Invariant measures for intermittent transport

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Abstract: *We are interested in the existence and properties of limits of invariant measures for Brownian diffusions started at distance ϵ from the boundary of a given domain and stopped when they hit back this boundary, when ϵ goes to* 0.

1 Introduction

The motivation of the following work has its origin in experimental physics. Some long molecules are solvable in a liquid (for instance imogolite in water or DNA in lithium) and the molecules forming the liquid show an intermittent dynamics, alternating diffusion in the bulb and adsorption on the long molecules. For the physicist's point of view, it is very important to have as precise as possible knowledge of the statistics of these brownian flights.

In [GKL+06] a connection is established between the statistics of the long flight lengths and the geometry of the long molecules (more precisely their Minkowski dimension). This connection has been made rigorous in [BLZ11],[BZ10].

Nevertheless, the initial distribution of the starting point for these statistics is particularly important.The aim of this paper is to describe the stable starting distributions. We will show that the uniform distributionis stationnary with respect to Brownian flights and therefore the associated statistics are relevant.

To prove our theorems we will suitably discretize Brownian motion, following [BL96] and [LS84] and apply an adapted version of the Perron-Frobenius theorem to a finite Markov chain.

2 Some Backgound

In the sequel, Ω will always denote a domain in $\mathbb{R}^d$ with compact boundary. The crucial tool we need to use is the notion of Whintey cubes. We thus recall the

Proposition 2.1 *(cf. [Gra08], p. 463) Given any non-empty open proper subset Ω of $\mathbb{R}^d$, there exists a family $\mathcal{W}$ of closed dyadic cubes $\{Q_j\}_j$ such that*

- $\bigcup_j Q_j = \Omega$ *and the cubes Q_j's have disjoint interiors*

- $\sqrt{d}\ell(Q_j) \leq$ *dist*$(Q_j, \partial\Omega) \leq 4\sqrt{d}\ell(Q_j)$
- *if* Q_j *and* Q_k *touch then* $\ell(Q_j) \leq 4\ell(Q_k)$
- *for a given Whitney cube* Q_j *there are at most* 12^d *Whitney cubes* Q_k*'s that touch* Q_j.

In this statement, $\ell(Q)$ stands for the side-length of the cube Q and, for $\lambda > 0$, λQ is the cube of the same center and of sidelength $\lambda\ell(Q)$. For $k \in \mathbb{Z}$, we denote by $\mathcal{Q}_k$, the collection of Whitney cubes Q_j with $\ell(Q_j) = 2^k$. We also recall the definition of the Minkowski sausage: for $r > 0$,

$$M_r = \{x \in \Omega \; ; \; \; \mathrm{dist}(x, \partial\Omega) \leq r\}$$

and

$$\Gamma_r = \{x \in \Omega \; ; \; \; \mathrm{dist}(x, \partial\Omega) = r\}$$

We then define $\mathcal{S}_r$ as the collection of Whitney cubes intersecting Γ_r. Notice that $\mathcal{S}_r$ is a finite set.

Definition 2.1 *Let* $\varepsilon > 0$. *We will call Brownian flight the random process* $F_t, t \geq 0$ *consisting in picking at random with equiprobability one of the dyadic Whitney cubes of* $\mathcal{S}_\varepsilon$ *and starting from the center of the cube a Brownian motion* g_t *killed once it reaches* $\partial\Omega$. *We denote by* $\tau_\Omega = \inf\{t \; ; \; F_t \notin \Omega\}$ *the lifetime of this process.*

Definition 2.2 *The Minkowski dimension of* K *is*

$$d_M(K) = \limsup_{j\to\infty} \frac{\log_2(N_j)}{j}$$

We can define similarly the Whitney dimension of $\partial\Omega$ as

$$d_W = d_W(\partial\Omega) = \limsup_{j\to\infty} \frac{\log_2(W_j)}{j}, \tag{1}$$

where W_j is the number of elements of $\mathcal{Q}_j$.

Under very mild conditions (see [Tri83], [Bis96], [JK82], [BLZ11]) these two dimensions coincide. If the boundary of Ω has some self similarity we can moreover say that there is a constant $c > 0$ such that

$$\frac{1}{c}\varepsilon^{d_M} \leq \#\mathcal{S}_\varepsilon \leq c\varepsilon^{d_M}, \tag{2}$$

for all $\varepsilon \leq R_\Omega$, where $d_M = d_M(\partial\Omega)$.

We also suppose that the domain Ω satisfies so-called Δ-regularity condition (see also [JW88], [Anc86], [HK93]): there exists $L > 0$ such that for all $x \in \Omega$, if $d_x = \mathrm{dist}(x, \partial\Omega) < R_\Omega$ then

$$\omega^x_{\mathbb{B}(x,2d_x)\cap\Omega}(\partial\Omega) \geq L, \tag{3}$$

where $\omega^x_{\mathbb{B}(x,2d_x)\cap\Omega}$ is the distribution law of the hitting point of Brownian motion starting at x and killed when reaching the boundary of $\mathbb{B}(x,2d_x)\cap\Omega$. This is a very mild condition (satisfied, for instance, by all domains in $\mathbb{R}^2$ with non-trivial connected boundary) that appears frequently in related literature in various forms (for instance "uniform capacity condition" or Hardy inequality).

The following result has been proven in [BLZ11]:

Theorem 2.3 *Let $\varepsilon < r < R_\Omega$. The probability that the hitting point of F is at distance greater than r from the starting point x is comparable to*

$$\left(\frac{\#\mathcal{S}_r}{\#\mathcal{S}_\varepsilon}\right)^{d_M}\left(\frac{r}{\varepsilon}\right)^{d-2} \tag{4}$$

Notice that we do not assume (2) for this theorem. If we do, we have

$$\left(\frac{\#\mathcal{S}_r}{\#\mathcal{S}_\varepsilon}\right)^{d_M}\left(\frac{r}{\varepsilon}\right)^{d-2} \sim \left(\frac{r}{\varepsilon}\right)^{d_M-(d-2)} \tag{5}$$

3 Discretization of Brownian Motion

We will modify the continuous diffusion process into a discrete one, with the same potential theory. In this section, Ω is a Green domain, $\mathcal{B}_t$ stands for Brownian motion in Ω, τ_Ω is the exit time (for brownian motion) of Ω, ie. the hitting time of $\partial\Omega$.

If G denotes the Green function of the domain $\Omega \subset \mathbb{R}^d$ and Q is a cube in Ω, recall that there exist a constant C such that for all $y \in Q$

$$\log\frac{\ell(Q)}{|x_Q - y|} \le G(x_Q, y) \le \log\frac{C\ell(Q)}{|x_Q - y|},$$

C depending on $\Omega \subset \mathbb{R}^2$ and

$$\frac{1}{||x_Q - y||^{d-2}} - \frac{1}{\ell(Q)^{d-2}} \le G(x,y) \le \frac{1}{||x_Q - y||^{d-2}},$$

for domains $\Omega \subset \mathbb{R}^d$, $d \ge 3$. Moreover,

$$\log\frac{\ell(Q)}{2|x_Q - y|} \le G_Q(x_Q, y) \le \log\frac{2\ell(Q)}{|x_Q - y|},$$

and

$$\frac{1}{||x_Q - y||^{d-2}} - \frac{C}{\ell(Q)^{d-2}} \le G_Q(x,y) \le \frac{1}{||x_Q - y||^{d-2}} - \frac{\sqrt{d}}{\ell(Q)^{d-2}},$$

for $d \ge 3$, G_Q being the Green function of the cube Q.

We denote by $\mathcal{N}$ be the collection of the centers of cubes in $\mathcal{W}$ and we consider the complete graph $\mathcal{G}$ associated. Let $x_Q \in \mathcal{N}$ be the center of a Whitney cube $Q \in \mathcal{W}$.

3.1 Planar domains

We consider separately planar domains not (only) because of the recurrence of brownian motion in $\mathbb{R}^2$ but in order to better explain the ideas of the proof.

Let $F_Q(\eta) = \{y \in \Omega \ ; \ G_Q(x_Q, y) \geq \eta\}$. Clearly, $F_Q(\eta)$ is a compact connected set, such that $x_Q \in F_Q$. Furthermore, by the preceeding observations and the definition of Whitney cubes we can deduce that, for η big enough, $F_Q = F_Q(\eta) \subset \mathring{Q}$ and that there is a constant $c_0 < 1$ not depending on Q such that $c_0 Q \subset \mathring{F}_Q$, where cQ will denote the (contracted) cube centered at x_Q but of sidelength $\ell(cQ) = c\ell(Q)$.

The triplet $(\mathcal{N}, \mathbf{F}, \mathbf{W})$, where $\mathbf{F} = \{F_Q \ ; \ Q \in \mathcal{W}\}$, and $\mathbf{W} = \{\mathring{Q} \ ; \ Q \in \mathcal{W}\}$ is a *balanced Lyons-Sullivan data*, defined in [BL96]. For convienience of the reader we remind hereby the principal facts of this paper.

1. The collection $\mathbf{F}$ is recurrent for Brownian motion in Ω, ie.
$$\mathbb{P}_x(\exists t < \tau_\Omega \ ; \ \mathcal{B}_t \in \bigcup_{\mathbf{F}} F_Q) = 1 \quad \text{for all } x \in \Omega.$$
2. $x_Q \in F_Q \subset \mathring{Q}$, for all $Q \in \mathcal{W}$,
3. $F_Q \cap Q' = \emptyset$, for all $Q \neq Q' \in \mathcal{W}$,
4. there exists a constant c such that for all $Q \in \mathcal{W}$, any positive harmonic function h in $\mathring{Q}$ and all $z \in F_Q$ we have
$$\frac{1}{c} h(x_Q) \leq h(z) \leq c h(x_Q)$$

Following [BL96] we define a Markov chain X on $\mathcal{N}$: for $y \in F = \bigcup_{\mathbf{F}} F_Q$ denote by $\phi(y) \in \mathcal{N}$ the center of the unique cube $Q = Q_y \in \mathcal{W}$ containing y. For a path ξ in the space of brownian paths Ξ starting at $y \in F$, let $S_0(\xi)$ be the exit time of ξ from Q_y. Recursively, we define the stopping times R_n and S_n in the following way

- $R_n(\xi) = \inf\{t > S_{n-1}(\xi) \ ; \ \xi(t) \in F\}$
- $S_n(\xi) = \inf\{t > R_{n-1}(\xi) \ ; \ \xi(t) \notin \mathring{Q}_{\xi(R_{n-1}(\xi))}\}$.

Recall that, if V is an open set and for any $x \in V$ we denote by ω_V^x the harmonic measure of V at x. By our hypothesis, there exist C such that for all $Q \in \mathcal{W}$ and all $y \in F_Q$,
$$\frac{1}{C} d\omega_{\mathring{Q}}^{x_Q} \leq d\omega_{\mathring{Q}}^{y} \leq C d\omega_{\mathring{Q}}^{x}.$$

Let now
$$\kappa_n(\xi) = \frac{1}{C} \frac{d\omega_{\mathring{Q}_{\phi(\xi(R_n(\xi)))}}^{\phi(\xi(R_n(\xi)))}(\xi(S_n(\xi)))}{d\omega_{\mathring{Q}_{\phi(\xi(R_n(\xi)))}}^{\xi(R_n(\xi))}(\xi(S_n(\xi)))} \leq 1$$

Using these stopping times Ballmann and Ledrappier consider the probability space

$$(\tilde{\Xi} = \Xi \times [0,1]^{\mathbb{N}}, \tilde{\mathbb{P}}_y = \mathbb{P}_y \otimes \lambda^{\mathbb{N}}),$$

λ being the Lebesgue measure in $[0,1]$. For $(\xi, \alpha) \in \tilde{\Xi}$ define recursively

- $N_0(\xi, \alpha) = 0$
- $N_k(\xi, \alpha) = \inf\{n > N_{k-1}(\xi, \alpha) \; ; \; \alpha_n < \kappa_n(\xi)\}$

One can then define a Markov chain (discrete random walk) X_i on $\mathcal{N}$ the centers of cubes in $\mathcal{W}$ with time homogeneous transition probabilities

$$p_{Q,Q'} = \tilde{\mathbb{P}}_{x_Q}(\xi(N_1(\xi, \alpha)) = x_{Q'}).$$

Let g be the Green function of this Markov chain on $\mathcal{N}$. The Markov chain is hence irreducible and aperiodic.

In [BL96, LS84] it is shown that for all $x = x_Q \in \mathcal{N}$ and all $y \neq x$

$$g(y,x) = \frac{1}{C} \sum_{n \in \mathbb{N}} \mathbb{P}_y(\xi(R_n(\xi)) \in F_Q) \tag{6}$$

and also that

$$G(y,x) = \sum_{n \in \mathbb{N}} \int_{F_Q} G_{\mathring{Q}}(z,x) \mathbb{P}_y(\xi(R_n(\xi)) \in dz) \tag{7}$$

By the choice of $F_Q = F_Q(\eta)$ and relations (6) and (7) we deduce that

$$g(x,y) = C\eta G(x,y) \tag{8}$$

and, moreover, that the transition probabilities of the Markov chain $p_{Q,Q'}$ are symmetric in Q, Q', ie. $p_{Q,Q'} = p_{Q',Q}$.

3.2 Domains in higher dimensions

We consider now bounded domains $\Omega \subset \mathbb{R}^d$, $d \geq 3$. In this setting we can not choose the sets $F_Q(\eta)$ in the same way. Such a choice would be in contradiction with the fourth definition property of Lyons-Sullivan data.

We will choose $\eta = \eta(Q)$ proportionnal to the distance of Q to the boundary. To start with, remark that, by the definition of Whitney cubes, if $Q \cap \Gamma_s \neq \emptyset$, then necessarily $Q \cap \Gamma_{s/4\sqrt{d}} = \emptyset$. Let $b = 1/(4\sqrt{d})$. For $Q \in \mathcal{W}$ put $\eta(Q) = b^{n(d-2)}$ if $Q \cap \Gamma_{b^n} \neq 0$ for some n and $\eta(Q) = \ell(Q)^{d-2}$ otherwise.

All previous definitions and properties stay valid except for (8). This equality must now be replaced by the following one : $\forall x \neq y \in \mathcal{N}$ such that for some $n \in \mathbb{N}$ both Q_x, Q_y are in $\mathcal{S}_{b^n}$ (ie. $Q_x \cap \Gamma_{b^n} \neq \emptyset$ and $Q_x \cap \Gamma_{b^n} \neq \emptyset$) we have

$$g(x,y) = Cb^n G(x,y) \tag{9}$$

and transition probabilities $p_{Q,Q'}$ are symmetric under the same conditions.

Potential theory for this Markov chain is equivalent to the potential theory for Brownian motion in Ω: in fact, the positive harmonic functions of the Markov chain are precisely the traces on $\mathcal{N}$ of positive harmonic functions in Ω, [Anc90].

4 An equivalent discrete model for Brownian flights

Let us now modify the initial model to make it "compatible" with discretized Brownian motion. The idea is to adapt the following remark (in fact Perron-Frobenius theorem) : if we replace brownian motion by X_k, a symmetric simple random walk on a graph, say $T = (\mathbb{Z}/n)^d$, we can consider the random process that consists on picking a boundary point x of T with probability distribution μ , starting random walk at this point and consider the first time τ the random walk gets back to the boundary of T. Clearly, the uniform measure ν on the boundary of T is invariant by the process $\nu(y) = \sum_x \nu(x)\mathbb{P}_x(X_\tau = y)$.

Recall that $\mathcal{S}_{2^{-n}}$ is the collection of Whitney cubes intersecting $\Gamma_{2^{-n}}$ (essentially the cubes at distance 2^{-n} to the boundary). Let us also assume, for the moment, that $\partial\Omega$ is bounded, of diameter say 1.

The dynamical system we are interested in is the following. Given a (discrete) probability measure μ on $\mathcal{S}_{2^{-n}}$, choose a cube $Q \in \mathcal{S}_{2^{-n}}$ with probability $\mu(Q)$. Consider the Markov chain $({}^Q X_k)$ defined above started at the center of Q, $X_0 = x_Q$. Since Ω is Greenian (random walk and Brownian motion are transient) so is the X_k on $\mathcal{N}$. Therefore, there is, almost surely, a finite time $\tau_n = \sup\{k \geq 0 \ ; \ {}^Q X_k \in \mathcal{S}_{2^{-n}}\}$, the last exit time of the random walk from the union of cubes in $\mathcal{S}_{2^{-n}}$.

We consider the function π assigning at every μ the exit distribution of ${}^Q X_{\tau_n}$. It is now clear that there is a discrete invariant measure for this function, μ_n (we have identified the cubes in $\mathcal{S}_{2^{-n}}$ with their centers $\mathcal{N} \cap \mathcal{S}_{2^{-n}}$).

The same tools used in [BLZ11] can now be used to prove the analogue of theorem 2.3:

Theorem 4.1 *Choose Q at random with uniform law within $\mathcal{S}_{2^{-n}}$. The probability that the distance $||x_Q - {}^Q X_{\tau_n}|| > r$ is comparable to*

$$\left(\frac{\#\mathcal{S}_r}{\#\mathcal{S}_{2^{-n}}}\right)^{d_M} (r2^n)^{d-2} \tag{10}$$

Recall that the domain Ω is assumed to verify the Δ-regularity condition (3). Under the same hypothesis we also have the main result:

Theorem 4.2 *There is a constant γ independent of n such that for all $Q \in \mathcal{S}_{2^{-n}}$,*

$$\frac{1}{\gamma\#\mathcal{S}_{2^{-n}}} \leq \mu_n(Q) \leq \frac{\gamma}{\#\mathcal{S}_{2^{-n}}},$$

ie. the measure μ_n is uniformly equivalent to the uniform measure on $\mathcal{S}_{2^{-n}}$. Moreover, for any measure μ on $\mathcal{S}_{2^{-n}}$ we have that $\lim_k \pi^k(\mu) = \mu_n$.

Proof For $Q, Q' \in \mathcal{S}_{2^{-n}}$, denote by

$$g_{Q,Q'} = g(x_Q, x_{Q'}) = \delta_{x_Q}(x'_Q) + \sum_{k=1}^{\infty} \mathbb{P}_{x_Q}({}^{Q}X_k = x_{Q'})$$

the mean time the random walk ${}^{Q}X_k$ started at x_Q spends inside Q'.

It follows on the construction of the random walk that there is a constant $\eta > 0$ such that $g_{Q,Q'} = g_{Q',Q} = \eta G(x_Q, x_{Q'})$. Let us point out here that for domains in higher dimension we need to pay attention that all $Q \in \mathcal{S}_{2^{-n}}$ have the same η.

Let us now consider, for $Q \in \mathcal{S}_{2^{-n}}$ the probability r_n^Q that random walk definitely leaves $\mathcal{S}_{2^{-n}}$ immediately after reaching Q, that is $r_n^Q = \mathbb{P}_Q(\tau_n = 0)$ by the Markov property.

Lemma 4.3 *There exists a constant $c > 0$ such that for all $n \in \mathbb{N}$ and all $Q \in \mathcal{S}_{2^{-n}}$, $r_n^Q \geq c$.*

We assume this lemma for the moment. The random walk being transient on $\mathcal{G}$, using the Markov property we get :

$$\begin{aligned}
1 &= \sum_{k=0}^{\infty} \sum_{Q' \in \mathcal{S}_{2^{-n}}} \mathbb{P}_{x_Q}({}^{Q}X_k = x_{Q'}\ ,\ \tau_n = k) \\
&= \sum_{k=0}^{\infty} \sum_{Q' \in \mathcal{S}_{2^{-n}}} \mathbb{P}_{x_Q}(\tau_n = k \mid {}^{Q}X_k = x_{Q'}) \mathbb{P}_{x_Q}({}^{Q}X_k = x_{Q'}) \\
&= \sum_{k=0}^{\infty} \sum_{Q' \in \mathcal{S}_{2^{-n}}} \mathbb{P}_{x_{Q'}}(\tau_n = 0) \mathbb{P}_{x_Q}({}^{Q}X_k = x_{Q'}) \\
&= \sum_{Q' \in \mathcal{S}_{2^{-n}}} \sum_{k=0}^{\infty} \mathbb{P}_{x_{Q'}}(\tau_n = 0) \mathbb{P}_{x_Q}({}^{Q}X_k = x_{Q'}) \\
&= \sum_{Q' \in \mathcal{S}_{2^{-n}}} \mathbb{P}_{x_{Q'}}(\tau_n = 0) \sum_{k=0}^{\infty} \mathbb{P}_{x_Q}({}^{Q}X_k = x_{Q'}) \\
&= \sum_{Q' \in \mathcal{S}_{2^{-n}}} g_{Q,Q'} r_n^{Q'},
\end{aligned}$$

for all $Q \in \mathcal{S}_{2^{-n}}$. Consider the measure μ_n on $\mathcal{S}_{2^{-n}}$ defined by

$$\mu_n(Q) = \frac{r_n^Q}{\sum_{Q' \in \mathcal{S}_{2^{-n}}} r_n^{Q'}}.$$

The probability that the probability that random walk started at Q definitely leaves $\mathcal{S}_{2^{-n}}$ through $\tilde{Q}$ is given by

$$b_{Q,\tilde{Q}} = \sum_{k=1}^{\infty} \mathbb{P}_{x_Q}({}^{Q}X_k = x_{\tilde{Q}}\ ,\ \tau_n = k).$$

Clearly, for any $\tilde{Q} \in \mathcal{S}_{2^{-n}}$

$$\begin{aligned}
\sum_{Q \in \mathcal{S}_{2^{-n}}} \mu_n(Q) b_{Q,\tilde{Q}} r_n^{\tilde{Q}} &= \sum_{Q \in \mathcal{S}_{2^{-n}}} \frac{r_n^Q}{\sum_{Q' \in \mathcal{S}_{2^{-n}}} r_n^{Q'}} b_{Q,\tilde{Q}} r_n^{\tilde{Q}} \\
&= \frac{r_n^{\tilde{Q}}}{\sum_{Q' \in \mathcal{S}_{2^{-n}}} r_n^{Q'}} \sum_{Q \in \mathcal{S}_{2^{-n}}} g_{Q,\tilde{Q}} r_n^Q \\
&= \frac{r_n^{\tilde{Q}}}{\sum_{Q' \in \mathcal{S}_{2^{-n}}} r_n^{Q'}} \sum_{Q \in \mathcal{S}_{2^{-n}}} g_{\tilde{Q},Q} r_n^Q,
\end{aligned}$$

because $g_{\tilde{Q},Q} = g_{Q,\tilde{Q}}$. Since $\sum_{Q \in \mathcal{S}_{2^{-n}}} g_{\tilde{Q},Q} r_n^Q = 1$ we get that μ_n is invariant:

$$\sum_{Q \in \mathcal{S}_{2^{-n}}} \mu_n(Q) g_{Q,\tilde{Q}} r_n^{\tilde{Q}} = \mu_n(\tilde{Q}).$$

By lemma 4.3, for all $Q \in \mathcal{S}_{2^{-n}}$, $c \leq r_n^Q \leq 1$. Hence, there is a constant $\gamma = \frac{1}{c}$ such that

$$\frac{1}{\gamma \# \mathcal{S}_{2^{-n}}} \leq \mu_n(Q) \leq \frac{\gamma}{\# \mathcal{S}_{2^{-n}}},$$

which is the first claim of the theorem.

The second claim follows on the fact that $(g_{Q,Q'} r_{Q'})_{Q,Q' \in \mathcal{S}_{2^{-n}}}$ is a stochastic matrix with strictly positive coefficients. •

We now turn to the proof of the lemma which strongly relies on the Δ-regularity hypothesis.

Proof of lemma 4.3 First observe that, by the definition of Whitney cubes, $\forall Q \in \mathcal{W}$ there is a Whitney cube $Q' \subset 8\sqrt{d}Q$ such that $\frac{\ell(Q)}{64\sqrt{d}^2} \leq \ell(Q') \leq \frac{\ell(Q)}{16\sqrt{d}}$.

Moreover, if $Q \in \mathcal{S}_{2^{-n}}$ and Q' as above, there is a constant $c > 0$ depending only on dimension such that the probability that Brownian motion stating at x_Q hits F at Q' for the first time, $\omega_{\Omega \setminus F}^{x_Q}(F_{Q'})$ is greater than c. Hence, there is a constant $c' > 0$ (depending only on the Lyons-Sullivan data) such that $\mathbb{P}_{x_Q}(X_1 = x_{Q'}) \geq c'$.

On the other hand, it follows on (3) that,

$$\omega^{x_{Q'}}_{\Omega \cap \mathbb{B}(x_{Q'}, 8\sqrt{d}\ell(Q'))} (\partial\Omega) \geq L.$$

We deduce that there exist c'' such that

$$\mathbb{P}_{x_{Q'}}(X_n \in \mathbb{B}(x_{Q'}, 8\sqrt{d}\ell(Q')) \ , \ \forall n \in \mathbb{N}) \geq c''.$$

And finally, by Markov's property $r_n^Q \geq c'c''$, which is the claim of the lemma. •

Theorem 4.2 has a continuous version for quasi-disks ; Let $\Omega \subset \mathbb{C}$ be a quasiconformal perturbation of the disc with small quasiconformal norm. In this case it is shown in [BLZ11] that the level lines Γ_r are locally Lipschitz graphs.

Consider the following operation. For $x \in \Gamma_r$, let $\mathcal{B}^x$ be Brownian motion started at x and killed on hitting $\partial\Omega$. Let τ be the hitting time of $\partial\Omega$ and $\sigma = \sup\{t < \tau \ ; \ \mathcal{B}^x_t \in \Gamma_r\}$. Consider the random variable $\mathcal{B}^x_\sigma$ that assigns to x the exit point of $\mathcal{B}^x$ from Γ_r and the distribution function associated ν_x. By usual fixed point arguments we get that there is an invariant measure ν_r :

$$\nu_r(dz) = \int_{\Gamma_r} \nu_x(dz)\nu_r(dx).$$

Theorem 4.4 *There is a constant γ such that, for $r > 0$, the invariant measure ν_r verifies $\frac{1}{\gamma}|dx| \leq \nu_r(dx) \leq \gamma|dx|$, where $|dx|$ stands for the (normalized) surface measure on Γ_r.*

Let us point out that, under (3), the probability $\mathbb{P}_x\left(|\mathcal{B}^x_\sigma - \mathcal{B}_\tau| > s\right) \leq cp^{s/r}$, where $p < 1$ and c are constants depending only on L (see [BZ10] for a complete proof). Therefore, the point $\mathcal{B}^x_\sigma$ is very close to the exit point $\mathcal{B}^x_\tau$.

Moreover, in the case of quasidisks we can define the mass transport differently. First of all remark that if the case of quasi-disks, for any $x \in \Gamma_r$ there exist an internal ray (Green line) γ_x intersecting Γ_r at x only and with a well-identified (unique) endpoint on $\partial\Omega$. The set of all endpoints of Green lines forms a set of harmonic measure 1. Instead of taking $\mathcal{B}^x_\sigma$ we associate to x the unique point $\mathcal{B}^x_{\tau\uparrow}$ on Γ_r lying on the Green line (internal ray) that has its endpoint at $\mathcal{B}_\tau$. The invariant measure ν'_r associated will be equivalent to ν_r defined above. The proof of this statement follows on the arguments of theorem 4.2 and on remarks in [BLZ11].

The proof of theorem 4.4 relies on theorem 4.2 but needs a few more arguments.

Proof We first show that for a dyadic Whitney cube Q centered on Γ_r we have that

$$\frac{1}{\gamma}|Q \cap \Gamma_r| < \nu_r(Q) < \gamma|Q \cap \Gamma_r|. \tag{11}$$

If we assume this inequality we use Harnack's principle to conclude : Observe first that by the definition of Whitney cubes, we have $\sqrt{d}Q \cap \partial\Omega = \emptyset$, for all cubes $Q \in \mathcal{W}$.

Therefore, there exist $C > 0$ such that for every positive harmonique function h in $\sqrt{d}Q$ and all $x, y \in \partial(\frac{\sqrt{d}+1}{2})$ we get $\frac{1}{C}h(y) \leq h(x) \leq Ch(y)$.

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The following result is a corollary of theorem 4.2.

Theorem 4.5 *Suppose that $\partial\Omega$ is an Alhfors s-regular set of finite Hausdorff measure $\mathcal{H}_s$. There is a constant γ such that every weak limite μ of the sequence μ_n satisfies*

$$\frac{1}{\gamma}r^s \leq \mu(\mathbb{B}_r) \leq \gamma r^s,$$

where $\mathbb{B}_r$ is any ball of radius r centered on $\partial\Omega$.

The uniqueness of the weak limit is false in general. Nevertheless, if the boundary is self-similar it is probable that the limit exists. Let us point out that, is $\partial\Omega$ is smooth enough, the above limit exists and is equal to the normalized surface measure. We must also cite here the results of [GS03] in the same vein.

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