A polynomial endomorphism of $\mathbb{C}^2$ with a wandering Fatou component

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Plan

Julia and Fatou sets

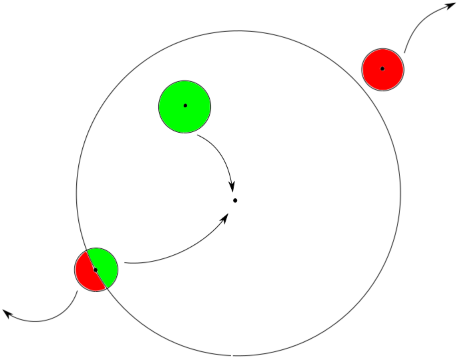
General setting

Let M be a complex manifold, and $f : M \to M$ a holomorphic map.

Definition

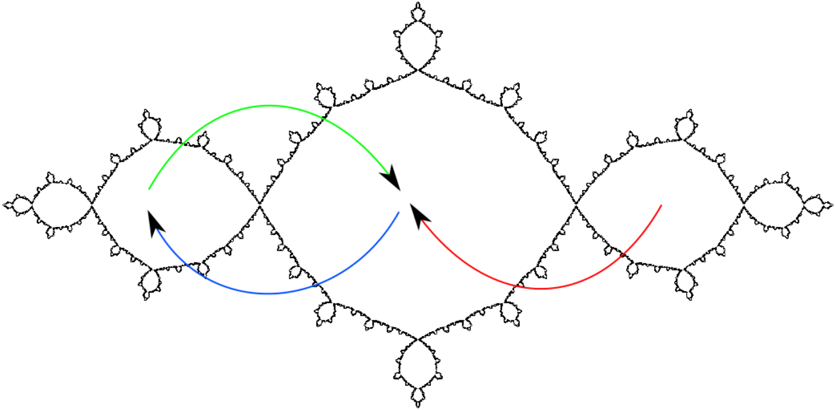
- The Fatou set is the largest open set on which the iterates $\{f^n, n \in \mathbb{N}\}$ form a normal family.
- The Julia set is the complement of the Fatou set.
- A connected component of the Fatou set is called a Fatou component.

Example 1: Julia and Fatou sets of $z \mapsto z^2$



Julia set : $J = S^1$

Example 2: Julia and Fatou sets of $z \mapsto z^2 - 1$



No Wandering Domain Theorem of Sullivan

Theorem (Sullivan, 1985)

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geq 2$. Every Fatou component of f is preperiodic.

Consequence :

Together with the classification theorem, we get a complete description of the dynamics in the Fatou set.

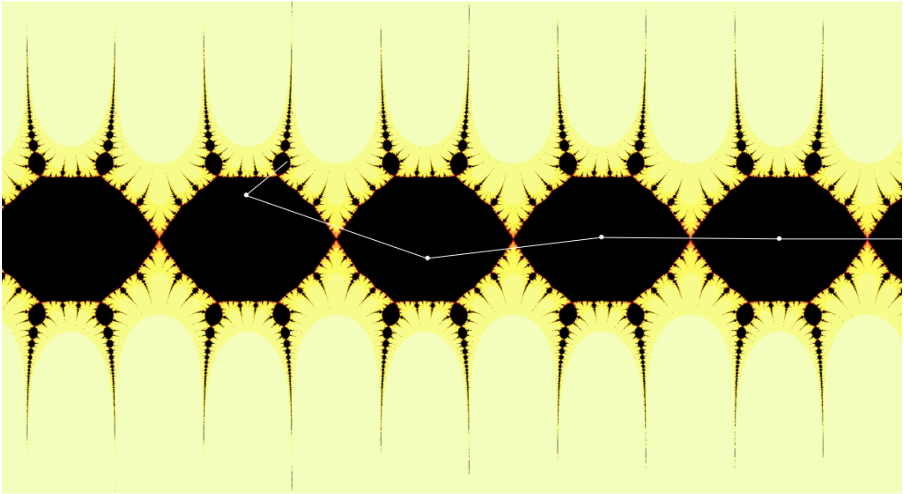
More generally :

If $f : M \to M$ is holomorphic, and M is a complex manifold, must every Fatou component be preperiodic ?

Answer :

- $M = \mathbb{C}$ and f a transcendental map : no (Baker, 1976)
- $M = \mathbb{P}^1(\mathbb{C})$: yes (Sullivan, 1985)
- $M = \mathbb{C}$ and f transcendental with only finitely many singular values : yes (Eremenko-Lyubich 1992, Goldberg-Keen 1986)
- $M = \mathbb{C}^2$, f biholomorphic and transcendental : no (Fornaess-Sibony, 1998)
- $M = \mathbb{C}$ and f transcendantal with bounded set of singular values : no (Bishop, 2012)
- $M = \mathbb{P}^2(\mathbb{C})$, f polynomial : no (A-Buff-Dujardin-Peters-Raissy, 2014).

A transcendantal example : $z \mapsto z - \sin(z) + 2\pi$



Main theorem

Theorem (A.-Buff-Dujardin-Peters-Raissy)

There exists a polynomial map $P : \mathbb{C}^2 \to \mathbb{C}^2$ with a wandering Fatou component.

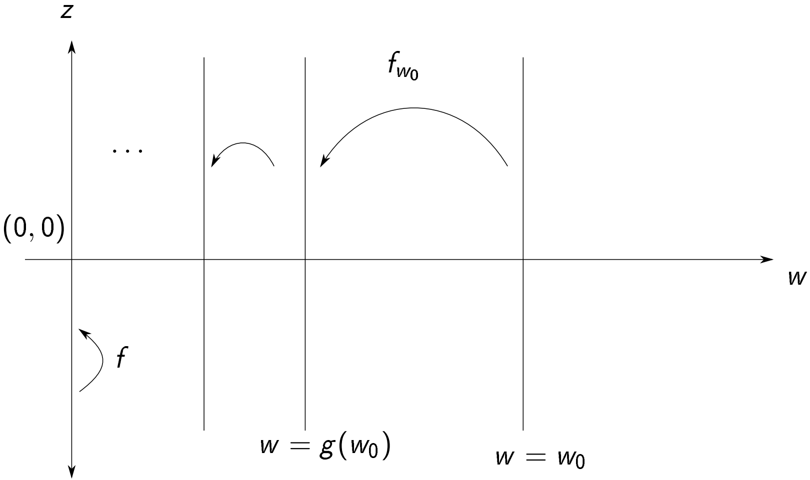
More precisely :

There exists polynomials f of the form $f(z) = z + z^2 + O(z^3)$ such that for any polynomial g of the form $g(w) = w - w^2 + O(w^3)$, the skew-product

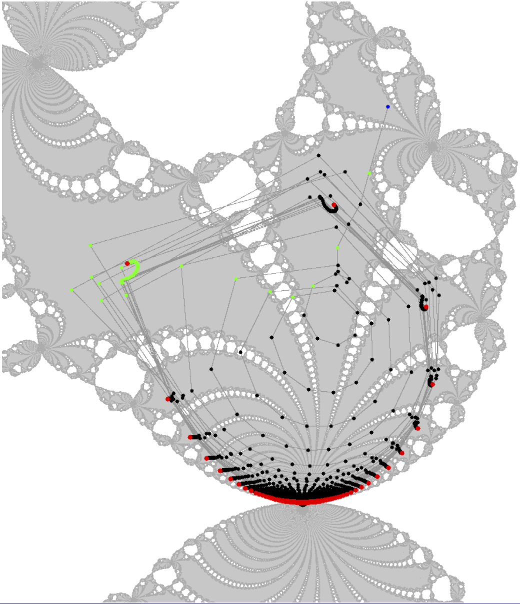
$$P(z, w) = \left(f(z) + \frac{\pi^2}{4} w, g(w) \right)$$

has a wandering domain, that accumulates the complex line $\{w = 0\}$.

Sketch of the dynamics of $P(z,w) = (f(z) + \frac{\pi^2}{4}w, g(w))$



Projection of a wandering orbit on $\{w = 0\}$



Strategy of the construction (idea : M. Lyubich)

Property wanted for $P(z, w) = (f(z) + \frac{\pi^2}{4} w, g(w))$

We want to chose f so that there is a subsequence m_k and an open set U such that for all $(z, w) \in U$,

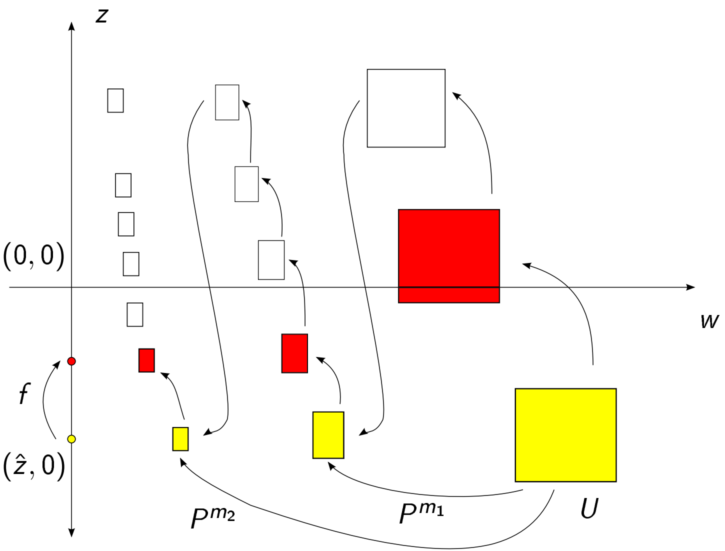
$$P^{m_k}(z, w) \to (\hat{z}, 0)$$

and $(\hat{z}, 0)$ is not preperiodic for P.

Proposition

If P satisfies that property, then P has a wandering domain.

Sketch



Sketch of the proof

Let U be a connected open set such that $\forall (z,w) \in U$, $P^{m_k}(z,w) \to (\hat{z},0)$ $(*)$

- U is a subset of a Fatou component Ω_0
- We have $(*)$ on all of Ω_0 by analytic continuation
- If $\Omega_i = P^i(\Omega_0)$, then for all $(z,w) \in \Omega_i$, $P^{m_k}(z,w)$ converges to $(f^i(\hat{z}),0)$
- If $i \neq j$, we have $f^i(\hat{z}) \neq f^j(\hat{z})$
- Therefore if $i \neq j$, $\Omega_i \neq \Omega_j$ and so $(\Omega_n)_{n\in\mathbb{N}}$ is wandering.

Dynamics near $\{w = 0\}$

Notation

Let $f_w(z) = f(z) + \frac{\pi^2}{4} w$. Then :

$$P^n(z, w_0) = \left(f_{w_{n-1}} \circ \ldots \circ f_{w_0}(z), g^n(w_0)\right)$$

with $w_k = g^k(w_0) \sim \frac{1}{k}$.

Idea

Understanding the dynamics of P near $\{w = 0\}$ amounts to understanding the non-autonomous compositions $f_{w_n} \circ \ldots \circ f_{w_k}$, with $f_{w_i}(z) \simeq f(z)$

Fatou coordinates and Lavaurs map

Notations

- $f(z) = z + z^2 + O(z^3)$ has a parabolic fixed point at 0
- $\mathcal{B}_f$ is its parabolic basin
- ϕ_f is the (normalized) attracting Fatou coordinate

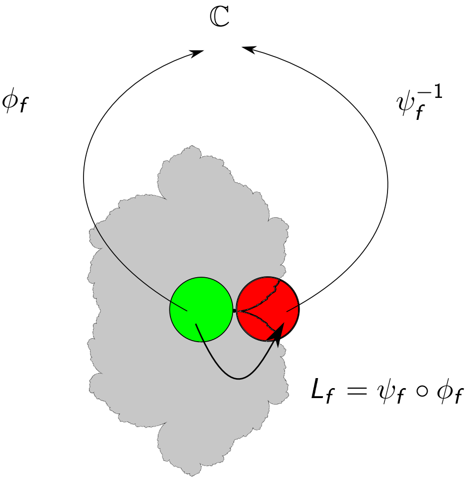
$$\phi_f \circ f = \phi_f + 1$$

- ψ_f is the (normalized) repelling Fatou parametrization

$$f \circ \psi_f(Z) = \psi_f(Z + 1)$$

- $L_f = \psi_f \circ \phi_f$ is the (phase 0) Lavaurs map

Lavaurs map



Parabolic implosion

Theorem (Lavaurs)

Let $\epsilon_k \to 0$ and $n_k \to +\infty$, such that $\frac{\pi}{\sqrt{\epsilon_k}} - n_k \to 0$. Then $(f+\epsilon_k)^{\circ n_k}$ converges to the Lavaurs map L_f uniformly on compacts of $\mathcal{B}_f$.

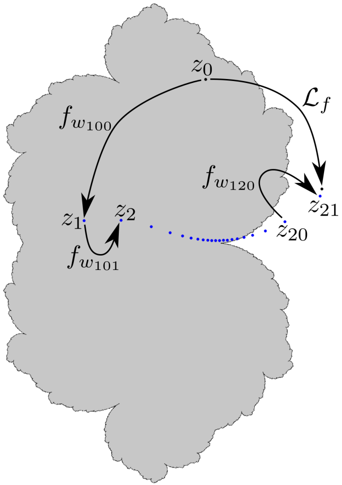
Consequence

With $\epsilon_k = \frac{\pi^2}{4} w_{k^2} \sim \frac{\pi^2}{4k^2}$ and $n_k = 2k$, the sequence $\underbrace{f_{w_{k^2}} \circ \ldots \circ f_{w_{k^2}}}_{2k}$ converges locally uniformly to L_f (with local uniformity in w as well).

Key proposition (A.-Buff-Dujardin-Peters-Raissy)

For all $w \in \mathcal{B}_g$, the sequence $\underbrace{f_{w_{(k+1)^2-1}} \circ \ldots \circ f_{w_{k^2}}}_{2k+1}$ converges locally uniform L_f (with local uniformity in w as well).

Lavaurs' theorem



Precise statement

Theorem (A-Buff-Dujardin-Peters-Raissy, 2014)

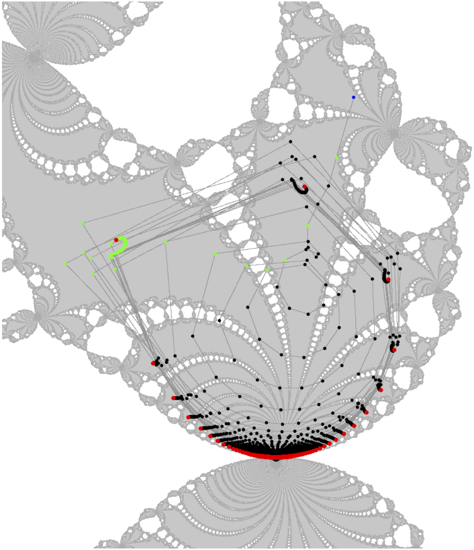
Suppose $f(z) = z + z^2 + O(z^3)$ is such that L_f has an attracting fixed point. Let $g(w) = w - w^2 + O(w^3)$. Then

$$P(z, w) = \left(f(z) + \frac{\pi^2}{4} w, g(w) \right)$$

has a wandering domain.

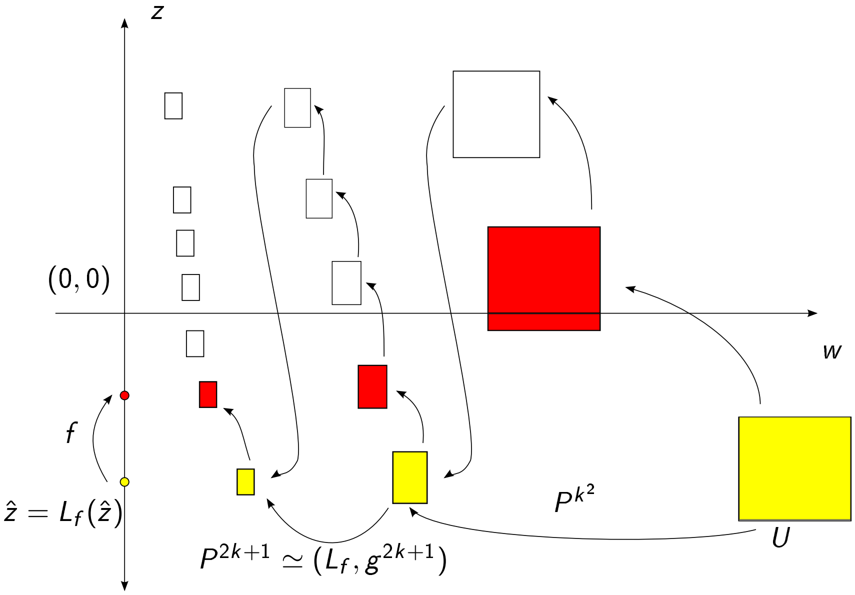
Numerical example :

$$P(z, w) = (z + z^2 + 0.95z^3 + \tfrac{\pi^2}{4} w, w - w^2)$$



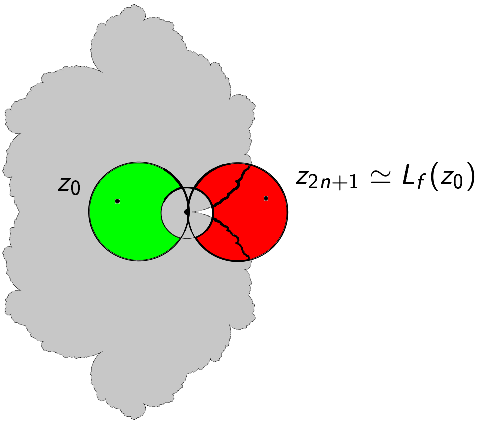
Red dots : attracting fixed points of L_f

Grey part : points that do not escape under $\langle f, L_f \rangle$ (Julia-Lavaurs set of f)



z
$(0,0)$
w
f
$\hat{z} = L_f(\hat{z})$
P^{k^2}
U
$P^{2k+1} \simeq (L_f, g^{2k+1})$

Idea of the proof of the key proposition



Notation : $P^k(z, g^{n^2}(w)) = \left(z_k, g^{n^2+k}(w)\right)$

The end

Thank you for your attention!