A polynomial endomorphism of $\mathbb{C}^2$ with a wandering Fatou component

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Plan

1. Introduction
2. Strategy of the construction
3. Parabolic implosion
Julia and Fatou sets

General setting
Let $M$ be a complex manifold, and $f : M \to M$ a holomorphic map.

Definition
- The Fatou set is the largest open set on which the iterates $\{f^n, n \in \mathbb{N}\}$ form a normal family.
- The Julia set is the complement of the Fatou set.
- A connected component of the Fatou set is called a Fatou component.
Example 1: Julia and Fatou sets of $z \mapsto z^2$

Julia set: $J = S^1$
Example 2: Julia and Fatou sets of $z \mapsto z^2 - 1$
Theorem (Sullivan, 1985)
Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geq 2$. Every Fatou component of $f$ is preperiodic.

Consequence:
Together with the classification theorem, we get a complete description of the dynamics in the Fatou set.
More generally:

If $f : M \to M$ is holomorphic, and $M$ is a complex manifold, must every Fatou component be preperiodic?

Answer:

- $M = \mathbb{C}$ and $f$ a transcendental map: no (Baker, 1976)
- $M = \mathbb{P}^1(\mathbb{C})$: yes (Sullivan, 1985)
- $M = \mathbb{C}$ and $f$ transcendental with only finitely many singular values: yes (Eremenko-Lyubich 1992, Goldberg-Keen 1986)
- $M = \mathbb{C}^2$, $f$ biholomorphic and transcendental: no (Fornaess-Sibony, 1998)
- $M = \mathbb{C}$ and $f$ transcendental with bounded set of singular values: no (Bishop, 2012)
A transcendental example: $z \mapsto z - \sin(z) + 2\pi$
Main theorem

Theorem (A.-Buff-Dujardin-Peters-Raissy)

There exists a polynomial map $P : \mathbb{C}^2 \to \mathbb{C}^2$ with a wandering Fatou component.

More precisely:

There exists polynomials $f$ of the form $f(z) = z + z^2 + O(z^3)$ such that for any polynomial $g$ of the form $g(w) = w - w^2 + O(w^3)$, the skew-product

$$P(z, w) = \left(f(z) + \frac{\pi^2}{4}w, g(w) \right)$$

has a wandering domain, that accumulates the complex line $\{w = 0\}$.
Sketch of the dynamics of $P(z, w) = (f(z) + \frac{\pi^2}{4} w, g(w))$
Projection of a wandering orbit on $\{w = 0\}$
Property wanted for $P(z, w) = (f(z) + \frac{\pi^2}{4} w, g(w))$

We want to chose $f$ so that there is a subsequence $m_k$ and an open set $U$ such that for all $(z, w) \in U$,

$$P^{m_k}(z, w) \to (\hat{z}, 0)$$

and $(\hat{z}, 0)$ is not preperiodic for $P$.

Proposition

If $P$ satisfies that property, then $P$ has a wandering domain.
Sketch

$w$}$z$

$(0, 0)$

$f$

$(\hat{z}, 0)$

$p_{m_2}$

$p_{m_1}$

$U$
Let $U$ be a connected open set such that $\forall (z, w) \in U, \ P^m_k(z, w) \to (\hat{z}, 0)$ (*)

- $U$ is a subset of a Fatou component $\Omega_0$
- We have (*) on all of $\Omega_0$ by analytic continuation
- If $\Omega_i = P^i(\Omega_0)$, then for all $(z, w) \in \Omega_i$, $P^m_k(z, w)$ converges to $(f^i(\hat{z}), 0)$
- If $i \neq j$, we have $f^i(\hat{z}) \neq f^j(\hat{z})$
- Therefore if $i \neq j$, $\Omega_i \neq \Omega_j$ and so $(\Omega_n)_{n \in \mathbb{N}}$ is wandering.
Dynamics near \( \{w = 0\} \)

**Notation**

Let \( f_w(z) = f(z) + \frac{\pi^2}{4} w \). Then:

\[
P^n(z, w_0) = (f_{w_{n-1}} \circ \ldots \circ f_{w_0}(z), g^n(w_0))
\]

with \( w_k = g^k(w_0) \sim \frac{1}{k} \).

**Idea**

Understanding the dynamics of \( P \) near \( \{w = 0\} \) amounts to understanding the non-autonomous compositions \( f_{w_n} \circ \ldots \circ f_{w_k} \), with \( f_{w_i}(z) \sim f(z) \).
fatou coordinates and Lavaurs map

Notations

- \( f(z) = z + z^2 + O(z^3) \) has a parabolic fixed point at 0
- \( \mathcal{B}_f \) is its parabolic basin
- \( \phi_f \) is the (normalized) attracting Fatou coordinate
  \[
  \phi_f \circ f = \phi_f + 1
  \]
- \( \psi_f \) is the (normalized) repelling Fatou parametrization
  \[
  f \circ \psi_f(Z) = \psi_f(Z + 1)
  \]
- \( L_f = \psi_f \circ \phi_f \) is the (phase 0) Lavaurs map
Lavaurs map

\[ L_f = \psi_f \circ \phi_f \]
Theorem (Lavaurs)

Let \( \epsilon_k \to 0 \) and \( n_k \to +\infty \), such that \( \frac{\pi}{\sqrt{\epsilon_k}} - n_k \to 0 \). Then \((f + \epsilon_k)^{n_k}\) converges to the Lavaurs map \( L_f \) uniformly on compacts of \( B_f \).

Consequence

With \( \epsilon_k = \frac{\pi^2}{4} w_k^2 \sim \frac{\pi^2}{4k^2} \) and \( n_k = 2k \), the sequence \( f_{w_{k2}} \circ \ldots \circ f_{w_{k2}} \) converges locally uniformly to \( L_f \) (with local uniformity in \( w \) as well).

Key proposition (A.-Buff-Dujardin-Peters-Raissy)

For all \( w \in B_g \), the sequence \( f_{w_{(k+1)^2-1}} \circ \ldots \circ f_{w_{k2}} \) converges locally uniform \( L_f \) (with local uniformity in \( w \) as well).
Lavaurs’ theorem
Theorem (A-Buff-Dujardin-Peters-Raissy, 2014)

Suppose \( f(z) = z + z^2 + O(z^3) \) is such that \( L_f \) has an attracting fixed point. Let \( g(w) = w - w^2 + O(w^3) \). Then

\[
P(z, w) = \left( f(z) + \frac{\pi^2}{4} w, g(w) \right)
\]

has a wandering domain.

Numerical example:

\[
P(z, w) = (z + z^2 + 0.95z^3 + \frac{\pi^2}{4} w, w - w^2)
\]
Red dots: attracting fixed points of $L_f$
Grey part: points that do not escape under $\langle f, L_f \rangle$ (Julia-Lavaurs set of $f$)
\[
\hat{z} = L_f(\hat{z})
\]

\[
p^{2k+1} \simeq (L_f, g^{2k+1})
\]

\[
U
\]
Idea of the proof of the key proposition

\[ z_0 \sim z_{2n+1} \simeq L_f(z_0) \]

Notation: \( P^k(z, g^{n^2}(w)) = (z_k, g^{n^2+k}(w)) \)
Thank you for your attention!