdorff, this proves that it is a topological manifold. Moreover, the diagram precisely means that π is a fibre bundle. □

We now prove that there is a complex structure on $\mathrm{Teich}(f)$ making the quotient map π holomorphic. This will be handled through the following lemma:

Lemma 5.5. *Let X be a topological manifold, M a complex manifold, and $h : X \to M$ a continuous map such that for all $x \in X$ there are parametrization ϕ and ψ making the*

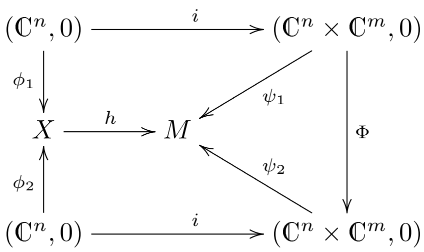
[2] Note however that the Beltrami coefficient of ψ_t needs not a priori depend continuously on t.

following diagram commute:



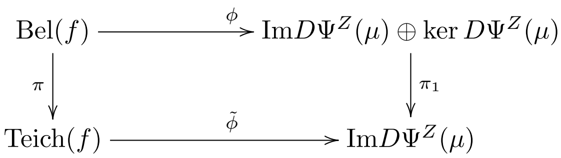
(the map i is the linear inclusion). Then there is a complex structure on X making the map f holomorphic.

Proof of lemma 5.5. Let ϕ_1, ϕ_2 and ψ_1, ψ_2 be such parametrizations of X and M respectively, with overlapping ranges. We shall prove that the transition map $\phi_2^{-1} \circ \phi_1$ is holomorphic. Indeed, we have the following commutative diagram:



Therefore we have $\Phi(x, 0) = (\phi(x), 0)$ for all $x \in \mathbb{C}^n$ (close enough to 0), where $\phi = \phi_2^{-1} \circ \phi_1$ is the transition map for X, and $\Phi = \psi_2^{-1} \circ \psi_1$ is the (holomorphic) transition map for M. In particular, ϕ is holomorphic. This proves that the charts ϕ_i form a complex atlas. Clearly for this atlas f is holomorphic. □

Let us now finish the proof of the theorem. According to lemma 5.3, the map Υ^Z : $\text{Teich}(f) \to \text{Rat}_d$ satisfy the conditions of lemma 5.5, where the parametrizations of $X = \text{Teich}(f)$ are those of lemma 5.4. Therefore there is a complex structure on Teich(f) making the map $\Upsilon^Z : \text{Teich}(f) \to \text{Rat}_d$ holomorphic. Moreover, the following diagram:



proves that π is a split holomorphic submersion for this complex structure. Clearly there can be at most one such complex structure on Teich(f). □

We can finally prove the Main Theorem :

Main Theorem. The map Υ^Z : $\text{Teich}(f) \to \text{Rat}_d$ is an immersion, whose image is transverse to $\mathcal{O}(f)$.

Proof. By corollary 6 it is enough to show that it is an immersion at 0. By definition, we have $\Psi^Z = \Upsilon^Z \circ \pi$, and therefore

$$D\Psi^Z(0) = D\Upsilon^Z([0]) \circ D\pi(0)$$

Injectivity of $D\Upsilon^Z([0])$ is then equivalent to the property $\ker D\Psi^Z(0) = \ker D\pi(0)$.

By proposition 5, $\ker D\Psi^Z(0) = N_f(\Omega_f)$, and by corollary 7, $\ker D\pi(0) = N_f(\Omega_f)$, which concludes the proof. □

We can now finally prove Corollary 3 concerning the description of the tangent and cotangent spaces $T_0\mathrm{Teich}(f)$ and $T_0^*\mathrm{Teich}(f)$.

Proof of Corollary 3. The first statement is a direct consequence of corollary 7.

Since $\mathrm{Teich}(f)$ is a finite-dimensional manifold, it is enough to prove that

$$\left(Q(\Lambda_f)/\overline{\nabla_f Q(\Lambda_f)}\right)^*$$

identifies to $T_0\mathrm{Teich}(f)$.

Let $E = Q(\Lambda_f)$, and $F = \overline{\nabla_f Q(\Lambda_f)}$. Since F is closed, the (topological) dual of E/F is $F^\perp \subset E^*$. We have that $E^* = \mathrm{Bel}(\mathbb{P}^1)/N(\Omega_f)$. Moreover, $F^\perp = \ker \Delta_f$, because $\Delta_f : E^* \to E^*$ is the transpose of ∇_f. Therefore :

$$(E/F)^* = \{\mu \in \mathrm{bel}(f)\}/\{\mu \in N_f(\Omega_f)\} = T_0\mathrm{Teich}(f)^*$$

□

Note that we obtain that $Q(\Lambda_f)/\overline{\nabla_f Q(\Lambda_f)}$ has finite dimension which is less than or equal to $2d-2$.

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