Supplementary Material to the article "Towards a classification of bifurcations in Vlasov equations"

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JORDAN BLOCK LINEAR STRUCTURE FOR A 1D VLASOV EQUATION

The article presents a formal study of the structure of the linearized operator at a critical non homogeneous stationary state with 0 eigenvalue, for a degenerate non classical Hamiltonian system. We specify here this computation for a generic 1D Vlasov equation, with bounded spatial domain with periodic boundary condition, taken to be $[0, 2\pi)$ without loss of generality.

In position-momentum (q, p) variables, Vlasov equation for the phase space density $F(q, p, t)$ reads

$$\partial_t F + p\partial_q F - \partial_q V \partial_p F = 0 \tag{1}$$

$$\text{with } V[F](q) = \iint v(q - q')F(q', p')dq'dp', \tag{2}$$

where $v(q)$ is the two-body interaction potential. Thanks to the periodic boundary condition, spatial Fourier series are a natural expansion for the interaction potential and the density; in an unbounded setting, one would have to use other expansions. The 2-body potential is

$$v(q) = \sum_{k\in\mathbb{Z}} v_k e^{ikq}$$

and must be even from the law of action and reaction. We can take $v_0 = 0$ without loss of generality. We are interested in functions whose total integral over phase space is 1. This implies in particular that we will consider perturbations with vanishing integral. We consider a stationary state

$$F_{\text{stat}}(q, p) = F_0(h(q, p)) \ , \quad h(q, p) = \frac{p^2}{2} + V[F_{\text{stat}}](q).$$

The one particle Hamiltonian h is integrable, hence we can introduce the associated angle-action variables (θ, J). h is a function of J, so that we can write the stationary state under the form $F_0(J)$. Writing $F = F_0 + f(\theta, J, t)$, the linearized evolution equation for f is, from (1):

$$\partial_t f = \mathcal{L} \cdot f \ , \quad \mathcal{L} \cdot f = -\Omega_0(J)\partial_\theta f + F_0'(J)\partial_\theta V[f], \tag{3}$$

where $\Omega_0(J)$ is the frequency of a trajectory with action J, in the potential created by the stationary distribution. We want to study the spectral structure of $\mathcal{L}$ when the stationary state F_0 is critical; hence we study the eigenvalue problem with eigenvalue 0. For a function g defined on the phase space, introducing the Fourier transform with respect to the angle variable is natural:

$$g(\theta, J) = \sum_{\alpha\in\mathbb{Z}} g_\alpha(J)e^{i\alpha\theta}. \tag{4}$$

We will need to switch between position-momentum and angle-action coordinates; we introduce for this purpose the functions

$$c_{k,\alpha}(J) = \frac{1}{2\pi}\int_0^{2\pi} e^{ikq(\theta,J)}e^{-i\alpha\theta}d\theta. \tag{5}$$

Before entering into the details of the computations, we give two remarks. The first remark is on the (θ, J) notation. The phase space may be divided into several parts in each of which the angle-action variables are defined separately. For instance, in the case $v(q) = -\cos q$ and when the mean-field potential does not vanish, the phase space has a separatrix and is divided into three parts [1]: inside the separatrix, the upper side of separatrix, and the lower side of separatrix. The above expressions such as (4) or (5) then use the notation (θ, J) as a convenient short-hand for this more complicated structure. To give a precise example, in a one-dimensional Hamiltonian system, we have to understand the definition (5) of the function $c_{k,\alpha}$ as follows: a periodic orbit φ corresponds to an iso-J line, and $c_{k,\alpha}$ is actually a function of this orbit

$$c_{k,\alpha}(\varphi) = \frac{\Omega(\varphi)}{2\pi}\int_0^{T(\varphi)} e^{ik\varphi_q(t)}e^{-i\alpha\Omega(\varphi)t}dt, \tag{6}$$

where $T(\varphi)$ and $\Omega(\varphi)$ are the period and the frequency of the orbit $\varphi = (\varphi_q, \varphi_p)$, $\varphi_q(t), \varphi_p(t)$ are the spatial and momentum coordinates along the orbit φ and we assumed $\Omega(\varphi) \neq 0$.

The second remark is on the symmetry of $c_{k,\alpha}(J)$, which is revealed by considering the time-reversed orbit $R[\varphi](t) = (\varphi_q(-t), -\varphi_p(-t))$. The time-reversed orbit exists thanks to the symmetry $h(q,-p) = h(q,p)$. Between the two orbits φ and $R[\varphi]$, we have the symmetry

$$c_{k,\alpha}(R[\varphi]) = c_{k,-\alpha}(\varphi). \tag{7}$$

Later we will find integrations over the action J. This integration has to be understood as an integration over the orbits foliating the phase space. Changing the foliation from $\{\varphi\}$ to $\{R[\varphi]\}$, roughly speaking, the symmetry (7) permits to replace $c_{k,\alpha}$ with $c_{k,-\alpha}$.

Computation of the eigenvector and generalized eigenvectors

We rewrite (3) using the Fourier transform with respect to θ:

$$\left(\mathcal{L}\cdot f\right)_\alpha = -i\alpha\Omega_0(J)f_\alpha(J) + i\alpha F_0' \sum_{k\in\mathbb{Z}} v_k Y_k[f] c_{k,\alpha}(J), \tag{8}$$

where

$$Y_k[f](J) = 2\pi \sum_{\alpha\in\mathbb{Z}} \int f_\alpha(J) c^*_{k,\alpha}(J) dJ. \tag{9}$$

We assume that F_0 is a critical stationary state: it has a bifurcating 0 eigenvalue. We look for the associated eigenvector $\psi^{(0)}$ which induces the generalized eigenvectors $\psi^{(n)}$ and makes a Jordan block as large as possible. To obtain a solution to $\mathcal{L}\cdot\psi^{(m+1)} = \psi^{(m)}$, for $\alpha = 0$, $\psi^{(m)}_{\alpha=0} = 0$ is necessary because we always have $\left(\mathcal{L}\cdot\psi^{(m+1)}\right)_{\alpha=0} = 0$.

The first step is to find $\psi^{(0)}$, solving $\mathcal{L}\cdot\psi^{(0)} = 0$. From (8), we see that the equation for $\alpha = 0$ is always satisfied; hence we may choose any function for $\psi^{(0)}_{\alpha=0}(J)$. As commented above, we take $\psi^{(0)}_{\alpha=0}(J) = 0$ in order to be able to find a first generalized eigenvector later. Notice that this choice of $\psi^{(0)}_0$ corresponds to perturbations that do not modify the values of the Casimir invariants at linear order. For $\alpha \neq 0$, we have:

$$\psi^{(0)}_\alpha(J) = \frac{F_0'(J)}{\Omega_0(J)} \sum_{l\in\mathbb{Z}} v_l Y_l[\psi^{(0)}] c_{l,\alpha}(J). \tag{10}$$

The weak resonance hypothesis ensures that the above expression is regular (or at least integrable): there are no or few particles with 0 frequency (Ω_0^{-1} may have a logarithmic divergence; this indeed happens whenever there is a separatrix trajectory, and may be called "weak resonance"). Both sides of (10) contain $\psi^{(0)}$, and therefore, $Y_l[\psi^{(0)}]$ must be determined self-consistently. Inserting (10) in (9) with the choice of $\psi^{(0)}_0 = 0$, we obtain

$$\sum_{l\in\mathbb{Z}} \Lambda_{kl} Y_l[\psi^{(0)}] = 0 \tag{11}$$

with

$$\Lambda_{kl} = \delta_{kl} - 2\pi v_l \sum_{\alpha\in\mathbb{R}} \int \frac{F_0'(J)}{\Omega_0(J)} c^*_{k,\alpha}(J) c_{l,\alpha}(J) dJ. \tag{12}$$

The condition to obtain a non trivial solution eigenvector is $\mathrm{Ker}(\Lambda)$ non trivial, which is consistent with the criticality assumption of F_0. The generic case is that $\mathrm{Ker}(\Lambda)$ is of dimension 1, which we now assume. We call (y_k) an element of this kernel. Then we have for any $\alpha \neq 0$:

$$\psi^{(0)}_\alpha(J) = \frac{F_0'(J)}{\Omega_0(J)} \sum_{l\in\mathbb{Z}} v_l y_l c_{l,\alpha}(J). \tag{13}$$

We now look for a generalized eigenvector $\psi^{(1)}$, solving $\mathcal{L}\cdot\psi^{(1)} = \psi^{(0)}$. From (8), we see that the equation for $\alpha = 0$ is again always satisfied thanks to the choice $\psi^{(0)}_{\alpha=0} = 0$, and $\psi^{(1)}_{\alpha=0}$ is free again. We choose again $\psi^{(1)}_{\alpha=0} = 0$ to search a second generalized eigenvector. For $\alpha \neq 0$, we have:

$$\psi^{(1)}_\alpha(J) = \frac{F_0'}{\Omega_0} \sum_{l\in\mathbb{Z}} v_l Y_l[\psi^{(1)}] c_{l,\alpha} - \frac{F_0'}{i\alpha\Omega_0^2} \sum_{l\in\mathbb{Z}} v_l y_l c_{l,\alpha}. \tag{14}$$

Inserting (14) into (9), we obtain

$$\sum_{l\in\mathbb{Z}} \Lambda_{kl} Y_l[\psi^{(1)}] = -2\pi \sum_{l\in\mathbb{Z}} v_l y_l \sum_{\alpha\neq 0} \int \frac{F_0'}{i\alpha\Omega_0^2} c^*_{k,\alpha} c_{l,\alpha} dJ. \tag{15}$$

Let us remember the remark after (7). In the right-hand-side, changing the orbit φ to $R[\varphi]$ does not change $F_0'(J)/(i\alpha\Omega_0^2)$ but $c^*_{k,\alpha}c_{l,\alpha}$ becomes $c^*_{k,-\alpha}c_{l,-\alpha}$. This implies that the sum over $\alpha \neq 0$ vanishes for any $l \in \mathbb{Z}$. A solution $Y_l[\psi^{(1)}]$ must be, therefore, chosen from $\mathrm{Ker}(\Lambda)$ and the first term of (14) is proportional to $\psi^{(0)}$. Therefore, we may choose $Y_l[\psi^{(1)}] \equiv 0$ and, for $\alpha \neq 0$,

$$\psi^{(1)}_\alpha(J) = \frac{F_0'(J)}{-i\alpha\Omega_0(J)^2} \sum_{l\in\mathbb{Z}} v_l y_l c_{l,\alpha}(J). \tag{16}$$

We now look for a further generalized eigenvector $\psi^{(2)}$, solving $\mathcal{L}\cdot\psi^{(2)} = \psi^{(1)}$. From (8), we see that the equation for $\alpha = 0$ is again always satisfied, and $\psi^{(2)}_{\alpha=0}$ is free again. We will choose $\psi^{(2)}_{\alpha=0}$ later. For $\alpha \neq 0$, we have:

$$\psi^{(2)}_\alpha(J) = \frac{F_0'}{\Omega_0} \sum_{l\in\mathbb{Z}} v_l Y_l[\psi^{(2)}] c_{l,\alpha} + \frac{F_0'}{(-i\alpha)^2\Omega_0^3} \sum_{l\in\mathbb{Z}} v_l y_l c_{l,\alpha}. \tag{17}$$

The self-consistent equation for $Y_l[\psi^{(2)}]$ is

$$\sum_{l\in\mathbb{Z}} \Lambda_{kl} Y_l[\psi^{(2)}] = 2\pi \sum_{l\in\mathbb{Z}} v_l y_l \sum_{\alpha\neq 0} \int \frac{F_0'}{(i\alpha)^2\Omega_0^3} c^*_{k,\alpha} c_{l,\alpha} dJ + 2\pi \int \psi^{(2)}_0 c^*_{k,0} dJ. \tag{18}$$

If we choose $\psi_0^{(2)} = 0$, the right-hand-side of (18) is non zero and the linear equation (18) does not have a solution in general. However, it is possible to choose appropriately the function $\Psi_0^{(2)}(J)$ in order to ensure that the sum in the right-hand-side vanishes, so that a solution exists, we can take $\forall k$, $Y_k[\psi^{(2)}] = 0$. This only requires that the $c_{k,0}$ form a free family of functions. We have thus built a second generalized eigenvector for the eigenvalue 0.

We can now summarize:

$$\psi^{(0)} = \begin{pmatrix} 0 \\ \frac{F_0'(J)}{\Omega_0(J)} \sum_{l\in\mathbb{Z}} v_l y_l c_{l,\alpha}(J) \end{pmatrix} \quad (19)$$

$$\psi^{(1)} = \begin{pmatrix} 0 \\ \frac{F_0'(J)}{-i\alpha\Omega_0^2(J)} \sum_{l\in\mathbb{Z}} v_l y_l c_{l,\alpha}(J) \end{pmatrix} \quad (20)$$

$$\psi^{(2)} = \begin{pmatrix} \psi_0^{(2)}(J) \\ \frac{F_0'(J)}{(i\alpha)^2\Omega_0^3(J)} \sum_{l\in\mathbb{Z}} v_l y_l c_{l,\alpha}(J) \end{pmatrix}, \quad (21)$$

where, in each eigenvector, the upper line represents the element for $\alpha = 0$, and the lower one contains the expression for $\alpha \neq 0$. Crucially, $\psi_0^{(2)} \neq 0$, which has two consequences:

- The equation $\mathcal{L}\cdot\psi^{(3)} = \psi^{(2)}$ has no solution, hence the characteristic space is only of dimension 3.
- $\psi^{(2)}$ has a non zero component in the direction that modifies the values of the Casimir invariants, $\alpha = 0$.

At the bifurcation point, the linearized operator $\mathcal{L}$ restricted to the subspace $\mathrm{Span}\{\psi^{(0)}, \psi^{(1)}, \psi^{(2)}\}$ is represented by the 3D Jordan block

$$L_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (22)$$

as found in the main text on the basis of genericity arguments.

Computations of the projections: adjoint problem

In order to concretely perform the non linear computations, we need to know the projection onto the generalized eigenspace we have just built. For this reason, we study here in details the adjoint linear problem. The adjoint linear operator of $\mathcal{L}$ with respect to the standard scalar product is

$$\mathcal{L}^\dagger \cdot h = \Omega_0 \frac{\partial h}{\partial\theta} - \sum_{k\in\mathbb{Z}} v_k e^{ikq} \int \frac{\partial h}{\partial\theta} F_0' e^{-ikq'} dq' dp'. \quad (23)$$

Making use of

$$e^{ikq} = \sum_{\alpha\in\mathbb{Z}} c_{k,\alpha}(J) e^{i\alpha\theta},$$

this yields

$$\left(\mathcal{L}^\dagger \cdot h\right)_\alpha = i\alpha\Omega_0(J) h_\alpha(J) - \sum_{k\in\mathbb{Z}} v_k c_{k,\alpha}(J) \sum_{\beta\in\mathbb{Z}} i\beta \int h_\beta(J') F_0'(J') c_{k,\beta}^*(J') dJ'.$$

We first look for $\phi^{(0)}$ such that $\mathcal{L}^\dagger\cdot\phi^{(0)} = 0$. For $\alpha = 0$, this imposes that for all k such that $v_k \neq 0$

$$\sum_{\beta\in\mathbb{Z}} i\beta \int \phi_\beta^{(0)}(J') F_0'(J') c_{k,\beta}^*(J') dJ' = 0.$$

Generically, this requires $\phi_\alpha^{(0)} = 0$, for all $\alpha \neq 0$. Hence $\phi^{(0)}(\theta, J)$ only has an $\alpha = 0$ component, and $\phi_0^{(0)}(J)$ is undetermined at this stage. We now look for $\phi^{(1)}$ such that $\mathcal{L}^\dagger\cdot\phi^{(1)} = \phi^{(0)}$. We obtain:

$$\alpha = 0: \quad -\sum_{k\in\mathbb{Z}} v_k Z_k[\phi^{(1)}] c_{k,0}(J) = \phi_0^{(0)}(J) \quad (24)$$

$$\alpha \neq 0: \quad \phi_\alpha^{(1)}(J) = \frac{1}{i\alpha\Omega_0(J)} \sum_{k\in\mathbb{Z}} v_k Z_k[\phi^{(1)}] c_{k,\alpha}(J) \quad (25)$$

$$\text{with} \quad Z_k[h] = 2\pi \sum_{\beta\in\mathbb{Z}} i\beta \int h_\beta(J') F_0'(J') c_{k,\beta}^*(J') dJ' \quad (26)$$

Eq. (25) rewrites as

$$\forall k \in \mathbb{Z}\ , \quad \sum_{l\in\mathbb{Z}} \Lambda_{kl} Z_l[\phi^{(1)}] = 0, \quad (27)$$

where the Λ infinite matrix has been introduced in (12); (27) then has a non trivial solution (y_k). Then (24) fixes the previously undetermined $\phi_0^{(0)}(J)$. $\phi_0^{(1)}(J)$ is a priori undetermined. We now look for $\phi^{(2)}$ such that $\mathcal{L}^\dagger\cdot\phi^{(2)} = \phi^{(1)}$. We obtain:

$$\alpha = 0: \quad -\sum_{k\in\mathbb{Z}} v_k Z_k[\phi^{(2)}] c_{k,0}(J) = \phi_0^{(1)}(J) \quad (28)$$

$$\alpha \neq 0: \quad \phi_\alpha^{(2)}(J) = \frac{1}{i\alpha\Omega_0(J)} \sum_{k\in\mathbb{Z}} v_k Z_k[\phi^{(2)}] c_{k,\alpha}(J) + \frac{1}{[i\alpha\Omega_0(J)]^2} \sum_{k\in\mathbb{Z}} v_k y_k c_{k,\alpha}(J) \quad (29)$$

Using (7), (29) rewrites as

$$\forall k \in \mathbb{Z}\ , \quad \sum_{l\in\mathbb{Z}} \Lambda_{kl} Z_l[\phi^{(2)}] = 0.$$

We may choose the solution $\forall l\ ,\ Z_l[\phi^{(2)}] = 0$. (28) then implies that $\phi_0^{(1)} = 0$. We can now summarize the eigenvector and generalized eigenvectors of $\mathcal{L}^\dagger$:

$$\phi^{(0)} = \begin{pmatrix} \sum_{l\in\mathbb{Z}} v_l y_l c_{l,0}(J) \\ 0 \end{pmatrix} \quad (30)$$

$$\phi^{(1)} = \begin{pmatrix} 0 \\ \frac{1}{i\alpha\Omega_0(J)} \sum_{l\in\mathbb{Z}} v_l y_l c_{l,\alpha}(J) \end{pmatrix} \quad (31)$$

$$\phi^{(2)} = \begin{pmatrix} \phi_0^{(2)}(J) \\ \frac{1}{[i\alpha\Omega_0(J)]^2} \sum_{l\in\mathbb{Z}} v_l y_l c_{l,\alpha}(J) \end{pmatrix}. \quad (32)$$

Furthermore, we want to impose that $\langle \phi^{(i)}, \psi^{(j)} \rangle = 0$ if $i + j \neq 2$, in order to ensure that the operator Π defined below has the nice structure (34). This imposes a constraint on $\phi_0^{(2)}$, which was fully undetermined until now. The projection operator Π onto the subspace $\mathrm{Span}\{\psi^{(0)}, \psi^{(1)}, \psi^{(2)}\}$ is written as

$$\Pi \cdot f = \sum_{j=0}^{2} \psi^{(j)} \langle \phi^{(j)}, f \rangle, \tag{33}$$

On the subspace $\mathrm{Span}\{\psi^{(0)}, \psi^{(1)}, \psi^{(2)}\}$, the representation matrix of Π, $\Pi_{ij} = \langle \phi^{(i)}, \psi^{(j)} \rangle$, has the structure

$$(\Pi_{ij}) = \begin{pmatrix} 0 & 0 & \star \\ 0 & \star & 0 \\ \star & 0 & 0 \end{pmatrix} \tag{34}$$

where the $\star$ are non zero elements. The $\star$ could be chosen to be 1 with an appropriate normalization of the eigenvectors and eigenprojections.

Specification to the case $v(q) = -\cos q$

The particular case $v(q) = -\cos q$ (HMF model) is used as an example in the paper. We give here the expressions of the eigenvectors and eigenprojections in this case, which are effectively used to derive an explicit reduced model and compare with the full dynamics. We use then $v_k = 0$ unless $k = \pm 1$, and $v_1 = v_{-1} = -1/2$. It will be useful to introduce the notation

$$C_\alpha = \frac{1}{2\pi} \int \cos q e^{-i\alpha\theta} d\theta = \frac{1}{2}(c_{1,\alpha} + c_{-1,\alpha}) \tag{35}$$
$$M[g] = \int g(q,p) \cos q \; dq \; dp = 2\pi \sum_{\alpha \in \mathbb{Z}} \int C_\alpha(J) g_\alpha(J) dJ.$$

$M[g]$ is the "magnetization" associated to the phase space distribution g. Recalling (12), the only non trivial part of the Λ_{kl} matrix is for $l = \pm 1$. Hence we have to look at the block $\Lambda_{kl}, k, l = \pm 1$. Making use of the symmetries of the trajectories in a cosine potential, we conclude that the determinant of this 2×2 matrix vanishes if and only if

$$\chi = 1 + 2\pi \sum_{\alpha \neq 0} \int \frac{F_0'(J)}{\Omega_0(J)} C_\alpha^2(J) dJ = 0 \; ; \tag{36}$$

this convenient criterion makes the search for critical stationary states easier. Furthermore, under the criticality condition $\xi = 0$, it turns out that $\Lambda_{1,1} + \Lambda_{1,-1} = 0$. Hence we can take $y_1 = y_{-1} = 2$ as a representative of the kernel of Λ (now considered as a 2×2 matrix). Then the eigenvectors and generalized eigenvectors are

$$\psi^{(0)} = \begin{pmatrix} 0 \\ -\frac{F_0'(J)}{\Omega_0(J)} C_\alpha(J) \end{pmatrix} \tag{37}$$
$$\psi^{(1)} = \begin{pmatrix} 0 \\ \frac{F_0'(J)}{i\alpha\Omega_0^2(J)} C_\alpha(J) \end{pmatrix} \tag{38}$$
$$\psi^{(2)} = \begin{pmatrix} \psi_0^{(2)}(J) \\ -\frac{F_0'(J)}{(i\alpha)^2\Omega_0^3(J)} C_\alpha(J) \end{pmatrix}. \tag{39}$$

The normalization choice for y_1, y_{-1} ensures that $M[\psi^{(0)}] = 1$. $M[\psi^{(1)}] = 0$ because the α and $-\alpha$ terms cancel each other. We impose $M[\psi^{(2)}] = 0$ by choosing

$$\psi_0^{(2)}(J) = a_0^{(2)} C_0(J) \; , \text{ with } a_0^{(2)} = \frac{b_2}{2\pi \int C_0^2(J) dJ},$$

where

$$b_2 = 2\pi \sum_{\alpha \in \mathbb{Z}^*} \frac{1}{(i\alpha)^2} \int \frac{F_0'(J)}{\Omega_0^3(J)} C_\alpha^2(J) dJ.$$

The adjoint eigenvectors and generalized eigenvectors are

$$\phi^{(0)} = \begin{pmatrix} C_0(J)/b_2 \\ 0 \end{pmatrix} \tag{40}$$
$$\phi^{(1)} = \begin{pmatrix} 0 \\ \frac{-1}{i\alpha\Omega_0(J)} C_\alpha(J)/b_2 \end{pmatrix} \tag{41}$$
$$\phi^{(2)} = \begin{pmatrix} \phi_0^{(2)}(J) \\ -\frac{1}{[i\alpha\Omega_0(J)]^2} C_\alpha(J)/b_2 \end{pmatrix}, \tag{42}$$

The normalization are chosen so that

$$\forall k, l \in \{0,1,2\} \; , \; \langle \phi^{(k)}, \psi^{(l)} \rangle = \delta_{k,2-l}.$$

To enforce the normalization for $k = 2, l = 0$, we choose

$$\phi_0^{(2)}(J) = b_0^{(2)} C_0(J) \; ,$$

with

$$b_0^{(2)} = \frac{-2\pi \sum_{\alpha \neq 0} \frac{1}{\alpha^4} \int \frac{F_0'(J)}{\Omega_0^5(J)} C_\alpha^2(J) dJ}{b_2^2}.$$

These expressions can now be inserted into the non linear computations of the next section, in order to obtain a reduced model close to the bifurcation, with explicitly computable coefficients.

NON LINEAR COMPUTATION AT QUADRATIC ORDER FOR A 1D VLASOV EQUATION

Building on the results of the first section, we derive now the reduced dynamics at quadratic order for a generic 1D Vlasov equation. We perform here a standard central manifold computation, check that no divergence appears (at variance with the resonant case [2, 3]),

and remark in the end that the final equations have the structure of a 3D bifurcating non canonical Hamiltonian system, as was obtained in Eq. (8) of the main paper using genericity arguments. The present computations of course provide explicit expressions for the coefficients of the reduced Hamiltonian system, allowing for a quantitative check of the theory.

Central manifold computation

The generalized eigenspace $E_0 = \mathrm{Span}(\psi^{(0)}, \psi^{(1)}, \psi^{(2)})$ is invariant for the linearized dynamics. It is natural to look for a 3D manifold $\mathcal{M}$, invariant for the whole dynamics, tangent to E_0 at the origin, which can be defined at least locally, and represented as a graph over E_0. Then, any $g \in \mathcal{M}$ can be written as

$$g(\theta, J) = \sum_{i=0}^{2} A_i \psi^{(i)} + H[A_0, A_1, A_2](\theta, J),$$

with H of order $(A_0, A_1, A_2)^2$. We consider now a family of stationary states F_μ, with $F_{\mu=0}$ the critical stationary state. We are interested in the regime $\mu > 0$ and small, when there is a single small unstable eigenvalue. The linearized Vlasov operator then writes $\mathcal{L}_\mu = \mathcal{L}_0 + \mu\delta\mathcal{L} + O(\mu^2)$, where $\mathcal{L}_\mu$ was introduced in the previous section (denoted there for simplicity $\mathcal{L}$), and E_0 is the generalized eigenspace of $\mathcal{L}_0 = \mathcal{L}_{\mu=0}$. The $O(\mu^2)$ terms will be neglected. The Vlasov equation for the perturbation g reads

$$\partial_t g = \mathcal{L}_0 \cdot g + \mu\delta\mathcal{L} \cdot g + B(g, g)$$

where the nonlinear term is given by the quadratic operator

$$B(g, h) = \frac{\partial g}{\partial J}\frac{\partial V[h]}{\partial \theta} - \frac{\partial g}{\partial \theta}\frac{\partial V[h]}{\partial J} \tag{43}$$

Expanding in angle Fourier series yields

$$B(g,h)_\alpha = \sum_{k\in\mathbb{Z}} v_k Y_k[h] \sum_\beta i\beta \frac{\partial g_{\alpha-\beta}}{\partial J} c_{k,\beta} - \sum_{k\in\mathbb{Z}} v_k Y_k[h] \sum_\beta i\beta g_\beta \frac{\partial c_{k,\alpha-\beta}}{\partial J}$$

and $Y_k[f]$ has been defined in (9). At quadratic order, we have to compute for $0 \le i, j, k \le 2$:

$$C_{ijk} = \langle \phi^{(i)}, B(\psi^{(j)}, \psi^{(k)}) \rangle.$$

Using (19),(20),(21), (30),(31),(32), $Y_k[\psi^{(1)}] = 0$, $Y_k[\psi^{(2)}] = 0$ and the symmetry (7), we can see that many coefficients vanish. The non zero ones are $C_{100}, C_{210}, C_{120}, C_{010}$. We also need to compute the contributions of $\mu\delta\mathcal{L}$. We have (emphasizing that all terms have leading order μ):

$$\mu\delta\mathcal{L} \cdot g = -\mu\ \delta\Omega\ \partial_\theta g - \mu\ \delta F_0'\ \partial_\theta V[g].$$

We have a priori 9 terms to compute, $\langle \phi^{(i)}, \delta\mathcal{L} \cdot \psi^{(j)} \rangle$. The non zero ones are

$$a = \langle \phi^{(1)}, \delta\mathcal{L}\cdot\psi^{(0)} \rangle\ ,\ b = \langle \phi^{(2)}, \delta\mathcal{L}\cdot\psi^{(1)} \rangle\ ,\ c = \langle \phi^{(1)}, \delta\mathcal{L}\cdot\psi^{(2)} \rangle\ .$$

The final reduced equations are

$$\begin{aligned} \dot{A}_0 &= (1+\mu b)A_1 + C_{210}A_0A_1, \\ \dot{A}_1 &= (1+\mu c)A_2 + \mu a A_0 + C_{100}A_0^2 + C_{120}A_0A_2, \\ \dot{A}_2 &= C_{010}A_0A_1, \end{aligned} \tag{44}$$

where the coefficients have to be computed numerically.

Hamiltonian form

We perform successively the near identity changes of variables

$$\begin{cases} X_0 = A_0 - \left(\frac{1}{2}C_{210} + \frac{1}{6}C_{120}\right)A_0^2, \\ X_1 = A_1 - \frac{1}{3}C_{120}A_0A_1, \\ X_2 = A_2 + C_{100}A_0^2 - \frac{1}{3}C_{120}A_1^2 + \frac{2}{3}C_{120}A_0A_2, \end{cases} \tag{45}$$

then

$$\begin{cases} Y_0 = \dfrac{1}{(1+\mu c)(1+\mu b)}X_0 \\ Y_1 = \dfrac{1}{1+\mu c}X_1 \\ Y_2 = X_2 + \mu a Y_0 \end{cases} \tag{46}$$

and finally

$$\begin{cases} Z_0 = Y_0 \\ Z_1 = Y_1 \\ Z_2 = Y_2 - \frac{1}{2}rY_0^2 - \mu a Y_0 \end{cases} \tag{47}$$

with $r = C_{010} + 2C_{100}$. Truncating the equations of motion at quadratic order, and neglecting terms of order μZ^2 or $\mu^2 Z$, they become

$$\begin{cases} \dot{Z}_0 = Z_1 \\ \dot{Z}_1 = Z_2 + \frac{1}{2}rZ_0^2 + \mu a Z_0 \\ \dot{Z}_2 = 0 \end{cases} \tag{48}$$

This can be rewritten as a non canonical Hamiltonian system, with degenerate Poisson operator:

$$\dot{Z} = J\nabla H(Z), \tag{49a}$$

with

$$Z = \begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{49b}$$

$$H(Z) = \frac{1}{2}Z_1^2 - \frac{1}{6}rZ_0^3 - \frac{1}{2}\mu a Z_0^2 - Z_0 Z_2. \qquad (49c)$$

Eq.(49c) is indeed of the same form as Eq.(8) in the main paper.

EXPLICIT REDUCTION FOR THE HMF MODEL AND NUMERICAL DETAILS

We provide here details on the simulations used in the article. We compare the reduced dynamics Eq. (49) with Direct Numerical Simulations (DNS) of the Vlasov-HMF equation, Eqs. (1,2) of the main text, simulated via the algorithm of Ref. [4] based on a second order time-split algorithm with a local modified cubic-spline interpolation.

In the DNS simulations the phase space (q, p) is divided into $N_q \times N_p$ grid elements with $N_q = N_p = 4096$. We have $q \in [-\pi, \pi]$ and cut off the velocity region to $p_{\max} = -p_{\min} = 2$ (which is enough to well capture the whole F density). It corresponds to $\Delta q = 2\pi/N_q \simeq 1.5 \cdot 10^{-3}$ and $\Delta p = 2p_{\max}/N_p \simeq 9.8 \cdot 10^{-4}$. The time step is $\Delta t = 5 \cdot 10^{-3}$.

We choose as a family F_μ of stationary states the non homogeneous "Fermi-Dirac like" distributions

$$F_\mu(H) = \mathcal{N}^{-1} \frac{1}{1 + e^{\beta[H-(\kappa-\mu)]}}, \qquad (50)$$

and fix $\beta = 40$, $\kappa = 0.6693$. Then $\mu = 0$ corresponds to a bifurcation point, with a real positive eigenvalue appearing for $\mu > 0$ [7]. The critical stationary state at $\mu = \mu_c = 0$ has magnetization $M_0 = 0.3361$. In the main text, we disturb a slightly unstable stationary state with $\mu = 1.44 \cdot 10^{-4}$, $M_\mu = 0.3360$ (it has an associated unstable eigenvalue $\lambda = 8.62 \cdot 10^{-3}$) as

$$F(t=0) - F_\mu = F_T = \varepsilon \cos q e^{-\beta_T p^2}, \qquad (51)$$

with different perturbations ε and $\beta_T = 10$. We then plot the time evolution of the magnetization $M(t) - M_\mu$ (see Fig. 2 of the main text) and compare it to the amplitude $Z_0(t)$ of the reduced model, Eq. (49), which is obtained analytically in the next section Eq. (54).

In the following we identify three challenges making the comparison between the reduced model (3D) Eq. (49) and the DNS difficult.

(I) *Numerical error of DNS:* The choices of the grid elements number N_q, N_p, maximum velocity $p_{\max}$ and time step Δt are crucial. Indeed, a large number of grid elements is needed to resolve the fine scales in the region of interest. In that sense a small $p_{\max}$ is good but at the same time $p_{\max}$ has to be large enough to capture the tails of the density $F(q, p, t)$ which might affect the dynamics. In addition, all the numerical parameters must respect some Courant-Friedrichs-Lewy stability condition for the numerical scheme to be stable. In order to know which parameters gives a result closer to the "truth" we rely on the computation of some conservation law. Namely the energy conservation $\mathrm{err}_\mathcal{E} = (\mathcal{E}(t) - \mathcal{E}(0))/\mathcal{E}(0)$ and the norm conservation $\mathrm{err}_\mathcal{N} = (\mathcal{N}(t) - \mathcal{N}(0))/\mathcal{N}(0)$ where $\mathcal{E}(t)$ is the energy associated with the density $F(q, p, t)$ and $\mathcal{N}(t)$ the norm of $F(t)$. The Vlasov equation should exactly conserves these quantities, hence we compute and find the parameters that minimize these errors. Typically the errors at $t = 200$ are $\mathrm{err}_\mathcal{N} \sim 10^{-13}$ and $\mathrm{err}_\mathcal{E} \sim 10^{-9}$.

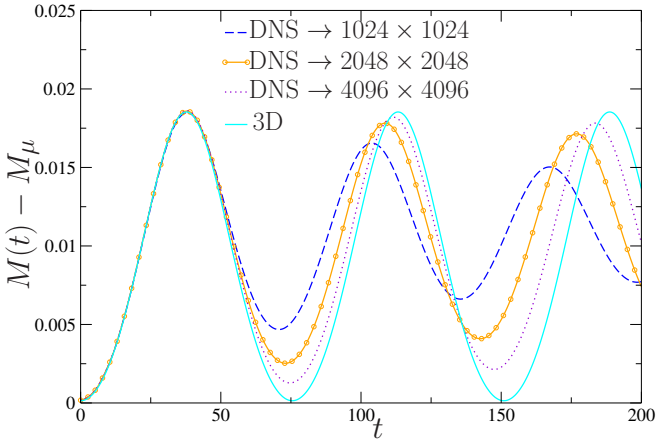

FIG. 1. Magnetization as a function of time: comparison at $\varepsilon = 1 \cdot 10^{-4}$ between direct numerical simulations (DNS) of Eq. (1) for different phase space grid number $N_q \times N_p$ and the analytic theory (3D) of Eq.(49).

This way, we can verify that fewer grid elements increases the numerical damping/shifting (all other parameters remaining constant) caused by the numerical solver (and well known in Vlasov simulations [5]). We illustrate the phenomenon, in Fig. 1, where we clearly see that higher N_q, N_p diminishes the effect of numerical damping and shifting. The DNS seem to converge toward the reduced model with higher grid number N_q, N_p.

(II) *Initial Conditions:* In order to have a quantitative comparison between DNS and reduced model, we need to compute the initial condition $(Z_0(0), Z_1(0), Z_2(0))$ corresponding to the perturbation F_T into the DNS. They can be explicitly computed by the following scalar products $A_0(t=0) = \langle \phi^{(0)}, F_T \rangle$, $A_1(t=0) = \langle \phi^{(1)}, F_T \rangle = 0$ (by symmetry of F_T) and $A_2(t=0) = \langle \phi^{(2)}, F_T \rangle$. Using that $Z(t=0) = A(t=0) + O(\text{quadratic}) \simeq A(t=0)$, we obtain the initial condition for the model Eq. (49). One problem is that the actual initial perturbation of the DNS is discretized onto the phase space grid which can affect the effective initial values. Ideally we could choose an initial perturbation along the eigenvectors $\psi^{(i)}$, however, they have singular $C_\alpha(J)$ contributions at the separatrix which is in practice hard to reproduce at the DNS grid level. Hence, we focus on a smooth perturbation F_T, as in Eq. (51).

(III) *Asymptotic validity:* The reduced model Eq. (49) is valid close to the bifurcation when $\mu \to 0$ and when

the amplitude of the vector $Z(t)$ remains small (so that neglecting higher orders is valid). Hence, the steady state F_μ^0 determined by (μ, M_μ) must stay close to the bifurcation point $(0, M_0)$. Moreover, the initial perturbation F_T from F_μ must also be small. Since $M[F](t) - M_\mu = Z_0(t) + O(\text{quadratic})$, $Z_0(t)$ gives directly the magnetization only in the small amplitude regime.

Keeping in mind these potential issues, one can from Eq. (50) compute the coefficients in Eq. (44) associated with the linear term at $(\mu,\ M_\mu)$: $\mu a \simeq 7.44 \cdot 10^{-5}$, as well as the ones involved in the quadratic terms $C_{010} = -0.409$, $C_{100} \simeq -0.181$. The initial conditions are found to be $Z_0(0) \simeq 1.42\varepsilon$, $Z_1(0) = 0$, $Z_0(0) \simeq 0.445\varepsilon$. We see that changing ε simultaneously affects $Z_0(0)$ and $Z_2(0)$. Note that all coefficients and initial condition are explicitly computed with their derived expression and thus we directly compare without any fitting parameters the DNS and the reduced dynamics.

ANALYTIC SOLUTION OF THE REDUCED DYNAMICS

We derive the explicit solution $Z_0(t)$ for the reduced dynamics Eq. (48) in terms of the Weierstrass elliptic function [8, 9]. First using that $Z_2(t) = Z_2(0))$ is constant and $\dot{Z}_0 = Z_1$, we get

$$\ddot{Z}_0 = Z_2 + \mu a Z_0 + \frac{1}{2} r Z_0^2,$$

which, by simple manipulation and variable changes $T = \sqrt{-r/12}t$ and $y = -(Z_0 + \frac{\mu a}{r})$, can be cast into

$$\left(\frac{dy}{dT}\right)^2 (T) = 4y^3(T) - g_2 y(T) - g_3, \tag{52}$$

with

$$g_2 = \frac{12}{r^2}\left((\mu a)^2 - 2rZ_2\right) \tag{53a}$$

$$g_3 = \frac{12}{r^2}\left(\left(\frac{\mu a}{r} + Z_0(0)\right)\left((\mu a)^2 - 2rZ_2\right) - \frac{r^2}{3}\left(\frac{\mu a}{r} + Z_0(0)\right)^3 + rZ_1(0)\right). \tag{53b}$$

Eq. (52) is sometimes used as a definition for the doubly-periodic Weierstrass elliptic $\wp-$function and has for solution $y(T) = \wp(T - T_0; g_2, g_3)$ where T_0 is determined by the initial conditions [8, 9]. Hence,

$$Z_0(t) = -\left(\wp\left(2\sqrt{\frac{3}{-r}}t - T_0; g_2, g_3\right) + \frac{\mu a}{r}\right). \tag{54}$$

In our case $Z_1(0) = 0$, $g_2 \gg g_3$ and $r < 0$, one can show that $T_0 = \omega_1 + \omega_2$ where ω_1, ω_2 are the two half periods of the $\wp-$function associated with g_2 and g_3. When $g_2 > 0$ (corresponding for our numerical experiment to $\varepsilon > 0$) the oscillation period of $Z_0(t)$ is asymptotically $\tau = 2\sqrt{-12/r}\omega_1 \sim 3^{1/4}\sqrt{\pi}\frac{\Gamma(1/4)}{\Gamma(3/4)}\frac{1}{((\mu a)^2 - 2rZ_2)^{1/4}}$ (in this case ω_2 is purely imaginary and has no clear physical meaning). Moreover in that case $(Z_0)_{\max} \sim -(\sqrt{3((\mu a)^2 - rZ_2)} + \mu a)/r$. When $Z_2 \ll (\mu)^2$, e.g. when the perturbation is taken exactly along Ψ_0 then $Z_2 = 0$, we have $(Z_0)_{\max} \propto \mu$ so that $(Z_0)_{\max} \propto \lambda^2$. When $g_2 < 0$ (i.e. $\varepsilon < 0$), the function $\wp(T - T_0)$ despite still being periodic encounters a pole in finite time and thus no longer has physical meaning.

OBTENTION OF THE NORMAL FORM FOR THE REDUCED HAMILTONIAN

The procedure to simplify a Hamiltonian in order to obtain a normal form is classical. For self consistency, we give here more details on how to use it to obtain Hamiltonian (10) in the main text.

We consider the normal form of the noncanonical Hamiltonian system

$$\frac{du}{dt} = J\nabla_u H(u, \mu), \quad u = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}, \tag{55}$$

around the origin $u = 0$, which is assumed to be stationary. The matrix J is defined in (49b). From the stationarity of the origin and the use of the Casimir invariants u_2, we may assume $\nabla_u H(0, 0) = 0$.

The idea to obtain the normal form is to simplify $H(u, \mu)$ by using the coordinate transform

$$u = T(U), \tag{56}$$

which gives the transformed Hamiltonian

$$\bar{H}(U, \mu) := H(T(U), \mu). \tag{57}$$

To keep the Poisson structure expressed by the J matrix, we introduce the constraint

$$[DT(U)]^{-1} J [DT(U)]^{-\mathrm{T}} = J \tag{58}$$

for the transform T, where $DT(U)$ is the Jacobian matrix of $T(U)$ and the superscript $-\mathrm{T}$ represents the transposition of the inverse matrix. Under the constraint (58), the transformed equations of motion is written as

$$\frac{dU}{dt} = J\nabla_U \bar{H}(U, \mu), \quad U = \begin{pmatrix} U_0 \\ U_1 \\ U_2 \end{pmatrix}. \tag{59}$$

We compute the normal form $\bar{H}(U, \mu)$ up to the cubic order of U. For this purpose, we expand the transform as

$$T(U) = U + T_2(U) + O(|U|^2), \tag{60}$$

the original Hamiltonian as

$$H(u,\mu) = H_2(u,\mu) + H_3(u,\mu) + O(|u|^4) \tag{61}$$

from $\nabla_u H(0,0) = 0$, and the transformed Hamiltonian as

$$\bar{H}(U,\mu) = \bar{H}_2(U,\mu) + \bar{H}_3(U,\mu) + O(|U|^4). \tag{62}$$

Substituting (60) into (61), we have

$$\bar{H}_2(U,\mu) = H_2(U,\mu) \tag{63}$$

and

$$\bar{H}_3(U,\mu) = H_3(U,\mu) + \nabla_u H_2(U,\mu)\cdot T_2(U). \tag{64}$$

Our job is to simplify $\bar{H}_3(U,\mu)$ for a given $H_2(u,\mu)$ by using the transform T_2 under the constraint of

$$(DT_2)J - [(DT_2)J]^{\mathrm{T}} = O_3, \tag{65}$$

which comes from (58) and O_3 is the zero matrix of size 3.

We first determine the explicit form of $T_2(U)$. The general form of $T_2(U)$ is written as

$$T_2(U) = \begin{pmatrix} a_1U_0^2 + a_2U_1^2 + a_3U_2^2 + a_4U_0U_1 + a_5U_1U_2 \\ \quad + a_6U_2U_0 \\ b_1U_0^2 + b_2U_1^2 + b_3U_2^2 + b_4U_0U_1 + b_5U_1U_2 \\ \quad + b_6U_2U_0 \\ c_1U_0^2 + c_2U_1^2 + c_3U_2^2 + c_4U_0U_1 + c_5U_1U_2 \\ \quad + c_6U_2U_0 \end{pmatrix} \tag{66}$$

where a_j, b_j and c_j are real parameters. The constraint (65) requires

$$\begin{cases} (2a_1 + b_4)U_0 + (a_4 + 2b_2)U_1 + (a_6 + b_5)U_2 = 0 \\ 2c_1U_0 + c_4U_1 + c_6U_2 = 0 \\ 2c_2U_1 + c_4U_0 + c_5U_2 = 0 \end{cases} \tag{67}$$

and therefore,

$$T_2(U) = \begin{pmatrix} a_1U_0^2 + a_2U_1^2 + a_3U_2^2 + a_4U_0U_1 + a_5U_1U_2 \\ \quad + a_6U_2U_0 \\ b_1U_0^2 + b_2U_1^2 + b_3U_2^2 + b_4U_0U_1 + b_5U_1U_2 \\ \quad + b_6U_2U_0 \\ c_3U_2^2 \end{pmatrix} \tag{68}$$

with

$$b_2 = -\frac{a_4}{2}, \quad b_4 = -2a_1, \quad b_5 = -a_6. \tag{69}$$

Now, $a_1, \cdots, a_6, b_1, b_3, b_6$ and c_3 are the free parameters. The simplification of $\bar{H}_3(U,\mu)$, (64), is realized by eliminating terms of $H_3(U,\mu)$ by using these free parameters included in $\nabla_u H_2(U,\mu)\cdot T_2(U)$.

The next step is to determine the quadratic Hamiltonian $H_2(u,\mu)$. We set

$$H_2(u,\mu) = \frac{u_1^2 - \mu u_0^2}{2} - u_0u_2, \tag{70}$$

which gives the linear part as

$$J\nabla_u H_2(u,\mu) = \begin{pmatrix} 0 & 1 & 0 \\ \mu & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}. \tag{71}$$

Thus, the choice of (70) permits to have the 3D Jordan block at $\mu = 0$, which is predicted in the general setting, and to change stability of the origin at $\mu = 0$ (unstable for $\mu < 0$ and stable for $\mu > 0$).

Before computing the normal form, we remark that $|\mu|$ is small and the dominant part of $\bar{H}_3(U,\mu)$ is given by $\bar{H}(U,0)$. Therefore, we use $\nabla_u H_2(U,0)\cdot T_2(U)$ to eliminate terms of $H_3(U,0)$. The quadratic Hamiltonian $H_2(u,\mu)$, (70), with the transform $T_2(U)$, (68), give

$$\begin{aligned} &\nabla_u H_2(U,0)\cdot T_2(U) \\ &= \left(\frac{-a_4}{2}\right) U_1^3 + (-a_3)U_2^3 + (-a_4 + b_6)U_0U_1U_2 \\ &+ b_1U_0^2U_1 - a_1(U_0^2U_2 + 2U_1^2U_0) + (-a_2 - a_6)U_1^2U_2 \\ &+ (-a_6 - c_3)U_2^2U_0 + (-a_5 + b_3)U_2^2U_1. \end{aligned} \tag{72}$$

Thanks to the 10 free parameters, we can eliminate almost all terms of $H_3(U,0)$, but U_0^3 can not be eliminated. In addition, one of the two terms $U_0^2U_2$ or $U_1^2U_0$ survives since they share the free parameter a_1.

Finally, considering the scaling of U in the equations of motion (59) with the scaling of t, we have the normal form of $\bar{H}$ up to the cubic order as

$$\bar{H}_{\mathrm{norm}} = \frac{U_1^2 - \mu U_0^2}{2} - U_0U_2 + U_0^3 + rU_0^2U_2 \tag{73}$$

or

$$\bar{H}_{\mathrm{norm}} = \frac{U_1^2 - \mu U_0^2}{2} - U_0U_2 + U_0^3 + rU_1^2U_0, \tag{74}$$

where r is the undetermined parameter. In the system (73), we have two different stationary points if and only if

$$(\mu - 2rU_2)^2 + 12U_2 > 0. \tag{75}$$

Introducing the cubic order coordinate transform $T_3(U)$ and using the free parameters in $T_2(U)$ and $T_3(U)$, we find the terms of U_0^4 and $U_0^2U_1^2$ in the quartic order normal form, $\bar{H}_4$. Depending on their signs, these terms may confine orbits in a bounded region. Notice however that i) a full reduction of the dynamics up to quartic order would require to compute the curvature of the central manifold, whereas considering its tangent space is enough

up to cubic order; ii) while large amplitude periodic trajectories appearing at quartic order are reminiscent of the numerical results (see inset of Fig.2 of the article), they go beyond the perturbative regime, so that we cannot expect a quantitive agreement between reduced and exact dynamics.

[1] J. Barré, A. Olivetti, and Y. Y. Yamaguchi, Dynamics of perturbations around inhomogeneous backgrounds in the HMF model, J. Stat. Mech. (2010) P08002.

[2] N. J. Balmforth, P. J. Morrison, and J.-L. Thiffeault, Pattern formation in Hamiltonian systems with continuous spectra; a normal-form single-wave model, arXiv:1303.0065.

[3] J. D. Crawford, Universal Trapping Scaling on the Unstable Manifold for a Collisionless Electrostatic Mode, Phys, Rev. Lett. **73**, 656 (1994).

[4] T.M. Rocha Filho. Solving the Vlasov equation for one-dimensional models with long range interactions on a GPU. Computer Physics Communications **184**, 34 (2013).

[5] J. Banks, and J. Hittinger. A new class of nonlinear finite-volume methods for Vlasov simulation. *Ieee Transactions On Plasma Science*. **38**, 2198 (2010).

[6] D. Silantyev, P. Lushnikov, and H. Rose. Langmuir wave filamentation in the kinetic regime. I. Filamentation instability of Bernstein-Greene-Kruskal modes in multidimensional Vlasov simulations. *Physics of Plasmas* **24**, 042104 (2017).

[7] J. Barré, D. Métivier, and Y. Y. Yamaguchi, Trapping scaling for bifurcations in the Vlasov systems, Phys. Rev. E **93**, 042207 (2016).

[8] J. Snape. Applications of Elliptic Functions in Classical and Algebraic Geometry. (University of Durham, 2004).

[9] G. Pastras. Four Lectures on Weierstrass Elliptic Function and Applications in Classical and Quantum Mechanics. Arxiv E-prints arXiv:1706.07371