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à la mémoire de mon frère

στη μνήμη του αδελφού μου

et à ma famille

και στην οικογένειά μου

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Il est pourtant nécessaire que je me lance...

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Για τους γονείς μου : η διατριβή αυτή σας ανήκει.

Summary

This thesis, presented to obtain the "Habilitation à Diriger des Recherches" diploma of the University of Orleans, is mainly concerned with problerms arising from random diffusion processes and potential theory, as the distribution of the harmonic measure in fractal domains, the multifractal analysis of measures and Brownian flights. It is divided in 5 thematical parts and contains a small appendix of basic notions.

Resumé

Cette thèse, présentée afin d'obtenir le diplôme d'"Habilitation à Diriger des Recherches" de l'Université d'Orléans, s'articule autour de problèmes émanant de la théorie du potentiel et des processus de diffusion aléatoires, comme la distribution de la mesure harmonique dans les domaines fractals, l'analyse multifractale et les vols browniens. Elle est séparée en 5 parties thématiques et elle se clôt par un petit rappel des notions fondamentales.

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Introduction

J'ai souhaité, plutôt que de reprendre mes résultats en un texte unique, adopter une démarche légèrement différente. Je présente d'abord cinq domaines de recherche dans lesquels j'ai travaillé et les résultats correspondants dans l'ordre chronologique. Il s'agit d'une présentation des problématiques, des résultats obtenus (y compris par moi) et des grandes lignes de leurs démonstrations. Les parties sont organisées en sous-sections thématiques. A la fin de chaque section j'aborde les questions que je me suis posées et qui restent ouvertes. Naturellement, ce sont des points que j'aimerais éclaircir dans l'avenir.

A la fin de la synthèse, en appendice, sont donnés quelques rappels de notions mentionnées fréquemment dans le texte (frontière de Martin, mesures et dimensions de Hausdorff, décomposition de Whitney, mouvement brownien réfléchi, discrétisation du mouvement brownien). Une bibliographie (non-exhaustive) suit cette appendice et enfin mes travaux sont annexés (en anglais).

La première partie, issue de mon doctorat [Bat97], est une adaptation de méthodes et résultats de C. Bishop [Bis91] en dimension supérieure. La relation avec des outils de la théorie du potentiel classique [Anc90, MP91] est ainsi mise en avant. Ces résultats, qui n'ont pas encore été soumis pour publication, font intervenir, outre les notions classiques de la théorie du potentiel, celles de dimension et de mesure de Hausdorff. Un extrait du premier chapitre de ma thèse est ajouté donc à la liste des publications : sa rédaction est détaillée et en français.

C'est précisément la notion de dimension de Hausdorff qui fait le lien avec le deuxième chapitre. Celui-ci contient ma contribution à l'analyse fine des mesures (analyse multifractale). Les résultats présentés sont extraits de trois articles, dont le premier est co-signé avec Yanick Heurteaux [BH02] et le dernier avec Benoît Testud [BT09]. Le premier contient un résultat général comparant deux notions de dimension : la dimension de Hausdorff et celle de Rényi ; le deuxième s'intéresse à la validité du formalisme multifractal dans le cadre des mesures dites de Markov tandis que le dernier se focalise sur les produits de Bernoulli inhomogènes.

La troisième partie de mon apport se concentre sur l'étude fine de la mesure harmonique. En particulier, je m'intéresse à la portée, aux limites et aux extensions possibles des célèbres résultats de N. Makarov [Mak85], de P. Jones et T. Wolff [JW88], [Wol93], [Wol95]. J'étudie ainsi la dimension de la mesure harmonique en tant que fonction du domaine soumis à des perturbations et j'établis quelques résultats de continuité. Cette partie contient aussi des résultats récents concernant la mesure harmonique associée au problème mixte de Dirichlet-Neumann et qui font partie d'un article commun avec Hung Nguyen [BN10].

Enfin, la quatrième partie est consacrée à l'étude de ce que nous avons appelé "vols Browniens". Il s'agit d'un modèle de phénomènes physiques et biologiques tels que la dilution de macromolécules dans l'eau, la quête de cible sur l'ADN par les protéines etc. Les travaux correspondants sont extraits d'articles cosignés avec P. Levitz et M. Zinsmeister ([BZ10a], [BLZ11], [BZ10b]) et ont été partiellement financés par l'ANR DYOPTRI.

La cinquième partie est, en réalité, un projet de recherche sur la croissance des villes que nous avons abordé dans le cadre d'un contrat avec la région. Nous utilisons des processus de croissance aléatoire (percolation, DLA) paramétrés et adaptés. Les résultats obtenus sont essentiellement numériques et font partie d'un travail commun avec M. Zinsmeister et Nga Nguyen, étudiante en thèse co-encadrée par M. Zinsmeister et moi-même.

Les références sont communes aux cinq parties. Du point de vue chronologique, la numérotation des parties correspond à l'ordre dans lequel je me suis intéressé aux différents problèmes. La quatrième partie utilise des outils introduits auparavant et renforce les liens qui unissent les trois premières ; la dernière partie concerne des travaux très récents et embryonnaires.

Partie I

Théorie du potentiel

Ce chapitre est divisé en deux sections, la première contenant des résultats généraux et la deuxième se focalisant sur certains domaines particuliers. Le point de départ de mon travail de thèse fut l'étude des domaines dits "de Poisson" et l'article de C. Bishop [Bis91].

1.1 Domaines de Poisson

Commençons par un rappel de la notion du domaine de Poisson.

Définition 1.1.1 *Un domaine $\Omega \subset \mathbb{R}^d$, $d \geq 2$ est dit domaine de Poisson si toute fonction u harmonique bornée dans Ω s'écrit sous la forme $u(x) = \int_{\partial\Omega} f(y)\omega_x(dy)$, où ω_x est la mesure harmonique de Ω évaluée en x et f est une fonction borélienne bornée sur la frontière de Ω.*

C. Bishop a proposé des conditions nécessaires et suffisantes pour qu'un domaine soit "poissonien".

Proposition 1.1.2 [Bis91] *Un domaine $\Omega \subset \mathbb{R}^d$ est poissonien si et seulement si pour tous sous-domaines disjoints Ω_1 et Ω_2 de Ω, les mesures harmoniques ω_1 et ω_2 correspondantes sont mutuellement singulières sur $\partial\Omega$.*

Il existe une caractérisation équivalente, due à Mountfort et Port, des domaines de Poisson en termes de la frontière de Martin. C'est probablement à travers celle-ci qu'on mesure le mieux la portée de la définition ci-dessus.

Théorème 1.1.3 [MP91] *Un domaine $\Omega \subset \mathbb{R}^d$ admettant une fonction de Green est poissonien si et seulement s'il existe un sous-ensemble mesurable Δ de la frontière de Martin minimale $\partial\Omega^M$ de Ω, un sous-ensemble mesurable E de la frontière $\partial\overline{\Omega}$ de Ω dans le compactifié d'Alexandroff de $\widehat{\mathbb{R}}^d$ de $\mathbb{R}^d$ et une application $\pi : \Delta \to E$ vérifiant :*

i) π est bijective et bi-mesurable.

ii) $\nu(\Delta) = 1$, où ν est la mesure harmonique de Martin relativement à un point $X_0 \in \Omega$ quelconque mais fixe.

iii) $\omega(E) = 1$, *où* ω *est la mesure harmonique de* Ω *évaluée en* X_0.
iv) Pour tout ensemble mesurable $F \subset E$ *on a* $\omega(F) = \nu(\pi^{-1}(F))$.

Il existe une caractérisation de domaines de Poisson impliquant une condition de type "série de capacités de Wiener" ; cette caractérisation a été le point de départ de ma thèse. Introduisons tout d'abord quelques notations.

Notation 1.1.4 Notons $\mathbb{S}_d$ la sphère unité de $\mathbb{R}^d$. Pour $x \in \mathbb{R}^d$, $\epsilon > 0$ et $\theta \in \mathbb{S}_d$ nous notons $C(x, r, \epsilon, \theta)$ le cône tronqué

$$C(x, r, \epsilon, \theta) = \{y\ ;\ y = x + r'\theta'\ ,\ \theta' \in \mathbb{S}_d\ ,\ ||\theta - \theta'|| < \epsilon\ ,\ 0 < r' < r\},$$

et $W(x, r, \epsilon, \theta) = C(x, r, \epsilon, \theta) \setminus \mathbb{B}(x, r/2)$. Pour $d > 0$ nous désignons par $\mathcal{H}_d$ la mesure de Hausdorff de dimension d.

Définition 1.1.5 *Soit* Ω *un domaine dans* $\mathbb{R}^d$, $d \geq 2$. *On dit que le point* $x \in \partial\Omega$ *satisfait à une condition de cône double faible (CDF) par rapport à* Ω *s'il existe* $\theta = \theta(x) \in \mathbb{S}_d$ *et* $\epsilon = \epsilon(x) > 0$ *tels que*

$$\sum_{n\in\mathbb{N}} \left\{ \operatorname{cap}\left\{2^n\left(W(x, 2^{-n}, \epsilon, \theta) \setminus \Omega\right)\right\} + \operatorname{cap}\left\{2^n\left(W(x, 2^{-n}, \epsilon, -\theta) \setminus \Omega\right)\right\} \right\} < +\infty, \qquad (1.1)$$

où cap *est*

- *la capacité de Green dans* $\mathbb{R}^d$, *pour* $d \geq 3$
- *la capacité de Green dans la boule* $\mathbb{B}(x, 1)$ *centrée en* x *et de rayon* 1, *quand* $d = 2$.

C. Bishop [Bis91] a proposé la condition CDF et a montré que, dans $\mathbb{R}^2$, un domaine Ω est de Poisson si et seulement si l'ensemble des points de $\partial\Omega$ satisfaisant à une condition CDF par rapport à Ω est de mesure de Hausdorff $\mathcal{H}_1$ nulle.

J'ai partiellement étendu ce résultat en dimension supérieure.

Théorème 1.1.6 *Soit* Ω *un domaine dans* $\mathbb{R}^d$ *dont l'ensemble des points frontières qui satisfont à une condition CDF par rapport à* Ω *est de* $\mathcal{H}_{d-1}$ *mesure strictement positive. Alors* Ω *n'est pas un domaine de Poisson.*

La preuve s'inspire de l'approche de Mountfort et Port ; en utilisant la condition CDF on construit un graphe Lipschitzien dont l'intersection avec $\partial\Omega$ est de $\mathcal{H}_{d-1}$ mesure positive. En suite, la condition CDF est traduite en une condition d'effilement minimal (cf. appendice) à l'aide d'un résultat de Zhang [Zha87]. Puis, on construit deux sous-domaines disjoints Ω_1 et Ω_2 de Ω tels que le graphe Lipschitzien intersecte leur frontière commune en un ensemble F de $\mathcal{H}_{d-1}$ mesure positive. En utilisant la condition d'effilement minimal on montre que les mesures harmoniques de ces domaines sont équivalentes à $\mathcal{H}_{d-1}$ sur F, et le résultat se déduit de la proposition 1.1.2.

Perspectives 1.1.7 La condition CDF est donc nécessaire pour qu'un domaine soit de Poisson. Est-elle suffisante ? La réponse est affirmative dans $\mathbb{R}^2$ mais ouverte en dimension supérieure. En fait, la démonstration est suspendue à l'affirmation suivante (dont on ne connait pas la validité)

"*Si les mesures harmoniques de deux domaines disjoints Ω_1 et Ω_2 sont équivalentes sur un ensemble $F \subset \partial\Omega_1 \cap \partial\Omega_2$ alors F est de $\mathcal{H}_{d-1}$ mesure positive et est inclus dans une réunion dénombrable de graphes lipschitziens* ".

Le théorème suivant, dû à Friedland et Hayman, apporte un début de réponse : les mesures ω_1 et ω_2 seront nécessairement absolument continues par rapport à $\mathcal{H}_{d-1}$.

Théorème 1.1.8 [FH76] *Si Ω_1, Ω_2 sont deux domaines disjoints de $\mathbb{R}^d$ et si ω_1, ω_2 sont leurs mesures harmoniques en deux points $x_1 \in \Omega_1$ et $x_2 \in \Omega_2$ respectivement, alors il existe une constante $C > 0$ (dépendant de x_1 et x_2) telle que pour tout $x \in \mathbb{R}^d$ et tout $r > 0$, on ait*

$$\omega_1 (B(x, r))\, \omega_2 (B(x, r)) \leq C^2 r^{2(d-1)}.$$

Kenig, Preis et Toro [KPT09] ont répondu par l'affirmative à cette question sous l'hypothèse que Ω est un domaine localement non-tangentiellement accessible (NTA) "des deux côtés" et que la mesure de Hausdorff $\mathcal{H}_{d-1}$ est localement de Radon sur $\partial\Omega$.

Plus généralement on obtient aussi le résultat suivant :

Proposition 1.1.9 *Si Ω_1 et Ω_2 sont deux domaines disjoints dont les mesures harmoniques sont strictement positives et équivalentes sur un ensemble E de mesure $\mathcal{H}_{d-1}$ σ-finie, alors elles sont équivalentes à $\mathcal{H}_{d-1}$ sur un sous-ensemble de E de $\mathcal{H}_{d-1}$ mesure strictement positive. En particulier, si F est un ensemble de $\mathcal{H}_{d-1}$ mesure nulle alors $\mathbb{R}^d \setminus F$ est un domaine de Poisson.*

La question de l'existence de tels domaines Ω_1 et Ω_2 lorsque la frontière de Ω est une surface de Van-Koch a été posée dans [LVV05] : la réponse est positive et a été apportée par Kenig, Preiss et Toro dans [KPT09].

1.2 Quelques domaines géométriques particuliers

Un domaine est dit *de Denjoy* s'il est de la forme $\mathbb{R}^d \setminus F$, où F est un sous-ensemble fermé d'un graphe Lipschitzien Γ. La proposition suivante, établie en collaboration avec A. Ancona, a été un outil pivot pour la preuve des résultats précédents. Elle présente en elle même un intérêt.

Proposition 1.2.1 *Soit $\Omega = \mathbb{R}^d \setminus F$ un domaine de Denjoy. Alors, $\Gamma \setminus F$ est effilé au sens minimal (relativement à $\mathbb{R}^d \setminus F$) en $\mathcal{H}_{d-1}$-presque tout point de F.*

On peut aussi traiter le cas des domaines dits “champagne bubbles”.

Théorème 1.2.2 *Soit Γ le graphe d'une fonction lipschitzienne et soit F un sous-ensemble fermé de Γ. Si $\mathcal{B}$ est une collection de boules fermées disjointes contenues dans $\mathbb{R}^d \setminus F$ telles que $E = \bigcup_{B \in \mathcal{B}} B \cup F$ soit fermé, alors le domaine $\Omega = \mathbb{R}^d \setminus E$ est de Poisson si et seulement si l'ensemble des points du bord de Ω satisfaisant une condition CDF par rapport à Ω est de $\mathcal{H}_{d-1}$-mesure nulle.*

Notons enfin le résultat suivant concernant les domaines dont la frontière est un ensemble *Besicovitch-irrégulier*, c'est-à-dire un ensemble dont l'intersection avec tout graphe lipschitzien est de $\mathcal{H}_{d-1}$-mesure nulle.

Proposition 1.2.3 *Soit E un ensemble compact Besicovitch-irrégulier et soit $\Omega = \mathbb{R}^2 \setminus E$. Si ω est la mesure harmonique du domaine Ω évaluée à l'infini, alors ω est singulière par rapport à $\mathcal{H}_1$ sur l'ensemble Δ des points doubles de E (au sens du compactifié de Martin).*

Rappelons au passage que l'ensemble des points au moins triples d'un domaine du plan est de mesure harmonique nulle : cette remarque nous a été faite par A. Ancona.

Perspectives 1.2.4 La proposition 1.2.3 répond partiellement à la question suivante de C. Bishop :

Si Ω un domaine de $\mathbb{R}^2$ et si E est un sous-ensemble de $\partial\Omega$ Besicovitch-irrégulier , est-il vrai que la mesure harmonique ω du domaine Ω est singulière par rapport à $\mathcal{H}_1$ sur E ?

Peter Jones a répondu par l'affirmative à cette conjecture sous une condition de capacité uniforme ; le théorème de Makarov [Mak85] et son amélioration due à Pommerenke [Pom86] permet d'étendre cette réponse au cas des domaines Ω simplement connexes.

Partie 2

Analyse fine des mesures

Cette partie concerne l'étude des propriétés géométriques des mesures (analyse multifractale). Je me suis intéressé à l'analyse fine des mesures à la suite de mes travaux de thèse concernant la mesure harmonique et présentés dans le chapitre suivant. Les techniques développées pour étudier la mesure harmonique avaient été reprises par Y. Heurteaux [Heu98] pour établir un lien entre la dimension d'une mesure et la dérivée de la fonction τ correspondante.

Nous précisons ici ces différentes notions et donnons quelques énoncés de résultats ; si, pour des raisons « pédagogiques », nous nous restreignons au cadre des espaces euclidiens les notions peuvent s'étendre au cas d'un espace métrique général.

2.1 Dimensions

Soit μ une mesure borélienne sur $\mathbb{R}^d$ (ou tout autre espace métrique). On appelle *dimension de Hausdorff* supérieure $\overline{\dim_{\mathcal{H}}}(\mu)$ de la mesure μ la quantité suivante

$$\overline{\dim_{\mathcal{H}}}(\mu) = \inf\{\dim_{\mathcal{H}}(E) \,; E \text{ mesurable et } \mu(\mathbb{R}^d \setminus E) = 0\}.$$

De façon similaire la dimension de Hausdorff inférieure $\underline{\dim_{\mathcal{H}}}(\mu)$ de μ est donnée par

$$\underline{\dim_{\mathcal{H}}}(\mu) = \inf\{\dim_{\mathcal{H}}(E) \,; E \text{ mesurable et } \mu(E) > 0\}.$$

Nous avons clairement $0 \leq \underline{\dim_{\mathcal{H}}}(\mu) \leq \overline{\dim_{\mathcal{H}}}(\mu) \leq d$. Une mesure est dite unidimensionelle si ces quantités sont égales ; dans ce cas nous utiliserons la notation $\dim_{\mathcal{H}}(\mu)$ pour désigner la valeur commune.

Ces quantités caractérisent la taille du support "effectif" de la mesure. Plus elles sont proches de 0 moins la mesure est "diffuse", la mesure de Lebesgue étant de dimension d et les mesures atomiques de dimension 0.

Des quantités équivalentes peuvent être définies en remplaçant la dimension de Hausdorff avec la dimension de packing ; elles sont notées respectivement $\overline{\dim_{\mathcal{P}}}(\mu)$, $\underline{\dim_{\mathcal{P}}}(\mu)$ et $\dim_{\mathcal{P}}(\mu)$.

Ces différentes dimensions sont difficiles à calculer : elles font intervenir une très grande collection d'ensembles E puis des infima et des suprema sur tous les recouvrements possibles de chaque ensemble E. Il est donc indispensable d'avoir une méthode de calcul numériquement programmable et aussi précise que possible.

Nous pouvons atteindre cet objectif dans certains cas de mesures à support compact auto-similaire à l'aide de la notion d'entropie. Afin de présenter les résultats dans un cadre suffisamment général, je considère un alphabet fini $\Sigma = \{0, ..., N-1\}$ et $\mathbb{K}$ l'ensemble de suites (a_i) où $a_i \in \Sigma$, pour tout $i \in \mathbb{N}$. Nous munissons $\mathbb{K}$ de la métrique ultra-métrique habituelle et nous considérons une mesure μ sur $\mathbb{K}$. Soient $\mathcal{F}_n$ la collection des cylindres de longueur n (et de diamètre N^{-n}) ; pour $x \in \mathbb{K}$ on note $I_n(x)$ le cylindre de $\mathcal{F}_n$ contenant x. On pose $h_n(\mu) = \dfrac{1}{n} \sum_{I \in \mathcal{F}_n} \mu(I) |\log_N \mu(I)|$, puis

$$\underline{h}(\mu) = \liminf_n h_n(\mu) \text{ et } \overline{h}(\mu) = \limsup_n h_n(\mu).$$

Il est bien connu (cf [Bil65], [Mat95]) que

$$\underline{\dim_{\mathcal{H}}}(\mu) = \inf \text{ess}_\mu \liminf_n \left| \frac{\log_N \mu(I_n(x))}{n} \right|$$

$$\text{et } \overline{\dim_{\mathcal{H}}}(\mu) = \sup \text{ess}_\mu \liminf_n \left| \frac{\log_N \mu(I_n(x))}{n} \right|,$$

où $\inf \text{ess}_\mu$ et $\sup \text{ess}_\mu$ sont les valeurs d'infimum et supremum μ-presque partout.

Pour la dimension de packing nous obtenons des égalités similaires en remplaçant la liminf par une limsup ([Heu98]). Une application directe du lemme de Fatou aux fonctions $S_n(x) = |\log_N \mu(I_n(x))|$, donne

$$\underline{\dim_{\mathcal{H}}}(\mu) \leq \underline{h}(\mu) \leq \overline{h}(\mu) \leq \overline{\dim_{\mathcal{P}}}(\mu).$$

Il est clair que si $\underline{\dim_{\mathcal{H}}}(\mu) = \underline{h}(\mu)$ alors μ est uni-dimensionnelle ; de même pour la mesure de packing, si $\overline{h}(\mu) = \overline{\dim_{\mathcal{P}}}(\mu)$ alors $\dim_{\mathcal{P}}(\mu)$ existe. Un encadrement de ces dimensions à l'aide de la dérivée de la fonction τ (rappelée en appendice) du formalisme multifractal est aussi proposé dans [BH02].

Notation 2.1.1 Il convient de signaler un changement de notation par rapport aux articles [BH02] et [Bat06b]. Il faut lire $\overline{\dim_{\mathcal{H}}} = \dim^*$, $\underline{\dim_{\mathcal{H}}} = \dim_*$ et $\underline{h} = h_*$, $\overline{h} = h^*$. Idem pour la dimension de packing $\dim_{\mathcal{P}}$, qui est notée Dim dans ces articles.

Dans [BH02] nous avons démontré les théorèmes suivants

Théorème 2.1.2 *Soit μ une mesure de probabilité sur $\mathbb{K}$. Les propriétés suivantes sont équivalentes :*

(i) $\underline{\dim_{\mathcal{H}}}(\mu) = \underline{h}(\mu)$

(ii) $\underline{\dim_{\mathcal{H}}}(\mu) = \overline{\dim_{\mathcal{H}}}(\mu) = \underline{h}(\mu)$

(iii) *Il existe une suite d'indices $(n_k)_{k\geq 1}$ telle que pour μ presque tout $x \in \mathbb{K}$,*

$$\lim_{k\to+\infty} \frac{\log \mu(I_{n_k}(x))}{-n_k \log N} = \underline{\dim_{\mathcal{H}}}(\mu) \ .$$

Le deuxième résultat est l'analogue pour la dimension de packing.

Théorème 2.1.3 *Soit μ une mesure de probabilité sur $\mathbb{K}$. On a*

$$\overline{\dim_{\mathcal{P}}}(\mu) \leq \overline{h}(\mu).$$

De plus, les propriétés suivantes sont équivalentes :

(i) $\overline{\dim_{\mathcal{P}}}(\mu) = \overline{h}(\mu)$

(ii) $\underline{\dim_{\mathcal{P}}}(\mu) = \overline{\dim_{\mathcal{P}}}(\mu) = \overline{h}(\mu)$

(iii) *Il existe une suite d'indices $(n_k)_{k\geq 1}$ telle que pour μ presque tout $x \in \mathbb{K}$,*

$$\lim_{k\to+\infty} \frac{\log \mu(I_{n_k}(x))}{-n_k \log N} = \overline{\dim_{\mathcal{P}}}(\mu) \ .$$

Les démonstrations reposent essentiellement sur un argument de convergence dominée de (sous-suites des) fonctions S_n.

L'existence de mesures ne satisfaisant pas les propriétés équivalentes de ces théorèmes est moins évidente. Nous proposons un tel exemple construit sur $[0,1]$ muni de la filtration dyadique. Nous construisons une famille de mesures ν_t indexées par $t \in]0,1[$ dont les entropies valent $t\alpha+(1-t)\beta$, où $0 < \alpha < \beta < 1$ sont deux valeurs fixées mais quelconques.

La propriété remarquable de ces mesures ν_t est qu'elles sont équivalentes entre elles. Ceci implique que leurs dimensions de Hausdorff et de packing sont égales et fixes. Il existe alors un unique $t_0 \in]0,1[$ vérifiant $\dim_{\mathcal{H}}(\nu_{t_0}) = \underline{h}(\nu_{t_0})$ et $\dim_{\mathcal{P}}(\nu_{t_0}) = \overline{h}(\nu_{t_0})$.

Motivé par une question de *rigidité* de la dimension de la mesure harmonique j'ai voulu étudier le même problème dans des cas concrets de mesures. Avec la même notation on a le corollaire des théorèmes 2.1.2 et 2.1.3 suivant.

Remarque 2.1.4 Si les variables aléatoires $X_n(x) = \log_N \frac{\mu(I_{n+1}(x))}{\mu(I_n(x))}$ satisfont à une loi faible des grands nombres alors

$$\dim_{\mathcal{H}} \mu = \underline{h}(\mu) \text{ et } \dim_{\mathcal{P}} \mu = \overline{h}(\mu). \tag{2.1}$$

Sont donc concernées par cette remarque les mesures de Bernoulli, et leurs convolutions même inhomogènes. Plus généralement, si les variables aléatoires $\mathbb{E}_\mu(X_n|\mathcal{F}_{n-1})$ sont constantes, la mesure μ satisfait les conclusions de la remarque.

Inspiré par cette remarque, qui s'applique aussi -modulo quelques modifications- aux mesures harmoniques d'ensembles autosimilaires, j'ai souhaité savoir si l'absence de ce qu'il conviendrait d'appeler "mémoire longue" de la suite des variables X_n était suffisante pour la validité des formules (2.1). Naturellement donc, je me suis intéressé aux mesures pour lesquelles la suite (X_n) forme une chaîne de Markov.

Pour simplifier les notations prenons le cas particulier $N = 2$. Le cas général n'est pas plus compliqué.

Notons $I = I_{\epsilon_1,\ldots,\epsilon_n}$ les cylindres de $\mathcal{F}_n$ où $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$: si J désigne le un-cylindre de $\mathcal{F}_1$ $I_{\epsilon_{n+1}}$ avec $\epsilon_{n+1} \in \{0, 1\}$, on notera $IJ = I_{\epsilon_1,\ldots,\epsilon_n,\epsilon_{n+1}}$ le sous-cylindre de I appartenant à $\mathcal{F}_{n+1}$.

On se donne deux suites $(p_n)_{n\geq 0}$ et $(q_n)_{n\geq 1}$ de poids, à valeurs dans $[0, 1]$ et on considère la mesure de probabilité μ sur $\mathbb{K}$ définie selon le procédé suivant : $\mu(I_0) = p_0$, $\mu(I_1) = 1 - p_0$ et, lorsque $\mu(I) \neq 0$

$$\frac{\mu(IJ)}{\mu(I)} = \begin{cases} p_n \mathbf{1}_{\{\epsilon_{n+1}=0\}} + (1 - p_n)\mathbf{1}_{\{\epsilon_{n+1}=1\}} \text{ , si } \epsilon_n = 0 \\ q_n \mathbf{1}_{\{\epsilon_{n+1}=0\}} + (1 - q_n)\mathbf{1}_{\{\epsilon_{n+1}=1\}} \text{ , si } \epsilon_n = 1 \end{cases}. \tag{2.2}$$

Quand $\mu(I) = 0$ on pose $\mu(IJ) = 0$.

Lorsque les suites (p_n) et (q_n) sont constantes la chaîne de Markov sous-jacente est homogène et il n'est pas difficile de voir que μ satisfait la propriété (2.1). De même, si $p_n = q_n$ la mesure est une convolution inhomogène de Bernoulli et (2.1) est une conséquence de la loi des grands nombres.

Dans [Bat06b] je montre que

Théorème 2.1.5 *Si μ satisfait (2.2), alors*

$$\dim_{\mathcal{H}} \mu = \underline{h}(\mu) \ \textit{et} \ \dim_{\mathcal{P}} \mu = \overline{h}(\mu).$$

Sous l'hypothèse de minoration des suites (p_n) et (q_n) par une constante $\epsilon > 0$, la chaîne de Markov X_n est une perturbation d'une chaîne de Markov homogène. Le théorème découle alors de la loi des grands nombres. Je dois signaler ici qu'après avoir rédigé une première version de cet article, Bisbas m'a indiqué que le cas particulier de la perturbation d'une chaîne de Markov homogène était déjà considéré dans [BK90].

Revenons au problème initial ; la question de rigidité de la dimension d'une mesure se traduit :

la dimension de la mesure μ est-elle continue par rapport aux variables (X_n), relativement à la distance ℓ^∞ ?

Un réponse positive partielle est donnée par

Théorème 2.1.6 *Soient μ et μ' deux mesures définies par (2.2) et soient $(p_n, q_n)_{n\in\mathbb{N}}$ et $(p'_n, q'_n)_{n\in\mathbb{N}}$ les suites de poids correspondantes. Alors $\dim_{\mathcal{H}}(\mu) \to \dim_{\mathcal{H}}(\mu')$ et $\dim_{\mathcal{P}}(\mu) \to \dim_{\mathcal{P}}(\mu')$ lorsque $(p_n, q_n)_{n\in\mathbb{N}} \to (p'_n, q'_n)_{n\in\mathbb{N}}$ au sens de la distance ℓ^∞.*

Pour la démonstration de 2.1.6 la validité de (2.1) joue un rôle crucial et le résultat est faux en général.

Dans [Bat06b] je construis, pour tout $\varepsilon > 0$, un exemple de deux mesures μ et μ' sur $[0, 1]$, doublantes sur la filtration des intervalles dyadiques, telles que

$$\left| \log_2 \frac{\mu(I_{n+1}(x))}{\mu(I_n(x))} - \log_2 \frac{\mu'(I_{n+1}(x))}{\mu'(I_n(x))} \right| < \varepsilon$$

avec, cependant, $|\dim_{\mathcal{H}}(\mu) - \dim_{\mathcal{H}}(\mu')| \geq \frac{1}{2}$.

Perspectives 2.1.7 Il est connu que, de manière général,

$$-\tau'_+(1) \leq \underline{\dim_{\mathcal{H}}}(\mu) \leq \underline{h}(\mu) \tag{2.3}$$

où τ est la fonction du formalisme multifractal rappelée en appendice page 44, et $-\tau'_+$ est sa derivée à droite (bien définie puisque $-\tau$ est convexe). L'égalité est établie quand la fonction τ est dérivable en 1 ([Nga97]) ce qui est le cas par exemple lorsque la mesure μ est quasi-bernoulli (cf. [Heu98]).

Les mesures introduites ci-dessus échappent aux conditions suffisantes précitées. Nous souhaiterions étudier le cas de la double égalité $-\tau'_+(1) = \underline{\dim_{\mathcal{H}}}(\mu) = \underline{h}(\mu)$.

2.2 Le spectre multifractal et la fonction τ

Cette section concerne essentiellement les résultats de [BT09]. Nous avons étudié la validité du formalisme multifractal dans le cas de produits de Bernoulli inhomogènes : on dit qu'une mesure μ portée par $\mathbb{K} = \{0, 1\}^{\mathbb{N}}$ est un *produit de Bernoulli* s'il existe une suite $(p_n)_n$ de poids telle que

$$\mu(I_{\epsilon_1 \ldots \epsilon_n}) = \prod_{j=1}^{n} p_j^{1-\epsilon_j}(1 - p_j)^{\epsilon_j}. \tag{2.4}$$

Cette définition se généralise aisément à toute alphabet Σ ; cependant on garde $\Sigma = \{0, 1\}$ encore une fois pour faciliter la lecture.

Si la suite (p_n) est constante ou périodique la mesure μ satisfait le formalisme multifractal. Il en est de même, mais c'est un peu plus difficile à voir, si la suite (p_n) converge vers une certain $p \in]0, 1[$. Si la suite (p_n) n'est pas convergente ni périodique le produit de Bernoulli est dit *inhomogène*.

Pour $x \in \mathbb{K}$ on pose

$$\alpha(x) = \liminf_{n\to+\infty} \alpha_n(x) = \liminf_{n\to+\infty} -\frac{\log \mu(I_n(x))}{n \log 2}.$$

On s'intéresse aux ensembles de niveau

$$\underline{E}_\alpha = \{x \; ; \; \alpha(x) \leq \alpha\} \, , \; \overline{F}_\alpha = \left\{x \; ; \; \limsup_{n\to\infty} \alpha_n(x) \geq \alpha\right\} ,$$

et

$$E_\alpha = \{x \; ; \; \alpha(x) = \alpha\} \, , \; F_\alpha = \left\{x \; ; \; \limsup_{n\to\infty} \alpha_n(x) = \alpha\right\} .$$

Considérons le spectre L^q de μ,

$$\tau(q) = \limsup_{n\to+\infty} \tau_{\mu,n}(q) \qquad \text{où} \qquad \tau_{\mu,n}(q) = \frac{1}{n\log 2} \log \left(\sum_{I \in \mathcal{F}_n} \mu(I)^q \right) ,$$

cf appendice, page 44. On obtient

Théorème 2.2.1 *Si μ est un produit (inhomogène) de Bernoulli sur $\mathbb{K}$, alors pour $q > 0$ on a :*

$$\liminf_{n\to\infty} -q\tau'_{\mu,n}(q) + \tau_{\mu,n}(q) \leq \dim \left(\underline{E}_{-\tau'(q^-)} \cap \overline{F}_{-\tau'(q^+)}\right) \leq \inf \left\{\tau^*(-\tau'(q^+)), \tau^*(-\tau'(q^-))\right\} .$$

Dans le cas où la suite $\tau_{\mu,n}$ converge nous pouvons être plus précis.

Théorème 2.2.2 *Supposons que la suite $(\tau_{\mu,n}(q))$ converge au point $q > 0$. Si, de plus, $\tau'_\mu(q)$ existe, alors pour $\alpha = -\tau'_\mu(q)$, on a*

$$\dim (E_\alpha \cap F_\alpha) = \tau^*_\mu(\alpha) = \alpha q + \tau_\mu(q). \tag{2.5}$$

Les démonstrations de ces théorèmes nécessitent, encore une fois, la construction de mesures de Gibbs portées par les ensembles de niveau dont on veut estimer la dimension.

Le résultat suivant, plus original, est négatif. Nous proposons un exemple de mesure μ qui ne vérifie pas le formalisme multifractal. Cette mesure est un produit de Bernoulli inhomogène.

Un point q est appelé point de *transition de phase* si la derivée $\tau'_\mu(q)$ n'existe pas.

Théorème 2.2.3 *Il existe des produits inhomogènes de Bernoulli μ dont les transitions de phase forment un ensemble dense dans $\mathbb{R}_+$.*

La construction de mesures admettant des transitions de phase n'est certainement pas chose nouvelle. La particularité de cet exemple réside sur la simplicité de la forme de la mesure (produit de Bernoulli) mais surtout sur la densité de l'ensemble des transitions de phase. La fonction τ étant concave, elle est dérivable sur $\mathbb{R}_+$ en dehors d'un ensemble dénombrable de points ; de ce point de vue, la fonction τ de notre exemple est aussi peu régulière que possible.

Perspectives 2.2.4 Dans le cadre strict des résultats cités ci-dessus, la question évidente qui se pose est celle de la valeur de la fonction $f(\alpha)$. Car, si l'existence d'une infinité de points q de non-dérivabilité du spectre L^q est possible, on ne sait rien de la valeur du spectre multifractal f dans l'intervalle $[-\tau'_-(q), -\tau'_+(q)]$.

Par ailleurs, nous souhaitons aborder le problème, beaucoup plus général, des mesures μ vérifiant une *condition de Markov faible*, ie. telles que pour tout $n, m \in \mathbb{N}$ et tous cylindres $J, K \in \mathcal{F}_n$, $L \in \mathcal{F}_m$ on ait

$$\frac{1}{C}\frac{\mu(JL)}{\mu(J)} \leq \frac{\mu(KL)}{\mu(K)} \leq C\frac{\mu(JL)}{\mu(J)},$$

où C est une constante donnée.

Toutes les mesures précédemment étudiées satisfont cette relation et (2.1). Qu'en est-il des mesures satisfaisant la condition de Markov faible ? Que peut-on dire de leur spectre multifractal f ?

Partie 3

Mesure harmonique de domaines fractals

Ce chapitre décrit mes résultats concernant la mesure harmonique. Il existe plusieurs façons d'aborder cette notion. La première approche est historiquement celle du problème de Dirichlet et de la théorie du potentiel classique. Si Ω est un domaine de Green et f une fonction continue sur $\partial\Omega$, on peut lui associer une fonction harmonique u_f dans Ω en utilisant la théorie de Perron-Wiener-Brelot (citons p.ex. [AG01], [Doo84], [GM05]). Pour $x \in \Omega$, l'opérateur $f \mapsto u_f(x)$ est linéaire et continu (positif et markovien) ; d'après le théorème de Riesz il existe alors une mesure de probabilité ω_x, *la mesure harmonique de* Ω *en* x, portée par $\partial\Omega$ telle que $u_f(x) = \int f d\omega_x$. Dans ce qui suit, cette mesure sera aussi notée $\omega(x, ., \Omega)$.

L'approche probabiliste facilite l'intuition ([Doo84], [Bas95]). Considérons le mouvement brownien $\mathcal{B}_t$ dans Ω et notons τ le temps de sortie de $\mathcal{B}_t$ de Ω. La mesure harmonique de Ω en $x \in \Omega$ est la distribution de sortie de $\mathcal{B}_\tau$, partant de x.

Une troisième approche, limitée aux domaines simplement connexes du plan, se fait via les applications conformes. Si Ω est un domaine simplement connexe, par le théorème de Riemann, il existe une fonction univalente g du disque unité sur Ω telle que $g(0) = x$. Notons σ la mesure de Lebesque normalisée sur le cercle unité ; la mesure harmonique ω_x est alors la projection de σ sur $\partial\Omega$, induite par g.

Notons que, par le principe de Harnack, le support de la mesure harmonique ne dépend pas du point de référence $x \in \Omega$ choisi. Depuis plus de trente ans le support de la mesure harmonique est l'objet d'un très grand nombre de travaux dont il n'est pas question de faire la liste exhaustive ici. Je propose à la place un très bref historique de l'avancement de la recherche dans ce domaine. Le premier résultat, au parfum probabiliste, est dû à Øksendal [Øks81] : il démontre que, quelque soit le domaine $\Omega \subset \mathbb{R}^2$, on peut trouver une partie F du bord dont la mesure harmonique est 1 et telle que $\dim_{\mathcal{H}} F < 2$. Il a aussi conjecturé qu'il existe un sous-ensemble F de $\partial\Omega$ de dimension au plus 1 et de mesure harmonique pleine. Avec la terminologie du chapitre précédent cette conjecture s'écrit $\dim_{\mathcal{H}} \omega \leq 1$.

Dans le célèbre article [Mak85], Makarov a montré que cette conjecture était vraie pour les domaines simplement connexes. En même temps, Carleson [Car85], proposait une approche ergodique dynamique pour les domaines fractals autosimilaires, approche reprise par Makarov et Volberg [MV86] avec l'introduction du formalisme thermodynamique. La conjecture a finalement été prouvée dans sa généralité par Jones et Wolff [JW88], [Wol93]. D'autres résultats, dans le prolongement de [MV86] ont été obtenus par Urbanski et Zdunik [UZ02]. Le même problème dans le cas général de domaines fractals engendrés par un système d'itérations de fonctions conformes est toujours ouvert.

En dimension supérieure à 3, Bourgain [Bou87] a montré qu'il existe une constante $\epsilon(d) > 0$ telle que la dimension de la mesure harmonique d'un domaine quelconque soit majorée par $d-\epsilon(d)$. Wolff [Wol95] a proposé un exemple prouvant que $\epsilon(d)$ est strictement inférieure à 1. Un raffinement des résultats de Wolff est proposé par J. Lewis, Nyström et Poggi-Corradini dans [LVV05].

Des résultats du même type concernant la mesure harmonique du p-Laplacien sont obtenus dans [LNPC] ; en particulier les auteurs ont montré que la dimension de la mesure p-harmonique de domaines simplement connexes du plan (qui vaut 1 si $p = 2$ par le théorème de Makarov) devient plus grande que 1 lorsque $p < 2$ et plus petite que 1 quand $p > 2$.

3.1 *Dimension de la mesure harmonique d'ensembles fractals non-autosimilaires*

Mes premiers travaux portaient sur la mesure harmonique d'ensembles de Cantor non-autosimilaires. L'objectif initial était d'épurer au maximum la stratégie de Carleson de ses contraintes d'autosimilarité. Je me suis donc naturellement tourné vers les ensembles de Cantor non-autosimilaires.

L'exemple-type que j'ai considéré est le suivant :

Soit $(a_j)_{j\in\mathbb{N}}$ une suite de nombres réels compris entre deux constantes $\underline{A}, \overline{A}$:

$$0 < \underline{A} \leq a_j \leq \overline{A} < \frac{1}{2} \ , \ \forall j \in \mathbb{N} \tag{3.1}$$

A partir de la suite $(a_j)_{j\in\mathbb{N}}$ on construit un ensemble de Cantor comme suit.

On part de $I = [0,1]^2$. Dans une première étape de la construction nous remplaçons I par quatre carrés de longueur a_1 situés aux quatre coins de I. Ensuite, dans une deuxième étape, nous replaçons chacun de ces carrés par quatre carrés de longueur $a_1 a_2$ situés à ses quatre coins. Dans une n-ième étape nous remplaçons chaque carré existant J par quatre carrés de longueur $a_1 ... a_n$ situés au quatre coins de J, fig. 3.1.

On utilise le codage habituel des carrés de la n-ième génération en leur associant les cylindres correspondants $\tilde{I}_{i_1 ... i_n}$ avec $i_j \in \{1,2,3,4\}$ du Cantor abstrait $\{1,...,4\}^{\mathbb{N}}$.

Bien entendu, les carrés $\tilde{I}_{i_1 ... i_n}$ sont contenus dans le carré $\tilde{I}_{i_1 ... i_{n-1}}$, pour tout $i_n = 1,2,3,4$.

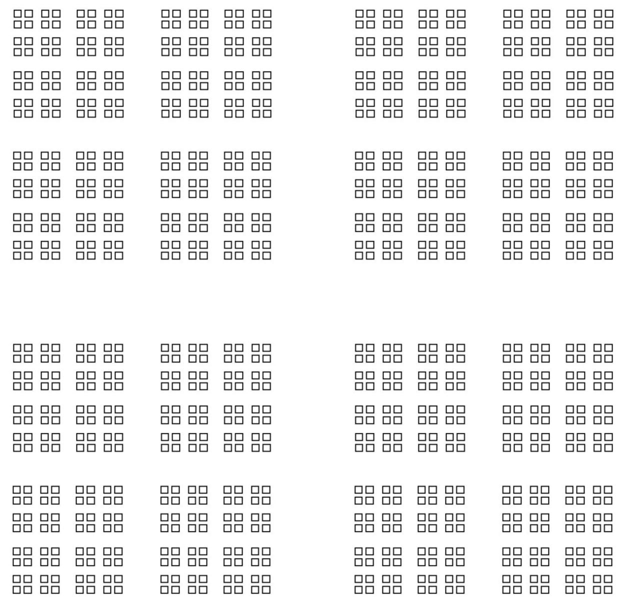

FIGURE 3.1 – Un ensemble de Cantor de type 4-coins

Le premier résultat obtenu (cf. [Bat96]) a été :

Théorème 3.1.1 *Soit $\mathbb{K}$ un ensemble de Cantor construit à partir d'une suite $(a_n)_{n\in\mathbb{N}}$ comme ci-dessus, et posons $\Omega = \mathbb{R}^2 \setminus \mathbb{K}$. Alors, la dimension de la mesure harmonique ω de Ω est strictement inférieure à la dimension de Hausdorff de $\mathbb{K}$, $\dim_{\mathcal{H}} \omega < \dim_{\mathcal{H}} \mathbb{K}$.*

Ce résultat, repris dans [GM05], utilise un mélange des méthodes développées dans [Car85] et [Bou87] ainsi que des outils classiques de la théorie du potentiel mais aucune technique provenant de la théorie ergodique ou des probabilités.

Un des points clefs de la démonstration est une adaptation du principe de Harnack dans des anneaux conformes successifs. Le point de départ est le suivant : Soit Ω un domaine de $\mathbb{R}^2$ tel qu'il existe $A_1 \subset B_1 \subset A_2 \subset ... \subset A_n \subset B_n \subset \Omega$ des disques conformes tels que $B_i \setminus A_i \subset \Omega$, $i = 1, ..., n$.

Proposition 3.1.2 *([Car85], [MV86]) Si les modules des anneaux $B_i \setminus A_i$ sont minorés par une constante $c > 0$ et si $z \in \Omega \setminus B_n$, alors il existe deux constantes C, q avec $q < 1$, ne dépendant que de c, telles que pour toute paire de fonctions harmoniques positives u, v s'annulant sur $\partial\Omega \setminus A_1$ et pour tout $x \in \Omega \setminus B_n$ nous ayons :*

$$\left| \frac{u(x)}{v(x)} : \frac{u(z)}{v(z)} - 1 \right| < Cq^n \tag{3.2}$$

Ce lemme, associé à un argument de symétrie de $\mathbb{K}$, nous permet de montrer que la mesure harmonique ω de Ω devient de moins en moins "uniforme". Puis, en utilisant une idée de [Bou87], reprise dans un contexte différent par Olsen [Ols95], on conclut que $\dim_{\mathcal{H}} \omega < \dim_{\mathcal{H}} \sigma$, où σ est la mesure uniforme sur $\mathbb{K}$.

Dans ce même travail nous montrons que la mesure harmonique est unidimensionnelle (cf section 2.1). Ce résultat, comme l'égalité entre les dimensions de Hausdorff et de Rényi, est une conséquence immédiate d'un article postérieure [Bat06a].

D'autres résultats sont aussi établis : si la suite $a_n = a$ est constante, nous notons $\mathbb{K}_a$ l'ensemble de Cantor associé et ω_a la mesure harmonique de $\mathbb{R}^2 \setminus \mathbb{K}$. A. Ancona prouve que $\dim_{\mathcal{H}} \omega_a \to 1$ quand $a \to 1/2$. Auparavant, il avait été montré dans [MV86] que $\lim_{a\to 0} \frac{\dim_{\mathcal{H}} \omega_a}{\dim_{\mathcal{H}} \mathbb{K}_a} = 1$.

Bishop avait proposé un exemple de domaine Ω dont la mesure harmonique est de dimension égale à la dimension du bord. Dans [Bat96] nous en proposons une construction différente de tels exemples pour chaque valeur de dimension du bord dans $]0, 1]$.

Perspectives 3.1.3 Il est conjecturé que si Ω est un domaine du plan dont le bord est de dimension $\alpha < 1$ alors la mesure de Hausdorff $\mathcal{H}_\alpha$ et la mesure harmonique sont singulières sur $\partial\Omega$. Ma construction comme celle de Bishop ne permet pas d'affirmer si cette conjecture est vraie, sauf dans le cas de domaines à bord autosimilaire.

Remarque 3.1.4 La démonstration du théorème 3.1.1 (et donc les conclusions) s'appliquent en l'état aux ensembles de Cantor du même type en dimension supérieure *ainsi qu'au tapis de Sierpinski.* C'est, à notre connaissance, l'unique résultat connu portant sur la dimension de la mesure harmonique du complémentaire d'ensembles auto-affines.

Notons, enfin, que les conclusions du théorème 3.1.1 restent valables si l'ensemble de Cantor est soumis à des petites perturbations non symétriques : notamment s'il existe une suite (a_n) de rapports et $\varepsilon > 0$ suffisamment petit tels que, pour tout $n \in \mathbb{N}$,

$$\left| \frac{\ell(I_{i_1 \dots i_n})}{\ell(I_{i_1 \dots i_{n-1}})} - a_n \right| < \varepsilon,$$

$\ell(I)$ étant la longueur du carré I, alors $\dim_{\mathcal{H}} \omega < \dim_{\mathcal{H}} \mathbb{K}$.

3.2 Variations de la dimension de la mesure harmonique

Une question naturelle est celle de la dépendance de la dimension de la mesure harmonique par rapport à la frontière du domaine. En particulier, est-il vrai que des petites "perturbations" de la frontière induisent des petits changements de la dimension ?

Il faut d'abord préciser ce que nous entendons par perturbations de la frontière. Il n'est pas difficile de vérifier que la métrique de Hausdorff n'est pas adaptée puisque la dimension de la mesure harmonique n'est pas une fonction continue de la frontière par rapport à cette métrique. Pour ce faire, il suffit par exemple de considérer un ensemble de Cantor de dimension < 1 et remplacer les cylindres de la n-ième génération par les carrés pleins $I_{i_1,\dots,i_n}$, en gardant la notation introduite ci-dessus. Pour n grand, les ensembles compacts $\mathbb{K}_n$ ainsi créés convergent au sens de la métrique de Hausdorff vers le Cantor $\mathbb{K}$ tandis que la dimension de la mesure harmonique du complémentaire de $\mathbb{K}_n$ vaut 1.

Pour les ensembles définis par récurrence et issus de systèmes dynamiques il est aisé de donner un sens au terme "perturbation de la frontière". Si $\mathbb{K}$ est un ensemble invariant par un système de fonctions $\{f_1, \dots, f_n\}$, il serait raisonnable de considérer perturbations ℓ^∞ des fonctions $f_1, \dots, f_n$.

Le problème, posé en toute généralité, ne peut avoir de réponse simple : si l'on considère les ensembles de Julia de polynômes quadratiques dans $\mathbb{C}$, l'implosion parabolique [DSZ97] nous fournit un premier exemple de discontinuité. Cependant, dans le cas des ensembles de Cantor présentés plus haut, l'énoncé est nettement plus simple à formuler et j'ai pu obtenir un certain nombre de résultats.

Soient $\mathbb{K}_{(a_n)}$ l'ensemble de Cantor construit comme décrit ci-dessus à l'aide de la suite $(a_n)_{n\in\mathbb{N}}$ et $\mathbb{K}'_{(a'_n)}$ une deuxième ensemble de Cantor, associé à la suite $(a'_n)_{n\in\mathbb{N}}$.

Notons ω la mesure harmonique de $\mathbb{R}^2 \setminus \mathbb{K}_{(a_n)}$ et ω' celle de $\mathbb{R}^2 \setminus \mathbb{K}'_{(a'_n)}$. Le théorème suivante est tiré de [Bat00] et a été généralisé dans [Bat06a].

Théorème 3.2.1 *Supposons que la suite (a_n) est constante égale à a.* *(H)*

Pour tout $\epsilon > 0$ il existe $\delta = \delta(a, \epsilon) > 0$ tel que si $\sup_n |a'_n - a| < \delta$, on ait $|\dim_{\mathcal{H}} \omega - \dim_{\mathcal{H}} \omega'| < \epsilon$.

Remarquons que ce théorème peut être vu comme un résultat de rigidité au voisinage d'ensembles de Cantor autosimilaires. Plus loin nous verrons que l'hypothèse *(H)* peut-être omise à la lumière des résultats de la section 2.1.

Un des points clefs de la preuve est le lemme de comparaison des mesures harmoniques des cylindres. Ici, la suite (a_n) n'est pas nécessairement constante.

Théorème 3.2.2 *Pour tout $\epsilon > 0$ il existe $\delta > 0$ tel que*

$$\sup_{n\in\mathbb{N}} |a_n - a'_n| < \delta \Rightarrow \left| \frac{\omega(I)}{\omega(\widehat{I})} : \frac{\omega'(I')}{\omega'(\widehat{I'})} - 1 \right| < \epsilon, \tag{3.3}$$

quelque soient $I \in \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ et $I' \in \bigcup_{n\in\mathbb{N}} \mathcal{F}'_n$ avec $I \overset{\text{cod}}{\sim} I'$,

où $I \overset{\text{cod}}{\sim} I'$ signifie que les carrés I et I' partagent le même encodage $i_1 \dots i_n$.

Si la suite (a_n) est constante, la mesure ω est équivalente à une mesure invariante par rapport à l'opérateur du décalage sur le Cantor abstrait $\{1, 2, 3, 4\}^{\mathbb{N}}$. En utilisant le théorème de Shannon-Mc Millan-Breiman la dimension $\dim_{\mathcal{H}} \omega$ de ω est donnée par sa dimension de Rényi. Plus précisément, à l'aide d'un découpage en "paquets" nous montrons que la suite

$$|S_n - S'_n| = \left| \frac{\log \omega(I_{i_1 \dots i_n})}{n} - \frac{\log \omega'(I'_{i_1 \dots i_n})}{n} \right|$$

est majorée par une fonction $\eta(\epsilon)$qui tend vers 0 avec ϵ.

En utilisant des estimations classiques de la dimension des mesures on conclut le théorème 3.2.1.

Le même résultat reste valable si l'on remplace $\mathbb{K}$ par n'importe quel ensemble de Cantor d'un système d'itération de similitudes.

Dans [Bat06a] j'ai réussi à me libérer de l'hypothèse *(H)*. La preuve s'inspire des techniques développées dans [Bat06b] : même si la mesure harmonique ne satisfait pas (2.2) elle vérifie la condition de Markov faible avec une "perte de mémoire" exponentielle. J'utilise, ensuite, une loi des grands nombres adaptée et un contrôle, par récurrence inversée, des espérances conditionnelles de la suite (S_n).

La méthode permet aussi de conclure que la fonction qui associe à (a_n) la dimension de la mesure harmonique de $\mathbb{R}^2 \setminus \mathbb{K}_{(a_n)}$ est Lipschitz-continue par rapport à la métrique ℓ^∞.

3.3 Distribution de sortie du mouvement brownien partiellement réfléchi

Je commencerai par une approche physicienne de la mesure harmonique destinée aux applications. Prenons l'exemple des échanges d'oxygène et de CO_2 dans les alvéoles pulmonaires. Les alvéoles du poumon humain présentent une structure quasi-fractale. Leur superficie totale atteint cent mètres carrés, ainsi l'air inspiré y arrive avec une vitesse quasi nulle et toutes les échanges d'oxygène/dioxyde de carbone se font par diffusion : les molécules d'oxygène sont absorbées au contact des alvéoles qui libèrent simultanément le dioxyde.

Toutes les alvéoles ne sont pas mises à contribution de la même façon ; les plus "exposées" sont les premières à être touchées par les molécules d'oxygène. Mais celles-ci étant vite saturées, une deuxième série rentre en jeu et ainsi de suite.

Le modèle mathématique *en 2D* associé est particulièrement intéressant : les alvéoles sont représentées par une surface fractale (du type "snowflake") et les poumons sont le domaine borné délimité par cette surface. On y voit l'apport du théorème de Makarov : les alvéoles "exposées" correspondent aux points de la frontière x tels que la mesure harmonique de $\mathbb{B}(x, r)$ est comparable à r.

Mais comment décrire la deuxième "série" alvéolaire ? Et surtout, quelle est sa géométrie/taille ? Une modélisation qui semble pertinente est la suivante : on considère un domaine Ω (les poumons) dont la frontière (alvéoles) consiste en deux parties : la partie réfléchissante pour le mouvement brownien et la partie absorbante. La partie réfléchissante correspond aux alvéoles saturées et la partie absorbante au reste de la surface alvéolaire.

Ce modèle présente quelques défauts. D'abord la partie réfléchissante n'est pas fixe, en réalité, mais aléatoire. De plus, la partie absorbante n'est pas nécessairement fermée. Or, cette condition est indispensable afin de pouvoir définir un mouvement brownien à l'aide des formes de Dirichlet aux conditions frontières mixtes Dirichlet-Neumann. Néanmoins, le modèle correspond à une multitude de problèmes physiques (dépôts d'ions sur une plaque cuivrée, recherche de cibles sur l'ADN par des protéines etc) et mérite d'être étudié.

Une première question que nous avons voulu élucider est celle de la dimension de la mesure harmonique du mouvement brownien absorbé sur une partie du bord et tué sur le complémentaire. Quelques notations sont nécessaires : si Ω est un domaine borné et $F \subset \partial\Omega$ est une partie compacte de son bord, on note $\mathcal{R}_t$ le mouvement brownien réfléchi dans Ω (voir appendice) et τ_F le temps d'atteinte de F par $\mathcal{R}_t$.

Dans [BN10] nous montrons le résultat suivant :

Théorème 3.3.1 *Pour tout $\eta > 0$ il existe un domaine $\Omega \subset \mathbb{R}^2$ (que l'on peut choisir simplement connexe) et $F \subset \partial\Omega$ satisfaisant $\mathbb{P}_x(\tau_F < \infty) = 1$ et tels que pour tout $A \subset F$ de dimension $\dim A < 2 - \eta$ on ait,*

$$\mathbb{P}_x(\mathcal{R}_{\tau_F} \in A) = 0,$$

pour tout $x \in \Omega$.

Une autre façon d'écrire la conclusion du théorème précédent est que si ω_x est la distribution de $\mathcal{R}_{\tau_F}$ par rapport $\mathbb{P}_x$, alors la dimension $\dim_{\mathcal{H}} \omega_x$ est plus grande que $2 - \eta$. Les domaines concernés sont délimités par une frontière de dimension proche de 2. Cependant, la preuve ne permet pas de conclure l'égalité $\dim_{\mathcal{H}} \omega_x = \dim_{\mathcal{H}} \partial\Omega$. Notons aussi que, par le principe de Harnack, les mesures ω_x et ω_y sont encore équivalentes pour tout $x, y \in \Omega$ et de ce fait nous noterons ces mesures ω sans préciser le point de référence x.

Cet théorème qui va à l'encontre des résultats de Makarov et de Jones-Wolff se situe a mi-chemin entre le cas absorbé ([Mak85, JW88]) et le cas totalement réfléchi [BCR04]. En effet, Benjamini, Chen et Rohde ont montré que sous certaines conditions la trace du mouvement brownien réfléchi sur la frontière $\partial\Omega$ d'un domaine est un ensemble de dimension égale à $\dim_{\mathcal{H}} \partial\Omega$. Les conditions sont celles reprises en appendice et ne servent qu'à assurer que le mouvement brownien réfléchi est bien défini.

L'exemple proposé est construit à partir d'un ensemble de Cantor $\mathbb{K}$ de type 4-coins de grande dimension (proche de 2). Cette ensemble servira de partie absorbante de $\partial\Omega$. Le domaine Ω sera délimité par $\mathbb{K}$, un cercle réfléchissant de grand diamètre qui assurera la

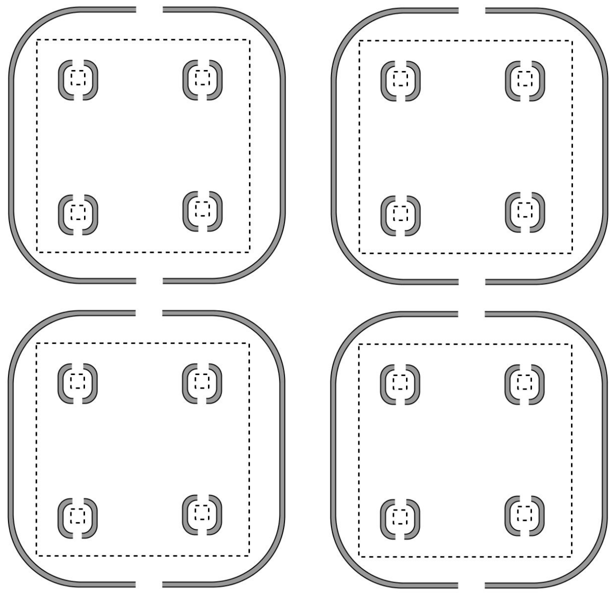

FIGURE 3.2 – Le domaine du théorème 3.3.1

condition « Ω borné » et par une réunion dénombrable d'arcs de cercles autour des carrés de construction de $\mathbb{K}$, cf fig. 3.2.

Perspectives 3.3.2 Cette partie de mon travail est à la fois la plus récente et la plus active. Voici une liste des question qui m'intéressent actuellement :

- Adapter le modèle pour englober les ensembles réfléchissants $F \subset \partial\Omega$ évoluant avec le temps (en particulier croissants). Des liens avec les processus de croissance (DLA, Hastings-Levitov,...) sont à explorer (voir aussi chapitre 5).
- Modifier l'exemple du théorème (en utilisant les techniques de [BCM06]) afin de montrer qu'il existe des domaines Ω dont la frontière est de dimension de Hausdorff arbitraire > 1 et tels que $\dim_{\mathcal{H}} \omega = \dim_{\mathcal{H}} \partial\Omega$.
- Trouver une modélisation paramétrable des problèmes physiques précités, impliquant des processus stochastiques allant du mouvement brownien simple au mouvement brownien réfléchi tels que les dimensions des distributions de sortie varient continûment entre 1 et $\dim_{\mathcal{H}} \partial\Omega$ (voir aussi chapitre 4).
- Dans [GS03] Gregoriu et Samorodnitsky explorent le lien entre les vols browniens et le mouvement brownien réfléchi. Il serait intéressant d'étudier l'itération des vols browniens en liaison avec l'objectif précédent. Notamment, on peut considérer des vols browniens itérés jusqu'à un temps d'arrêt convenablement choisi qui pourrait servir de paramètre. Une approche dans la même veine mais du point de vue “physique” est adoptée par Grebenkov ([Gre10], [Gre06]).

Partie 4

Vols Browniens

Cette thématique est fortement liée aux problèmes présentés dans les paragraphes précédents. Dans la pratique, les particules en diffusion (atomes d'oxygène, protéines, molécules d'eau,...) ne peuvent être repérées par les instruments de mesure que lorsque elles se trouvent à une certaine distance de l'interface (alvéoles, ADN, polyélectrolytes,...). Leur dynamique est donc décomposée en deux phases qui ont lieu en alternance : une phase de diffusion dans l'espace et une phase dite *d'adsorption*, pendant laquelle elles sont confondues avec l'interface. De même, seules les diffusions loin de la surface sont détectables ; nous cherchons donc à modéliser des diffusions de particules, partant près d'une interface et évoluant à distance "détectable" de celle-ci avant d'être adsorbées à nouveau.

Dernier problème technique : on observe des nuages de particules et l'on ne peut poser les questions autrement que "statistiquement". Pire, on n'a aucun accès aux trajectoires mais seulement aux durées de chaque phase.

4.1 Statistiques du premier vol

Dans [DPP+08, GKL+06] ainsi que dans [RCL+09], [HT06] une modélisation est suggérée à la base de ce que nous avons appelés « vols browniens ». Voici une première esquisse : on se donne une interface $\partial\Omega$ délimitant un domaine $\Omega \subset \mathbb{R}^d$. Pour $\varepsilon > 0$, on considère Γ_ε l'ensemble des $x \in \Omega$ tels que $\operatorname{dist}(x, \partial\Omega) = \varepsilon$.

On se donne ensuite une loi μ sur Γ_ε et on choisit $x \in \Gamma_\varepsilon$ selon μ. On considère un mouvement brownien $\mathcal{B}_t$ issu de x que l'on arrêtera au moment τ_Ω du contact de $\partial\Omega$. Nous nous sommes intéressés aux statistiques de ce "premier vol" noté dans la suite $\mathcal{F}_t^\mu$: on souhaite connaître les distributions des variables aléatoires

$$\tau = \tau_\Omega \text{ et } D_\Omega = |\mathcal{F}_0^\mu - \mathcal{F}_\tau^\mu|.$$

Les expériences suggèrent un lien fort de ces statistiques avec la géométrie de la surface via la dimension de Hausdorff. Les simulations numériques ont confirmé que, quand μ est la mesure uniforme sur Γ_ε, la probabilité $\mathbb{P}(\tau_\Omega > t)$ décroit en t^α et $\mathbb{P}(D_\Omega > s)$ en s^β, où les exposants α et β dépendent de la dimension de $\partial\Omega$.

Plus rigoureusement, on peut à titre d'exemple, considérer deux cas simples :

– *Le cas d'un hyperplan.*
  Notre domaine étant invariant par translation, le point de départ ne joue pas de rôle particulier. Calculons donc la probabilité que le mouvement brownien, partant de $(0, \varepsilon) \in \mathbb{R}^n \times \mathbb{R}_+$, revienne -pour la première fois- dans l'hyperplan $\mathbb{R}^n \times \{0\}$ en dehors de la boule $\mathbb{B}(0, R)$.
  Un calcul direct (cf. [Fel71]) donne

$$\mathbb{P}(|(0,\varepsilon) - \mathcal{B}_\tau| > R) \simeq \int_{|x|>R} c_n \frac{\varepsilon}{(|x|^2 + \varepsilon^2)^{\frac{n+1}{2}}} dx \sim \frac{\varepsilon}{R}.$$

– *Le cas d'un cylindre dans* $\mathbb{R}^3$. On considère un cylindre de révolution de rayon 1 autour de l'axe des z. On peut décomposer le mouvement brownien B_t de $\mathbb{R}^3$ en un mouvement brownien 2-dimensionnel B_t^2 sur le plan xy et un mouvement brownien réel B_t^1 sur l'axe des z. La distance de B_t à l'axe des z vaut $|B_t^2|$.
  D'autre part $b_t = \ln |B_t^2|$ est un mouvement brownien 1-dimensionnel et donc la probabilité que b_t partant de $\ln(r)$ atteigne $\ln(R)$ avant de revenir en 0 est égale à $\ln(r)/\ln(R)$.
  Pour conclure nous affirmons que la probabilité d'"atterrir" à distance R du point de départ est équivalente à la probabilité de s'éloigner d'une distance $\geq R$ du cylindre, ie. que $|b_t|$ atteigne $\ln(R)$ avant de revenir en 0. Ce point est l'objet d'un lemme général de [BLZ11].

Revenons au problème de modélisation. En utilisant les inégalités de Harnack nous pouvons remplacer la loi μ qui guide le choix de x sur Γ_ε par une loi discrète sur la famille des cubes dyadiques de Whitney (cf. ci-dessous) $\mathcal{Q}_\varepsilon$ qui rencontrent Γ_ε : le mouvement brownien partira alors du centre du cube choisi.

Notations On considère un cube Q. On note
– $\ell(Q)$ la longueur de ses cotés,
– λQ le cube dilaté par un coefficient $\lambda > 0$.

Rappels

☐ *Décomposition de Whitney* (cf. [Gra08], p. 463)
Pour tout ensembe ouvert Ω de $\mathbb{R}^d$, il existe une collection de cubes dyadiques $\{Q_j\}_j$ telle que
– $\bigcup_j Q_j = \Omega$ et les les intérieurs de Q_j sont deux à deux disjoints.
– $\sqrt{d}\ell(Q_j) \leq \text{ dist}(Q_j, \partial\Omega) \leq 4\sqrt{d}\ell(Q_j)$
– si Q_j et Q_k sont en contact alors $\ell(Q_j) \leq 4\ell(Q_k)$
– pour tout cube de Whitney Q_j il existe au plus 12^d cubes de Whitney Q_k en contact avec Q_j.

Notons aussi $\mathcal{Q}_k$ la sous-collection de cubes de Whitney Q_j tels que $\ell(Q_j) = 2^k$, où $k \in \mathbb{Z}$.

□ *Saucisse de Minkowski*
Pour $r > 0$ on écrira

$$M_r = \{x \in \Omega\ ;\ \ \mathrm{dist}(x, \partial\Omega) \leq r\}$$

□ Dimension de Whitney
Nous noterons $\mathcal{S}_r$ la sous-collection de cubes de Whitney qui intersectent Γ_r. Si $\partial\Omega$ est compact alors $\mathcal{S}_r$ est un ensemble fini. La dimension de Whitney est alors

$$d_W(\partial\Omega) = \limsup_{r\to 0} \frac{\log \#\mathcal{S}_r}{|\log r|}$$

On suppose que Ω est un domaine à bord compact satisfaisant la *condition de Δ-régularité* : il existe $L > 0$ tel que pour tout $x \in \Omega$ avec $d_x = \mathrm{dist}(x, \partial\Omega) < 1$

$$\omega(x, \mathbb{B}(x, 2d_x) \setminus \Omega, \mathbb{B}(x, 2d_x) \cap \Omega) \geq L, \tag{4.1}$$

où $\omega(x, \mathbb{B}(x, 2d_x)\setminus\Omega, \mathbb{B}(x, 2d_x)\cap\Omega)$ est la mesure harmonique en x de l'ensemble $\mathbb{B}(x, 2d_x)\setminus \Omega$ dans l'ouvert $\mathbb{B}(x, 2d_x) \cap \Omega)$.

Le résultat principal de [BLZ11] est le suivant.

Théorème 4.1.1 *On choisit Q selon la loi uniforme dans $\mathcal{S}_\varepsilon$. La probabilité que le mouvement brownien partant de (n'importe quel) $x \in Q$ atteigne Γ_r avant de sortir de Ω est comparable à* $\dfrac{\#\mathcal{S}_r}{\#\mathcal{S}_\varepsilon}\left(\dfrac{r}{\varepsilon}\right)^{n-2}$.

Notons ici que dans [BLZ11] les collections $\mathcal{S}$ sont désignées par $\mathcal{Q}$.

Un domaine Ω satisfait la *condition "tire-bouchon"* s'il existe $r_0 > 0$ et $c > 0$ tels que pour tout $r < r_0$ et tout $x \in \partial\Omega$ on puisse trouver $y \in \Omega$ avec $cr < \mathrm{dist}(x, y) < r$ et $\mathrm{dist}(y, \partial\Omega) > cr$.

Comme corollaire du théorème 4.1.1, en notant $\mathcal{F}_t$ le processus $\mathcal{F}_t^\mu$ associé à la mesure μ uniforme sur Γ_ε nous obtenons :

Théorème 4.1.2 *Soit $0 < \varepsilon < r$. Si Ω satisfait*

1. *La condition "tire-bouchon"*
2. *la condition de Δ-régularité*
3. *et si $\partial\Omega$ admet une dimension exacte de Minkowski d_M,*

alors pour tout $\eta > 0$ il existe une constante $c_{\eta,d} > 0$ (qui ne dépend pas de ε, r) telle que

$$\frac{1}{c_{\eta,n}}\left(\frac{r}{\varepsilon}\right)^{d-d_M-2+\eta} \leq \mathbb{P}\left(D_\Omega > r\right) \leq c_{\eta,n}\left(\frac{r}{\varepsilon}\right)^{d-d_M-2-\eta}.$$

Observons que la probabilité $\mathbb{P}\left(D_\Omega > r\right) = \mathbb{P}\left(|\mathcal{F}_0^\mu - \mathcal{F}_\tau^\mu| > r\right)$ est la probabilité que le vol brownien sorte de Ω à distance $> r$ du point de départ.

La condition “tire-bouchon”, qui apparaît dans les hypothèses du théorème 4.1.2, sert à identifier les dimensions de Whitney et de Minkowski. Des hypothèses plus faibles assurant l'égalité des deux dimensions peuvent être trouvées dans [Tri83] et [Bis96].

La preuve du théorème 4.1.1 s'appuie sur la remarque suivante :

Remarque 4.1.3 La probabilité que le mouvement brownien touche $\partial\Omega$ pour la première fois à distance κr du point de départ, *sans quitter* la saucisse de Minkowski M_r, décroit exponentiellement avec κ.

Ainsi, d'après les conclusions du théorème, la probabilité que le mouvement brownien sorte de Ω à distance κr du point de départ est essentiellement égale de s'éloigner κr de $\partial\Omega$.

Nous nous intéressons alors à la probabilité que le mouvement brownien, partant de $x \in Q \in \mathcal{S}_r$ atteigne $\mathcal{S}_{\kappa r}$ sans quitter Ω. En utilisant la fonction de Green et sa symétrie nous montrons que cette probabilité est approximativement κ^{d-d_M-2} fois la probabilité de toucher Γ_r partant d'un point de $\Gamma_{\kappa r}$. Cette probabilité est minorée (elle est même égale à 1 pour les domaines de $\mathbb{R}^2$) puisque $\partial\Omega$ est de mesure harmonique non nulle.

Nous nous tournons maintenant vers l'estimation du temps de parcours de ces événements rares que sont les ensembles de trajectoires du mouvement brownien partant près du bord et touchant le bord pour la première fois loin du point de départ.

Dans [BZ10b] nous avons obtenu le

Théorème 4.1.4 *Si Ω satisfait*

1. *La condition “tire-bouchon”*
2. *la condition de Δ-régularité*
3. *et si $\partial\Omega$ admet une dimension exacte de Minkowski d_M,*

alors il existe $c > 0$ ne dépendant pas de ε telle que

$$\frac{1}{c}\left(\frac{\varepsilon}{\sqrt{t}}\right)^{d_M+2-d} \leq \mathbb{P}(\tau_\Omega > t) \tag{4.2}$$

et

$$\mathbb{P}(\tau_\Omega > t) \leq c\left(\frac{\varepsilon}{\sqrt{t}}\right)^{d_M+2-d}\left|\log\left(\frac{\varepsilon}{\sqrt{t}}\right)\right|^{2d}, \tag{4.3}$$

pour tout $\varepsilon^2 < t < R_\Omega^2$.

Il est utile de “tester” ce théorème dans le cas d'un intervalle réel, afin de dissiper quelques interrogations légitimes.

Soient $[0, a]$ un segment et $x \in (0, a)$. La probabilité $\mathbb{P}(\tau_x > t)$ que le mouvement brownien, partant de x n'ait touché ni 0 ni a au moment t est donnée par les formules suivantes ([Fel71], pg. 342)

$$\mathbb{P}(\tau_x > t) = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{t}}^{-x/\sqrt{t}} \exp\left(-\frac{1}{2}y^2\right) dy + \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} (-1)^k \int_{(ka-x)\sqrt{t}}^{(ka+x)\sqrt{t}} \exp\left(-\frac{1}{2}y^2\right) dy$$

et

$$\mathbb{P}(\tau_x > t) = \frac{4}{\pi} \sum_{k=1}^{\infty} \exp\left(-\frac{(2n+1)^2\pi^2}{2a^2} t\right) \sin \frac{(2n+1)\pi x}{a}.$$

On peut supposer, par symétrie, que $x \leq \frac{a}{2}$. Si $\frac{a}{\sqrt{t}}$ n'est pas trop grand, p.ex. $\frac{a}{\sqrt{t}} < 1/2$, nous avons par la première formule : $\mathbb{P}(\tau_x > t) \sim \frac{x}{\sqrt{t}}$. D'autre part, si $\frac{a}{\sqrt{t}}$ n'est pas trop petit, la deuxième formule donne : $\mathbb{P}(\tau_x > t) \leq \exp\left(-\frac{\pi^2}{2a^2} t\right)$.

Il existe alors un changement de régime pour la décroissance avec t de $\mathbb{P}(\tau_x > t)$, passant de la décroissance polynômiale à l'exponentielle. Pour des temps relativement petits $\frac{x}{\sqrt{t}}$ est la bonne décroissance.

La preuve du théorème s'appuie sur le théorème 4.1.2 et sur le lemme suivant ([BZ10b]), déjà exploité sous une autre forme dans [BLZ11].

Lemme 4.1.5 *Si Ω satisfait l'hypothèse de Δ-régularité (4.1) alors, pour tout $j \in \mathbb{Z}_-$, la probabilité que le mouvement brownien visite plus de N cubes de Whitney distincts de taille 2^j avant de quitter Ω décroit en Cp^N, où $0 < p < 1$ et $C > 0$ ne dépendent pas de j.*

A l'aide de ce lemme nous montrons que les temps de parcours de trajectoires du théorème 4.1.2 sont essentiellement donnés par le changement de r en $\sqrt{t}$.

Perspectives 4.1.6 Bien que la mesure uniforme soit un choix naturel puisque, comme nous le verrons par la suite, l'itération de vols browniens admet la mesure uniforme comme état stationnaire, on peut se demander comme varient les statistiques des vols browniens si le point x est choisi selon une loi μ.

Il existe un fort lien avec le travail de N. Makarov [Mak99] : si $\mu = \delta_x$ est une masse de Dirac le problème est lié à la géométrie locale de $\partial\Omega$. De même, si x est choisi dans un ensemble de niveau E_α, cf appendice, il est probable que les statistiques seront affectés par le choix de α.

Une autre question que nous nous sommes posée est celle de la diffusion au bord. Que se passe-t-il si on associe aux vols browniens une phase de diffusion sur le bord de $\partial\Omega$? Comment les statistiques sont-elles modifiées ?

4.2 Distributions stables pour l'itération des vols browniens

Dans ce qui précède nous avons focalisé notre attention sur le premier vol. De surcroît, nous avons fixé le choix de la mesure initiale en prenant la loi uniforme. Ce choix est pertinent puisque nous montrons qu'il existe un état stationnaire ν pour les vols browniens et que cet état est fortement équivalent (avec des constantes ne dépendant pas de ε) à la mesure uniforme. De plus, quelque soit la mesure initiale μ, la distribution de sortie d'une itération de vols browniens convergera vers ν.

Afin d'énoncer rigoureusement un théorème nous devons définir l'itération de vols browniens. Car la mesure initiale de $\mathcal{F}_t$ est portée par Γ_ε tandis que la distribution de sortie est portée par $\partial\Omega$; en particulier, il n'existe pas de distribution stable.

Nous avons deux options : la première consiste à projeter la distribution de sortie sur Γ_ε en suivant les lignes de Green. La deuxième option (que nous avons retenue) consiste à remonter sur les trajectoires jusqu'au dernier contact avec Γ_ε. Les deux approches ne sont pas très éloignées : d'après la remarque 4.1.3, le point de sortie de Ω est ε-proche du point de dernier contact de la trajectoire avec Γ_ε avec une très grande probabilité.

C'est la seconde approche qui nous parait la plus naturelle et la plus adaptée aux applications et cela pour deux raisons :

1. la projection de la distribution de sortie sur Γ_ε suivant les lignes de Green est mal définie sur un grand nombre de domaines.
2. il est plus aisé de discrétiser le deuxième choix ; or, cette discrétisation est utile pour les applications mais aussi l'outil clef de notre démonstration.

Remarque 4.2.1 Si nous considérons $\Omega \cap (\mathbb{Z}/n)^d$ un réseau dans Ω et si nous remplaçons le mouvement brownien par la marche aléatoire simple sur ce réseau alors, le théorème de Perron-Frobenius permet de conclure que le seul état invariant est celui donné par la mesure uniforme sur les sommets frontières.

Inspirés par cette remarque nous avons cherché à discrétiser le mouvement brownien en une marche aléatoire de façon à préserver les objets étudiés. Cette discrétisation, dont la version adaptée aux variétés est due à Lyons et Sullivan [LS84], présente les propriétés nécessaires : la marche aléatoire induite est symétrique et a les “mêmes” trajectoires que le mouvement brownien [LS84, BL96]. De surcroît, les fonctions harmoniques positives associées à cette marche aléatoire sont exactement les traces sur le réseau des fonctions harmoniques positives de Ω, [LS84, Anc90].

Notre réseau sera celui des centres des cubes de Whitney, noté $\mathcal{N}$. Dans l'appendice j'ai ajouté un bref descriptif de la discrétisation ainsi que les principaux outils utilisés dans [BL96].

On considère le processus aléatoire qui consiste à choisir un cube de Whitney Q dans $\mathcal{S}_\varepsilon$ (ie. tel que $Q \cap \Gamma_\varepsilon \neq \emptyset$) selon une loi μ et à considérer le mouvement brownien discrétisé

X_k sur $\mathcal{N}$ partant de x_Q. Soit τ_ε le dernier passage de X_k dans $\mathcal{S}_\varepsilon$. Notons π_μ la mesure sur $\mathcal{S}_\varepsilon$ donnée par

$$\pi_\mu(Q') = \mathbb{P}_\mu(X_{\tau_\varepsilon} \in Q') = \sum_{Q \in \mathcal{S}_\varepsilon} \mu(Q)\mathbb{P}_{x_Q}(X_{\tau_\varepsilon} \in Q').$$

Nous montrons le

Théorème 4.2.2 *Si Ω satisfait la condition de Δ-régularité (4.1), alors pour tout $\varepsilon > 0$ il existe une unique mesure de probabilité ν_ε satisfaisant $\pi_{\nu_\varepsilon} = \nu_\varepsilon$; De plus il existe $C > 0$ indépendant de ε telle que*

$$\frac{1}{c}\frac{1}{\#\mathcal{S}_\varepsilon} \leq \nu_\varepsilon(Q) \leq c\frac{1}{\#\mathcal{S}_\varepsilon}\ , \quad \textit{quelque soit } Q \in \mathcal{S}_\varepsilon.$$

En d'autres termes, la seule mesure invariante par le processus aléatoire est fortement équivalente à la mesure uniforme, indépendamment de ε. Ce processus aléatoire est une version discrète de celui, continu, décrit dans la section précédente : si l'on remplace la marche aléatoire par $\mathcal{F}_t$, avec la même définition que ci-dessus les mesures $\pi(\mu)$ pour X_k et pour $\mathcal{F}_t$ sont équivalentes (et les constantes multiplicatives d'équivalence ne dépendent ni de μ ni de ε).

Je donne une esquisse de la preuve de ce théorème dans le cas particulier où Ω est un domaine borné du plan. Le théorème est prouvé dans sa généralité dans [BZ10a].

Rappelons que si Ω est un domaine borné du plan, pour $(\mathcal{N}, \mathbf{F}, \mathbf{V})$ choisis comme dans la remarque 5.0.10 la marche aléatoire X sur $\mathcal{N}$ discrétisant le mouvement brownien est symétrique, ie. $p_{x,y} = p_{y,x}$ pour tout $x, y \in \mathcal{N}$. Pour $x = x_Q \in \mathcal{N}$ centre du cube Q de Whitney, posons $r_Q = \mathbb{P}_x(\tau_\varepsilon = 0)$. Sous l'hypothèse de Δ-régularité, il existe une constante $c_1 > 0$ telle que $r_Q > c_1$ pour tout $Q \in \mathcal{W}$.

La suite de la preuve reprend les arguments du théorème de Perron-Frobenius. Avec la notation ci-dessus, pour tout $x = x_Q$ avec $Q \in \mathcal{S}_\varepsilon$, on a

$$1 = \sum_{Q' \in \mathcal{S}_\varepsilon} \sum_{k=0}^{\infty} \mathbb{P}_x(X_k = x_{Q'}, \tau_\varepsilon = k)$$

et d'après la propriété forte de Markov, on peut écrire

$$1 = \sum_{Q' \in \mathcal{S}_\varepsilon} \sum_{k=0}^{\infty} \mathbb{P}_x(X_k = x_{Q'})\mathbb{P}_{x_{Q'}}(\tau_\varepsilon = 0)$$

ou encore

$$1 = \sum_{Q \in \mathcal{S}_\varepsilon} r_Q \sum_{k=0}^{\infty} \mathbb{P}(X_k = x_{Q'})$$

et finalement à

$$1 = \sum_{Q' \in \mathcal{S}_\varepsilon} r_{Q'} g(x, x_{Q'}),$$

g étant la fonction de Green de la chaîne de Markov X. Il s'ensuit que la mesure ν_ε donnée par

$$\nu_\varepsilon(Q) = \frac{r_Q}{\sum_{Q' \in \mathcal{S}_\varepsilon} r_{Q'}}$$

vérifie $\pi(\nu_\varepsilon) = \nu_\varepsilon$ et puisque $r_Q > c_1$ pour tout $Q \in \mathcal{W}$ le théorème est prouvé.

Il est naturel de se poser la question de l'existence et de la valeur de la limite des mesures ν_ε, quand $\varepsilon \to 0$. Nous obtenons le corollaire suivant.

Corollaire 4.2.3 *Sous les conditions du théorème 4.2.2 et si $\partial\Omega$ est Ahlfors-régulier par rapport à $\mathcal{H}_\alpha$, alors il existe une constante c_1 telle que toute limite faible ν de la famille ν_ε lorsque $\varepsilon \to 0$ vérifie*

$$\frac{1}{c}\mathcal{H}_\alpha \leq \nu \leq c\mathcal{H}_\alpha$$

Rappelons ici qu'un ensemble F est dit Ahlfors-régulier par rapport à $\mathcal{H}_\alpha$ s'il existe $c > 0$ tel que pour tout $x \in F$ et tout $r > 0$ on ait $\frac{1}{c}r^\alpha \leq \mathcal{H}_\alpha(\mathbb{B}(x,r)) \leq cr^\alpha$. Clairement, si F est compact et Ahlfors-régulier il est de mesure $\mathcal{H}_\alpha$ positive et finie.

Il existe une version continue du théorème 4.2.2 dans le cadre de déformations quasi-conformes du disque. On suppose que Ω est un quasi-disque, proche du disque : l'application quasi-conforme associée a une norme suffisamment petite. Il a été remarqué dans [BLZ11] que les ensembles Γ_ε sont localement des graphes lipschitziens.

On choisit un point $x \in \Gamma_\varepsilon$ suivant une mesure μ. On considère le mouvement brownien $\mathcal{B}_t$ partant de x et arrêté au temps τ de contact de $\partial\Omega$. Comme précédemment nous notons ce processus aléatoire $\mathcal{F}_t^\mu$. Posons $\tau_\varepsilon = \sup\{t \geq 0 \ ; \ \mathcal{F}_t \in \Gamma_\varepsilon\}$ et notons $\pi(\mu)$ la distribution de $\mathcal{F}^\mu_{\tau_\varepsilon}$.

Nous avons le

Théorème 4.2.4 *Si Ω est un quasi-disque de petite norme, il existe une constante $c > 0$ telle que pour tout $\varepsilon > 0$ il existe une unique mesure de probabilité ν_ε satisfaisant $\pi(\nu_\varepsilon) = \nu_\varepsilon$; cette mesure satisfait*

$$\frac{1}{c}\sigma_\varepsilon \leq \nu_\varepsilon \leq c\sigma_\varepsilon,$$

où σ_ε désigne la mesure uniforme sur Γ_ϵ.

Ce résultat fait partie d'un travail en cours.

Perspectives 4.2.5 Nous souhaitons

- faire le lien entre la “limite” des vols browniens quand $\varepsilon \to 0$ et le mouvement brownien réfléchi. Ce lien a déjà été mis en lumière dans [GS03] quand le bord du domaine est lisse.
- ajouter une phase de diffusion sur la frontière et étudier les statistiques et l’existence de régimes stationnaires ([BMTV08],[BLMV08],...). Concernant la diffusion sur la frontière, plusieurs modélisations semblent pertinentes : les diffusions sur les fractales, un mouvement brownien réfléchi à l’intérieur d’une saucisse de Minkowski projeté sur le bord, ou un temps d’arrêt de mouvement brownien réfléchi dans Ω (cf. [Gre10]).
- étudier (et comparer) l’itération des vols browniens lorsque l’on remplace le point de “sortie” $\mathcal{F}_{\tau_\varepsilon}$ par le point $y \in \Gamma_\varepsilon$ dont la ligne de Green touche $\partial\Omega$ au point $\mathcal{F}_\tau$.

Les travaux de cette partie ont été motivés et partiellement financés par l’ANR DYOPTRI.

Partie 5

Croissance

Cette partie est en cours d'élaboration et n'est à ce titre qu'un projet de recherche. Avec des collègues géographes, physiciens et spécialistes de l'aménagement du territoire nous nous sommes proposés d'étudier la croissance de villes et des réseaux en utilisant les idées et techniques émanant des processus de croissance aléatoire tels que le DLA (diffusion limited agregation) , le processus de Hastings-Levitov ou la percolation du gradient.

L'idée d'utiliser le DLA pour modéliser la croissance des villes n'est pas nouvelle mais elle est peu satisfaisante : l'impulsion dans cette direction provient d'un article de H.Makse, Andrade et al ([MJB+98]). L'idée est de modifier la percolation du gradient en ajoutant une corrélation entre les sites afin de mieux coller à la réalité de la croissance des agglomérations. Les illustrations ci-dessous sont extraites du mémoire de Master 2 de Thi-Thuy-Nga Nguyen, étudiante en thèse co-encadrée par M. Zinsmeister et moi même.

Percolation au gradient

On se donne une grille dans R^d (ou plus généralement un pavage) et on fixe une fonction f de cette grille à valeurs dans $[0, 1]$. On colorie chaque nœud x avec probabilité $f(x)$. Dans le cas particulier où $f(x)$ dépend seulement de la distance du nœud x à une interface Γ le modèle est appelé "percolation au gradient" ; il a été introduit par Sapoval, Rosso et Gouyet, [BSG85] et étudié par Nolin [Nol08].

Ce modèle et ses variantes n'est pas évolutif. Les images obtenues sont fixes (mais intéressantes, cf. la première image de figure 5.1).

Afin de tenir compte des effets de corrélation entre sites le modèle est modifié de la façon suivante.

A chaque nœud x de la grille nous attribuons de façon indépendante une quantité aléatoire $X(x)$. La variable aléatoire $x \mapsto X(x)$ est distribuée selon une loi u choisie au préalable (ici nous avons utilisé une Gaussienne) ; nous modifions ensuite la variable aléatoire X afin de prendre en compte des phénomènes de corrélation entre les nœuds : on définit la variable aléatoire Y :

$$Y = C_\alpha * X,$$

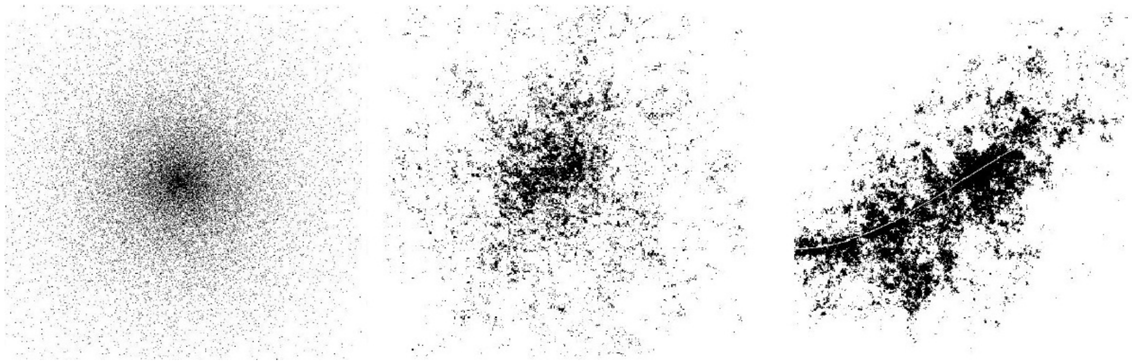

FIGURE 5.1 – Percolation 1. au gradient avec $f = f(r)$ exponentielle 2. avec corrélation 3. avec axe de transport/fleuve

où $C_\alpha(\ell) = (1+\ell^2)^{-\alpha}$, et $\ell = |x - x'|$ correspond à la distance euclidienne entre deux nœuds x, x' .

Ensuite, nous décidons de la coloration de chaque nœud. On se donne une fonction à deux variables $(t, x) \mapsto f(t, x)$, strictement croissante en t, telle que $f(0, .) = 0$. En temps t, le nœud x est coloré en noir si $Y(x) \leq f(t, x)$. De cette façon nous obtenons un processus aléatoire croissant. Dans notre exemple, la dépendance de f en x est radiale : $f(t, x) = g(t, |x|)$, où $|x|$ est la distance de x à un ensemble donné (p. ex. le centre-ville) et g une fonction strictement croissante en t et en $|x|$. Les images obtenues avec ce modèle, selon les paramètres choisis, sont du type illustré dans figure 5.1, deuxième image.

La troisième image est obtenue en ajoutant l'effet d'un axe de transport : la fonction $|x|$ est une fonction de la distance du nœud au centre ville mais aussi de distance de l'axe(fleuve, autoroute etc).

Nous souhaitons étudier ces processus : nous avons déjà montré que dans le cas radial la distance moyenne du centre à la frontière du cluster et égale à r_t, où r_t est tel que $g(t, r_t) = p_c$, la probabilité critique de la percolation simple sur la grille. Nous aimerions préciser ces estimations et ajuster/enrichir les paramètres pour obtenir un modèle aussi proche de la réalité que possible. La thèse de Nguyen s'inscrit dans ce projet.

Appendice

Frontière de Martin, points simples et multiples

Soit Ω un domaine de $\mathbb{R}^d$, muni d'une fonction de Green G. Notons $\mathbf{H}_+$ l'ensemble des fonctions harmoniques positives sur Ω. Soit $X_0 \in \Omega$ un point fixé. Pour $\zeta \in \Omega \setminus \{X_0\}$ nous définissons le noyau de Martin $K_\zeta : \Omega \to \mathbb{R}$ associé à ζ :

$$K_\zeta(x) = \frac{G(\zeta, x)}{G(\zeta, X_0)} \tag{5.1}$$

Remarquons que, pour tout $\zeta \in \Omega \setminus \{X_0\}$, la fonction K_ζ est une fonction surharmonique sur Ω, harmonique sur $\Omega \setminus \{\zeta\}$ et vérifie $K_\zeta(X_0) = 1$. On appellera *point de Martin* de Ω toute fonction u harmonique sur Ω de la forme

$$u = \lim_n K_{\zeta_n},$$

pour une suite $(\zeta_n)_n \subset \Omega$.

Remarquons que la définition ne dépend pas du choix de X_0. De plus,

α) u est une fonction harmonique positive sur Ω, normalisée par $u(X_0) = 1$.

β) d'après le principe de Harnack, toute suite tendant vers l'infini dans Ω admet une sous-suite convergeant vers un point de Martin.

Il existe alors une compactification $\widehat{\Omega} = \Omega \cup \partial\widehat{\Omega}$ de Ω, métrisable, ayant les propriétés suivantes :

α) la suite $(x_n)_{n\in\mathbb{N}}$ de points de Ω converge vers un point de $\partial\widehat{\Omega}$ si et seulement si elle converge vers un point de Martin.

β) deux suites de points de Ω convergent vers le même point de $\partial\widehat{\Omega}$ si et seulement si elles définissent la même fonction harmonique.

La compactification ne dépend pas du choix de point de normalisation X_0. La topologie induite par cette compactification, *appelée topologie de Martin*, coïncide sur Ω avec la topologie euclidienne. La frontière $\partial\widehat{\Omega}$ du compactifié de Ω est appelé *frontière de Martin de* Ω.

Le théorème suivant est l'analogue, dans le cadre de la théorie de Martin, de la représentation intégrale de Poisson des fonctions harmoniques positives dans une boule.Pour $\zeta \in \partial\widehat{\Omega}$, on note K_ζ la fonction harmonique associée.

Théorème 5.0.6 *(Martin) Pour toute fonction harmonique $u \in \mathbf{H}_+$, il existe une mesure borélienne et positive ν_u sur $\partial\widehat{\Omega}$ telle que*

$$u(x) = \int_{\zeta \in \partial\widehat{\Omega}} K_\xi(x)\nu_u(d\xi) \tag{5.2}$$

Cette mesure est unique lorsque l'on impose à son support d'être inclus dans l'ensemble des points de Martin ζ tels que K_ζ soit minimale. Notons ν_1 la mesure associée à la fonction constante égale à 1.

Dans les domaines que nous étudions il existe une projection continue π de $\widehat{\Omega}$ sur le compactifié d'Alexandroff $\overline{\Omega}$ prolongeant l'identité sur Ω. Cette projection, transfère la mesure ν_1 sur la mesure harmonique habituelle.

Proposition 5.0.7 *Soient Ω un domaine de Green . Notons ν_1 la mesure de Martin de Ω associée à la fonction constante égale à 1 relativement à un point $X_0 \in \Omega$. Supposons qu'il existe une application $\pi : \widehat{\Omega} \to \overline{\Omega}$, continue, prolongeant l'identité de Ω. Soit $x \in \Omega$. Si ω_x est la mesure harmonique de Ω en x alors pour tout ensemble $F \subset \partial\Omega$ mesurable*

$$\omega_x(F) = \int_{\pi-1(F)} K_\zeta(x)\nu_1(d\zeta) \tag{5.3}$$

Un point $x \in \partial\Omega$ est dit *simple* si $\pi^{-1}(x)$ est un singleton et *multiple* sinon.

On dit que un ensemble $F \subset \Omega$ est *effilé au sens minimal* au point de Martin ζ, si K_ζ est minimale et si la réduite de K_ζ sur F,

$$\mathbf{R}^F_{K_\zeta} = \inf\{s : \Omega \to \mathbb{R}_+ \ ; \ s \geq K_\zeta \ , \ s \text{ surharmonique}\},$$

vérifie $\mathbf{R}^F_{K_\zeta} \neq K_\zeta$.

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Dimensions et formalisme multifractal

Dans cette section je propose un bref historique/rappel du formalisme multifractal (très loin d'être exhaustif) et un rappel de la notion de dimension de Minkowski que j'ai utilisé au chapitre 4.

Dimension de Minkowski

Soit F un sous-ensemble compact de $\mathbb{R}^d$. On note

$$\mathcal{N}_\epsilon = \inf\{\#\mathcal{R}_\epsilon \ ; \ \mathcal{R}_\epsilon \text{ recouvrement (fini) de } F \text{ par de boules de rayon } \epsilon\}.$$

La dimension de Minkowski d_M de F est alors

$$d_M = \limsup_{\epsilon \to 0} \frac{\log \mathcal{N}_\epsilon}{|\log \epsilon|}.$$

On dira que F admet une dimension de Minkowski *exacte* si cette limite existe. Rappelons que la dimension de Minkowski, qui est aussi connue sous l'appelation "box counting" dimension, n'en est pas une ! En effet elle ne satisfait pas la condition minimale : la dimension d'une réunion dénombrable d'ensembles n'est pas le supremum des dimensions (ex. $\mathbb{Q} \cap [0,1]$).

Formalisme multifractal

Avec les notations du chapitre 2, si μ est une mesure sur $\mathbb{K} = \{0,1\}^{\mathbb{N}}$, on définit la *dimension locale* (ou exposant de Hölder local) de la mesure μ en $x \in \mathbb{K}$ la quantité :

$$\alpha(x) = \liminf_{n \to +\infty} \alpha_n(x) = \liminf_{n \to +\infty} -\frac{\log \mu(I_n(x))}{n \log 2}.$$

L'analyse multifractale vise à étudier la dimension de Hausdorff $\dim_{\mathcal{H}}(E_\alpha)$ des ensembles de niveau $E_\alpha = \{x : \alpha(x) = \alpha\}$, $\alpha > 0$. La fonction $f(\alpha) = \dim(E_\alpha)$ est *le spectre des singularités* (ou *spectre multifractal*) de μ.

Le concept a été introduit par plusieurs auteurs dans des contextes différents : [Man74], [FP85],[BPPV84], [MCSW86]. Afin d'étudier la fonction $f(\alpha)$, Hentschel et Procaccia [HP83] ont introduit les dimensions généralisées D_q par

$$D_q = \lim_{n \to +\infty} \frac{1}{q-1} \frac{\log\left(\sum_{I \in \mathcal{F}_n} \mu(I)^q\right)}{n \log 2},$$

(cf. [GP83, Gra83]). Halsey et al. [HJK⁺86] ont suggéré que le spectre multifractal $f(\alpha)$ et les dimensions D_q sont liés à travers la transformée de Legendre :

$$f(\alpha) = \dim(E_\alpha) = \tau^*(\alpha) = \inf(\alpha q + \tau(q),\ q \in \mathbb{R}), \tag{5.4}$$

où

$$\tau(q) = \limsup_{n \to +\infty} \tau_n(q) \quad \text{et} \quad \tau_n(q) = \frac{1}{n \log 2} \log\left(\sum_{I \in \mathcal{F}_n} \mu(I)^q\right).$$

(les cylindres de mesure nulle sont à exclure de la somme).

La fonction $\tau(q)$ est le spectre L^q de μ. Si la limite existe on obtient $\tau(q) = (q-1)D_q$.

La relation (5.4), utilisée aussi par le formalisme thermodynamique dans le cadre de l'opérateur du transfert par Bowen [Bow75] et Ruelle [Rue78], est au cœur du formalisme multifractal.

On dit qu'un mesure μ satisfait le formalisme multifractal si son spectre f vérifie la formule (5.4). Il est bien connu que les mesures de Bernoulli satisfont le formalisme multifractal (e.g. [Fal97]) ; il en est de même pour les mesures autosimilaires [CM92, LN99, Ols95], et quasi-Bernoulli [BMP92, Heu98, Tes06]. Des conditions pour qu'une mesure associée à un système dynamique satisfasse le formalisme multifractal sont proposées dans [Col88, Fan94, Ran89].

La minoration de $\dim(E_\alpha)$ est souvent liée à l'existence d'une mesure μ_q invariante par le shift, ergodique, telle que

$$\forall n,\ \forall I \in \mathcal{F}_n, \quad \frac{1}{C}\mu(I)^q 2^{-n\tau(q)} \leq \mu_q(I) \leq C\mu(I)^q 2^{-n\tau(q)},$$

où la constante $C > 0$ ne dépend pas de n ni de I. Cette mesure est appelée "mesure de Gibbs" [Mic83]. La majoration est en général l'objet d'un calcul direct.

Si la fonction τ est dérivable en q, la mesure μ_q est portée par $E_{-\tau'(q)}$. Dans ce cas, Brown, Michon et Peyrière [BMP92, Pey92] ont montré l'existence de mesures de Gibbs et en ont déduit que

$$\dim_{\mathcal{H}}(E_{-\tau'(q)}) = \tau^*(-\tau'(q)) = -q\tau'(q) + \tau(q).$$

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Mouvement brownien réfléchi

La définition du mouvement brownien réfléchi est bien connue dans les domaines à frontière régulière. Dans les domaines irréguliers ainsi que sur les fractals l'approche passe par les formes de Dirichlet.

Rappelons d'abord la notion de (ϵ, δ)-domaines, due à P. Jones [Jon81] : le domaine Ω est dit (ϵ, δ) *ou localement uniforme* s'il existe des constantes ϵ et δ telles que pour tout $x, y \in \Omega$ avec $|x - y| < \delta$ il existe une courbe rectifiable γ joignant x et y satisfaisant

1. $\epsilon\ell(\gamma) \leq |x - y|$
2. $\epsilon \min\{|x - z|, |y - z|\} \leq \mathit{dist}(z, \partial\Omega)$

Notons $W^{1,2}(\Omega) = \{f \in L^2(\Omega)\ ;\ \nabla f \in L^2(\Omega)\}$ l'espace de Sobolev sur Ω munie de la norme de Sobolev $||f||_{1,2} = ||f||_2 + ||\nabla f||_2$. Les domaines (ϵ, δ) satisfont la *condition d'extension $W^{1,2}$ de Sobolev*, cf [Jon81] :

« *il existe un opérateur linéaire borné $T : W^{1,2}(\Omega) \to W^{1,2}(\mathbb{R})$ prolongeant l'identité de $W^{1,2}(\Omega)$* ».

Pour $f, g \in W^{1,2}(\Omega)$ on définit

$$\mathcal{E}(f, g) = \int_\Omega < \nabla f, \nabla g > dx,$$

et

$$\mathcal{E}_1(f,g) = \mathcal{E}(f,g) + \int_\Omega fg dx.$$

La forme de Dirichlet $(\mathcal{E}, W^{1,2}(\Omega))$ est dite régulière dans $\overline{\Omega}$ si $W^{1,2}(\Omega) \cap C(\overline{\Omega})$ est dense à la fois dans $(W^{1,2}(\Omega), \mathcal{E}_1^{\frac{1}{2}})$ et dans $(C(\overline{\Omega}), ||.||_\infty)$. Si Ω est un domaine localement uniforme, la forme de Dirichlet $(\mathcal{E}, W^{1,2}(\Omega))$ est régulière dans $\overline{\Omega}$.

Nous pouvons maintenant définir (voir aussi [Che93], [BCR04]) le mouvement brownien réfléchi dans Ω.

Si Ω est localement uniforme, il existe un processus de Markov (fort) $\mathcal{R}_t$ associé à $(\Omega, \mathcal{E})$, ayant des trajectoires continues. De surcroit, nous pouvons construire une famille de distributions $(\mathcal{R}_t^x)_t$ pour ce processus, partant de x, quelque soit $x \in \overline{\Omega}$ (pour les détails voir le livre de Fukushima, Oshima et Takeda, [FOT94]).

Considérons un fermé $F \subset \partial\Omega$ et posons τ_F le temps d'atteinte de F pour le processus $\mathcal{R}_t^x$. En supposant que $\mathbb{E}_x[\tau_F] < +\infty$ pour (au moins un) $x \in \Omega$, on obtient que pour tout $f \in C(F)$, la fonction

$$u : x \mapsto \mathbb{E}_x\left[f(\mathcal{R}_{\tau_F})\right]$$

est harmonique bornée dans Ω et converge vers f en tout point régulier de F. Cette hypothèse est vérifiée dans le cadre de l'exemple proposé en section 3.3 et est indispensable pour mettre en œuvre une théorie du potentiel.

Si l'on fait l'hypothèse supplémentaire que $\partial\Omega \setminus F$ est régulier, alors u est la solution du problème de Dirichlet avec des conditions au bord mixtes de type Dirichlet-Neumann :

$$\begin{cases} u \text{ harmonique dans } \Omega \\ \dfrac{\partial u}{\partial \vec{\eta}} = 0 \text{ sur } \partial\Omega \setminus F \\ u = f \text{ sur } F \end{cases}, \tag{5.5}$$

où $\vec{\eta}$ est le vecteur normal à la frontière $\partial\Omega$.

Remarque 5.0.8 Soit $C_F(\overline{\Omega})$ l'ensemble des fonctions continues sur $\overline{\Omega}$, nulles sur F. Si $W^{1,2}(\Omega) \cap C_F(\overline{\Omega})$ est dense dans $C_F(\overline{\Omega}), ||.||_\infty)$, alors on peut, comme ci-dessus, définir le processus stochastique $\mathcal{R}_t^F$ associé. Ce processus coïncide avec $\mathcal{R}_{t\mathbf{1}_{t<\tau_F}+\tau_F\mathbf{1}_{t\geq\tau_F}}$ dans les domaines localement uniformes (see also [AB10]).

Notons $\omega_.$ la mesure harmonique associée à cette diffusion, ie. pour $x \in \Omega$ et $A \subset \partial\Omega$,

$$\omega_x(A) = \mathbb{P}_x(\mathcal{R}_{\tau_F} \in A).$$

La relation (5.5) implique que pour $A \subset \partial\Omega$ mesurable, l'application $x \mapsto \omega_x(A)$ est positive, harmonique dans Ω, de valeur 1 sur A, nulle sur $F \setminus A$ et de dérivée normale nulle sur la partie régulière de $\partial\Omega \setminus F$.

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Discrétisation du mouvement brownien

Cette brève description contient essentiellement des résultats de [BL96, Anc90, LS84] adaptés au contexte de ce mémoire.

Soit Ω un domaine de Green dans $\mathbb{R}^d$; le mouvement brownien est alors transient dans Ω, ie. sortira définitivement de tout compact de Ω en temps fini presque sûrement. On dira qu'un sous ensemble fermé F de Ω est *récurrent* si pour tout $x \in \Omega$ le mouvement brownien partant de x touche $\mathbb{P}_x$-presque-sûrement l'ensemble F avant $\partial\Omega$.

Considérons un sous-ensemble discret $\mathcal{N}$ et deux familles d'ensembles

$$\mathbf{F} = \{F_x \text{ fermé} \ ; \ x \in \mathcal{N}\} \text{ et } \mathbf{V} = \{V_x \text{ ouvert} \ ; \ x \in \mathcal{N}\}$$

satisfaisant :

- $x \in \overset{\circ}{F}_x$ et $F_x \subset V_x$, pour tout $x \in \mathcal{N}$,
- $F_x \cap V_z = \emptyset$ pour tout $x \neq z \in \mathcal{N}$,
- l'ensemble $F = \bigcup_{x \in \mathcal{N}} F_x$ est récurrent,
- il existe $C > 0$ telle que pour tout $x \in \mathcal{N}$ et toute fonction h harmonique positive dans V_x on ait
$$\frac{1}{C} h(x) \leq h(z) \leq C h(x), \text{ pour tout } z \in F_x.$$

On se réfère à la dernière propriété sous le nom de *condition d'anneau de Harnack*.

En reprenant la terminologie de [BL96] le triplet $(\mathcal{N}, \mathbf{F}, \mathbf{V})$ est appelé *« Lyons Sullivan data »*.

Si nous supposons, en plus, qu'il existe une constante $\alpha > 0$ telle que $G_{V_x}(x, z) = \alpha$, pour tout $x \in \mathcal{N}$ et tout $z \in \partial F_x$ le triplet $(\mathcal{N}, \mathbf{F}, \mathbf{V})$ est appelé *« balanced Lyons Sullivan data »*, que je traduis ici "structure réversible de Lyons-Sullivan".

Le théorème qui suit est extrait de [BL96, LS84].

Théorème 5.0.9 *Si Ω est un domaine de Green admettant une structure réversible de Lyons-Sullivan $(\mathcal{N}, \mathbf{F}, \mathbf{V})$ alors il existe une chaîne de Markov irréductible X sur $\mathcal{N}$ discrétisant le mouvement brownien , ie. :*

1. *il existe une constante positive κ telle que $g(x, z) = \kappa G(x, z)$ pour tout $x, z \in \mathcal{N}$, où g désigne la fonction de Green de la chaîne de Markov X.*
2. *les probabilités de transition de X sont symétriques, ie. $p_{x,z} = p_{z,x}$ pour tout $x, z \in \mathcal{N}$.*
3. *l'espace des fonctions harmoniques pour X et positives sur $\mathcal{N}$ est isomorphe à l'espace des fonctions harmoniques positives sur Ω.*

La remarque suivante nous permettra de mieux comprendre le cas d'un domaine plan.

Remarque 5.0.10 Considérons la collection $\mathbf{V}$ des cubes de Whitney dyadiques et soient $\mathcal{N}$ leurs centres. Pour $a > 0$ on note, $\mathbf{F}$ la collection des ensembles $F_x = \{z \in \Omega \ ; \ G_{V_x}(x,z) = a\}$, $x \in \mathcal{N}$. Si le domaine $\Omega \subset \mathbb{R}^2$ est borné (de diamètre normalisé égal à 1), alors pour tout $a > 0$ le triplet $(\mathcal{N}, \mathbf{F}, \mathbf{V})$ est une structure réversible de Lyons-Sullivan. Pour établir cela il suffit de vérifier la condition d'anneaux de Harnack : il existe $C > 0$ telle que pour tout $x \in \mathcal{N}$ toute fonction h harmonique positive dans V_x on ait

$$\frac{1}{C} h(x) \leq h(z) \leq C h(x), \text{ pour tout } z \in F_x.$$

Or, il est clair que $-\log[\ell(V_x)|x-y|/2] \leq G_{V_x}(x,y) \leq -\log[2\ell(V_x)|x-y|]$, où $\ell(V_x)$ désigne la longueur de côté du carré V_x. Il s'ensuit que, pour $a > 0$ fixe, il existe $0 < \rho_1 < \rho_2 < 1$ tels que $\mathbf{F}_x \subset \rho_2 V_x \setminus \rho_1 V_x$ et que donc la constante de Harnack associée à $V_x \setminus F_x$ est majorée par celle de l'anneau conforme $\rho_2 V_x \setminus \rho_1 V_x$.

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1. Athanasios Batakis et Viet Hung Nguyen, “On the dimension of harmonic measure of partially reflected diffusions”, prépublication (2010).
2. Athanasios Batakis et Michel Zinsmeister, “ Invariant measures for Intermittent Transport”, prépublication (2010).
3. Athanasios Batakis et Michel Zinsmeister, “ On the time-table of Brownian flights”, prépublication (2009), soumis.
4. Athanasios Batakis et Benoît Testud, “Multifractal Analysis of inhomogeneous Bernoulli products”, prépublication (2008), soumis.
5. Athanasios Batakis et Michel Zinsmeister, “ Brownian flights”, à paraitre dans Pure and Applied Mathematics Quaterly, 2011.
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Autres Ouvrages

1. Guy Allain et Athanasios Batakis. “Formes de Dirichlet et diffusions : le cas du mouvement Brownien Partiellement réfléchi”, en rédaction.
2. Athanasios Batakis, “A survey on harmonic measure”, Actes Séminaire SCAM.
3. Athanasios Batakis et Yanick Heurteaux, “Multifractal analysis and harmonic measure”, Cours de 3ème cycle, Université de Monastir, Tunisie.

Travaux de la partie I

extraits de mon doctorat

Sur les domaines de Poisson

L'objectif de ce chapitre est de donner une condition nécessaire pour qu'un domaine de $\mathbb{R}^d$, $d \geq 3$, soit Poissonien. Cette condition (voir définition 0.1.6) que nous noterons CDF (condition faible de cône double) a été proposée par Bishop dans [Bis91] pour les domaines du plan et est semblable à la condition d'effilement de Wiener. Nous proposons en outre quelques exemples de situations où cette condition est également suffisante.

0.1 Cadre, résultats liés.

Bishop [Bis91] a introduit la condition CDF et il a montré qu'un domaine du plan est Poissonien si et seulement si l'ensemble des points satisfaisant à une condition CDF est de mesure de Hausdorff linéaire nulle. D'autres conditions pour qu'un domaine soit Poissonien ont également été énoncées par Mountford et Port [MP91]. Mountford et Port font appel à la théorie de probabilités, tandis que Bishop s'appuie sur la théorie du Potentiel classique.

En dimension 2, Bishop utilise une caractérisation de deux domaines disjoints dont les mesures harmoniques ne sont pas mutuellement singulières, pour montrer que la condition CDF est suffisante pour qu'un domaine soit Poissonien. Cette caractérisation est énoncée par Bishop, Carleson, Garnett et Jones dans [BCGJ89] pour les domaines du plan. La preuve s'appuie sur un résultat de Makarov [Mak85], amélioré par Pommerenke [Pom86], qui fournit une comparaison de la mesure harmonique des domaines simplement connexes du plan et de la mesure de Hausdorff linéaire. La démonstration de Makarov exploitant les propriétés des fonctions conformes du plan, ne peut pas être appliquée en dimension supérieure. En outre, Wolff [Wol95] a montré que les résultats de Makarov ne sont pas adaptables en dimension supérieure à deux. Ainsi, la question de savoir si la condition CDF est suffisante pour qu'un domaine de $\mathbb{R}^d$, $d \geq 3$, soit Poissonien reste-t-elle ouverte.

La démonstration de la nécessité de la condition CDF que nous proposons pour les domaines de $\mathbb{R}^d$, $d \geq 3$ utilise la théorie de Martin. La voie suivie nous a été proposée par A. Ancona.

0.1.1 Définitions, propriétés des domaines de Poisson

Donnons tout d'abord la définition d'un domaine de Poisson.

Définition 0.1.1 *Un domaine $\Omega \subset \mathbb{R}^d$ est appelé domaine de Poisson, ou domaine Poissonien, si pour toute fonction u harmonique bornée dans Ω il existe une fonction mesurable $f : \partial\Omega \to \mathbb{R}$ telle que :*

$$u(x) = H_f^\Omega = \int_{\partial\Omega} f(z)\omega(x, dz) \tag{1}$$

où ω est la mesure harmonique de Ω.

Donnons l'interprétation de la définition en termes de frontière de Martin ([MP91], théorème 9).

Théorème 0.1.2 *([MP91]) Un domaine de Green $\Omega \subset \mathbb{R}^d$ est Poissonien si et seulement s'il existe un sous-ensemble Δ de la frontière de Martin minimale $\partial\Omega^M$ de Ω, un sous-ensemble E de la frontière $\partial\overline{\Omega}$ de Ω dans $\widehat{\mathbb{R}}^d$ et une application $\pi : \Delta \to E$ vérifiant :*

i) π est bijective et bimesurable.

ii) $\nu(\Delta) = 1$, où ν est la mesure harmonique de Martin relativement à un point $X_0 \in \Omega$.

iii) $\omega(E) = 1$, où ω est la mesure harmonique de Ω évaluée en X_0.

iv) Pour tout ensemble mesurable $F \subset E$ et pour tout $X_0 \in \Omega$, si ν est la mesure harmonique de Martin relativement à X_0 et ω la mesure harmonique de Ω évaluée en X_0, alors $\omega(F) = \nu(\pi^{-1}(F))$.

Les propriétés suivantes des domaines de Poisson ont également été démontrées dans [MP91].

α) Toute composante connexe de l'intersection de deux domaines de Poisson est un domaine de Poisson.

β) Soient Ω_1 et Ω_2 deux domaines Poissoniens de $\mathbb{R}^d$ tels que $\Omega = \Omega_1 \cup \Omega_2$ soit connexe. Soit ω la mesure harmonique de Ω. Si $\omega(\partial\Omega_1 \cap \partial\Omega_2) = 0$, alors Ω est Poissonien.

Donnons quelques exemples :

Exemple 0.1.3 Par le théorème de Herglotz-Riesz les boules de $\mathbb{R}^d$ sont des domaines de Poisson. •

Exemple 0.1.4 Donnons un exemple de domaine non-Poissonien. Soit $\widehat{\mathbb{C}}$ le compactifié du plan complexe avec le point à l'infini. Dans $\widehat{\mathbb{C}}$ considérons le complémentaire du segment $[0, 1]$. Le domaine $\Omega = \widehat{\mathbb{C}} \setminus [0, 1]$ est simplement connexe et est l'image par l' application conforme

$$\phi(z) = \frac{(1-z)^2}{(1-z)^2 - (1+z)^2}$$

du disque unité $\mathbb{B}$. l'application ϕ se prolonge de manière continue sur $\partial\mathbb{B}$. On peut également vérifier que les arcs L et M du cercle unité $\partial\mathbb{B}$ définis par

$$M = \partial\mathbb{B} \cap \{z \in \widehat{\mathbb{C}}\ ;\ \mathrm{Im} z > 0\} \text{ et } L = \partial\mathbb{B} \cap \{z \in \widehat{\mathbb{C}}\ ;\ \mathrm{Im} z < 0\}$$

vérifient $\phi(L) = \phi(M) =]0, 1[$.

Soit $u = H^{\partial\mathbb{B}}_{\mathbf{1}_M}$ la solution du problème de Dirichlet dans $\mathbb{B}$ pour la fonction indicatrice $\mathbf{1_M}$ de M et soit v l'image par l'application conforme ϕ de la fonction u. Montrons que v est une fonction harmonique bornée qui ne s'écrit pas sous la forme (1) ce qui montrera que Ω n'est pas Poissonien.

Si $x \in M$ et $y \in L$ sont deux points de $\partial\mathbb{B}$ avec $\zeta = \phi(x) = \phi(y)$ et si V est un voisinage de ζ suffisament petit alors $\phi^{-1}(V)$ est la réunion d'un voisinage V_1 de x et d'un voisinage V_2 de y dans $\overline{\mathbb{B}}$, V_1 et V_2 disjoints (voir fig. 1).

Remarquons que u converge vers 1 en tout point de M et vers 0 en tout point de L. Par symétrie, si v est la solution du problème de Dirichlet (généralisé) pour une fonction frontière $f : [0, 1] \to \mathbb{R}$, alors $v(z)$ converge vers la même valeur quand z tend vers ζ, $\mathrm{Im} z < 0$ et quand z tend vers ζ, $\mathrm{Im} z > 0$. Or, ceci n'est pas le cas.

Une autre façon de procéder est en utilisant le théorème de Hunt et Wheeden [HW70]. Si z_n tend vers ζ non-tangentiellement alors $v(z_n)$ converge vers $f(\zeta)$, pour $\mathcal{H}_1$-presque tout ζ in $]0, 1[$. Or, la limite inférieure non-tangentielle de v est 0 tandis que sa limite supérieure est 1 en tout point de $]0, 1[$. Il est donc clair qu'il n'existe pas de fonction $f : [0, 1] \to \mathbb{R}$ telle que $v = H^{\Omega}_f$. •

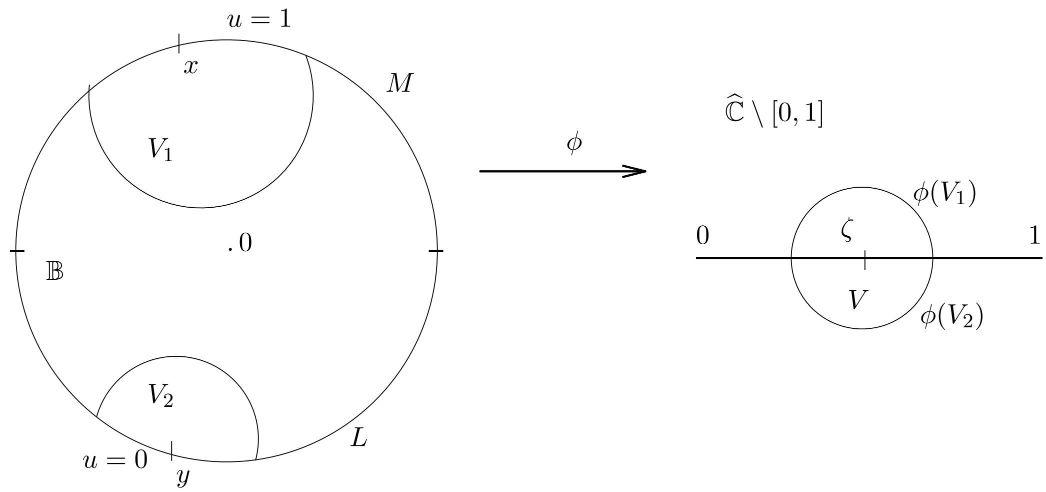

Figure 1: L'application ϕ de l'exemple 0.1.4.

0.1.2 La condition CDF

Nous introduisons à présent la condition CDF, proposée par Bishop.

Notation 0.1.5 Notons $\mathbb{S}_d$ la sphère unité de $\mathbb{R}^d$. Pour $x \in \mathbb{R}^d$, $\epsilon > 0$ et $\theta \in \mathbb{S}_d$ nous notons $C(x, r, \epsilon, \theta)$ le cône

$$C(x, r, \epsilon, \theta) = \{y \; ; \; y = x + r'\theta' \; , \; \theta' \in \mathbb{S}_d \; , \; ||\theta - \theta'|| < \epsilon \; , \; 0 < r' < r\}.$$

Nous écrivons $W(x, r, \epsilon, \theta) = C(x, r, \epsilon, \theta) \setminus \mathbb{B}(x, r/2)$ et nous notons $\theta_d = (0, ..., 0, 1) \in \mathbb{S}_d$.

Définition 0.1.6 *Soit Ω un domaine dans $\mathbb{R}^d$, $d \geq 2$. On dit que le point $x \in \partial\Omega$ satisfait à une condition CDF par rapport à Ω s'il existe $\theta = \theta(x) \in \mathbb{S}_d$ et $\epsilon = \epsilon(x) > 0$ tels que*

$$\sum_{n \in \mathbb{N}} \left\{ \mathrm{cap}\Big\{2^n (W(x, 2^{-n}, \epsilon, \theta) \setminus \Omega)\Big\} + \mathrm{cap}\Big\{2^n (W(x, 2^{-n}, \epsilon, -\theta) \setminus \Omega)\Big\} \right\} < +\infty, \tag{2}$$

cap *signifiant la capacité de Green dans $\mathbb{R}^d$, pour $d \geq 3$ et la capacité de Green dans la boule $\mathbb{B}(x, 1)$ centrée en x et de rayon 1, quand $d = 2$.*

Dans la suite nous supposons $d \geq 3$; le cas $d = 2$ peut être traité de manière similaire, en se ramenant à des ensembles contenus dans une boule.

Remarque 0.1.7 Si $\Omega' \subset \Omega$ sont deux domaines de $\mathbb{R}^d$, avec $\partial\Omega \subset \partial\Omega'$ et si un point $x \in \partial\Omega \cap \partial\Omega'$ satisfait à une condition CDF par rapport à Ω', alors x satisfait à une condition CDF par rapport à Ω.

0.2 Une condition nécessaire pour qu'un domaine de $\mathbb{R}^d$, $d \geq 3$, soit Poissonien

C.J. Bishop a proposé la condition CDF et a montré que, dans $\mathbb{R}^2$, un domaine Ω est Poissonien si et seulement si l'ensemble des points de $\partial\Omega$ satisfaisant à une condition CDF par rapport à Ω est de $\mathcal{H}_1$-mesure nulle [Bis91]. Le théorème suivant étend partiellement le résultat de Bishop en dimension supérieure.

Théorème 0.2.1 *Soit Ω un domaine dans $\mathbb{R}^d$ et supposons que la $\mathcal{H}_{d-1}$-mesure des points de sa frontière qui satisfont à une condition CDF par rapport à Ω est strictement positive. Alors Ω n'est pas Poissonien.*

Le reste de la section est consacré à la démonstration de ce théorème.

0.2.1 Préliminaires

La démonstration du théorème s'effectue en 4 étapes :

A) Nous montrons dans un premier temps que si l'ensemble des points de $\partial\Omega$ satisfaisant à une condition CDF est de mesure $\mathcal{H}_{d-1}$ positive, alors il existe un sous-ensemble de la frontière de Ω, de mesure $\mathcal{H}_{d-1}$ positive, inclus dans une surface lipschitzienne dont tous les points satisfont à une condition CDF.

B) Dans un deuxième temps, nous affirmons que si Γ est le graphe d'une fonction lipschitzienne si $x \in \Gamma$ et si B est une partie d'un cône de sommet x disjoint de Γ, alors l'effilement au sens minimal en x relativement à $\mathbb{R}^d \setminus \Gamma$ de B est équivalent à une condition de type CDF.

C) Ensuite nous construisons deux sous-domaines de Ω dont les frontières intersectent une surface lipschitzienne sur un ensemble K de mesure $\mathcal{H}_{d-1}$ positive et qui sont effilées au sens minimal relativement à $\mathbb{R}^d \setminus K$ en $\mathcal{H}_{d-1}$-presque tout point de K.

D) Finalement, en utilisant l'effilement au sens minimal des frontières de ces domaines en $\mathcal{H}_{d-1}$-presque tout point de K, nous montrons que leurs mesures harmoniques sont équivalentes à $\mathcal{H}_{d-1}$ sur K. La preuve est completée par la proposition suivante :

Proposition 0.2.2 *([Bis91]) Un domaine $\Omega \in \mathbb{R}^d$ est Poissonien si et seulement si pour tous sous-domaines disjoints Ω_1 et Ω_2 de Ω, les mesures harmoniques ω_1 et ω_2 correspondantes sont mutuellement singulières sur $\partial\Omega$.*

0.2.2 Démonstration du théorème 0.2.1

A) Nous proposons quelques lemmes géométriques pour démontrer la première étape de la preuve du théorème. Les notions d'ensembles régulier et irrégulier au sens de Besicovitch sont définies en appendice.

Nous considérons un domaine $\Omega \subset \mathbb{R}^d$.

Proposition 0.2.3 *Soit $E \subset \partial\Omega$ un ensemble qui vérifie $0 < \mathcal{H}_{d-1}(E) < \infty$. Supposons que $\mathcal{H}_{d-1}$-presque tout point de E satisfait à une condition CDF par rapport à Ω. Alors E est régulier au sens de Besicovitch.*

La démonstration de la proposition que nous proposons est comparable à celle du lemme 4.3 de [Bis91].

Lemme 0.2.4 *Sous les hypothèses de la proposition 0.2.3, il existe $\epsilon_0 > 0$ et $\theta_0 \in \mathbb{S}_d$ tels que l'ensemble F des points qui satisfont une condition CDF par rapport à Ω avec $\epsilon = \epsilon_0$ et $\theta = \theta_0$ soit de $\mathcal{H}_{d-1}$-mesure positive.*

Démonstration Pour $n \in \mathbb{N}$, posons

$$S_n = \{x \in E \,;\, x \text{ satisfait à une condition CDF avec } \frac{1}{n+1} \leq \epsilon < \frac{1}{n}\}.$$

Clairement, $E = \cup_{n=1}^{+\infty} S_n$ et donc il existe S_m tel que $\mathcal{H}_{d-1}(S_m) > 0$.

Considérons une partition de $\mathbb{S}_d$ en un nombre fini d'ensembles $\{S^k\}_{k=1}^n$ de diamètre inférieure à $\frac{1}{100m}$. Pour $k = 1, ..., n$, soit S_m^k l'ensemble des points de S_m satisfaisant à une condition CDF par rapport à Ω avec $\theta \in S^k$.

Il est clair que $\displaystyle\bigcup_{1\le k\le n} S_m^k = S_m$. Il existe donc un ensemble $S_m^{k_0}$ parmi ceux-ci, avec $\mathcal{H}_{d-1}(S_m^{k_0}) > 0$. Posons $F = S_m^{k_0}$ et soit $\theta_0 \in S^{k_0}$. Alors, tout $x \in F$ satisfait la condition CDF par rapport à Ω avec $\theta(x) = \theta_0$ et $\epsilon = \frac{1}{100m}$, le cône $C(x, r, \frac{1}{100m}, \theta_0)$ étant inclus dans le cône $C(x, r, \epsilon(x), \theta(x))$. •

Lemme 0.2.5 *Soit E un ensemble irrégulier au sens de Besicovitch tel que $0 < \mathcal{H}_{d-1}(E) < +\infty$. Alors il existe un sous-ensemble $F \subset E$ avec $0 < \mathcal{H}_{d-1}(F) < +\infty$ et une constante $C > 0$ tels que*

$$\alpha) \quad \frac{\mathcal{H}_{d-1}(B(x,r) \cap E)}{r^{d-1}} \le C, \qquad \forall x \in F\,,\ 0 < r < 1 \tag{3}$$

$$\beta) \quad \textit{Pour tout } \epsilon > 0 \ \ \textit{et} \ \ \theta \in \mathbb{S}_d \tag{4}$$

$$\limsup_{r\to 0} \frac{\mathcal{H}_{d-1}(F \cap C(x,r,\epsilon,\theta))}{r^{d-1}} + \limsup_{r\to 0} \frac{\mathcal{H}_{d-1}(F \cap C(x,r,\epsilon,-\theta))}{r^{d-1}} > 0$$

pour $\mathcal{H}_{d-1}$-presque tout $x \in F$.

Démonstration Ce lemme est un corollaire de deux résultats classiques de la théorie de la mesure géométrique (cf. [Mat95]) :

Théorème 0.2.6 *Soit $E \in \mathbb{R}^d$, avec $0 < \mathcal{H}_{d-1}(E) < +\infty$. Si E est irrégulier au sens de Besicovitch alors, pour tout $\epsilon > 0$, tout $\theta \in \mathbb{S}_d$ et pour $\mathcal{H}_{d-1}$-presque tout $x \in E$*

$$\limsup_{r\to 0} \frac{\mathcal{H}_{d-1}(E \cap C(x,r,\epsilon,\theta))}{r^{d-1}} + \limsup_{r\to 0} \frac{\mathcal{H}_{d-1}(E \cap C(x,r,\epsilon,-\theta))}{r^{d-1}} > 0$$

D'autre part, pour tout ensemble $E \subset \mathbb{R}^d$ avec $0 < \mathcal{H}_{d-1}(E) < +\infty$,

$$\limsup_{r\to 0} \frac{\mathcal{H}_{d-1}(E \cap B(x,r))}{r^{d-1}} \le 1 \text{ pour } \mathcal{H}_{d-1} \text{ presque tout } x \in E$$

Il existe alors un sous-ensemble F de E, de $\mathcal{H}_{d-1}$-mesure positive, sur lequel la condition (3) soit satisfaite pour une constante $C > 0$. L'ensemble F est également irrégulier au sens de Besicovitch et donc, par le théorème précédent, la relation (4) est valable sur F. •

Démonstration de la proposition 0.2.3.

Nous pouvons écrire E comme $E = E_1 \cup E_2$ avec E_1 rectifiable, E_2 purement non-rectifiable. Raisonnons par l'absurde et supposons que E_2 soit de $\mathcal{H}_{d-1}$ mesure strictement positive, $0 < \mathcal{H}_{d-1}(E_2) < +\infty$. Quitte à diminuer E, nous pouvons donc supposer que E est irrégulier au sens de Besicovitch et la condition CDF est satisfaite $\mathcal{H}_{d-1}$-presque partout sur ce nouveau E, $\mathcal{H}_{d-1}(E) > 0$.

Nous utilisons les lemmes 0.2.4 et 0.2.5 pour faire une estimation de la capacité des anneaux coniques $W(x, 2^{-n}, \epsilon, \theta) \setminus \Omega$ qui, à son tour, aboutira à une contradiction du fait que la condition CDF est satisfaite sur E.

D'après le lemme 0.2.4, il existe $\epsilon > 0$ et $\theta \in \mathbb{S}_d$ tels que l'ensemble F des points $x \in E$ satisfaisant à une condition CDF avec $\epsilon(x) = \epsilon$ et $\theta(x) = \theta$ soit de mesure $\mathcal{H}_{d-1}$ strictement positive. L'ensemble F est irrégulier au sens de Besicovitch donc, quitte à se restreindre à un sous-ensemble de F, nous pouvons supposer qu'il vérifie les relations (3) et (4).

Nous montrons que pour $\mathcal{H}_{d-1}$-presque tout $x \in F$, il existe un nombre infini de tranches $W(x, 2^{-n}, \epsilon, \theta)$ du cône $C(x, r, \epsilon, \theta)$, notées W_n, telles que

$$\mathcal{H}_{d-1}(2^n(F \cap W(x, 2^{-n}, \epsilon, \theta))) \ge c_1 \tag{5}$$

où $c_1 = c_1(x)$ est une constante positive.

Fixons $x \in F$. Quitte à remplacer eventuellement θ par $-\theta$ nous pouvons supposer que x vérifie

$$1 \geq c_0 = \limsup_{r \to 0} \frac{\mathcal{H}_{d-1}(F \cap C(x,r,\epsilon,\theta))}{r^{d-1}} > 0.$$

Prenons $0 < r < 1$ tel que $\dfrac{\mathcal{H}_{d-1}(F \cap C(x,r,\epsilon,\theta))}{r^{d-1}} > \dfrac{c_0}{2}$. Or, par la relation (3), pour $\tilde{r} < r$

$$\mathcal{H}_{d-1}(F \cap C(x,r,\epsilon,\theta) \setminus C(x,\tilde{r},\epsilon,\theta)) \geq \frac{c_0}{2} r^{d-1} - C\tilde{r}^{d-1}.$$

Posons $\tilde{r} = \left(\dfrac{c_0}{2}\right)^{\frac{1}{d-1}} \dfrac{r}{4C}$. Nous avons

$$\mathcal{H}_{d-1}(F \cap C(x,r,\epsilon,\theta) \setminus C(x,\tilde{r},\epsilon,\theta)) \geq c r^{d-1},$$

pour une constante $c = c(c_0) > 0$. Soit $N_0 = \log_2 \dfrac{r}{\tilde{r}}$. Remarquons que $N_0 > 0$ ne dépend que de c_0 et de C. Il existe, alors, une tranche

$$W_n = W(x, 2^{-n}, \epsilon, \theta) \subset C(x,r,\epsilon,\theta) \setminus C(x,\tilde{r},\epsilon,\theta)$$

telle que $\mathcal{H}_{d-1}(2^n(W_n \cap F)) > c_1 = 2^{-N_0(d-1)} \dfrac{c_0}{N_0}$.

L'itération de l'argument montre qu'il existe un nombre infini d'anneaux coniques $W_n = W(x, 2^{-n}, \epsilon, \theta)$ vérifiant (5). Pour ces anneaux nous avons le lemme suivant.

Lemme 0.2.7 *Il existe une constante $c_2 = c_2(c_1, c_0)$ telle que*

$$\text{cap}(W_n \cap F) \geq c_2 > 0 \tag{6}$$

Démonstration La capacité d'un ensemble $F \subset \mathbb{R}^d$ est donnée par

$$\text{cap}F = \sup\{I^{-1}(\mu) \; ; \; ||\mu|| = 1 \; ; \; \text{supp}\mu \subset F\},$$

où $I(\mu) = \displaystyle\int_F \int_F \frac{1}{|x-y|^{d-2}} \mu(dx)\mu(dy)$.

Pour W_n vérifiant (5), posons $\tilde{F}_n = 2^n(W_n \setminus F)$ et soit $\lambda = \frac{1}{\mathcal{H}_{d-1}(2^n F_n)} \mathcal{H}_{d-1}$. Nous obtenons :

$$\begin{aligned} I(\lambda) &= \int_{\tilde{F}_n} \int_{\tilde{F}_n} \frac{1}{||x-y||^{d-2}} \lambda(dx)\lambda(dy) \\ &= \frac{1}{\mathcal{H}_{d-1}(\tilde{F}_n)^2} \int_{\tilde{F}_n} \int_{\tilde{F}_n} \frac{1}{||x-y||^{d-2}} \mathcal{H}_{d-1}(dx)\mathcal{H}_{d-1}(dy). \end{aligned}$$

En utilisant la relation (3), nous montrons que l'intégrale est uniformément majorée par $c_2^{-1} < +\infty$, $c_2 > 0$ étant une constante indépendante de n :

$$\begin{aligned} &\int_{\tilde{F}_n} \int_{\tilde{F}_n} \frac{1}{||x-y||^{d-2}} \lambda(dx)\lambda(dy) \leq \\ &\leq \int_{\tilde{F}_n} \sum_{s=0}^{+\infty} 2^{-s(d-2)} \lambda(\tilde{F}_n \cap \{y \in \tilde{F}_n \; ; \; 2^{-s-1} \leq \text{dist}(y,x) \leq 2^{-s}\})\lambda(dx) \\ &\leq \frac{1}{\mathcal{H}_{d-1}(W_n \cap F)} \int_{\tilde{F}_n} 2\, C\lambda(dx) \leq 2c_1^{-1}C = c_2^{-1} \end{aligned}$$

et donc $\text{cap}(\tilde{F}_n) \geq c_2$. •

Ce lemme étant démontré, nous en déduisons que la série des capacités $\sum_{n=1}^{+\infty} \operatorname{cap}(\tilde{F}_n)$ diverge.

Or, $\sum_{n=1}^{+\infty} \operatorname{cap}[2^n(W(x,2^{-n},\epsilon,\theta)\setminus\Omega)] \geq \sum_{n=1}^{+\infty} \operatorname{cap}(\tilde{F}_n) = +\infty$.

Il est clair que le même raisonnement donne, pour $\mathcal{H}_{d-1}$-presque tout $x \in F$, un nombre infini d'anneaux coniques $W_n(x) = W(x,2^{-n},\epsilon,\theta)$ vérifiant (6), ce qui est en contradiction avec notre hypothèse. •

Soient Ω un domaine Greenien de $\mathbb{R}^d$ et E l'ensemble des points de la frontière de Ω satisfaisant à une condition CDF par rapport à Ω. Supposons que $\mathcal{H}_{d-1}(E)$ soit strictement positive. Il existe, alors, un sous-ensemble F de E de mesure positive, finie. La proposition 0.2.3, montre alors que l'ensemble F est régulier au sens de Besicovitch. Il existe alors un graphe lipschitzien Γ tel que $\mathcal{H}_{d-1}(F\cap\Gamma) > 0$.

Nous pouvons améliorer la proposition 0.2.3 de la manière suivante.

Remarque 0.2.8 Il existe un sous-ensemble F de E inclus dans une surface lipschitzienne Γ tel que $0 < \mathcal{H}_{d-1}(F)$ et pour tout $x \in F$ le double cône

$$C(x,1,\epsilon(x),\theta(x)) \cup C(x,1,\epsilon(x),-\theta(x))$$

n'intersecte pas Γ. Pour montrer ceci il suffit de remplacer les conditions 3 et 4 du lemme 0.2.5 par la condition de densité suivante : pour $\mathcal{H}_{d-1}$-presque tout $x \in F$

$$\lim_{r\to 0} \frac{\mathcal{H}_{d-1}(E\cap B(x,r))}{(2r)^{d-1}} = 1.$$

B) La démonstration de la deuxième étape s'effectue par une suite de lemmes, concernant les domaines lipschitziens, qui sont bien connus. Pour la notion d'ensemble effilé au sens minimal en un point de la frontière d'un domaine de Denjoy, nous renvoyons p. ex. à [Anc90b].

Il existe un autre type d'effilement équivalent, dans certains cas, à l'effilement au sens minimal.

Définition 0.2.9 *Soit $x \in \mathbb{R}^d$. L'ensemble $E \subset \mathbb{R}^d$ est dit effilé au sens interne en x si $x \notin \overline{E\setminus\{x\}}$ ou s'il existe une fonction s, surharmonique dans un voisinage euclidien de x, vérifiant*

$$\liminf_{\substack{y\to x \\ y\in E\setminus\{x\}}} s(y) > s(x).$$

Nous utilisons le lemme suivant dû à Y. Zhang [Zha87], pour montrer l'équivalence des deux types d'effilement, dans le cadre d'un domaine lipschitzien, et ensuite nous faisons appel au critère d'effilement au sens interne de Wiener pour lier l'effilement minimal à la condition CDF. Pour ces resultats voir aussi [Aik85].

Soit Γ le graphe d'une fonction lipschitzienne $f : \mathbb{R}^{d-1} \to \mathbb{R}$. Posons

$$\mathbb{H}_f^+ = \{(x_1,x_2,...,x_d) \in \mathbb{R}^d \;;\; f(x_1,...x_{d-1}) > x_d\}.$$

Soit $C(x,r,\epsilon,\theta)$ un cône contenu dans $\mathbb{H}_f^+$.

Lemme 0.2.10 *([Zha87], théorème 1) Un ensemble $B \subset C(x,1,\epsilon,\theta)$ est effilé au sens minimal en x dans $\mathbb{H}_f^+$ si et seulement s'il est effilé au sens interne en x.*

Le théorème suivant propose une condition équivalente à l'effilement au sens interne (cf. [Doo84]).

Théorème 0.2.11 *(Critère de Wiener) Un ensemble B est effilé au sens interne en $x \in \mathbb{R}^d$ si et seulement si*

$$\sum_{n\in\mathbb{N}} \operatorname{cap}\{2^n(B\cap B(x,2^{-n})\setminus B(x,2^{-n-1}))\} < +\infty.$$

Résumons la situation :

Lemme 0.2.12 *Soient Γ et $\mathbb{H}_f^+$ comme ci-dessus et $B \subset \mathbb{H}_f^+$ un ensemble quelconque. Soit $C(x)$ un cône de révolution, de sommet $x \in \Gamma$, contenu dans $\mathbb{H}_f^+$. Alors $B \cap C(x)$ est effilé au sens minimal en x dans $\mathbb{H}_f^+$ si et seulement si*

$$\sum_{n\in\mathbb{N}} \mathrm{cap}\{2^n\{B \cap C(x) \cap (B(x, 2^{-n}) \setminus B(x, 2^{-n-1}))\}\} < +\infty. \tag{7}$$

C) Cette étape consiste à lier l'effilement au sens minimal et la mesure harmonique. La proposition suivante en donne un premier résultat. Soit f, Γ et $\mathbb{H}_f^+$ comme ci-dessus.

Proposition 0.2.13 *Soit Ω un domaine inclus dans $\mathbb{H}_f^+$. Alors, si ω est la mesure harmonique du domaine Ω, pour tout ensemble $E \subset \Gamma$ on a $\omega(E) = 0$ si et seulement si $\mathbb{H}_f^+ \setminus \Omega$ est non-effilé au sens minimal dans $\mathbb{H}_f^+$ en $\mathcal{H}_{d-1}$-presque tout point de E.*

Démonstration Un résultat de Dahlberg ([Dah77]) affirme que la mesure harmonique de $\mathbb{H}_f^+$ est équivalente à la mesure de Hausdorff $\mathcal{H}_{d-1}$. Il suffit alors de montrer que la frontière de Martin s'identifie à la frontière euclidienne de $\mathbb{H}_f^+$. Or, ceci est affirmé par le théorème de Hunt et Wheeden [HW70] et le résultat s'en déduit. •

La proposition suivante nous permettra de trouver deux sous-domaines disjoints de Ω inclus dans deux demi-espaces complémentaires dont les mesures harmoniques ne sont pas mutuellement singulières. La démonstration nous a été suggerée par A. Ancona.

Proposition 0.2.14 *Soit $F \subset \Gamma$ fermé et posons $\Omega = \mathbb{R}^d \setminus F$. Alors, $\Gamma \setminus F$ est effilé au sens minimal (relativement à $\mathbb{R}^d \setminus F$) en $\mathcal{H}_{d-1}$-presque tout point de F.*

N. Chevallier [Che89] a donné un exemple d'un ensemble F fermé contenu dans un graphe lipschitzien Γ et d'un point $x \in F$ de densité (pour la mesure $\mathcal{H}_{d-1}$ restreinte sur F) qui est un point simple pour $\mathbb{R}^d \setminus F$ et tel que $\Gamma \setminus F$ soit effilé en x dans $\mathbb{R}^2 \setminus F$. A. Ancona a montré dans [Anc90a] (en répondant par la négative à une question de Chevallier) que si F est un sous-ensemble fermé d'un graphe lipschitzien Γ et si $x \in F$ est un point double pour $\mathbb{R}^d \setminus F$, alors $\Gamma \setminus F$ n'est pas nécessairement effilé en x dans $\mathbb{R}^d \setminus F$.

Le lemme suivant est un lemme classique de la théorie de Martin (cf. [Anc84]).

Lemme 0.2.15 *Soient $\Omega \subset \mathbb{R}^d$ un domaine de Green, $\widehat{\Omega}$ son compactifié de Martin et ν sa mesure harmonique de Martin relativement à un point $X_0 \in \Omega$. Si K est un sous-ensemble compact de $\partial\widehat{\Omega}$ avec $\nu(K) = 0$ alors il existe une fonction s surharmonique dans Ω, positive, vérifiant*

$$\lim_{\substack{x\to\zeta \\ x\in\Omega}} s(x) = +\infty \quad , \qquad \textit{pour tout } \ \zeta \in K \tag{8}$$

et telle que $s(X_0) = 1$.

Démonstration Soit K un compact de $\partial\widehat{\Omega}$ de mesure ν nulle. Soit U un voisinage de K dans $\widehat{\Omega}$. Montrons tout d'abord que $\mathbf{R}_{\mathbf{1}}^U = \int_{\partial\Omega^M} \mathbf{R}_{K_\zeta}^U \nu(d\zeta)$ décroît vers 0 lorsque U décroît vers K. En écrivant

$$\mathbf{R}_{\mathbf{1}}^U = \int_U \mathbf{R}_{K_\zeta}^U \nu(d\zeta) + \int_{\partial\Omega^M\setminus U} \mathbf{R}_{K_\zeta}^U \nu(d\zeta)$$

et en utilisant le théorème de convergence dominée on obtient qu'il suffit de montrer que $\mathbf{R}_{K_\zeta}^U$ décroît vers 0 pour tout $\zeta \notin K$. Soit $\zeta \notin K$ et supposons qu'il existe une suite $(U_n)_{n\in\mathbb{N}}$ de voisinages de K décroissant vers K et un point $x \in \Omega$ tels que $q_n(x) = \mathbf{R}_{K_\zeta}^{U_n}(x) > c > 0$. Pour n assez grand, les ensembles U_n sont effilés au sens minimal en x et donc $(q_n)_{n\in\mathbb{N}}$ est une suite décroissante de potentiels. Pour $n \in \mathbb{N}$ la fonction q_n est harmonique dans $\Omega \setminus \overline{U}_n$. Si $u = \lim_{n\to\infty} q_n$, alors u est harmonique non-nulle dans Ω, ce qui est absurde.

Choisissons une suite $(U_n)_{n\in\mathbb{N}}$ de voisinages de K tels que $\mathbf{R}_{\mathbf{1}}^{U_n}(X_0) \leq 2^{-n}$ pour tout $n \in \mathbb{N}$. La fonction $s' = \sum_{n=1}^{+\infty} \mathbf{R}_{\mathbf{1}}^{U_n}$ est surharmonique et $0 < s'(X_0) \leq 1$. D'autre part, $\lim_{x\to\zeta} s(x) = +\infty$, pour tout $\zeta \in K$. En posant $s = \frac{1}{s'(X_0)} s'$ nous trouvons la fonction surharmonique cherchée. •

Démonstration de la proposition 0.2.14 Notons encore $\partial\Omega^M$ la frontière de Martin minimale de Ω. Soit $\Sigma = \Gamma \setminus F$ et posons

$$E = \{\zeta \in \partial\Omega^M \ ; \ \Sigma \text{ est effilé en } \zeta\} \text{ et } N = \{\zeta \in \partial\Omega^M \ ; \ \Sigma \text{ est non-effilé en } \zeta\}.$$

Soit $\mathbf{1}$ la fonction harmonique constante égale à 1. Si ν est la mesure harmonique de Martin relativement à un point X_0 de Ω, on a la formule suivante :

$$\widehat{\mathbf{R}}_{\mathbf{1}}^{\Sigma} = \widehat{\mathbf{R}}^{\Sigma}_{\int_{\partial\Omega^M} K_\zeta \nu(d\zeta)} = \widehat{\mathbf{R}}^{\Sigma}_{\int_N K_\zeta \nu(d\zeta)} + \widehat{\mathbf{R}}^{\Sigma}_{\int_E K_\zeta \nu(d\zeta)} = u + p,$$

où u est la fonction $u = \widehat{\mathbf{R}}^{\Sigma}_{\int_N K_\zeta \nu(d\zeta)} \leq 1$ et $p = \widehat{\mathbf{R}}^{\Sigma}_{\int_E K_\zeta \nu(d\zeta)}$. Remarquons que u est une fonction harmonique dans Ω car $\widehat{\mathbf{R}}^{\Sigma}_{\int_N K_\zeta \nu(d\zeta)} = \int_N \widehat{\mathbf{R}}^{\Sigma}_{K_\zeta} \nu(d\zeta) = \int_N K_\zeta \nu(d\zeta)$ et que p est un potentiel puisque $\widehat{\mathbf{R}}^{\Sigma}_{K_\zeta}$ est un potentiel dans Ω pour tout $\zeta \in E$.

D'après le théorème de Fatou pour la compactification de Martin, u admet une limite fine ν-presque partout sur $\partial\Omega^M$; cette limite est nulle en dehors de N et vaut 1 sur N (ν-presque partout).

Soit L la constante de lipschitz de la fonction f dont le graphe est Γ et soit $0 < \epsilon < \frac{\pi}{2} - \arctan L$. Notons N^+ l'ensemble des points $x \in N$ en lesquels $C(x, 1, \epsilon, \theta_d)$ est non-effilé et N^- l'ensemble des points $x \in N$ en lesquels $C(x, 1, \epsilon, -\theta_d)$ est non-effilé. Par les propriétés des ensembles effilés au sens minimal et en utilisant un lemme de [Che89], on a $N = N^+ \cup N^-$.

Par le théorème de Fatou u admet la limite fine 1 ν-presque partout sur N. Donc, pour ν-presque tout point ζ de N^+, il existe une suite de points $x_n \in \Omega$ qui tend vers ζ dans la topologie de Martin telle que $u(x_n) \to 1$.

Soit ϕ l'application qui envoie le compactifié de Martin $\widehat{\Omega}$ de Ω sur la fermeture $\overline{\Omega} \cup \{\infty\}$ de Ω dans $\widehat{\mathbb{R}}^d$. Pour ν-presque tout point $\zeta \in N^+$, si $\xi = \phi(\zeta)$, nous pouvons choisir la suite x_n parmi les points du cône $C(\xi, 1, \epsilon, \theta_d)$, celui-ci n'étant pas effilé au sens minimal en ζ [Che89].

D'autre part, la fonction u est harmonique dans $\mathbb{H}_f^+$. Nous avons

$$H^{\mathbb{H}_f^+}_{\mathbf{1_\Sigma u}} = \widehat{\mathbf{R}}^{\Sigma}_u = u,$$

la réduite étant considérée dans Ω. Or $u \leq 1$ donc $u = H^{\mathbb{H}_f^+}_{\mathbf{1_\Sigma u}} \leq H^{\mathbb{H}_f^+}_{\mathbf{1_\Sigma}}$.

En d'autre termes, u est majorée par la mesure harmonique de Σ dans $\mathbb{H}_f^+$. Le théorème de Hunt et Wheeden [HW70] affirme qu'alors u converge non-tangentiellement vers 0, $\mathcal{H}_{d-1}$-presque partout sur $\Gamma \setminus \Sigma$ et donc sur $\phi(N^+)$. Or, pour ν-presque tout ζ de N^+ il existe une suite $(x_n)_{n\in\mathbb{N}}$ qui tend vers $\phi(\zeta)$ non-tangentiellement telle que $\lim_{n\to\infty} u(x_n) = 1$. En notant ν^+ la mesure ν restreinte à N^+, nous obtenons que $\phi\nu^+$ est singulière par rapport à $\mathcal{H}_{d-1}$ sur $\phi(N^+)$. Montrons maintenant que ceci est impossible si $\mathcal{H}_{d-1}(\phi(N^+))$ est positif.

Supposons que $\mathcal{H}_{d-1}(\phi(N^+))$ soit positif. Dans ce cas, il existe un sous-ensemble K de $\phi(N^+)$ compact avec $0 < \mathcal{H}_{d-1}(K) < +\infty$ et $\phi\nu^+(K) = 0$. Par le lemme 0.2.15, il existe alors une fonction s surharmonique positive sur Ω telle que $\lim_{\substack{x\to\zeta \\ x\in\Omega}} s(x) = +\infty$ pour tout $\zeta \in K$ et $s(X_0) = 1$ pour un point $X_0 \in \mathbb{H}_f^+$.

Or, la fonction s admet une limite non-tangentielle finie $\mathcal{H}_{d-1}$-presque partout sur F, ce qui est en contradiction avec l'hypothèse.

Il s'ensuit que $\mathcal{H}_{d-1}(\phi(N^+)) = 0$. De même, $\mathcal{H}_{d-1}(\phi(N^-)) = 0$ et donc Σ est effilé au sens minimal en $\mathcal{H}_{d-1}$-presque tout point de F. •

D) Si Γ est le graphe d'une fonction lipschitzienne f de constante de lipschitz L notons, pour $x \in \Gamma$ et $a > 2L$, $C_a(x)$ le cône de révolution $C(x, 10, \frac{\pi}{2} - \arctan a, \theta_d)$. Pour $z \in \mathbb{H}_f^+$ et pour un ensemble $E \subset \Gamma$, nous notons $B_a(z) = \{x \in \Gamma \; ; \; z \in C_a(x)\}$ et $T_a^E(z) = B_a(z) \cap E$.

Lemme 0.2.16 *Soit E un sous-ensemble de Γ, $0 < \mathcal{H}_{d-1}(E) < +\infty$. Alors pour tout a, a' avec $2L < a' < a$ et tout $\epsilon > 0$ il existe un sous-ensemble F de E vérifiant*

α) $\mathcal{H}_{d-1}(F) > (1-\epsilon)\mathcal{H}_{d-1}(E)$

β) $\omega(z, T_{a'}^E(z)) \geq c$ *pour tout* $z \in \bigcup\{C_a(x) \; ; \; x \in F\}$ *avec* $\mathrm{dist}(z, \Gamma) < 4(L+1)$,
où ω est la mesure harmonique de $\mathbb{H}_f^+$ et c est une constante strictement positive ne dépendent pas de z.

Démonstration Fixons un point X_0 dans $\mathbb{H}_f^+$ avec $\mathrm{dist}(X_0, \Gamma) > 20$. Par le théorème de Dahlberg [Dah77] la mesure harmonique de $\mathbb{H}_f^+$ est équivalente à $\mathcal{H}_{d-1}$ sur Γ. Nous pouvons donc, pour tout $\epsilon > 0$, choisir un ensemble compact $F \subset E$ tel que $\mathcal{H}_{d-1}(F) > (1-\epsilon)\mathcal{H}_{d-1}(E)$ et

$$\omega(X_0, B(x,r) \cap E) \geq c_1 r^{d-1} \text{ , pour tout } x \in F \text{ , } 0 < r < 1 \tag{9}$$
$$\omega(X_0, B(x,r)) \leq c_2 r^{d-1} \text{ , pour tout } x \in F \text{ , } 0 < r < 1 \tag{10}$$

où les constantes strictement positives c_1 et c_2 ne dépendent pas de x et de r.

Soit $z = (z_1, .., z_d) \in \bigcup\{C_a(x) \; ; \; x \in F\}$ et notons $T(z) = T_{a'}^E(z)$. Quitte à effectuer une translation, nous pouvons supposer que $z_1 = ... = z_{d-1} = 0$ et que 0 appartient à Γ. Notons $K(z)$ le cylindre de révolution autour de l'axe des x_d de rayon $3\mathrm{diam}T(z)$, délimité par les "demi-espaces" $\{x = (x_1, ..., x_d) \; ; \; x_d = 2z_d\}$ et $\{x = (x_1, ..., x_d) \; ; \; x_d = -2z_d\}$.

En appliquant le principe de Harnack au bord [Anc78] nous obtenons l'existence d'une constante $c > 0$ telle que

$$\omega(z, T(z)) \geq c\,\omega(z', T(z)) \text{ , pour tout } z' \notin K(z),$$

la constante c ne dépendant pas de z et de z'.

Montrons maintenant qu'il existe une constante c_0 telle que $\omega(z, B_{a'}(z)) \geq c_0$ pour tout $z \in \mathbb{H}_f^+$. Considérons le domaine $K(z) \setminus \tilde{C}_{a'}(0)$, où $\tilde{C}_{a'}(0) = \{x = \in \mathbb{R}^d \; ; \; -x \in C_{a'}(0)\}$. La solution du problème de Dirichlet dans ce domaine pour la fonction de la frontière $\mathbf{1}_{\tilde{\mathbf{C}}_{\mathbf{a}'}(\mathbf{0})}$ est une fonction harmonique positive, inférieure à $\omega(., B_{a'}(z))$ par le principe de maximum et minorée par une constante strictement positive au point z (car la mesure harmonique d'un domaine est invariante par dilatation du domaine).

Il suit du principe de Harnack au bord que

$$\frac{\omega(z', T(z))}{\omega(z', B_{a'}(z))} \sim \frac{\omega(z, T(z))}{\omega(z, B_{a'}(z))} \text{ , pour tout } z' \notin K(z).$$

En posant $z' = X_0$ et en appliquant les formules (9) et (10) nous obtenons $\omega(z, T_{a'}(z)) \geq c > 0$ pour tout $z \in \bigcup\{C_a(x) \; ; \; x \in F\}$, avec $\mathrm{dist}(z, \Gamma) < 4$. La constante $c > 0$ est indépendante de z. •

Le théorème suivant est dû à Naïm.

Théorème 0.2.17 *([Naï57], théorème 15) Soit Ω un domaine de $\mathbb{R}^d$ et F un fermé de Ω. Si h est une fonction harmonique minimale de Ω et si F est effilé en h alors*

1) $h' = h - \widehat{\mathbf{R}}_h^F$ est une fonction harmonique minimale de $\Omega \setminus F$ et

2) l'ensemble $A \subset \Omega \setminus F$ est effilé en h si et seulement s'il est effilé en h'.

Procédons maintenant à la démonstration du théorème.

Démonstration du théorème 0.2.1 : Soit K un sous-ensemble de $\partial\Omega$, tel que tout point de K satisfait à une condition CDF et $0 < \mathcal{H}_{d-1}(K) < +\infty$. Par la proposition 0.2.3, il existe un sous-ensemble E de K et un graphe lipschitzien Γ tels que

$$\mathcal{H}_{d-1}(E) > 0 \text{ et } E \subset \Gamma \tag{11}$$

D'après le lemme 0.2.4 nous pouvons supposer que tout $x \in E$ satisfait la condition CDF avec la même angle $\theta = \theta_0 \in \mathbb{S}_d$ et la même ouverture $\epsilon = \epsilon_0$. De plus, par la remarque 0.2.8, nous pouvons choisir E fermé vérifiant :

$$\text{Il existe } r_0 > 0 \text{ tel que } \quad (C(x, r_0, \epsilon_0, \theta_0) \cup C(x, r_0, \epsilon_0, -\theta_0)) \cap \Gamma = \emptyset \;,\; \forall x \in E \tag{12}$$

Quitte à effectuer une rotation nous pouvons également supposer que $\theta_0 = \theta_d$. Notons f la fonction de Lispchitz dont le graphe est Γ et posons encore

$$\mathbb{H}_f^+ = \{z = (z_1, ..., z_d) \in \mathbb{R}^d \;;\; z_d > f(z_1, ..., z_{d-1})\}$$

$$\mathbb{H}_f^- = \{z = (z_1, ..., z_d) \in \mathbb{R}^d \;;\; z_d < f(z_1, ..., z_{d-1})\}.$$

Nous montrons dans un premier temps que si ω_1 est la mesure harmonique de $\Omega_1 = \Omega \cap \mathbb{H}_f^+$ alors, pour tout $\epsilon > 0$ il existe un ensemble $F \subset E$ tel que $\mathcal{H}_{d-1}(F) > (1-\epsilon)\mathcal{H}_{d-1}(E)$ et ω_1 soit équivalente avec $\mathcal{H}_{d-1}$ sur F. Remarquons que par le théorème de Dahlberg la mesure harmonique ω de $\mathbb{H}_f^+$ est équivalente à $\mathcal{H}_{d-1}$ sur Γ, et donc, par la monotonie de la mesure harmonique comme fonction du domaine, la mesure ω_1 est absolument continue par rapport à $\mathcal{H}_{d-1}$ sur Γ. Il suffit donc de démontrer que $\mathcal{H}_{d-1}$ est absolument continue par rapport à ω_1 sur un sous-ensemble F de Γ de $\mathcal{H}_{d-1}$ mesure positive.

Lemme 0.2.18 *Pour $\epsilon > 0$, soit $F \subset E$ l'ensemble donné par le lemme 0.2.16, pour $a = \tan(\frac{\pi}{2} - \frac{\epsilon_0}{2})$ et cet ϵ. Alors, pour tout $S \subset F$*

$$\mathcal{H}_{d-1}(S) > 0 \Longrightarrow \omega_1(S) > 0.$$

En d'autre termes, la mesure $\mathcal{H}_{d-1}$ est absolument continue par rapport à ω_1 sur F.

Démonstration Soit $S \subset F$ avec $\mathcal{H}_{d-1}(S) > 0$. Pour le domaine lipschitzien $\mathbb{H}_f^+$ la mesure harmonique de Martin ν s'identifie naturellement à la mesure harmonique ω de $\mathbb{H}_f^+$ qui est équivalente à $\mathcal{H}_{d-1}$. Nous montrons que $\mathbb{H}_f^+ \setminus \Omega$ est effilé au sens minimal en $\mathcal{H}_{d-1}$-presque tout point de F. Par la proposition 0.2.13 la mesure harmonique de S dans Ω_1 est alors positive, ce qui donne l'énoncé du lemme.

Soit $\mathcal{D} = \bigcup\{C_a(x), x \in F\}$. $\mathcal{D}$ est un domaine lipschitzien, car l'enveloppe inférieure de fonctions lipschitziennes uniformément minorées est une fonction lipschitzienne, voir fig. 2.

Par la proposition 0.2.14, $\partial\mathcal{D} \setminus F$ est effilé au sens minimal en $\mathcal{H}_{d-1}$-presque tout point de F (dans le compactifié de Martin de $\mathbb{R}^d \setminus F$). Une deuxième application de la proposition 0.2.14 et le lemme 0.2.17 donnent alors que $\mathbb{H}_f^+ \setminus \mathcal{D}$ est effilé au sens minimal dans $\mathbb{H}_f^+$, en $\mathcal{H}_{d-1}$-presque tout x de F. Puisque la réunion de deux ensemble effilés au sens minimal en un point x est effilé au sens

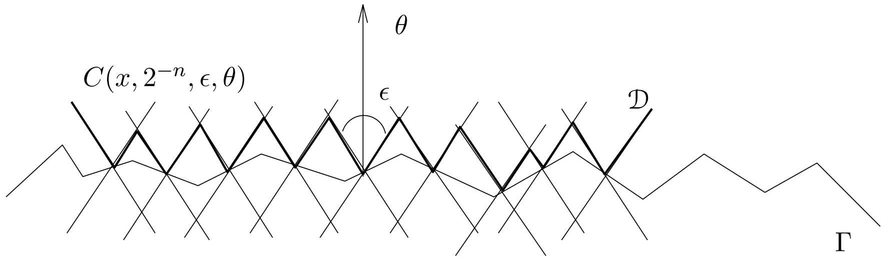

Figure 2: Le domaine $\mathcal{D}$ et le graphe lipschitzien Γ

minimal en x, il reste à montrer que $\mathcal{D} \setminus \Omega$ est effilé au sens minimal, dans $\mathbb{H}_f^+$, en $\mathcal{H}_{d-1}$-presque tout point de F.

Posons $a' = \tan(\frac{\pi}{2} - \epsilon_0)$. Les cônes $C_{a'}(x)$ contiennent les cônes $C_a(x)$, $x \in E$. Notons, pour $x \in E$, K_x la fonction de la frontière de Martin de $\mathbb{H}_f^+$ attachée à x (relativement à un point $X_0 \in \Omega_1$ ([HW70]) et considérons les fonctions $\widehat{\mathbf{R}}_{K_x}^{C_{a'}(x)\setminus\Omega}$, la réduite étant considérée dans $\mathbb{H}_f^+$. Ces fonctions sont des potentiels par le lemme 0.2.12 et la condition CDF.

Par les propriétés des ensembles effilés nous pouvons, pour chaque $x \in E$, substituer $C_{a'}(x) \setminus \Omega$ par un ensemble U_x ouvert, contenant $C_{a'}(x) \setminus \Omega$, tel que $p_x = \mathbf{R}_{K_x}^{U_x}$ soit un potentiel. Posons $p(z) = \int_E p_x(z)\nu(dx)$, ν étant la mesure harmonique de Martin relativement à X_0 ; il est facile de vérifier que $p(z)$ est un potentiel comme intégrale de potentiels.

D'autre part, lemme 0.2.16, pour $z \in \mathcal{D}$ si en notant $T(z) = \{x \in E \ ; \ z \in C_{a'}(x)\}$, la mesure $\omega(z, T(z), \mathbb{H}_f^+)$ est minorée par une constante c dépendant du graphe lipschitzien Γ, de ϵ_0 et de ϵ.

Soit $z \in \bigcup\{C_a(x) \setminus \Omega \ , \ x \in F\} = \mathcal{D} \setminus \Omega$. Remarquons que $p_x = K_x$ sur U_x. En appliquant le lemme 0.2.16, nous obtenons

$$p(z) = \int_E \mathbf{R}_{K_x}^{U_x}(z)\nu(dx) \geq \int_{T(z)\cap F} K_x(z)\nu(dx) = \omega(z, T(z)) > c.$$

D'autre part, p admet la limite fine 0 sur Γ, $\mathcal{H}_{d-1}$-presque partout (théorème de Fatou) et donc $\{z \in \mathbb{H}_f^+ \ ; \ p(z) > c\}$ est un ensemble effilé en K_x, pour $\mathcal{H}_{d-1}$-presque tout $x \in E$. Or, $\mathcal{D} \setminus \Omega = \bigcup\{C_a(x) \cap \mathcal{D} \setminus \Omega \ , \ x \in E\} \subset \{x \in \mathbb{H}_f^+ \ ; \ p(x) > c\}$. Il s'ensuit que $\mathcal{D} \setminus \Omega$ est effilé au sens minimal en $\mathcal{H}_{d-1}$ presque tout point de F. •

Les mêmes arguments sont valables pour la mesure harmonique ω_2 de $\Omega_2 = \mathbb{H}_f^- \cap \Omega$. En appliquant le lemme précédent pour $\epsilon < 1/2$, nous obtenons que les mesures ω_j sont équivalentes à $\mathcal{H}_{d-1}$ sur un ensemble $F \subset E$ de mesure $\mathcal{H}_{d-1}$ positive. Les mesures ω_1 et ω_2 sont alors équivalentes sur F et la proposition 0.2.2 affirme que dans ce cas Ω n'est pas Poissonien. •

0.3 Une réciproque partielle

Dans le cadre bidimensionnel, C.J. Bishop [Bis91] a montré qu'un domaine $\Omega \subset \mathbb{R}^2$ est Poissonien si et seulement si l'ensemble des points du bord de Ω satisfaisant à une condition CDF par rapport à Ω est de mesure $\mathcal{H}_1$-nulle. La preuve de ce résultat dépend de façon essentielle du résultat de Bishop, Carleson, Garnett et Jones [BCGJ89] suivant.

Théorème 0.3.1 *([BCGJ89]) Soient Ω_1, Ω_2 deux domaines disjoints de $\mathbb{R}^2$ et soient ω_1 et ω_2 leurs mesures harmoniques. Si ω_1 est équivalente à ω_2 sur un ensemble E de mesure ω_1 positive, alors il existe un ensemble $R \subset E$ régulier au sens de Besicovitch sur lequel les mesures ω_1 et ω_2 sont équivalentes à $\mathcal{H}_1$ et $\mathcal{H}_1(R) > 0$.*

La preuve de ce résultat fait appel aux applications conformes et au théorème de Makarov sur le support de la mesure harmonique des domaines simplement connexes dans $\mathbb{R}^2$ et n'est donc pas adaptable aux espaces de dimension supérieure à 2.

Nous donnons une réciproque du théorème 0.2.1 sous l'hypothèse supplémentaire sur le domaine Ω suivante (on renvoie également à la remarque 0.3.11).

HYPOTHÈSE : *si Ω_1 et Ω_2 sont deux sous-domaines disjoints de Ω et si les mesures harmoniques correspondantes ω_1 et ω_2 sont équivalentes sur un sous-ensemble de $\partial\Omega$ de mesure ω_1 strictement positive alors elles sont équivalentes sur un sous-ensemble de $\partial\Omega$ régulier au sens de Besicovitch de mesure ω_1 strictement positive.*

Nous énonçons la réciproque dans un cas spécial. Cependant la preuve s'adapte facilement à un cas plus général.

Théorème 0.3.2 *Soit Γ le graphe d'une fonction lipschitzienne f et soit $F \subset \Gamma$ fermé. Si $\mathcal{B}$ est une collection de boules fermées disjointes contenues dans $\mathbb{R}^d \setminus F$ telles que $E = \bigcup_{B\in\mathcal{B}} B \cup F$ soit fermé, alors le domaine $\Omega = \mathbb{R}^d \setminus E$ est Poissonien si et seulement si l'ensemble des points du bord de Ω satisfaisant une condition CDF par rapport à Ω est de mesure $\mathcal{H}_{d-1}$-nulle.*

Pour la preuve nous aurons besoin du résultat suivant.

Proposition 0.3.3 *Soit $F \subset \mathbb{R}^d$ avec $\mathcal{H}_{d-1}(F) = 0$. Alors, le domaine $\Omega = \mathbb{R}^d \setminus F$ est Poissonien.*

La démonstration de la proposition utilise un théorème, essentiellement contenu dans un article de Friedland et Hayman [FH76], qui donne une majoration du produit des mesures harmoniques de deux domaines disjoints (pour une preuve complète voir aussi [Bis92]). En utilisant ce résultat, nous montrons que pour tout couple de sous-domaines disjoints de $\Omega = \mathbb{R}^d \setminus F$ les mesures harmoniques correspondantes sont singulières sur F.

Théorème 0.3.4 *([FH76]) Si Ω_1, Ω_2 sont deux domaines disjoints de $\mathbb{R}^d$ et si ω_1, ω_2 sont leurs mesures harmoniques, évaluées en deux points fixés $x_1 \in \Omega_1$ et $x_2 \in \Omega_2$ respectivement, alors il existe une constante $C > 0$ telle que pour tout $x \in \mathbb{R}^d$ et tout $r > 0$,*

$$\omega_1 \left(B(x,r)\right) \omega_2 \left(B(x,r)\right) \leq C^2 r^{2(d-1)}.$$

Démonstration de la proposition 0.3.3 Soient Ω_1 et Ω_1 deux sous-domaines disjoints de Ω, et soient ω_1 et ω_2 leurs mesures harmoniques, évaluées en $x_1 \in \Omega_1$ et $x_2 \in \Omega_2$ respectivement. Par le théorème 0.3.4 il existe une constante $C > 0$ telle que pour tout $x \in \partial\Omega_1 \cap \partial\Omega_2$ nous ayons

$$\omega_1 \left(B(x,r)\right) \omega_2 \left(B(x,r)\right) \leq C^2 r^{2(d-1)}. \tag{13}$$

Soit $\epsilon > 0$ et $\mathcal{F}$ un recouvrement de $\partial\Omega$ par des boules telles que $\sum_{B\in\mathcal{F}} (\mathrm{diamB})^{d-1} < \epsilon$. Soient

$$\mathcal{F}_1 = \left\{ B = B(x,r) \in \mathcal{F} \; ; \; \omega_1 \left(B\right) > C r^{d-1} \right\} \text{ et}$$
$$\mathcal{F}_2 = \left\{ B = B(x,r) \in \mathcal{F} \; ; \; \omega_2 \left(B\right) > C r^{d-1} \right\}.$$

D'après le théorème 0.3.4, $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$, et

$$\omega_j \left(\bigcup \{ B \in \mathcal{F} \setminus \mathcal{F}_j \} \right) < 2^{d-1} C \epsilon \; , \; j = 1, 2.$$

Par conséquent, $\omega_1 \left(\bigcup \{ B \in \mathcal{F}_1 \} \right) > \omega_1(\partial\Omega) - 2^{d-1}C\epsilon$ et $\omega_2 \left(\bigcup \{ B \in \mathcal{F}_1 \} \right) < 2^{d-1}C\epsilon$.

Puisque ϵ est arbitraire, nous pouvons trouver une suite d'ensembles $(F_n)_{n\in\mathbb{N}}$ tels que $\omega_1(F_n) > \omega_1(\partial\Omega) - 2^{-n}$ et $\omega_2(F_n) < 2^{-n}$. En utilisant le lemme de Borel-Cantelli, si $F_\infty = \limsup_{n\to\infty} F_n$, nous obtenons $\omega_1(F_\infty) = \omega_1(\partial\Omega)$ et $\omega_2(F_\infty) = 0$, donc $\omega_1 \perp \omega_2$. Le résultat étant valable pour tout couple de sous-domaines disjoints de Ω, nous en déduisons, après application de la proposition 0.2.2, que Ω est Poissonien. $\bullet$

Corollaire 0.3.5 *Si Ω_1 et Ω_2 sont deux domaines disjoints dont les mesures harmoniques sont strictement positives et équivalentes sur un ensemble E de mesure $\mathcal{H}_{d-1}$ σ-finie, alors elles sont équivalentes à $\mathcal{H}_{d-1}$ sur un sous-ensemble de E de $\mathcal{H}_{d-1}$ mesure strictement positive.*

Démonstration Si les deux mesures étaient singulières par rapport à $\mathcal{H}_{d-1}$, il existerait un ensemble F de $\mathcal{H}_{d-1}$-mesure nulle qui porterait les deux mesures. Une application du théorème 0.3.4 aboutit alors à une contradiction. •

De la même façon nous pouvons montrer le corollaire suivant :

Corollaire 0.3.6 *Soient Ω_1 et Ω_2 deux domaines de $\mathbb{R}^d$ tels que la mesure $\mathcal{H}_{d-1}$ soit σ-finie sur $\partial\Omega_1 \cap \partial\Omega_2$. Soient ω_1, ω_2 leurs mesures harmoniques. Alors nous avons l'alternative suivante :*
- *ou bien $\omega_1 \perp \omega_2$*
- *ou il existe un ensemble $F \subset \partial\Omega_1 \cap \partial\Omega_2$ sur lequel les mesures ω_1 et ω_2 sont équivalentes à $\mathcal{H}_{d-1}$ et non-nulles.*

Nous pouvons maintenant démontrer le théorème énoncé.

Démonstration du théorème 0.3.2 Supposons que Ω ne soit pas Poissonien. Il existe alors deux sous-domaines Ω_1 et Ω_2 de Ω, disjoints tels que les mesures harmoniques correspondantes ω_1 et ω_2 soient équivalentes sur un ensemble $K \subset \partial\Omega$.

Montrons d'abord que $\cup\{B \in \mathcal{B}\} \cap K = \emptyset$. Remarquons que le complémentaire d'une boule fermée est un domaine de Poisson. Pour ceci il suffit d'appliquer une transformation de Kelvin pour se ramener à l'intérieur d'une boule et appliquer le théorème de Herglotz-Riesz de représentation intégrale des fonctions harmoniques bornées dans une boule.

Supposons que les mesures harmoniques ω_1 et ω_2 soient équivalentes et non-nulles sur le bord de la boule $B_0 \in \mathcal{B}$. Selon la proposition 0.2.2, ceci implique que le complémentaire de la boule n'est pas un domaine Poissonien, ce qui est absurde.

Le reste de la preuve s'applique également à la preuve du théorème 0.3.7.

Supposons maintenant que les mesures harmoniques des deux domaines soient équivalentes sur $K \subset \Gamma$. Par le corollaire 0.3.5, nous pouvons supposer que K est de $\mathcal{H}_{d-1}$ mesure positive et que les mesures harmoniques sont équivalentes à $\mathcal{H}_{d-1}$ sur K. D'après la proposition 0.2.14, $\Gamma \setminus K$ est effilé au sens minimal en $\mathcal{H}_{d-1}$-presque tout point de K. Or, dans le domaine $\tilde{\Omega} = \mathbb{R}^d \setminus K$, $\mathcal{H}_{d-1}$-presque tout point de K est double, c'est-à-dire qu'à presque tout point de K sont associés (au sens de la projection canonique de la frontière de Martin sur $\partial\Omega$) deux points de la fronti'ere de Martin minimale de $\tilde{\Omega}$.

La frontière de Martin considerée est dorénavant celle du domaine $\tilde{\Omega}$; nous notons ν la mesure harmonique de Martin de ce domaine relativement à un point $X_0 \in \Omega$ et π la projection de la frontière de Martin de $\tilde{\Omega}$ sur $\partial\tilde{\Omega} \cup \{+\infty\}$, cf. appendice. La mesure ν est projetée par π sur la mesure harmonique $\tilde{\omega}$ de $\partial\tilde{\Omega}$ évaluée en X_0 qui est équivalente à $\mathcal{H}_{d-1}$ sur un sous-ensemble de K de mesure $\mathcal{H}_{d-1}$ strictement positive (puisque ω_1 est absolument continue par rapport à $\tilde{\omega}$). Quitte à diminuer K, nous pouvons supposer que $\mathcal{H}_{d-1}(K)$ est positive et que les mesures ω_1, ω_2, $\tilde{\omega}$ et $\mathcal{H}_{d-1}$ sont équivalentes sur K.

Soient $\mathbb{H}_f^+$, $\mathbb{H}_f^-$ les "demi-espaces" $\mathbb{H}_f^+ = \{(x_1, x_2, ..., x_d) \in \mathbb{R}^d \; ; \; f(x_1, ...x_{d-1}) < x_d\}$ et $\mathbb{H}_f^- = \{(x_1, x_2, ..., x_d) \in \mathbb{R}^d \; ; \; f(x_1, ...x_{d-1}) > x_d\}$ et $A \subset K$ l'ensemble des points associés à deux points de la frontière de Martin. Quitte à se restreindre à un sous-ensemble A' de A de mesure $\mathcal{H}_{d-1}$ égale à $\mathcal{H}_{d-1}(A)$, nous pouvons supposer qu'à chaque point ζ de la frontière de Martin minimale $\partial\tilde{\Omega}^M$ de $\tilde{\Omega}$ avec $\pi(\zeta) \in A'$, un et un seul "demi-espace" parmi les $\mathbb{H}_f^+$, $\mathbb{H}_f^-$ est effilé au sens minimal en ζ. Soient

$$\tilde{\Omega}_+ = \{\zeta \in \pi^{-1}(A') \; ; \; \mathbb{H}_f^- \text{ est effilé au sens minimal en } \zeta\}$$

$$\tilde{\Omega}_- = \{\zeta \in \pi^{-1}(A') \; ; \; \mathbb{H}_f^+ \text{ est effilé au sens minimal en } \zeta\}.$$

Clairement,

$$\tilde{\Omega}_+ \cap \tilde{\Omega}_- = \emptyset \ , \ \text{et} \ \tilde{\Omega}_+ \cup \tilde{\Omega}_- = \{\zeta \in \partial\tilde{\Omega}^M \ ; \ \pi(\zeta) \in A'\}. \tag{14}$$

D'autre part, $\tilde{\Omega} \setminus \Omega_1$ ne peut pas être non-effilé au sens minimal en ν-presque tout point de $\pi^{-1}(A')$, car A' est de mesure harmonique positive dans Ω_1, cf. [Anc90b]. Or, $\pi(\nu) = \tilde{\omega}$ est équivalente à la mesure de Hausdorff $\mathcal{H}_{d-1}$, et donc il existe un ensemble $O \subset A'$ avec $\mathcal{H}_{d-1}(O) > 0$, tel que pour tout $x \in O$ l'ensemble $\tilde{\Omega} \setminus \Omega_1$ soit effilé en au moins un point de $\pi^{-1}(x)$.

Par la relation (14), quitte à permutter $\mathbb{H}_f^+$ et $\mathbb{H}_f^-$, nous pouvons supposer qu'il existe $Z \subset \pi^{-1}(O)$ tel que

1. $\mathbb{H}_f^-$ soit effilé au sens minimal en tout point de Z.

2. $\nu(Z) > 0$.

3. $\tilde{\Omega} \setminus \Omega_1$ soit effilé en tout point de Z.

Nous en déduisons, à l'aide du théorème 0.2.17, que $\mathbb{H}_f^+ \setminus \Omega_1$ est effilé au sens minimale dans $\mathbb{H}_f^+$ en tout point de $\pi(Z)$. Or, $\nu(Z)$ est positive donc, $\tilde{\omega}(\pi(Z)) > 0$. D'autre part, $\tilde{\omega}$ est équivalente à $\mathcal{H}_{d-1}$ sur A' et donc, par le théorème de Dahlberg, est équivalente à la mesure harmonique de $\mathbb{H}_f^+$. Nous obtenons $\omega(\pi(Z), \mathbb{H}_f^+ \cap \Omega_1) > 0$.

Remarquons que $\tilde{\Omega} \setminus \Omega_2 \supset \Omega_1$ est non-effilé au sens minimal en tout point de Z (dans $\tilde{\Omega}$). Puisque $\omega_2(\pi(Z))$ est équivalente à $\mathcal{H}_{d-1}$ sur $\pi(Z)$, l'ensemble $\tilde{\Omega} \setminus \Omega_2$ est effilé au sens minimal dans $\mathbb{H}_f^-$ en $\mathcal{H}_{d-1}$-presque tout point de $\pi(Z)$.

Le lemme 0.2.12 montre qu'alors $\mathcal{H}_{d-1}$-p.t. $x \in \pi(Z)$ satisfait à une CDF, ce qui achève la preuve du théorème. •

Le théorème 0.3.2 se généralise de la façon suivante :

Théorème 0.3.7 *Soit Ω un domaine de $\mathbb{R}^d$, $d \geq 2$ et soit π une la projection de la frontire de Martin sur $\partial\Omega$ ([Anc90b, MP91]). Soit $C \subset \partial\Omega$ l'ensemble des points qui satisfont à une condition CDF par rapport à Ω et $D \subset \partial\Omega$ l'ensemble des points au moins doubles (relativement à π) de $\partial\Omega$. Supposons que D est un ensemble régulier au sens de Besicovitch. Alors Ω est Poissonien si et seulement si $\mathcal{H}_{d-1}(C) = 0$.*

La preuve repose sur le lemme suivant.

Lemme 0.3.8 *Soit Ω un domaine de $\mathbb{R}^d$, $d \geq 2$ et soient Ω_1 et Ω_2 deux domaines disjoints inclus dans Ω. Soient ω_1 et ω_2 les mesures harmoniques de Ω_1 et Ω_2 respectivement. Si $S \subset \partial\Omega$ est l'ensemble des points simples pour Ω (relativement à une application π comme ci-dessus, alors ω_1 est singulière par rapport à ω_2 sur S.*

Démonstration Soient ω la mesure harmonique de Ω évaluée en un point $X_0 \in \Omega$ et ν la mesure harmonique du compactifié de Martin $\widehat{\Omega}$ de Ω relativement à X_0. Considérons la fermeture $\overline{\Omega}$ dans $\widehat{\mathbb{R}}^d$. Remarquons également que la fonction π est injective sur S.

Soit $\Phi \subset S$ l'ensemble des points $x \in S$ tels que $\Omega \setminus \Omega_1$ soit non-effilé au sens minimal en $\pi^{-1}(x)$. Alors,

$$\omega_1(.,\Phi) = \omega(.,\Phi) - \int_{\partial\Omega_1 \cap \Omega} \omega(z,\Phi)\omega_1(.,dz) = \omega(.,\Phi) - \int_{\partial\Omega_1 \cap \Omega} \int_{\pi^{-1}(\Phi)} K_x(z)\nu(dx)\omega_1(.,dz)$$

$$= \omega(.,\Phi) - \int_{\pi^{-1}(\Phi)} \mathbf{R}_{K_x}^{\partial\Omega_1 \cap \Omega}(z)\nu(dx) = \omega(.,\Phi) - \int_{\pi^{-1}(\Phi)} K_x(z)\nu(dx) = 0.$$

Les ensembles Φ et $S \setminus \Phi$ sont mesurables et $\omega_1(\Phi) = \omega_2(S \setminus \Phi) = 0$. Il s'ensuit que ω_1 est singulière par rapport à ω_2 sur S. •

Démonstration du théorème 0.3.7 Supposons que Ω ne soit pas Poissonien. Alors il existe deux domaines disjoints Ω_1 et Ω_2 inclus dans Ω, dont les mesures harmoniques sont équivalentes et non-nulles sur un ensemble $K \subset \partial\Omega$. Par le lemme précédent nous obtenons que les mesures ω_1 et ω_2 sont équivalentes et non-nulles sur un sous-ensemble de D et par conséquent sur un sous-ensemble d'un graphe lipschitzien. Le reste de la preuve est le même que pour le théorème 0.3.2. •

Donnons quelques exemples d'applications des théorèmes précédents :

Exemple 0.3.9 Si Γ est un graphe lipschitzien dans $\mathbb{R}^d$ et si $E \subset \Gamma$ est fermé , alors trivialement $\mathcal{H}_{d-1}(E) > 0$ si et seulement si $R^d \setminus E$ n'est pas Poissonien. •

Exemple 0.3.10 Reciproquement, si F est un ensemble de mesure $\mathcal{H}_{d-1}$ nulle, alors par la proposition 0.3.3, $\mathbb{R}^d \setminus F$ est Poissonien. •

Remarque 0.3.11 Pour généraliser le théorème 0.3.7 il nous suffirait de savoir que les mesures harmoniques de deux domaines disjoints sont équivalentes sur un ensemble régulier au sens de Besicovitch. Ce résultat (connu dans le cadre bidimensionel, voir théorème 0.3.1), n'est malheuresement pas connu en dimension ≥ 3.

0.4 Mesure harmonique et ensembles irréguliers de $\mathbb{R}^2$

Nous nous plaçons dans $\mathbb{R}^2$. Dans [Bis92], Bishop propose le problème suivant : Soit Ω un domaine de $\mathbb{R}^2$ et E est un ensemble irrégulier au sens de Besicovitch, $E \subset \partial\Omega$. Alors, si ω est la mesure harmonique du domaine Ω, est-il vrai que ω est singulière à $\mathcal{H}_1$ sur E?

Peter Jones a donné une preuve de la conjecture sous une condition de capacité uniforme, et le théorème de Makarov [Mak85], amélioré par Pommerenke [Pom86], permet de la démontrer si le domaine Ω est simplement connexe. En s'inspirant de la méthode de A. Ancona et M. Zinsmeister de [AZ89] nous montrons que si Ω est le complémentaire d'un ensemble E compact irrégulier au sens de Besicovitch, alors ω est singulière par rapport à $\mathcal{H}_1$ sur l'ensemble des points doubles de E. Rappelons que l'ensemble des points au moins triples d'un domaine du plan est de mesure harmonique nulle, remarque que nous devons à A. Ancona.

Le théorème de Pommerenke mentionné est le suivant.

Théorème 0.4.1 *([Pom86], corollaire 2) Soit Ω un domaine simplement connexe et soit ω sa mesure harmonique. Il existe une partition de $\partial\Omega$, $\partial\Omega = E_0 \cup E_1 \cup E_2$ avec les propriétés suivantes :*

1) $\mathcal{H}_1(E_0) = 0$

2) la mesure ω est équivalente à $\mathcal{H}_1$ sur E_1

3) E_1 est de mesure $\mathcal{H}_1$ σ-finie, et $\partial\Omega$ a une tangente en tout point de E_1

4) $\omega(E_2) = 0$

Rappelons-nous qu'un ensemble compact irrégulier au sens de Besicovitch n'a pas de tangentes $\mathcal{H}_1$-presque surement; on en déduit le résultat suivant.

Corollaire 0.4.2 *Soient E un ensemble compact irrégulier au sens de Besicovitch, Ω un domaine simplement connexe tel que $E \subset \partial\Omega$ et soit ω la mesure harmonique de Ω. Alors, ω est singulière à $\mathcal{H}_1$ sur E.*

Le lemme suivant est bien connu. Cependant nous présentons la démonstration pour des raisons de commodité.

Lemme 0.4.3 *Si E est un ensemble compact, totalement discontinu, et si $\Omega = \mathbb{R}^2 \setminus E$, alors il existe une projection continue π, du compactifié de Martin $\widehat{\Omega}$ sur la fermeture $\overline{\Omega}$ de Ω dans $\widehat{\mathbb{R}}^2$, prolongeant l'identité qui projette la mesure harmonique de Martin de Ω sur la mesure harmonique du domaine Ω.*

Démonstration Soit h une fonction harmonique minimale attachée à un point de la frontière de Martin $\partial\widehat{\Omega}$ de Ω. D'après la théorie de Martin, il existe $x \in E$ et une suite $(x_n)_{n\in\mathbb{N}} \subset \Omega$ tel que $h = \lim_{x_n \to x} K_{x_n}$. Soit $x' \in E$ et soit $\mathcal{V}$ est une base de voisinages de x' dans $\mathbb{R}^2 = \overline{\Omega}$ telle que pour tout $V \in \mathcal{V}$ la frontière ∂V de V soit une courbe de Jordan fermée, $\partial V \subset \Omega$ et $x \notin V$. Montrons que $\widehat{\mathbf{R}}_h^V$ est un potentiel et donc V est effilé en h, pour tout $V \in \mathcal{V}$.

Soit $x' \in E$, $x' \neq x$, et soit $\gamma \subset \Omega$ une courbe de Jordan fermée qui sépare x, x'. Soit V la composante connexe de $\mathbb{R}^2 \setminus \gamma$ qui contient x'. Soit X_0 le point de normalisation de $\widehat{\Omega}$, $K_{x_n}(X_0) = 1$, $\forall n \in \mathbb{N}$. Du principe de Harnack on obtient qu'il existe une constante $C > 0$ telle que $\frac{1}{C} \leq K_{x_n} \leq C$ sur γ, pour tout $n \in \mathbb{N}$. D'autre part K_{x_n} tend vers h uniformément sur tout compact de Ω. Il s'ensuit que $C\widehat{\mathbf{R}}_{K_{x_n}}^V \geq \widehat{\mathbf{R}}_h^V$ et donc $\widehat{\mathbf{R}}_h^V$ est un poteniel.

Soit $\pi : \widehat{\Omega} \to \widehat{\mathbb{R}^2}$ l'application qui prolonge l'identité de Ω en associant à chaque fonction $h \in \partial\widehat{\Omega}$ l'unique point $x \in \partial\Omega$ tel que pour tout voisinage V de x dans $\mathbb{R}^2$, $\widehat{\mathbf{R}}_h^V$ n'est pas un potentiel. L'application est clairement bijective. Montrons maintenant que π est continue. En effet, si $h_i \subset \widehat{\Omega}$ converge finement vers un point $h \in \partial\widehat{\Omega}$, alors le filtre $x_i \subset \overline{\Omega}$ correspondant tend vers le point x associé à h (s'il existait un deuxième point d'accumulation on pourrait trouver comme ci-dessus un voisinage de h effilé en h, absurde).

Finalement, cf [Anc90b, MP91], la mesure harmonique de Martin de Ω est projetée sur la mesure harmonique de Ω. •

Proposition 0.4.4 *Soit E un ensemble compact irrégulier au sens de Besicovitch et soit $\Omega = \mathbb{R}^2 \setminus E$. Si ω est la mesure harmonique du domaine Ω évaluée à l'infini, alors ω est singulière à $\mathcal{H}_1$ sur l'ensemble Δ des points doubles de E (relativement à la fonction π donnée par le lemme précédent) .*

Démonstration Posons $L = \pi^{-1}(\Delta)$. Raisonnons par l'absurde et supposons que ω n'est pas singulière à $\mathcal{H}_1$ sur Δ. T. Wolff a montré dans [Wol93], que les mesures harmoniques des domaines du plan sont portées par des ensembles de mesure $\mathcal{H}_1$ σ-finie. Nous pouvons donc supposer que ω est équivalente à $\mathcal{H}_1$ sur un sous-ensemble Δ' de Δ.

Quitte à diminuer Δ', nous pouvons trouver deux compacts L_1 et L_2 dans L disjoints, tels que $\pi(L_1) = \pi(L_2) = \Delta'$ et $\nu(L_2) > 0$, ν étant la mesure harmonique de Martin, relativement à l'infini, projetée par π sur ω.

Soient V_1 et V_2 deux voisinages de L_1 et L_2 respectivement. Rappelons que tout point de la frontière de Martin minimale de Ω admet un système de voisinages à intersection avec Ω connexe. Nous pouvons alors supposer, en diminuant eventuellement L_1, que $V_1 \cap \Omega$ est un ensemble ouvert connexe. En posant $F = \overline{V_1 \cap \Omega}$, nous en déduisons que chaque composante de $\widehat{\mathbb{R}^2} \setminus F$ est donc simplement connexe. Soit $(U_n)_{n\in\mathbb{N}}$ la collection des composantes connexes de $\widehat{\mathbb{R}^2} \setminus F$ et soit, pour $n \in \mathbb{N}$, ω_n la mesure harmonique de U_n. Posons $\tilde{\omega} = \sum_{n\in\mathbb{N}} 2^{-n}\omega_n$.

Remarquons que par construction F est effilé à chaque point de L_2. Soit $S \subset L_2$. Posons $u = \int_S K_x \nu(dx)$ et considérons la fonction $u - \mathbf{R}_u^F$. Si $\nu(S) > 0$, alors la fonction $u - \mathbf{R}_u^F$ est harmonique non-nulle dans $\Omega \setminus F$ (puisque F est effilé à chaque point de L_2). Il s'ensuit que $\pi(S)$ est de mesure harmonique positive dans une de composantes connexes de $\mathbb{R}^2 \setminus F$. On conclut que si $\tilde{\nu}$ est la restriction de ν sur L_2, la projection de la mesure $\tilde{\nu}$ sur Δ' est absolument continue par rapport à $\tilde{\omega}$.

Or, d'après le corollaire 0.4.2, la mesure $\tilde{\omega}$ est singulière par rapport à $\mathcal{H}_1$ sur Δ'. D'autre part la mesure $\tilde{\nu}$ est absolument continue par rapport à ω et donc ω n'est pas équivalente à $\mathcal{H}_1$ sur Δ', ce qui est absurde. •

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Publications de la partie 2

On relations between entropy and Hausdorff dimension of measures

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Abstract : *We characterize the measures for which the Hausdorff dimension can be calculated by an entropy formula. Several examples and counterexamples are proposed.*

1 Introduction.

Let D be an integer greater than 1 and m be a probability measure in $[0,1)^D$. Fix $\ell \geq 1$ and denote by $\mathcal{F}_n$ the family of ℓ-adic cubes of the n^{th} generation, that is

$$\mathcal{F}_n = \left\{ I = \prod_{i=1}^{D} [k_i/\ell^n, (k_i+1)/\ell^n) \ ; \ 0 \leq k_i < \ell^n \right\} .$$

For an arbitrary point x in $[0,1)^D$, let $I_n(x)$ be the unique cube $I \in \mathcal{F}_n$ such that $x \in I$.

We want to give estimates for the lower and upper dimensions of the measure m. These are respectively defined by

$$\begin{cases} \dim_*(m) = \inf(\dim(E) \ ; \ m(E) > 0) \\ \dim^*(m) = \inf(\dim(E) \ ; \ m(E) = 1) \end{cases} .$$

It is well known that there exist some relations between these quantities and the function τ which appears in the multifractal formalism. More precisely, if we let :

$$\tau_n(t) = \frac{1}{n \log \ell} \log \left(\sum_{I \in \mathcal{F}_n} m(I)^t \right) \quad \text{and} \quad \tau(t) = \limsup_{n \to +\infty} \tau_n(t) \ ,$$

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it is proved in [Heu98] that

$$-\tau'_+(1) \leq \dim_*(m) \leq h_*(m) \tag{1}$$

where $h_*(m)$ is the lower entropy (also called lower Rényi dimension) of the measure m, defined as

$$h_*(m) = \liminf_{n\to+\infty} h_n(m) \quad \text{where} \quad h_n(m) = -\tau'_n(1) = \frac{-1}{n\,\log\ell} \sum_{I\in\mathcal{F}_n} m(I)\,\log(m(I))\ .$$

In [Heu98], we also give sufficient conditions for the equality $-\tau'_+(1) = \dim_*(m)$ to hold.

In this paper we are interested in describing the measures m satisfying $\dim_*(m) = h_*(m)$. Theorem 2.1 states that $\dim_*(m) = h_*(m)$ holds if and only if there exists a subsequence n_k such that for dm-almost every $x \in [0,1)^D$,

$$\lim_{k\to+\infty} \frac{\log m(I_{n_k}(x))}{-n_k\,\log\ell} = \dim_*(m)\ .$$

In particular, such measures are unidimensional (i.e. $\dim_*(m) = \dim^*(m)$). If we denote by $\dim(m)$ this common value, the measure m is supported by a set of dimension $\dim(m)$ but every set of dimension strictly less than $\dim(m)$ is negligible. Nevertheless, unidimensionality is not a sufficient condition to have $\dim_*(m) = h_*(m)$. An example of a measure of exact dimension (i.e. unidimensional) for which $\dim_*(m) < h_*(m)$ is proposed in proposition 4.1.

Similar results can be established, comparing the upper entropy

$$h^*(m) = \limsup_{n\to+\infty} h_n(m)$$

and the packing dimension of the measure m. Following [Heu98] or [Fal97], we can introduce

$$\begin{cases} \mathrm{Dim}_*(m) = \inf(\mathrm{Dim}(E)\ ;\ m(E) > 0) \\ \mathrm{Dim}^*(m) = \inf(\mathrm{Dim}(E)\ ;\ m(E) = 1) \end{cases},$$

where $\mathrm{Dim}(E)$ is the packing dimension of the set E. As proved in [Heu98],

$$h^*(m) \leq \mathrm{Dim}^*(m) \leq -\tau'_-(1) \tag{2}$$

and we characterize in theorem 2.2 of the present paper the measures for which the equality $h^*(m) = \mathrm{Dim}^*(m)$ holds.

Several examples of measures satisfying $\dim_*(m) = h_*(m)$ and $\mathrm{Dim}^*(m) = h^*(m)$ are also proposed. In particular, this is the case in an ergodic situation (example 2.3), for quasi-Bernoulli measures (example 2.5) and in a context where the strong law of large numbers can be applied (example 2.6).

In the last section, we construct a measure m of exact dimension for which

$$\dim(m) = h_*(m) \quad \text{but} \quad \mathrm{Dim}(m) > h^*(m)\ .$$

2 Main results and examples.

The main result of this paper is the following.

Theorem 2.1 *Let m be a probability measure in $[0,1)^D$. Then*

$$\dim_*(m) \leq h_*(m) \ .$$

Moreover, the following properties are equivalent :

(i) $\dim_*(m) = h_*(m)$
(ii) $\dim_*(m) = \dim^*(m) = h_*(m)$
(iii) *There exists a subsequence $(n_k)_{k\geq 1}$ such that for dm-almost every $x \in [0,1)^D$,*

$$\lim_{k\to+\infty} \frac{\log m(I_{n_k}(x))}{-n_k \log \ell} = \dim_*(m) \ .$$

A similar result can also be established, comparing the upper entropy $h^*(m)$ of a measure m with its packing dimension $\mathrm{Dim}^*(m)$.

Theorem 2.2 *We also have*

$$h^*(m) \leq \mathrm{Dim}^*(m),$$

and the following properties are equivalent :

(i) $\mathrm{Dim}^*(m) = h^*(m)$
(ii) $\mathrm{Dim}_*(m) = \mathrm{Dim}^*(m) = h^*(m)$
(iii) *There exists a subsequence $(n_k)_{k\geq 1}$ such that for dm-almost every $x \in [0,1)^D$,*

$$\lim_{k\to+\infty} \frac{\log m(I_{n_k}(x))}{-n_k \log \ell} = \mathrm{Dim}^*(m) \ .$$

Remark. As was pointed out in the introduction, unidimensionality is not sufficient to establish the equalities $\dim_*(m) = h_*(m)$ and $\mathrm{Dim}^*(m) = h^*(m)$ (see propositions 4.1 and 5.1). In fact, the statements of theorems 2.1 and 2.2 stem from a deep homogeneity property.

Let us now give some useful examples of measures m for which the equalities $\dim_*(m) = h_*(m)$ and $\mathrm{Dim}^*(m) = h^*(m)$ hold.

Example 2.3 Suppose that the sequence

$$\frac{\log m(I_n(x))}{-n \log \ell} \tag{3}$$

is almost surely converging to a constant d. Then, the equivalent properties of theorems 2.1 and 2.2 are satisfied and we have

$$d = \dim(m) = \mathrm{Dim}(m) \ . \tag{4}$$

In particular, this is the case in an ergodic context. Let us denote the elements of $\mathcal{F}_n$ by $I_{\varepsilon_1,\dots,\varepsilon_n}$ with $\varepsilon_i \in \{0,\dots,\ell^D-1\}$ and

$$I_{\varepsilon_1,\dots,\varepsilon_{n+1}} \subset I_{\varepsilon_1,\dots,\varepsilon_n} \ .$$

Define $J(x)$ to be the unique element $(\varepsilon_i)_{i\geq 1}$ of the Cantor set $\{0,\dots,\ell^D-1\}^{\mathbb{N}^*}$ such that $\{x\} = \bigcap_n I_{\varepsilon_1,\dots,\varepsilon_n}$, and consider the image $\tilde{m}$ of m with respect to the application J. If we suppose that the mesure $\tilde{m}$ is invariant and ergodic with respect to the shift operator, Shannon-McMillan's theorem ensures that (3) admits an almost sure constant limit d and that (4) is satisfied (see [Heu98] for more details and [Zin97] for basic facts on ergodic theory). •

Example 2.4 The following situation was described by S.M. Ngai in [Nga97]. Suppose that the function τ of the multifractal formalism admits a derivative $\tau'(1)$. Then, using inequalities (1) and (2), we conclude that the measure m is unidimensional and satisfies

$$\dim(m) = \mathrm{Dim}(m) = h(m) = -\tau'(1)$$

where $h(m) = h_*(m) = h^*(m)$ is the genuine limit of the sequence $h_n(m)$. •

Example 2.5 The case of quasi-Bernoulli measures is related to examples 2.3 and 2.4. Suppose that there exists a constant $C > 0$ such that for every $\varepsilon_1,\dots,\varepsilon_{n+p}$ we have :

$$\frac{1}{C}\, m\left(I_{\varepsilon_1,\dots,\varepsilon_n}\right) m\left(I_{\varepsilon_{n+1},\dots,\varepsilon_{n+p}}\right) \leq m\left(I_{\varepsilon_1,\dots,\varepsilon_{n+p}}\right) \leq C\, m\left(I_{\varepsilon_1,\dots,\varepsilon_n}\right) m\left(I_{\varepsilon_{n+1},\dots,\varepsilon_{n+p}}\right) \ .$$

It is well known (see [Car85]) that such a measure is equivalent to an invariant and ergodic measure. Moreover, it is proved in [Heu98] that $\tau'(1)$ exists in this case. Let us also remember that the multifractal formalism is available for such measures (see [BMP92]). •

Example 2.6 Let us write

$$X_n(x) = \log\left(\frac{m(I_n(x))}{m(I_{n-1}(x))}\right)$$

and suppose that

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n}(X_i - \mathbb{E}(X_i)) = 0 \ \ dm \ a.s.$$

where the expectation is relative to the probability m. If we remark that

$$h_n(m) = \frac{-1}{n\,\log \ell} \sum_{i=1}^{n} \mathbb{E}(X_i),$$

we can easily conclude that the equalities $\dim_*(m) = h_*(m)$ and $\mathrm{Dim}^*(m) = h^*(m)$ are satisfied in this situation. In particular, using the strong law of large numbers, this is the case when the random variables X_i are bounded (according to [Wu98], the measure m is called ℓ-adic doubling) and uncorrelated.

A classical case (called Bernoulli product) of such measures is often described in the literature (see for example [BK90], [Bis95] or [Heu98]). Fix a sequence $(p_i)_{i\geq 1}$ of real numbers with $0 < p_i < 1$ and consider a sequence $(Y_i)_{i\geq 1}$ of independant random variables such that

$$\mathbb{P}(Y_i = 0) = p_i \quad \text{and} \quad \mathbb{P}(Y_i = 1) = 1 - p_i \ .$$

Then, the law m of the random variable

$$\sum_{i=1}^{+\infty} 2^{-i} Y_i$$

satisfies

$$\begin{cases} \dim(m) = h_*(m) = \liminf\limits_{n\to+\infty} \dfrac{1}{n} \sum\limits_{i=1}^{n} s(p_i) \\ \mathrm{Dim}(m) = h^*(m) = \limsup\limits_{n\to+\infty} \dfrac{1}{n} \sum\limits_{i=1}^{n} s(p_i) \end{cases} \tag{5}$$

where the function $s(t)$ is defined by

$$s(t) = -[t\,\log_2 t + (1-t)\,\log_2(1-t)] \quad \text{for} \quad t \in [0,1]$$

and $\log_2$ is the logarithm in base 2. Particular cases of this example will be used in section 4. •

Example 2.7 Example 2.6 can be seen as a particular case of a more general situation which is described in [Heu98], Corollaire 4.2. If we suppose that the sequence of functions $(\tau_n'(t))_{n\geq 1}$ is equicontinous at $t = 1$, then

$$\dim_*(m) = h_*(m) = -\tau_+'(1) \quad \text{and} \quad \mathrm{Dim}^*(m) = h^*(m) = -\tau_-'(1) \ .$$

•

3 Proof of theorems 2.1 and 2.2.

We prove theorem 2.1. Since the proof of theorem 2.2 is similar, we will only sketch it at the end of the section.

The inequality $\dim_*(m) \le h_*(m)$ is well known (see for example [Heu98] or [You82]). Nevertheless, let us give an elementary proof in order to make easier the study of the equality case. If $x \in [0,1)^D$, put :

$$\alpha_n(x) = \frac{\log m(I_n(x))}{-n\log \ell} \quad \text{and} \quad \underline{\alpha}(x) = \liminf_{n\to+\infty} \alpha_n(x) \ .$$

It is well known ([Fan94], [Fal97] or [Heu98]) that :

$$\dim_*(m) = \sup(\{\alpha \ge 0 \ ; \ \underline{\alpha} \ge \alpha \ \ dm \ a.s.\}) = \inf \operatorname{ess}(\underline{\alpha}) \ . \tag{6}$$

Using Fatou's lemma and the fact that m is a probability measure, we get

$$\dim_*(m) \le \int \underline{\alpha}(x)\, dm(x) \le h_*(m) \ . \tag{7}$$

Proof of (iii) $\Rightarrow$ (i). Suppose that there exists a subsequence $(n_k)_{k\ge 1}$ such that for dm-almost every $x \in [0,1)^D$,

$$\lim_{k\to+\infty} \alpha_{n_k}(x) = \dim_*(m) \ .$$

Using the dominated convergence theorem (see lemma 3.1 below), we obtain

$$\begin{aligned} \dim_*(m) &= \int \lim_{k\to+\infty} \alpha_{n_k}(x)\, dm(x) \\ &= \lim_{k\to+\infty} \int \alpha_{n_k}(x)\, dm(x) \\ &= \lim_{k\to+\infty} h_{n_k}(m) \\ &\ge h_*(m) \end{aligned}$$

and we are done. Observe that due to the following lemma, we may apply Lebesgue's dominated convergence theorem.

Lemma 3.1 *Let $\phi = \sup_{n\ge 1} \alpha_n$. Then $\phi \in L^1(m)$.*

Proof. It is sufficient to prove that the real variable function

$$t \longmapsto m(\{x \ ; \ \phi(x) > t\})$$

is integrable in a neighbourhood of $+\infty$ with respect to the Lebesgue's measure. But we know that $\alpha_n(x) > t$ if and only if $m(I_n(x)) < \ell^{-nt}$. According to the fact that the partition $\mathcal{F}_n$ contains ℓ^{nD} elements,

$$m(\{x \ ; \ \alpha_n(x) > t\}) \leq \ell^{nD}\ell^{-nt} \ .$$

If $t > D$, we obtain

$$\begin{aligned} m(\{x \ ; \ \phi(x) > t\}) &\leq \sum_{n=1}^{+\infty} m(\{x \ ; \ \alpha_n(x) > t\}) \\ &\leq \sum_{n=1}^{+\infty} \ell^{n(D-t)} \\ &= \frac{\ell^{D-t}}{1-\ell^{D-t}} \end{aligned}$$

which proves the integrability of ϕ.

Proof of (i) $\Rightarrow$ (ii). Suppose that $\dim_*(m) = h_*(m)$. Using (6) and (7), we remark that for dm-almost every point x, $\underline{\alpha}(x) = \dim_*(m)$. Then, we also have

$$\dim_*(m) = \inf(\{\alpha \geq 0 \ ; \ \underline{\alpha} \leq \alpha \ \ dm \ a.s.\}) = \sup \operatorname{ess}(\underline{\alpha})$$

and hence $\dim_*(m) = \dim^*(m)$ (see [Heu98] or [Fal97]). The measure m is thus unidimensional.

Proof of (ii) $\Rightarrow$ (iii). Let $d = \dim_*(m) = h_*(m)$. We will make use of the following lemma.

Lemma 3.2 *Let $\eta \in (0,1)$ and $n_0 \geq 1$. We can choose an integer $n_1 \geq n_0$ such that :*

$$m(\{x \ ; \ \alpha_{n_1} > d+\eta\}) \leq (2+d)\,\eta \ .$$

Proof. As was remarked before, the equality $\underline{\alpha} = d$ holds dm-almost surely. Using Egorov's argument, we can find $n_0' \geq n_0$ such that

$$m\left(\bigcap_{n\geq n_0'} \{x \ ; \ \alpha_n(x) > d-\eta^2\}\right) > 1-\eta^2 \ .$$

On the other hand, by hypothesis (ii), we can choose an integer $n_1 \geq n_0'$ such that

$$h_{n_1}(m) = \int \alpha_{n_1}(x)\, dm(x) < d+\eta^2 \ .$$

This integer n_1 will be the right one. Let us denote

$$A = \{x \ ; \ \alpha_{n_1}(x) > d - \eta^2\}$$

and try to estimate the mass of the set

$$B = \{x \ ; \ \alpha_{n_1}(x) > d + \eta\} \ .$$

Obviously, $B \subset A$. We can write

$$\begin{aligned} d + \eta^2 &\geq \int \alpha_{n_1}(x)\, dm(x) \\ &\geq \int_{A \setminus B} \alpha_{n_1}(x)\, dm(x) + \int_B \alpha_{n_1}(x)\, dm(x) \\ &\geq (d - \eta^2)\,(m(A) - m(B)) + (d + \eta)\, m(B) \\ &\geq (d - \eta^2)\,(1 - \eta^2) + (\eta + \eta^2)\, m(B) \ . \end{aligned}$$

We then get

$$m(B) \leq \frac{2\eta^2 + d\eta^2}{\eta + \eta^2} \leq (2 + d)\,\eta$$

which is the conclusion of the lemma.

By using lemma 3.2 with $\eta = 2^{-k}$, we can construct a subsequence $(n_k)_{k \geq 1}$ such that for all $k \geq 1$,

$$m(\{x \ ; \ \alpha_{n_k}(x) > d + 2^{-k}\}) \leq (2 + d)\, 2^{-k} \ .$$

By applying Borel Cantelli's lemma, we obtain that for almost every $x \in [0, 1)^D$, there exists $k_0 \geq 1$ such that for every $k \geq k_0$,

$$\alpha_{n_k}(x) \leq d + 2^{-k} \ .$$

Then we have

$$\limsup_{k \to +\infty} \alpha_{n_k}(x) \leq d \ \ dm \ a.s. \ .$$

On the other hand,

$$d = \underline{\alpha}(x) \leq \liminf_{k \to +\infty} \alpha_{n_k}(x) \ \ dm \ a.s. \ .$$

Thus, we have proved that the subsequence $(\alpha_{n_k})_{k \geq 1}$ converges almost surely to d. $\square$

To prove theorem 2.2, we introduce the function

$$\bar{\alpha}(x) = \limsup_{n \to +\infty} \alpha_n(x) \ .$$

As proved in [Heu98] and [Fal97], we have :

$$\mathrm{Dim}^*(m) = \inf(\{\alpha \geq 0 \ ; \ \bar{\alpha} \leq \alpha \ \ dm \ a.s.\}) = \sup \mathrm{ess}(\bar{\alpha}) \ .$$

The inequality $h^*(m) \leq \mathrm{Dim}^*(m)$ is a consequence of Fatou's lemma applied to the sequence $\phi - \alpha_n$. The study of the equality case uses the same ideas as before.

4 An example where $\dim(m) < h_*(m)$ and $\mathrm{Dim}(m) > h^*(m)$.

In this section, we take $D = 1$ and $\ell = 2$. We begin with the construction of a family of auxiliary measures which are of the type described in example 2.6. Let a and b be two real numbers with $0 < a, b < 1$ and fix a sequence of integers $(T_k)_{k\geq 1}$ such that

$$T_1 = 1, \quad T_k < T_{k+1} \quad \text{and} \quad \lim_{k\to+\infty} \frac{T_{k+1}}{T_k} = +\infty \ .$$

Then, we define the family of parameters $p_{a\,b}(i)$:

$$p_{a\,b}(i) = a \ \text{ if } \ T_{2n-1} \leq i < T_{2n} \quad \text{and} \quad p_{a\,b}(i) = b \ \text{ if } \ T_{2n} \leq i < T_{2n+1} \ .$$

If $(Y_i)_{i\geq 1}$ is a sequence of independant random variables such that

$$\mathbb{P}(Y_i = 0) = p_{a\,b}(i) \quad \text{and} \quad \mathbb{P}(Y_i = 1) = 1 - p_{a\,b}(i) \ ,$$

we denote by $m_{a\,b}$ the law of the random variable

$$\sum_{i=1}^{+\infty} 2^{-i} Y_i \ .$$

The choice of the integers T_k and the identities (5) ensure that for $dm_{a\,b}$-almost every $x \in [0, 1)$, we have :

$$\begin{cases} \liminf\limits_{n\to+\infty} \dfrac{\log m_{a\,b}(I_n(x))}{-n \log 2} = \inf(s(a), s(b)) \\ \limsup\limits_{n\to+\infty} \dfrac{\log m_{a\,b}(I_n(x))}{-n \log 2} = \sup(s(a), s(b)) \end{cases} . \tag{8}$$

We can now construct our counterexamples. Let us fix a parameter $\beta \in (0, 1/2]$ and two real numbers p and $\tilde{p}$ such that $0 < p < \tilde{p} \leq 1/2$. If $F(t)$ is defined by

$$F(t) = 2t \ \text{ when } \ t \in [0, 1/2) \quad \text{and} \quad F(t) = 2t - 1 \ \text{ when } \ t \in [1/2, 1),$$

we are interested in the measure m_β defined by

$$m_\beta(A) = \beta\, m_{p\,\tilde{p}}(F([0, 1/2) \cap A)) + (1 - \beta)\, m_{\tilde{p}\,p}(F([1/2, 1) \cap A)) \ .$$

In other words, the measure m_β assigns the mass β (resp. $1-\beta$) to the interval $[0, 1/2)$ (resp. $[1/2, 1)$) and is a copy, in this set, of the measure $m_{p\,\tilde{p}}$ (resp. $m_{\tilde{p}\,p}$)

Measures m_β are examples of unidimensional measures whose dimension can not be calculated with an entropy formula. More precisely, we have the following.

Proposition 4.1 *The measure m_β satisfies the following properties :*

1. $\dim_*(m_\beta) = \dim^*(m_\beta) = s(p)$
2. $\mathrm{Dim}_*(m_\beta) = \mathrm{Dim}^*(m_\beta) = s(\tilde{p})$
3. $h_*(m_\beta) = \beta\, s(\tilde{p}) + (1-\beta)\, s(p)$ *and* $h^*(m_\beta) = \beta\, s(p) + (1-\beta)\, s(\tilde{p})$.

In particular, $\dim(m_\beta) < h_*(m_\beta) \le h^*(m_\beta) < \mathrm{Dim}(m_\beta)$.

Remark. For such a measure m_β, unidimensionality ensures that for almost every point x, there exists a subsequence n_k such that $\alpha_{n_k}(x)$ converges to $\dim_*(m)$. Nevertheless, according to theorem 2.1, we can not find a subsequence n_k such that α_{n_k} converges almost surely to $\dim_*(m)$. A similar remark can be made concerning $\mathrm{Dim}^*(m)$.

If we consider two real numbers $\beta_1 \neq \beta_2$ and if $\mu = m_{\beta_1}$ and $\nu = m_{\beta_2}$ we also have the following corollary :

Corollary 4.2 *We can construct two unidimensional measures μ and ν such that :*

1. $\exists c > 0$; $\frac{1}{c}\,\nu \le \mu \le c\,\nu$
2. $h_*(\mu) \neq h_*(\nu)$ *and* $h^*(\mu) \neq h^*(\nu)$.

This result indicates that entropy is a bad concept of dimension (even for unidimensional measures) ; for a good concept of dimension, it is indeed reasonable to demand that two equivalent measures should have the same dimension.

Proof of proposition 4.1. Properties 1 and 2 are immediate consequences of (8). We only have to prove property 3. Let us put $p_i = p_{p\tilde{p}}(i)$ and $\tilde{p}_i = p_{\tilde{p}p}(i)$. We have

$$\begin{aligned}\tau_n(t) &= \frac{1}{n}\log_2\left(\sum_{I\in\mathcal{F}_n} m_\beta(I)^t\right)\\ &= \frac{1}{n}\log_2\left(\beta^t\prod_{i=1}^{n-1}(p_i^t + (1-p_i)^t) + (1-\beta)^t\prod_{i=1}^{n-1}(\tilde{p}_i^t + (1-\tilde{p}_i)^t)\right) .\end{aligned}$$

We can easily deduce that

$$h_n(m_\beta) = -\tau_n'(1) =$$

$$= \frac{1}{n}\left(\beta\left(-\log_2\beta + \sum_{i=1}^{n-1} s(p_i)\right) + (1-\beta)\left(-\log_2(1-\beta) + \sum_{i=1}^{n-1} s(\tilde{p}_i)\right)\right) . \quad (9)$$

If $n_0(n)$ is the number of integers $i \le n-1$ such that $p_i = p$ and if $n_1(n) = n - 1 - n_0(n)$, we can write

$$h_n(m_\beta) = \frac{n_0(n)}{n}\,(\beta\, s(p) + (1-\beta)\, s(\tilde{p})) + \frac{n_1(n)}{n}\,(\beta\, s(\tilde{p}) + (1-\beta)\, s(p)) + O(\frac{1}{n}) .$$

In particular, we have

$$\beta\, s(\tilde{p}) + (1-\beta)\, s(p) + O(\frac{1}{n}) \leq h_n(m_\beta) \leq \beta\, s(p) + (1-\beta)\, s(\tilde{p}) + O(\frac{1}{n}) \ .$$

To conclude, it suffices to remark that

$$\limsup_{n\to+\infty} \frac{n_0(n)}{n} = 1 \quad \text{and} \quad \limsup_{n\to+\infty} \frac{n_1(n)}{n} = 1 \ .$$

5 An example where $\dim(m) = h_*(m)$ and $\mathrm{Dim}(m) > h^*(m)$.

As observed in example 2.6, the equalities $\dim_*(m) = h_*(m)$ and $\mathrm{Dim}^*(m) = h^*(m)$ hold as soon as the sequence of random variables

$$X_n(x) = \log\left(\frac{m(I_n(x))}{m(I_{n-1}(x))}\right)$$

satisfies the strong law of large numbers

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n}(X_i - \mathbb{E}(X_i)) = 0 \ \ dm \ a.s. \ . \tag{10}$$

We are going to construct a unidimensional measure m for which $\dim_*(m) = h_*(m)$ but $\mathrm{Dim}^*(m) > h^*(m)$. Of course, such a measure does not satisfy (10).

Proposition 5.1 *There exists a probability measure m such that*

$$\dim_*(m) = \dim^*(m) = h_*(m) \quad \textit{and} \quad h^*(m) < \mathrm{Dim}^*(m) = \mathrm{Dim}_*(m) \ .$$

The notations are the same as in the previous section. Let $\beta = 1/2$. The measure $\mu_1 = m_{p\tilde{p}}$ is the same as before but we modify slightly the measure $m_{\tilde{p}p}$. In section 4, the measure $m_{\tilde{p}p}$ was constructed in such a way that

$$\forall n \geq 1, \quad \forall i \in \{T_{2n-1}, \dots, T_{2n} - 1\} \ , \quad \tilde{p}_i = p_{\tilde{p}p}(i) = \tilde{p} \ .$$

We modify the value of $\tilde{p}_i$ according to the parity of n and define

$$\begin{cases} \tilde{p}_i = \tilde{p} \ \text{ if } \ i \in \{T_{2n-1}, \dots, T_{2n} - 1\} \ \text{ and } \ n = 2k+1 \\ \tilde{p}_i = p \ \text{ if } \ i \in \{T_{2n-1}, \dots, T_{2n} - 1\} \ \text{ and } \ n = 2k \end{cases} .$$

We do not modifie the value of $\tilde{p}_i$ when $i \in \{T_{2n}, \dots, T_{2n+1} - 1\}$. Let μ_2 be the so constructed measure and put :

$$m(A) = \frac{1}{2}\,\mu_1(F([0,1/2) \cap A)) + \frac{1}{2}\,\mu_2(F([1/2,1) \cap A)) \ .$$

Similar computations as those made before ensure that :

$$\dim_*(m) = \dim^*(m) = s(p) \quad \text{and} \quad \mathrm{Dim}_*(m) = \mathrm{Dim}^*(m) = s(\tilde{p}) \ .$$

Let us see the evolutions of entropy. Since $p < \tilde{p}$, the new measure has smaller entropy than the measure $m_{1/2}$ (corresponding to the parameter $\beta = 1/2$) of proposition 4.1. Hence,

$$h^*(m) \leq \frac{1}{2}\left(s(p) + s(\tilde{p})\right) < \mathrm{Dim}^*(m) \ .$$

In fact, it is easy to prove that $h^*(m) = (s(p) + s(\tilde{p}))/2$.

Moreover, using formula (9) for the new measure m, we have :

$$\begin{aligned} h_{T_{4k}}(m) &= \frac{1}{2T_{4k}} \sum_{i=1}^{T_{4k}-1} \left(s(p_i) + s(\tilde{p}_i)\right) \ + O(\frac{1}{T_{4k}}) \\ &= \frac{T_{4k} - T_{4k-1}}{T_{4k}} \,.\, s(p) \ + O(\frac{T_{4k-1}}{T_{4k}}) + O(\frac{1}{T_{4k}}) \ . \end{aligned}$$

It follows that

$$h_*(m) \leq \lim_{k \to +\infty} h_{T_{4k}}(m) = s(p) = \dim_*(m),$$

which gives the non trivial inequality between these two numbers. Let us finaly remark that it is easy to prove that for almost every $x \in [0, 1)$,

$$\lim_{k \to +\infty} \frac{\log m\left(I_{T_{4k}}(x)\right)}{-T_{4k} \log 2} = s(p) \ .$$

6 An extension of theorems 2.1 and 2.2.

The ℓ-adic partition $\mathcal{F}_n$ of the cube $[0, 1)^D$ is not the only situation where theorems 2.1 and 2.2 make sense. Fix a sequence $(\ell_n)_{n \geq 1}$ of strictly positive real numbers and construct a family $(\mathcal{G}_n)_{n \geq 0}$ in such a way.

1. $\mathcal{G}_0 = \{B_0\}$ where B_0 is a Borel set in $\mathbb{R}^D$ with $B(0, 1/c) \subset B_0 \subset B(0, c)$ (we denote by $B(0, r)$ the ball with center 0 and radius r).

2. For every $n \geq 0$, $\mathcal{G}_n$ is a finite family of disjoint Borel sets which are similar to B_0 in the ratio ℓ_n.

3. For every $B \in \mathcal{G}_{n+1}$, there exists a unique $\tilde{B} \in \mathcal{G}_n$ such that $B \subset \tilde{B}$.

Let m be a measure supported by a borel set E and suppose that for every $n \geq 0$, E is a subset of $\bigcup_{B \in \mathcal{G}_n} B$. If we suppose that the sequence $\log(\ell_n)/n$ is bounded, then, coverings using elements of $\bigcup_n \mathcal{G}_n$ are sufficient to calculate the Hausdorff dimension of subsets of E. Conclusions of theorem 2.1 and theorem 2.2 are also true in this situation if we define $h_*(m)$ (resp. $h^*(m)$) as the $\liminf$ (resp. $\limsup$) of the sequence

$$h_n(m) = \frac{1}{\log \ell_n} \sum_{B \in \mathcal{G}_n} m(B) \log m(B) \ .$$

In particular, our results can be applied for measures supported by Cantor sets constructed in the same way as those described in [Bat96] or [Bat98].

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On entropy and Hausdorff dimension of measures defined through a non-homogeneous Markov process

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Abstract

In this work we study the Hausdorff dimension of measures whose weight distribution satisfies a markov non-homogeneous property. We prove, in particular, that the Hausdorff dimensions of this kind of measures coincide with their lower Rényi dimensions (entropy). Moreover, we show that the packing dimensions equal the upper Rényi dimensions.

As an application we get a continuity property of the Hausdorff dimension of the measures, when it is seen as a function of the distributed weights under the ℓ^∞ norm.

1 Introduction

Let us consider the dyadic tree (even though all the results in this paper can be easily generalised to any ℓ-adic structure, $\ell \in \mathbb{N}$), let $\mathbb{K}$ be its limit (Cantor) set and denote by $(\mathcal{F}_n)_{n\in\mathbb{N}}$ the associated filtration with the usual $0-1$ encoding. We are interested in Borel measures μ on $\mathbb{K}$ constructed in the following way: Take $(p_n, q_n)_{n\in\mathbb{N}}$ a sequence of couples of real numbers satisfying $0 \le p_n, q_n \le 1$.

Let $I = I_{\epsilon_1,\dots,\epsilon_n}$ be a cylinder of the nth generation, $J = I_{\epsilon_{n+1}}$ a cylinder of the first generation and $IJ = I_{\epsilon_1,\dots,\epsilon_n,\epsilon_{n+1}}$ the subcylinder of I of the $(n+1)$th generation, where $\epsilon_1, \dots, \epsilon_n, \epsilon_{n+1} \in \{0, 1\}$. The mass distribution of $\mu_{|I}$ will be as follows: $\mu(I_0) = p_0$, $\mu(I_1) = 1 - p_0$ and

$$\frac{\mu(IJ)}{\mu(I)} = \begin{cases} p_n \mathbf{1}_{\{\epsilon_{n+1}=0\}} + (1-p_n)\mathbf{1}_{\{\epsilon_{n+1}=1\}} \text{ , if } \epsilon_n = 0 \\ q_n \mathbf{1}_{\{\epsilon_{n+1}=0\}} + (1-q_n)\mathbf{1}_{\{\epsilon_{n+1}=1\}} \text{ , if } \epsilon_n = 1 \end{cases}, \tag{1}$$

where the extreme case $\mu(I) = 0$ (and hence $\mu(IJ) = 0$) is treated in the same way by convention.

We use the notation $\dim_{\mathcal{H}}$ for the Hausdorff dimension and $\dim_{\mathcal{P}}$ for the packing dimension.

Definition 1.1 *If μ is a measure on $\mathbb{K}$, we will denote by $h_*(\mu)$ the lower entropy of the measure :*

$$h_*(\mu) = \liminf_{n\to\infty} \frac{-1}{n} \sum_{I\in\mathcal{F}_n} \log \mu(I) \cdot \mu(I),$$

by $h^*(\mu)$ the upper entropy of the measure :

$$h^*(\mu) = \limsup_{n\to\infty} \frac{-1}{n} \sum_{I\in\mathcal{F}_n} \log \mu(I) \cdot \mu(I),$$

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by $\dim_*(\mu)$ *the lower Hausdorff dimension of* μ*:*

$$\dim_* \mu = \inf\{\dim_{\mathcal{H}} E\ ;\ E \subset \mathbb{K} \textit{ and } \mu(E) > 0\}$$

and by $\dim^*(\mu)$ *the upper Hausdorff dimension of* μ*:*

$$\dim^* \mu = \inf\{\dim_{\mathcal{H}} E\ ;\ E \subset \mathbb{K} \textit{ and } \mu(\mathbb{K} \setminus E) = 0\}.$$

In the same way we define the lower packing dimension of μ*:*

$$\mathrm{Dim}_*\mu = \inf\{\dim_{\mathcal{P}} E\ ;\ E \subset \mathbb{K} \text{ and } \mu(E) > 0\}$$

and by $\mathrm{Dim}^*(\mu)$ *the upper packing dimension of* μ*:*

$$\mathrm{Dim}^*\mu = \inf\{\dim_{\mathcal{P}} E\ ;\ E \subset \mathbb{K} \text{ and } \mu(\mathbb{K} \setminus E) = 0\}.$$

One can show that (see [Bat02],[BH02])

$$\dim_*(\mu) \le h_*(\mu) \le h^*(\mu) \le \mathrm{Dim}^*(\mu),$$

and there are examples of these inequalities being strict, even when the measure μ is rather "regular".

It is also well known (cf [Fal97], [Bil65], [Mat95], [Fan94], [You82], [Rén70] and [Heu98]) that

$$\dim_*(\mu) = \inf \mathrm{ess}_\mu \liminf_{n\to\infty} \frac{\log \mu(I_n(x))}{-n \log 2}$$

and

$$\dim^*(\mu) = \sup \mathrm{ess}_\mu \liminf_{n\to\infty} \frac{\log \mu(I_n(x))}{-n \log 2},$$

where $I_n(x)$ is the dyadic cylinder of the nth generation containing x, $\inf \mathrm{ess}_\mu$ is the essential infimum and $\sup \mathrm{ess}_\mu$ is the essential supremum, taken over μ-almost all $x \in \mathbb{K}$.

Whenever μ is a shift-invariant and ergodic measure, it is well known that all limits exist and $\lim_{n\to\infty} \frac{\log \mu(I_n(x))}{-n \log 2} = h_*(\mu) = h^*(\mu)$ which is the Breiman-Shanon-McMillan formula. This is also valid in several random settings (see for instance [Nas87], [Kah87], [KP76] and [Heu03]) and for products of Bernoulli measures (cf. [Bil65]).

In the case of measures defined by (1) we can use tools developed in [Bat96] and [Bat00] to prove they are exact, i.e. that $\dim_*(\mu) = \dim^*(\mu)$ or equivalently that

$$\liminf_{n\to\infty} \frac{\log \mu(I_n(x))}{-n \log 2} = \dim_*(\mu), \text{ for } \mu\text{-almost all } x \in \mathbb{K}$$

and therefore $\dim_*(\mu) = \dim^*(\mu)$. This is, for instance, the case of harmonic measure on homogeneous Cantor sets and on limit sets of a large class of iterated function systems as the ones considered in the articles mentionned above. Nevertheless, some kind of shift-invariance is needed in replacement of the Markov condition proposed in this work. We note, however, that theorem 1.2 implies that $\dim_*(\mu) = \dim^*(\mu)$.

There is no inequality relation between $h_*(\mu)$ and $\dim^*(\mu)$ holding for all measures. Furthermore, it is easy to construct measures μ satisfying (1) such that $h_*(\mu) \ne h^*(\mu)$ which shows that the sequence of functions $\frac{\log \mu(I_n(x))}{-n \log 2}$ does not necessarily converge (in any space).

The proof of theorem 1.2 implies that there is a sequence $(c_n)_{n\in\mathbb{N}}$ of real numbers such that

$$\lim_{n\to\infty}\left[\frac{\log\mu(I_n(x))}{-n\log 2}-c_n\right]=0,$$

where $c_n=\dfrac{-1}{n\log 2}\displaystyle\sum_{I\in\mathcal{F}_n}\log(\mu(I))\mu(I)$. This can be seen as a Breiman-Shannon-McMillan type theorem generalised to measures defined through non-homogeneous Markov chains.

Remark that the tools of [KP76] and [Kah87] can be applied to give the same results for *"almost every"* measure μ satisfying (1). Other results in this sense involving coloring of graphs are proposed in [Nas87].

A. Bisbas and C. Karanikas [BK94] have already partially proved the conclusions of theorem 1.2, for this kind of measure, under some assumptions on the sequences $(p_n,q_n)_{n\in\mathbb{N}}$. In particular they prove the theorem when the sequences $(p_n,q_n)_{n\in\mathbb{N}}$ are uniformly bounded away from 0 and 1, which is the case of a perturbation of an homogeneous Markov chain. We thank A. Bisbas for communicating to us this article.

Theorem 1.2 *If μ satisfies (1) then*

$$\dim_*(\mu)=\dim^*(\mu)=h_*(\mu)\text{ and }\mathrm{Dim}_*(\mu)=\mathrm{Dim}^*(\mu)=h^*(\mu).$$

Using the same type of arguments we also obtain the following continuity result.

Theorem 1.3 *Let μ and μ' be measures defined by (1) and the corresponding sequences $(p_n,q_n)_{n\in\mathbb{N}}$ and $(p'_n,q'_n)_{n\in\mathbb{N}}$ respectively. Then $|\dim_*(\mu)-\dim_*(\mu')|$ and $|\mathrm{Dim}_*(\mu)-\mathrm{Dim}_*(\mu')|$ go to 0 as $||(p_n,q_n)_{n\in\mathbb{N}}-(p'_n,q'_n)_{n\in\mathbb{N}}||_\infty$ tends to 0.*

2 Lemmas and preliminary results

Let us introduce some notation: for $p\in[0,1]$ we denote by

$$h(p)=p\log p+(1-p)\log(1-p)$$

and if $I=I_{\epsilon_1,\ldots,\epsilon_{n-1}}\in\mathcal{F}_n$, let us also set

$$\gamma(I,n)=\sum_{i=0,1}\log\left(\frac{\mu(II_i)}{\mu(I)}\right)\frac{\mu(II_i)}{\mu(I)}.$$

Note that $\gamma(I,n)=\mathbb{E}_I(X_n)$ with respect to notation of [Chu01], section 9.1, page 295. We also remark that for $n\in\mathbb{N}$ and $I\in\mathcal{F}_{n-1}$,$\gamma(I,n)$ is equal to $h(p_n)$ if $\epsilon_{n-1}=0$ and to $h(q_n)$ if $\epsilon_{n-1}=1$ and therefore $|\gamma(I,n)|\le\log 2$.

Let us start with the following easy lemma.

Lemma 2.1 *For all $n,k\in\mathbb{N}$ and all $I\in\mathcal{F}_{n-1}$ we can write*

$$\sum_{K\in\mathcal{F}_k}\log\left(\frac{\mu(IK)}{\mu(I)}\right)\frac{\mu(IK)}{\mu(I)}=\gamma(I,n)+\sum_{i=0,1}\frac{\mu(II_i)}{\mu(I)}\sum_{K\in\mathcal{F}_{k-1}}\log\left(\frac{\mu(II_iK)}{\mu(II_i)}\right)\frac{\mu(II_iK)}{\mu(II_i)}.\qquad(2)$$

where I_0 and I_1 are the two cylinders of the first generation.

Furthermore, if we denote by $a_n^k(I)$ and $b_n^k(I)$ respectively the quantities

$$a_n^k(I) = \sum_{K\in\mathcal{F}_{k-1}} \log\left(\frac{\mu(II_0K)}{\mu(II_0)}\right)\frac{\mu(II_0K)}{\mu(II_0)} \text{ and}$$

$$b_n^k(I) = \sum_{K\in\mathcal{F}_{k-1}} \log\left(\frac{\mu(II_1K)}{\mu(II_1)}\right)\frac{\mu(II_1K)}{\mu(II_1)}$$

then $a_n^k(I) = a_n^k(I')$ and $b_n^k(I) = b_n^k(I')$, for all $I, I' \in \mathcal{F}_n$.

Proof We have

$$\begin{aligned}
&\sum_{K\in\mathcal{F}_k} \log\left(\frac{\mu(IK)}{\mu(I)}\right)\frac{\mu(IK)}{\mu(I)} = \\
&\sum_{i=0,1}\sum_{K\in\mathcal{F}_{k-1}} \log\left(\frac{\mu(II_iK)}{\mu(I)}\right)\frac{\mu(II_iK)}{\mu(I)} = \\
&\sum_{i=0,1}\sum_{K\in\mathcal{F}_{k-1}} \log\left(\frac{\mu(II_iK)}{\mu(II_i)}\right)\frac{\mu(II_iK)}{\mu(I)} + \sum_{i=0,1}\log\left(\frac{\mu(II_i)}{\mu(I)}\right)\frac{\mu(II_i)}{\mu(I)} \qquad (3)
\end{aligned}$$

Since we have set

$$\gamma(I,n) = \sum_{i=0,1}\log\left(\frac{\mu(II_i)}{\mu(I)}\right)\frac{\mu(II_i)}{\mu(I)},$$

the equalities (3) give

$$\sum_{K\in\mathcal{F}_k} \log\left(\frac{\mu(IK)}{\mu(I)}\right)\frac{\mu(IK)}{\mu(I)} = \gamma(I,n) + \sum_{i=0,1}\frac{\mu(II_i)}{\mu(I)}\sum_{K\in\mathcal{F}_{k-1}} \log\left(\frac{\mu(II_iK)}{\mu(II_i)}\right)\frac{\mu(II_iK)}{\mu(II_i)}.$$

It is immediate that $0 \le -\gamma(I,n) \le \log 2$. By the construction of the measure μ, the quantities $a_n^k(I)$ and $b_n^k(I)$ do not depend on the cylinder I but only on the cylinder's generation n and this ends the proof. •

Remark 2.2 Since the quantities $a_n^k(I)$ and $b_n^k(I)$ depend only on the generation of I and on k, we can denote by $a_n^k = a_n^k(I)$ and $b_n^k = b_n^k(I)$ for $I \in \mathcal{F}_n$. We also denote by $\Delta_n^k = \frac{1}{k}|a_n^k - b_n^k|$.

The following lemma is easy to prove but helps to clarify the proof.

Lemma 2.3 *Take $\epsilon > 0$. There exists $\zeta > 0$ such that for all $p, q \in [0,1]$ we have either $|h(p) - h(q)| \le \epsilon/2$ or $|p - q| < 1 - \zeta$. For all $k > k_0 = \left[\frac{4\log 2}{\zeta\epsilon}\right]$ and all $\alpha > \epsilon/2$,*

$$\frac{|h(p)-h(q)|}{k} + |p-q|\left(1-\frac{1}{k}\right)\alpha < \left(1-\frac{1}{2k}\right)\alpha,$$

and hence, for all $\alpha > 0$

$$\frac{|h(p)-h(q)|}{k} + |p-q|\left(1-\frac{1}{k}\right)\alpha < \min\left\{\epsilon\,,\,\left(1-\frac{1}{2k}\right)\alpha\right\}.$$

The proof is elementary and therefore omitted. In the following we will denote by k_0 the positive integer defined in the previous lemma.

Proposition 2.4 *Let I, I' be two cylinders of the nth generation. Then*

$$\frac{1}{k}\left|\sum_{K\in\mathcal{F}_k}\log\left(\frac{\mu(IK)}{\mu(I)}\right)\frac{\mu(IK)}{\mu(I)}-\sum_{K\in\mathcal{F}_k}\log\left(\frac{\mu(I'K)}{\mu(I')}\right)\frac{\mu(I'K)}{\mu(I')}\right|<\eta(k)$$

where η is a positive function, not depending on n, such that $\eta(k)$ goes to 0 as k tends to ∞.

Proof Take any two cylinders $I = I_{\epsilon_1,\ldots\epsilon_n}, I' = I_{\epsilon'_1,\ldots\epsilon'_n}$ of the nth generation. If $\epsilon_n = \epsilon'_n$ then by definition of the measure μ we get

$$\frac{1}{k}\left|\sum_{K\in\mathcal{F}_k}\log\left(\frac{\mu(IK)}{\mu(I)}\right)\frac{\mu(IK)}{\mu(I)}-\sum_{K\in\mathcal{F}_k}\log\left(\frac{\mu(I'K)}{\mu(I')}\right)\frac{\mu(I'K)}{\mu(I')}\right|=0.$$

If $\epsilon_n \neq \epsilon'_n$, using lemma 2.1 and the notation therein we obtain:

$$\begin{aligned}
\Delta_{n-1}^{k+1} &= \left|\frac{1}{k+1}\sum_{K\in\mathcal{F}_{k+1}}\log\left(\frac{\mu(IK)}{\mu(I)}\right)\frac{\mu(IK)}{\mu(I)}-\frac{1}{k+1}\sum_{K\in\mathcal{F}_{k+1}}\log\left(\frac{\mu(I'K)}{\mu(I')}\right)\frac{\mu(I'K)}{\mu(I')}\right| = \\
&= \left|\frac{\gamma(I,n)-\gamma(I',n)}{k+1}+\frac{1}{k+1}\frac{\mu(II_0)}{\mu(I)}\sum_{K\in\mathcal{F}_k}\log\left(\frac{\mu(II_0K)}{\mu(II_0)}\right)\frac{\mu(II_0K)}{\mu(II_0)}+\right.\\
&+ \frac{1}{k+1}\frac{\mu(II_1)}{\mu(I)}\sum_{K\in\mathcal{F}_k}\log\left(\frac{\mu(II_1K)}{\mu(II_1)}\right)\frac{\mu(II_1K)}{\mu(II_1)}-\\
&- \frac{1}{k+1}\frac{\mu(I'I_0)}{\mu(I')}\sum_{K\in\mathcal{F}_k}\log\left(\frac{\mu(I'I_0K)}{\mu(I'I_0)}\right)\frac{\mu(I'I_0K)}{\mu(I'I_0)}-\\
&- \left.\frac{1}{k+1}\frac{\mu(I'I_1)}{\mu(I')}\sum_{K\in\mathcal{F}_k}\log\left(\frac{\mu(I'I_1K)}{\mu(I'I_1)}\right)\frac{\mu(I'I_1K)}{\mu(I'I_1)}\right| = \\
&= \left|\frac{h(p_n)-h(q_n)}{k+1}+\frac{1}{k+1}\left(\left(\frac{\mu(II_0)}{\mu(I)}-\frac{\mu(I'I_0)}{\mu(I')}\right)a_n^k+\left(\frac{\mu(II_1)}{\mu(I)}-\frac{\mu(I'I_1)}{\mu(I')}\right)b_n^k\right)\right| \leq \\
&\leq \frac{|h(p_n)-h(q_n)|}{k+1}+\left|\frac{1}{k+1}\left(\frac{\mu(II_0)}{\mu(I)}-\frac{\mu(I'I_0)}{\mu(I')}\right)(a_n^k-b_n^k)\right|. \qquad (4)
\end{aligned}$$

We can rewrite relation (4) in the following way

$$\frac{|a_{n-1}^{k+1}-b_{n-1}^{k+1}|}{k+1}\leq\frac{|h(p_n)-h(q_n)|}{k+1}+|p_n-q_n|\frac{|a_n^k-b_n^k|}{k}\left(1-\frac{1}{k+1}\right)$$

and thus,

$$\Delta_{n-1}^{k+1}\leq\frac{|h(p_n)-h(q_n)|}{k+1}+|p_n-q_n|\left(1-\frac{1}{k+1}\right)\Delta_n^k. \qquad (5)$$

Take $\epsilon > 0$. By lemma 2.3, for $k \geq k_0$ we have,

$$\Delta_{n-1}^{k+1}\leq\min\left\{\epsilon\,,\left(1-\frac{1}{2(k+1)}\right)\Delta_n^k\right\}. \qquad (6)$$

We use a recursion argument to finish the proof the lemma. First observe that if for some $\ell \in \{1, ..., k - k_0\}$ we have

$$\Delta_{n+\ell}^{k-\ell} < \epsilon \tag{7}$$

then we will also have $\Delta_{n+\ell-1}^{k-\ell+1} < \min\left\{\epsilon, \left(1 - \frac{1}{2(k+1)}\right)\Delta_{n+\ell}^{k-\ell}\right\} \leq \epsilon$ by relation (6), and therefore $\Delta_n^k < \epsilon$.

On the other hand, if inequality (7) does not hold for any $\ell \in \{1, ..., k - k_0\}$ then by (6) we get

$$\Delta_{n+\ell-1}^{k-\ell+1} \leq \left(1 - \frac{1}{2(k-\ell+1)}\right)\Delta_{n+\ell}^{k-\ell}$$

and finally

$$\Delta_n^k \leq \prod_{\ell=k_0}^{k+1}\left(1 - \frac{1}{2(\ell+1)}\right)\log 2, \tag{8}$$

which becomes strictly smaller than ϵ if k is big enough and the proof is complete. •

We will also use the following two theorems of [BH02] that we include without proof for the convenience of the reader (a straight forward proof -without use of these theorems- is possible but much longer).

Theorem 2.5 *[BH02] Let m be a probability measure in $[0,1)^D$ equipped with the filtration of ℓ-adic cubes, $\ell \in \mathbb{N}$. Then*

$$\dim_*(m) \leq h_*(m) .$$

Moreover, the following properties are equivalent :

1. $\dim_*(m) = h_*(m)$
2. $\dim_*(m) = \dim^*(m) = h_*(m)$
3. *There exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that for m-almost every $x \in [0,1)^D$,*

$$\lim_{k\to+\infty} \frac{\log m(I_{n_k}(x))}{-n_k \log \ell} = \dim_*(m) .$$

Theorem 2.6 *[BH02] We also have*

$$h^*(m) \leq \mathrm{Dim}^*(m),$$

and the following properties are equivalent :

1. $\mathrm{Dim}^*(m) = h^*(m)$
2. $\mathrm{Dim}_*(m) = \mathrm{Dim}^*(m) = h^*(m)$
3. *There exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that for m-almost every $x \in [0,1)^D$,*

$$\lim_{k\to+\infty} \frac{\log m(I_{n_k}(x))}{-n_k \log \ell} = \mathrm{Dim}^*(m) .$$

3 Proofs of the theorems

To prove theorem 1.2 we will use the following strong law of large numbers (cf. [HH80]).

Theorem 3.1 (**Law of Large Numbers**) *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of uniformly bounded in $\mathcal{L}^2$ real random variables on a probability space $(\mathbb{X}, \mathcal{B}, P)$ and let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be an increasing sequence of σ-subalgebras of $\mathbb{B}$ such that X_n is measurable with respect to $\mathcal{F}_n$, for all $n \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (X_k - \mathbb{E}(X_k|\mathcal{F}_{k-1})) = 0\,, \quad P\text{-almost surely} \tag{9}$$

We point out that the assumptions on the random variables are not optimal but it will be sufficient for our goal. The space here is $\mathbb{K}$, the filtration will be the dyadic one and μ will take the place of the probability measure P.

Proof of Theorem 1.2. Consider the random variables X_n, $n \in \mathbb{N}$, defined on $\mathbb{K}$, given by

$$X_n(x) = \log \frac{\mu\left(I_n(x)\right)}{\mu\left(I_{n-1}(x)\right)},$$

where, for $x \in \mathbb{K}$, we have denoted by $I_n(x)$ the unique element of $\mathcal{F}_n$ containing x. Theorem 3.1 implies that for all positive p's

$$\lim_{n\to\infty} \frac{1}{(n+1)} \sum_{j=1}^{n} \left(\frac{1}{p} \sum_{k=1}^{p} [X_{jp+k} - \mathbb{E}(X_{jp+k}|\mathcal{F}_{jp})] \right) = 0\ ,\ \mu\text{-almost surely.} \tag{10}$$

On the other hand, on each $I \in \mathcal{F}_n$, the conditional expectation $\frac{1}{p}\sum_{k=1}^{p} \mathbb{E}(X_{np+k}|\mathcal{F}_{np})$ is given by

$$\frac{1}{p}\sum_{k=1}^{p} \mathbb{E}(X_{np+k}|\mathcal{F}_{np}) = \frac{1}{p} \sum_{K\in\mathcal{F}_p} \log\left(\frac{\mu(IK)}{\mu(I)}\right) \frac{\mu(IK)}{\mu(I)}. \tag{11}$$

By proposition 2.4, for every $\epsilon > 0$ there exists $p \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and all I in $\mathcal{F}_{np}$

$$\left| \frac{1}{p} \sum_{K\in\mathcal{F}_p} \log\left(\frac{\mu(IK)}{\mu(I)}\right) \frac{\mu(IK)}{\mu(I)} - c_n \right| < \epsilon, \tag{12}$$

where $c_n = \frac{1}{p}\mathbb{E}\left\{ \sum_{K\in\mathcal{F}_p} \log\left(\frac{\mu(IK)}{\mu(I)}\right) \right\}$ is a constant depending only on n and on the chosen p but not on the cylinder I of $\mathcal{F}_n$.

It is also easy to see that the variables $(X_n)_{n\in\mathbb{N}}$ are uniformly bounded in $\mathcal{L}^2(\mu)$. We deduce, using the relations (10) and (11), that for every $\epsilon > 0$ there exists $p \in \mathbb{N}$ and a sequence $(c_n)_{n\in\mathbb{N}}$ of real numbers such that

$$\begin{aligned} -\epsilon &< \liminf_{n\to\infty} \frac{1}{(n+1)} \sum_{j=1}^{n} \left(\frac{1}{p} \sum_{k=1}^{p} X_{jp+k} - c_j \right) \le \\ &\le \limsup_{n\to\infty} \frac{1}{(n+1)} \sum_{j=1}^{n} \left(\frac{1}{p} \sum_{k=1}^{p} X_{jp+k} - c_j \right) < \epsilon, \end{aligned} \tag{13}$$

μ-almost everywhere on $\mathbb{K}$. This relation implies that

$$\liminf_{n\to\infty} \frac{-1}{(n+1)} \sum_{j=1}^{n} c_j - \epsilon < \liminf_{n\to\infty} \frac{-1}{p} \frac{1}{(n+1)} \sum_{j=1}^{n} \sum_{k=1}^{p} X_{jp+k} < \liminf_{n\to\infty} \frac{-1}{(n+1)} \sum_{j=1}^{n} c_j + \epsilon \quad (14)$$

and

$$\limsup_{n\to\infty} \frac{-1}{(n+1)} \sum_{j=1}^{n} c_j - \epsilon < \limsup_{n\to\infty} \frac{-1}{p} \frac{1}{(n+1)} \sum_{j=1}^{n} \sum_{k=1}^{p} X_{jp+k} < \limsup_{n\to\infty} \frac{-1}{(n+1)} \sum_{j=1}^{n} c_j + \epsilon \quad (15)$$

μ-almost everywhere on $\mathbb{K}$. If we denote by

$$\underline{c} = \liminf_{n\to\infty} \frac{-1}{(n+1)\log 2} \sum_{j=1}^{n} c_j \text{ and } \overline{c} = \limsup_{n\to\infty} \frac{-1}{(n+1)\log 2} \sum_{j=1}^{n} c_j,$$

we deduce from (14) and (15) that $\dim_* \mu = \underline{c}$ and $\mathrm{Dim}_* \mu = \overline{c}$.

Furthermore, the inequalities (13) imply that for every positive ϵ there is a strictly increasing sequence of natural numbers $(n_l)_{l\in\mathbb{N}}$ verifying

$$-\epsilon < \liminf_{l\to\infty} \frac{-1}{(n_l+1)} \sum_{j=1}^{n_l} \left(\frac{1}{p} \sum_{k=1}^{p} X_{jp+k} \right) - \underline{c} \leq \limsup_{l\to\infty} \frac{-1}{(n_l+1)} \sum_{j=1}^{n_l} \left(\frac{1}{p} \sum_{k=1}^{p} X_{jp+k} \right) - \underline{c} < \epsilon,$$

for μ-almost all $x \in \mathbb{K}$.

One easily proves (using, for instance, Cantor's diagonal argument) that there exists a strictly increasing sequence of natural numbers $(n_l)_{l\in\mathbb{N}}$ such that

$$\lim_{l\to\infty} \frac{-1}{n_l \log 2} \log \mu \left(I_{n_l}(x) \right) = \dim_*(\mu),$$

for μ-almost all $x \in \mathbb{K}$.

Similarly, there exists a strictly increasing sequence of natural numbers $(\hat{n}_l)_{l\in\mathbb{N}}$ such that

$$\lim_{l\to\infty} \frac{-1}{\hat{n}_l \log 2} \log \mu \left(I_{\hat{n}_l}(x) \right) = \mathrm{Dim}_*(\mu),$$

for μ-almost all $x \in \mathbb{K}$. We use theorems 2.5 and 2.6 to finish the proof. •

To prove theorem 1.3 we will use proposition 2.4 and lemma 3.1.

Proof of theorem 1.3 Take $\epsilon > 0$ and let $(p_n, q_n)_{n\in\mathbb{N}}$ and $(p'_n, q'_n)_{n\in\mathbb{N}}$ be two sequences of weights satisfying $0 < p_n, q_n, p'_n, q'_n < 1$ for all $n \in \mathbb{N}$ and

$$||(p_n, q_n)_{n\in\mathbb{N}} - (p'_n, q'_n)_{n\in\mathbb{N}}||_\infty < \zeta.$$

We denote by μ and μ' the measures corresponding to these two sequences of weights. We will show that

$$|\dim_*(\mu) - \dim_*(\mu')| < \epsilon,$$

if ζ is small enough.

It follows from proposition 2.4 that there exist a natural number p large enough and two sequences of real numbers $(c_n)_{n\in\mathbb{N}}, (c'_n)_{n\in\mathbb{N}}$ such that the following relations hold:

$$\left|\frac{1}{p}\sum_{K\in\mathcal{F}_p}\log\left(\frac{\mu(IK)}{\mu(I)}\right)\frac{\mu(IK)}{\mu(I)} - c_n\right| < \frac{\epsilon}{4}$$

and

$$\left|\frac{1}{p}\sum_{K\in\mathcal{F}_p}\log\left(\frac{\mu'(IK)}{\mu'(I)}\right)\frac{\mu'(IK)}{\mu'(I)} - c'_n\right| < \frac{\epsilon}{4}$$

for all cylinders $I \in \mathcal{F}_{np}$ and all $n \in \mathbb{N}$. Since p is a fixed finite number it suffices to take ζ small in order to have

$$\left|\frac{1}{p}\sum_{K\in\mathcal{F}_p}\log\left(\frac{\mu(IK)}{\mu(I)}\right)\frac{\mu(IK)}{\mu(I)} - \frac{1}{p}\sum_{K\in\mathcal{F}_p}\log\left(\frac{\mu'(IK)}{\mu'(I)}\right)\frac{\mu'(IK)}{\mu'(I)}\right| < \frac{\epsilon}{2},$$

for all $I \in \mathcal{F}_{np}$ and all $n \in \mathbb{N}$. Hence,

$$-\epsilon < \liminf_{n\to\infty}\frac{1}{(n+1)}\sum_{j=1}^{n}|c_j - c'_j| \le \limsup_{n\to\infty}\frac{1}{(n+1)}\sum_{j=1}^{n}|c_j - c'_j| < \epsilon.$$

we deduce from (14) and (15) that $|\dim_*(\mu) - \dim_*(\mu')| < \epsilon$ and $|\mathrm{Dim}_*(\mu) - \mathrm{Dim}_*(\mu')| < \epsilon$, which completes the proof. •

Hypothesis of theorem 1.3 canot be omitted as we show in the following section.

4 A counterexample

For every $\epsilon > 0$ we construct two dyadic doubling measures μ and ν on $\mathbb{K}$ such that if $X_n(x) = \log\dfrac{\mu(I_n(x))}{\mu(I_{n-1}(x))}$ and $Y_n(x) = \log\dfrac{\nu(I_n(x))}{\nu(I_{n-1}(x))}$, for $n \in \mathbb{N}$ then

$$\sup_{n\in\mathbb{N}}\Big|\Big|X_n - Y_n\Big|\Big|_{L^\infty} < \epsilon \tag{16}$$

and, nevertheless, $|\dim_*(\mu) - \dim_*(\nu)| > \dfrac{1}{4}$. A first example was proposed to us by Professor Alano Ancona; the proof provided here is of a similar nature.

The construction is carried out in two stages. We fix two Bernoulli measures satisfying (16) and we use a reccurent process to modify them in order to get the corresponding dimensions very different.

For $I \in \mathcal{F}_n$ we denote by $\widehat{I}$ the unique cylinder of the $(n-1)$th generation $\mathcal{F}_{n-1}$ contenant I. Relation (16) can now be reformulated in the following way

$$\left|\frac{\mu(I)}{\mu(\widehat{I})} : \frac{\nu(I)}{\nu(\widehat{I})} - 1\right| < \epsilon\ , \quad \text{for all cylinders } I \text{ of } \bigcup_{n\in\mathbb{N}}\mathcal{F}_n \tag{17}$$

4.0.1 The starting point

Take $\epsilon > 0$ and λ_0 the Lebesgue (uniform) measure (of dimension 1) on $\mathbb{K}$.

Consider the Bernoulli measure ρ_0 of weight variable $\frac{1}{2} - \epsilon$, i.e. such that for $I \in \mathcal{F}_n$, $n \in \mathbb{N}$,

$$\rho_0(II_0) = (\frac{1}{2} - \epsilon)\rho(I)\ ,\ \rho_0(II_1) = (\frac{1}{2} + \epsilon)\rho(I). \tag{18}$$

Put $\mu_0 = \lambda_0$ and $\nu_0 = \rho_0$. By construction the measures λ_0 and ρ_0 verify condition (16), are exact and doubling on the dyadics. Moreover, we have

$$\dim \rho_0 = h_*(\rho) = -\frac{\left(\frac{1}{2} - \epsilon\right)}{\log 2} \log\left(\frac{1}{2} - \epsilon\right) - \frac{\left(\frac{1}{2} + \epsilon\right)}{\log 2} \log\left(\frac{1}{2} + \epsilon\right)$$

It is clear that λ_0 and ρ_0 are singular. Furthermore by the Shannon-MacMillan formula (cf for instance [Zin97]),

$$\lim_{n\to\infty} \frac{\log \rho_0(I_n(x))}{n} = h_*(\rho_0)\ \ \rho_0\text{-almost evcerywhere on}\mathbb{K}.$$

Hence, we can find $n_1 \in \mathbb{N}$ and a partition $\{F_0, F_1\}$ of $\mathcal{F}_{n_1}$ verifying :

1. $F_0 \cup F_1 = \mathcal{F}_{n_1}$
2. $\left|\dfrac{\log \rho_0(I)}{n} + h_*(\rho_0)\right| < \epsilon$ for all $I \in F_1$
3. $\left|\dfrac{\log \lambda_0(I)}{n} + \log 2\right| < \epsilon$ for all $I \in F_0$
4. $\displaystyle\sum_{I \in F_1} \rho_0(I) > 1 - \epsilon$
5. $\displaystyle\sum_{I \in F_0} \lambda_0(I) > 1 - \epsilon$

Let us also define the Bernoulli measures λ_1 and ρ_1 on $\mathbb{K}$ in the following way

$$\begin{array}{ccc} \rho_1(I_0) = \delta & \text{and} & \rho_1(I_1) = 1 - \delta \\ \lambda_1(I_0) = \delta(1-\epsilon) & \text{and} & \lambda_1(I_1) = 1 - \delta(1-\epsilon) \end{array} \tag{19}$$

where $\delta > 0$ will be fixed later.

4.0.2 Going on with the construction

For $I_{i_1 \ldots i_n} \subset I \in F_1$ we put

$$\begin{aligned} \mu_1(I_{i_1\ldots i_n}) &= \mu_0(I_{i_1\ldots i_{n_1}})\lambda_1(I_{i_{n_1}\ldots i_n}) \\ \nu_1(I_{i_1\ldots i_n}) &= \nu_0(I_{i_1\ldots i_{n_1}})\rho_1(I_{i_{n_1}\ldots i_n}) \end{aligned} \tag{20}$$

and for $I_{i_1 \ldots i_n} \subset I \in F_0$

$$\mu_1(I_{i_1\ldots i_n}) = \mu_0(I_{i_1\ldots i_n}) \quad , \quad \nu_1(I_{i_1\ldots i_n}) = \nu_0(I_{i_1\ldots i_n}) \tag{21}$$

We remark that for $I = I_{i_1\dots i_n}$ with $n \le n_1$ we leave $\mu_1(I) = \mu_0(I)$ and $\nu_1(I) = \nu_0(I)$.

The restrictions of the measures μ_1 and ν_1 in the cylinders of $\mathcal{F}_{n_1} = F_0 \cup F_1$ are Bernoulli measures of different dimensions, so they are singulars. Therefore, we can find $n_2 \in \mathbb{N}$ and a partition $\{F_{00}, F_{01}, F_{10}, F_{11}\}$ of $\mathcal{F}_{n_2}$ such that

1. $I \in F_{j0} \cup F_{j1}$ if and only if there is $J \in F_j$ such that $I \subset J$, $j \in \{0, 1\}$.
2. $\left| \dfrac{\log \mu_1(I)}{n_2} + \log 2 \right| < \epsilon^2$ for all $I \in F_{00}$.
3. $\left| \dfrac{\log \nu_1(I)}{n_2} + h_*(\rho_1) \right| < \epsilon^2$ for all $I \in F_{11}$.
4. $\displaystyle\sum_{\substack{J \in F_{00} \\ J \subset I}} \mu_1(J) > (1-\epsilon^2)\mu_1(I)$ et $\displaystyle\sum_{\substack{J \in F_{01} \\ J \subset I}} \nu_1(J) > (1-\epsilon^2)\nu_1(I)$ pour $I \in F_0$
5. $\displaystyle\sum_{\substack{J \in F_{10} \\ J \subset I}} \mu_1(J) > (1-\epsilon^2)\mu_1(I)$ and $\displaystyle\sum_{\substack{J \in F_{11} \\ J \subset I}} \nu_1(J) > (1-\epsilon^2)\nu_1(I)$ pour $I \in F_1$

If $I \in F_{00} \cup F_{10}$ and $J \in \displaystyle\bigcup_{n \in \mathbb{N}} \mathcal{F}_n$, we put

$$\mu_2(IJ) = \mu_1(I)\lambda_0(J) \quad , \quad \nu_2(IJ) = \nu_1(I)\rho_0(J).$$

If $I \in F_{01} \cup F_{11}$ and $J \in \displaystyle\bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ we put

$$\mu_2(IJ) = \mu_1(I)\lambda_1(J) \quad , \quad \nu_2(IJ) = \nu_1(I)\rho_1(J).$$

Finally, for $I \in \mathcal{F}_n$, with $n \le n_2$ we keep the same mass distribution $\mu_2(I) = \mu_1(I)$ et $\nu_2(I) = \nu_1(I)$.

Suppose the measures μ_k, ν_k and the partition $\{F_{i_1\dots i_k}\ ,\ i_1, \dots, i_k \in \{0,1\}\}$ de $\mathcal{F}_{n_k}$ are constructed. As in the two first stages, the restrictions of the measures μ_k and ν_k on such cylinder of $\mathcal{F}_{n_k}$ are supposed to be Bernoulli measures: whether λ_0 and ρ_0 whether λ_1 and ρ_1, respectively.

The measures μ_k and ν_k are mutually singular. Hence, there is $n_{k+1} > n_k$ and a partition $\{F_{i_1\dots i_{k+1}}\ ,\ i_1, \dots, i_{k+1} \in \{0,1\}\}$ of $\mathcal{F}_{n_{k+1}}$ satisfying

1. $I \in F_{i_1\dots i_k 0} \cup F_{i_1\dots i_k 1}$ if and only if there is $J \in F_{i_1\dots i_k}$ such that $I \subset J$, with $i_1, \dots, i_k \in \{0, 1\}$
2. $\left| \dfrac{\log \mu_k(I)}{n_{k+1}} + \log 2 \right| < \epsilon^{k+1}$ for all $I \in F_{i_1\dots i_{k-1}00}$.
3. $\left| \dfrac{\log \nu_k(I)}{n_2} + h_*(\rho_1) \right| < \epsilon^{k+1}$ for all $I \in F_{i_1\dots i_{k-1}11}$.
4. $\displaystyle\sum_{\substack{J \in F_{i_1\dots i_{k-1}00} \\ J \subset I}} \mu_k(J) > (1-\epsilon^{k+1})\mu_k(I)$ and $\displaystyle\sum_{\substack{J \in F_{i_1\dots i_{k-1}01} \\ J \subset I}} \nu_k(J) > (1-\epsilon^{k+1})\nu_k(I)$,
   for all cylinders $I \in F_{i_1\dots i_{k-1}0}$.

5. $\displaystyle\sum_{\substack{J\in F_{i_1\ldots i_{k-1}10}\\ J\subset I}} \mu_k(J) > (1-\epsilon^{k+1})\mu_k(I)$ et $\displaystyle\sum_{\substack{J\in F_{i_1\ldots i_{k-1}11}\\ J\subset I}} \nu_k(J) > (1-\epsilon^{k+1})\nu_k(I)$,

for all cylinders $I \in F_{i_1\ldots i_{k-1}1}$.

If $I \in F_{i_1\ldots i_k 0}$, $i_1, ..., i_k \in \{0,1\}$, then for all $J \in \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ we put

$$\mu_{k+1}(IJ) = \mu_k(I)\lambda_0(J) \quad \text{and} \quad \nu_{k+1}(IJ) = \nu_k(I)\rho_0(J).$$

If $I \in F_{i_1\ldots i_k 1}$, $i_1, ..., i_k \in \{0,1\}$, then for all $J \in \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ we put

$$\mu_{k+1}(IJ) = \mu_k(I)\lambda_1(J) \quad \text{and} \quad \nu_{k+1}(IJ) = \nu_k(I)\rho_1(J).$$

4.0.3 Properties of the measures defined

It is clear that the sequences $(\mu_n)_{n\in\mathbb{N}}$ and $(\nu_n)_{n\in\mathbb{N}}$ converge towards two probability measures μ and ν respectively. By the construction μ and ν are doubling on the dyadics, exacts and satisfy (16).

On the other hand, clearly $\dim_* \mu = 1$ and it is not difficult to see that $\dim_* \nu \le \frac{1}{2}$, if δ is small enough, since $\displaystyle\liminf_{n\to\infty} \frac{-\log \nu(I_n(x))}{n \log 2} = \frac{h_*(\rho_1)}{\log 2}$, ν-almost everywhere. Evenmore, the measures μ and ν satisfy the conclusion of theorem 1.2. The counterexample is complete.

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Multifractal Analysis of inhomogeneous Bernoulli products

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Abstract We are interested to the multifractal analysis of inhomogeneous Bernoulli products which are also known as coin tossing measures. We give conditions ensuring the validity of the multifractal formalism for such measures. On another hand, we show that these measures can have a dense set of phase transitions.

Keywords : Hausdorff dimension, multifractal analysis, Gibbs measure, phase transition.

1 Introduction

Let us consider the dyadic tree $\mathbb{T}$ (even though all the results in this paper can be easily generalised to any ℓ-adic structure, $\ell \in \mathbb{N}$), let $\Sigma = \{0,1\}^{\mathbb{N}}$ be its limit (Cantor) set and denote by $(\mathcal{F}_n)_{n\in\mathbb{N}}$ the associated filtration with the usual $0-1$ encoding.

For $\epsilon_1, ..., \epsilon_n \in \{0,1\}$ we denote by $I_{\epsilon_1...\epsilon_n}$ the cylinder of the nth generation defined by $I_{\epsilon_1...\epsilon_n} = \{x = (i_1, ..., i_n, i_{n+1}, ...) \in \Sigma, \ ; \ i_1 = \epsilon_1, ..., i_n = \epsilon_n\}$. For every $x \in \Sigma$, $I_n(x)$ stands for the cylinder of $\mathcal{F}_n$ containing x.

If $(p_n)_n$ is a sequence of weights, $p_n \in (0,1)$, we are interested in Borel measures μ on Σ defined in the following way

$$\mu(I_{\epsilon_1...\epsilon_n}) = \prod_{j=1}^{n} p_j^{1-\epsilon_j}(1-p_j)^{\epsilon_j}. \tag{1}$$

A measure of this form will be referred to as an *inhomogeneous Bernoulli product.* The aim of this paper is to study multifractal properties of such measures.

The particular case where the sequence (p_n) is constant is well-known and provides an example of measure satisfying the multifractal formalism (see e.g [Fal97]). In the general case, Bisbas [Bis95] gave a sufficient condition on the sequence (p_n) ensuring that μ is a multifractal measure (i.e. the level sets are not empty). However, the work of Bisbas does not provide the dimension of the level sets E_α associated to the measure μ.

Let us give a brief description of multifractal formalism. For a probability measure m on Σ, we define the *local dimension* (also called Hölder exponent) of m at $x \in \Sigma$ by

$$\alpha(x) = \liminf_{n\to+\infty} \alpha_n(x) = \liminf_{n\to+\infty} -\frac{\log m(I_n(x))}{n \log 2}.$$

The aim of multifractal analysis is to find the Hausdorff dimension, $\dim(E_\alpha)$, of the level set $E_\alpha = \{x : \alpha(x) = \alpha\}$ for $\alpha > 0$. The function $f(\alpha) = \dim(E_\alpha)$ is called the *singularity spectrum* (or multifractal spectrum) of m and we say that m is a *multifractal measure* when $f(\alpha) > 0$ for several $\alpha's$.

The concepts underlying the multifractal decomposition of a measure go back to an early paper of Mandelbrot [Man74]. In the 80's multifractal measures were used by physicists to study various models arising from natural phenomena. In fully developped turbulence they were used by Frisch and Parisi [FP85] to investigate the intermittent behaviour in the regions of high vorticity. In dynamical system theory they were used by Benzi et al. [BPPV84] to measure how often a given region of the attractor is visited. In diffusion-limited aggregation (DLA) they were used by Meakin et al. [MCSW86] to describe the probability of a random walk landing to the neighborhood of a given site on the aggregate.

In order to determine the function $f(\alpha)$, Hentschel and Procaccia [HP83] used ideas based on Renyi entropies [Rén70] to introduce the generalized dimensions D_q defined by

$$D_q = \lim_{n\to+\infty} \frac{1}{q-1} \frac{\log\left(\sum_{I\in\mathcal{F}_n} m(I)^q\right)}{n \log 2},$$

(see also [GP83, Gra83]). From a physical and heuristical point of view, Halsey et al. [HJK+86] showed that the singularity spectrum $f(\alpha)$ and the generalized dimensions D_q can be derived from each other. The Legendre transform turned out to be a useful tool linking $f(\alpha)$ and D_q. More precisely, it was suggested that

$$f(\alpha) = \dim(E_\alpha) = \tau^*(\alpha) = \inf(\alpha q + \tau(q),\ q \in \mathbb{R}), \tag{2}$$

where

$$\tau(q) = \limsup_{n\to+\infty} \tau_n(q) \quad \text{with} \quad \tau_n(q) = \frac{1}{n\log 2} \log\left(\sum_{I\in\mathcal{F}_n} m(I)^q\right).$$

(The sum runs over the cylinders I such that $m(I) \neq 0$.) The function $\tau(q)$ is called the L^q-spectrum of m and if the limit exists $\tau(q) = (q-1)D_q$.

Relation (2) is called the multifractal formalism and in many aspects it is analogous to the well-known thermodynamic formalism developed by Bowen [Bow75] and Ruelle [Rue78]. In general, the main problem is to obtain the minoration $\dim(E_\alpha) \geq \tau^*(\alpha)$.

For number of measures, this formalism can be verified rigorously. In particular, if the sequence (p_n) is constant or periodic, the measure μ given by (1) satisfies the multifractal formalism (e.g. [Fal97]). It is also the case for invariant measures in some dynamical systems (e.g [Col88, Fan94, Ran89]), for self-similars measures under separation conditions (e.g [CM92, Fen03, LN99, Ols95, Rie95, Ye05]) and for quasiindependent measures(e.g [BMP92, Heu98, Tes06a]).

Despite all these investigations mentioned, the exact range of the validity of the multifractal formalism is still not known. Olsen [Ols95] give a rigorous approach of multifractal formalism in a general context. This work and the paper of Brown, Michon and Peyrière [BMP92] enlighten the link between the minoration $\dim(E_\alpha) \geq \tau^*(\alpha)$ and the existence of auxiliary measures m_q (the so-called *Gibbs measure* [Mic83]) satisfying

$$\forall n,\ \forall I \in \mathcal{F}_n, \quad \frac{1}{C} m(I)^q 2^{-n\tau(q)} \leq m_q(I) \leq C m(I)^q 2^{-n\tau(q)},$$

where the constant $C > 0$ is independent of n and I. In fact, it is shown in [Ben94, BBH02] that the existence of a measure m_q satisfying

$$m_q(I) \leq C m(I)^q 2^{-n\tau(q)},$$

is sufficient to obtain the minoration $\dim(E_\alpha) \geq \tau^*(\alpha)$ for $\alpha = -\tau'(q)$. In this situation, the values of α for which the multifractal formalism may fail lie in intervals $(-\tau'(q^+), -\tau'(q^-))$ where q is a point of non differentiability of τ ($\tau'(q^+)$ and $\tau'(q^-)$ stands for the right and the left derivatives respectively). Such a point q will be called a *phase transition*.

If the weights p_n are not all the same, the measure μ defined by (1) is in general no shift-invariant and we cannot apply classical tools of ergodic theory, as Shannon-McMillan theorem (e.g [Bil65]), to get a lower bound of $\dim(E_\alpha)$ and the differentiability of the function τ.

Let us introduce the other following level sets defined by

$$\underline{E}_\alpha = \{x\ ;\ \alpha(x) \leq \alpha\},\ \overline{F}_\alpha = \left\{x\ ;\ \limsup_{n\to\infty} \alpha_n(x) \geq \alpha\right\},$$

and

$$F_\alpha = \left\{x\ ;\ \limsup_{n\to\infty} \alpha_n(x) = \alpha\right\}.$$

We can now state our main results. In section 2, we prove the following.

Theorem 1.1 *Let μ be an inhomogeneous Bernoulli product on Σ and $q \in \mathbb{R}$. We have*

$$\liminf_{n\to\infty} -q\tau_n'(q) + \tau_n(q) \le \dim\left(\underline{E}_{-\tau'(q^-)} \cap \overline{F}_{-\tau'(q^+)}\right) \le \sup\left\{\tau^*(-\tau'(q^+)), \tau^*(-\tau'(q^-))\right\}.$$

The proof of the lower bound relies on the construction of a special inhomogeneous Bernoulli product which has the dimension of the level set studied.

In section 3 we study the case $\alpha = -\tau'(q)$. The existence of $\tau'(q)$ is not sufficient to ensure that the validity of multifractal formalism for such values of $\alpha's$. However, we prove that the multifractal formalism holds if the sequence $\tau_n(q)$ converges. More precisely, we have

Theorem 1.2 *Suppose that the sequence $(\tau_n(q))$ converges at a point $q \in \mathbb{R}$. If $\tau'(q)$ exists and if $\alpha = -\tau'(q)$, we have*

$$\dim(E_\alpha \cap F_\alpha) = \tau^*(\alpha) = \alpha q + \tau(q). \tag{3}$$

Theorem 1.2 leads us to study the differentiability of the L^q-spectrum $\tau(q)$. In sections 4, we will see that the L^q-spectrum of an inhomogeneous Bernoulli product may be a very irregular function. In particular,

Theorem 1.3 *There exist inhomogeneous Bernoulli products presenting a dense set of phase transitions on $(1, +\infty)$.*

The are several examples of measures presenting phase transitions (see for instance [Tes06a] and the references therein). The example we propose in this work differs from previous ones at three points : first the phase transitions are situated at points $q > 1$ and not at negative ones, where constructions are easier to carry out. Secondly, the set of transitions is dense in $[1, \infty)$, that means as « bad » as can be. And finally, the measure presenting this pathologie is just a Bernoulli product ! Let us also point out that with some minor modifications our method can also apply to create a dense set of phase transitions within $(0, 1)$.

2 Proof of theorem 1.1

We begin by a preliminary result.

Lemma 2.1 *If μ is an inhomogeneous Bernoulli product, then the functions $(\tau''_{\mu,n})$ are locally uniformly bounded on $(0, +\infty)$.*

Proof We denote by $\beta(p_i)$ the homogeneous Bernoulli measure of parameter p_i and by $\tau(p_i,q)$ it's τ function, $\tau(p_i,q)=\log_2(p_i^q+(1-p_i)^q)$. Using the fact that μ is the product of $\beta(p_i)$ we easily obtain

$$\tau_{\mu,n}(q)=\frac{1}{n}\sum_{i=1}^{n}\tau(p_i,q).$$

It is therefore sufficient to show that, for any $q_0>0$, there exists a constant $C=C(q_0)$ such that for all $p\in(0,1)$ and all $q>q_0$, $\dfrac{\partial^2\tau(p,q)}{\partial q^2}\leq C$. We have

$$\begin{aligned}
\frac{\partial^2\tau(p,q)}{\partial q^2} &= \frac{p^q(\log_2 p)^2+(1-p)^q(\log_2(1-p))^2}{p^q+(1-p)^q}-\frac{\left(p^q\log_2 p+(1-p)^q\log_2(1-p)\right)^2}{(p^q+(1-p)^q)^2}\\
&= \frac{p^q(1-p)^q\left((\log_2 p)^2+(\log_2(1-p))^2-2\log_2 p\log_2(1-p)\right)}{(p^q+(1-p)^q)^2}\\
&= \frac{p^q(1-p)^q\left(\log_2\frac{p}{1-p}\right)^2}{(p^q+(1-p)^q)^2}\leq[4p(1-p)]^q(\log_2\frac{p}{1-p})^2\\
&\leq [4p(1-p)]^{q_0}(\log_2\frac{p}{1-p})^2,
\end{aligned}$$

which is uniformly bounded on $p\in(0,1)$ and the proof is complete. •

Lemma 2.1 allows us to give estimates for the lower and the upper Hausdorff dimension of the measure μ. They are respectively defined by

$$\dim_*(\mu)=\inf\{\dim(E),\ \ \mu(E)>0\}\,;\ \dim^*(\mu)=\inf\{\dim(E),\ \ \mu(E)=1\}.$$

We say that μ is exact if $\dim_*(\mu)=\dim^*(\mu)$ and we note $\dim(\mu)$ the common value. In the same way, we can define the lower and the upper Packing dimension Dim of the measure μ. It is well known that there exist some relations between these quantities and the derivatives of the function $\tau_\mu(q)$ at $q=1$. More precisely, it is proved in [Fan94, Heu98] that

$$-\tau'_\mu(1+)\leq\dim_*(\mu)\leq h_*(\mu)\leq h^*(\mu)\leq\operatorname{Dim}^*(\mu)\leq-\tau'_\mu(1-),$$

where $h_*(\mu)$ and $h^*(\mu)$ stand for the lower and the upper entropy of the measure μ, defined as

$$h_*(\mu)=\liminf-\frac{1}{n\log 2}\sum_{I\in\mathcal{F}_n}\mu(I)\log\mu(I)=\liminf-\tau'_{\mu_n}(1)$$

and

$$h^*(\mu) = \limsup -\frac{1}{n\log 2}\sum_{I\in\mathcal{F}_n}\mu(I)\log\mu(I) = \limsup -\tau'_{\mu_n}(1).$$

By Lemma 2.1, we deduce (see [BH02, Heu98]) the following remark.

Remark 2.2 If μ is an inhomogeneous Bernoulli product then

$$\dim\mu = -\tau'_\mu(1^+) = h_*(\mu) = \liminf_{n\to\infty} -\tau'_{\mu_n}(1)$$

and

$$\operatorname{Dim}\mu = -\tau'_\mu(1^-) = h^*(\mu) = \limsup_{n\to\infty} -\tau'_{\mu_n}(1).$$

Fix $q\in\mathbb{R}$. To prove Theorem 1.1, we construct an auxiliary measure ν supported by the set $\underline{E}_{-\tau'(q^-)}\cap\overline{F}_{-\tau'(q^+)}$. More precisely, we consider a sequence of measures ν_n satisfying

$$\forall I\in\mathcal{F}_n,\quad \nu_n(I) = \frac{\mu(I)^q}{\sum_{I\in\mathcal{F}_n}\mu(I)^q} = \mu(I)^q|I|^{\tau_{\mu,n}(q)}. \tag{4}$$

($|I| = 2^{-n}$ stands for the diameter of I). The following lemma implies that the sequence (ν_n) converges in the weak* sense to a probability measure ν which is by construction an inhomogeneous Bernoulli product.

Lemma 2.3 *Let $n\in\mathbb{N}$ and $I\in\mathcal{F}_n$. If μ is an inhomogeneous Bernoulli product, we have $\nu_n(I)=\nu_{n+1}(I)$.*

Proof Take $n>0$ and $I\in\mathcal{F}_n$. We can compute

$$\nu_{n+1}(I) = \frac{\sum_{J\in\mathcal{F}_1}\mu(IJ)^q}{\sum_{I\in\mathcal{F}_n}\sum_{J\in\mathcal{F}_1}\mu(IJ)^q} = \frac{\mu(I)^q(p_{n+1}^q+(1-p_{n+1})^q)}{\sum_{I\in\mathcal{F}_n}(p_{n+1}^q+(1-p_{n+1})^q)\mu(I)^q}$$

and therefore $\nu_{n+1}(I)=\nu_n(I)$ for all $I\in\mathcal{F}_n$. •

By remark 2.2, we then deduce that the Hausdorff and the Packing dimension of ν are given by an entropy formula. In other terms, we have

$$\dim\nu = \liminf_{n\to\infty} -\tau'_{\nu,n}(1) = h_*(\nu)$$

and

$$\operatorname{Dim}\nu = \limsup_{n\to\infty} -\tau'_{\nu,n}(1) = h^*(\nu).$$

Now we can prove Theorem 1.1.

Proof of Theorem 1.1 The upper bound is a well known fact of multifractal formalism (see for instance [BMP92]). In fact we have

1. If $\alpha \leq -\tau'(0^+)$ then $\dim E_\alpha \leq \dim \underline{E}_\alpha \leq \tau^*(\alpha)$.
2. If $\alpha \geq -\tau'(0^-)$ then $\dim F_\alpha \leq \dim \overline{F}_\alpha \leq \tau^*(\alpha)$.
3. $-\tau'(0^+) \leq \alpha \leq -\tau'(0^-)$ then $\tau^*(\alpha) = \tau(0) = 1$ and the upper bound follows.

Relation (4) easily gives $\tau_{\nu,n}(s) = \tau_{\mu,n}(qs) - s\tau_{\mu,n}(q)$. From remark 2.2, using the inhomogeneous Bernoulli property of μ and ν, we deduce that

$$-\tau_\nu'(1^+) = \liminf -\tau_{\nu,n}'(1) = \liminf \left(-q\tau_{\mu,n}'(q) + \tau_{\mu,n}(q)\right).$$

The following lemma then implies the lower bound.

Lemma 2.4 *We have* $\nu\left(\underline{E}_{-\tau'(q^-)} \cap \overline{F}_{-\tau'(q^+)}\right) = 1$.

Remark 2.5 Contrary to more regular situations (e.g [BBH02, Heu98, Ols95]), we cannot obtain the more precise result $\nu\left(\overline{E}_{-\tau'(q^-)} \cap \underline{F}_{-\tau'(q^+)}\right) = 1$ where

$$\overline{E}_\alpha = \{x\ ;\ \alpha(x) \geq \alpha\},\ \underline{F}_\alpha = \left\{x\ ;\ \limsup_{n\to\infty} \alpha_n(x) \leq \alpha\right\}.$$

Proof of Lemma 2.4 For $\eta > 0$ we put $\beta = -\tau_\mu'(q^-) + \eta$ and we prove that $\nu(\Sigma \setminus \underline{E}_\beta) = 0$. In a similar way, it can be shown that $\nu(\Sigma \setminus \overline{F}_\gamma) = 0$ for $\gamma < -\tau_\mu'(q^+)$. The lemma then easily follows.

It suffices to show that $\Sigma \setminus \underline{E}_\beta = \left\{x \in \Sigma\ ;\ \liminf_{n\to\infty} \alpha_n(x) > \beta\right\}$ is of 0 ν-measure. Consider the collection $\mathcal{R}_n(\beta)$ of cylinders $I \in \mathcal{F}_n$ satisfying $\dfrac{\log \mu(I)}{\log |I|} > \beta$. It is clear that $\Sigma \setminus \underline{E}_\beta \subset \liminf_{n\to\infty} \tilde{\mathcal{R}}_n(\beta)$ with $\tilde{\mathcal{R}}_n(\beta) = \{x \in \Sigma\ ;\ I_n(x) \in \mathcal{R}_n(\beta)\}$.

Let $(\tau_{\mu,n_k})_{k\in\mathbb{N}}$ be the subsequence of $(\tau_{\mu,n})_{n\in\mathbb{N}}$ such that $\lim_{k\to\infty} \tau_{\mu,n_k}(q) = \tau_\mu(q)$. Using the convergence of $\tau_{\mu,n_k}(q)$ we can choose (and fix) $t < 0$ such that for k big enough

$$\tau_\mu(q+t) - \tau_{\mu,n_k}(q) < -\left(\beta - \frac{\eta}{2}\right)t = \left(\tau_\mu'(q^-) - \frac{\eta}{2}\right)t.$$

Since $\mu(I)^{-t}|I|^{\beta t} \leq 1$ if $I \in \mathcal{R}_n(\beta)$, we have

$$\begin{aligned}
\nu(\tilde{\mathcal{R}}_n(\beta)) = \sum_{I\in\mathcal{R}_{n_k}(\beta)} \nu(I) &= \sum_{I\in\mathcal{R}_{n_k}(\beta)} \mu(I)^q |I|^{\tau_{\mu,n_k}(q)} = \sum_{I\in\mathcal{R}_{n_k}(\beta)} \mu(I)^{q+t}|I|^{\tau_{\mu,n_k}(q)-\beta t}\mu(I)^{-t}|I|^{\beta t} \\
&\leq \sum_{I\in\mathcal{R}_{n_k}(\beta)} \mu(I)^{q+t}|I|^{\tau_{\mu,n_k}(q)-\beta t} \leq |I|^{-\frac{\eta}{4}t} \sum_{I\in\mathcal{F}_{n_k}} \mu(I)^{q+t}|I|^{\tau_\mu(q+t)-\frac{\eta}{4}t} \\
&\leq |I|^{-\frac{\eta}{4}t} \sum_{I\in\mathcal{F}_{n_k}} \mu(I)^{q+t}|I|^{\tau_{\mu,n_k}(q+t)} = |I|^{-\frac{\eta}{4}t}.
\end{aligned}$$

For the last inequality, we used the fact that $\tau_\mu(q+t) = \limsup \tau_{\mu,n}(q+t)$. We deduce that

$$\limsup_{n\to\infty} \nu(\tilde{\mathcal{R}}_n(\beta)) = 0$$

and the lemma easily follows.

•

The proof of Theorem 1.1 is now completed. •

3 Proof of Theorem 1.2

We will use the following result.

Proposition 3.1 *For $q \in \mathbb{R}$, let (τ_{μ,n_k}) be the subsequence of $(\tau_{\mu,n})$ such that*

$$\lim_{k\to\infty} \tau_{\mu,n_k}(q) = \limsup_{n\to\infty} \tau_{\mu,n}(q) = \tau_\mu(q).$$

Then, we have

$$\tau'_\mu(q^-) \leq \liminf_{k\to\infty} \tau'_{\mu,n_k}(q) \leq \limsup_{k\to\infty} \tau'_{\mu,n_k}(q) \leq \tau'_\mu(q^+)$$

where $\tau'_\mu(q^-)$ and $\tau'_\mu(q^+)$ stand for the left and the right hand dérivative of τ_μ at q.

Hence, if $\tau'_\mu(q)$ exists, we have

$$\lim_{k\to\infty} \tau'_{\mu,n_k} = \tau'_\mu(q).$$

Proof We only prove the inequality $\limsup_{k\to\infty} \tau'_{\mu,n_k}(q) \le \tau'_\mu(q^+)$. The proof of $\tau'_\mu(q^-) \le \liminf_{k\to\infty} \tau'_{\mu,n_k}(q)$ is similar.

Take $\epsilon > 0$ and $\tilde{q} > q$ such that

$$\left| \frac{\tau_\mu(\tilde{q}) - \tau_\mu(q)}{\tilde{q} - q} - \tau'_\mu(q^+) \right| < \epsilon/3.$$

We can chose k big enough to have

$$\frac{|\tau_{\mu,n_k}(q) - \tau_\mu(q)|}{|\tilde{q} - q|} \quad < \quad \epsilon/3$$

and

$$\tau_{\mu,n_k}(\tilde{q}) \quad \le \quad \tau_\mu(\tilde{q}) + (\tilde{q} - q)\epsilon/3.$$

We then obtain

$$\begin{aligned} \tau'_\mu(q^+) \quad &\ge \quad \frac{\tau_\mu(\tilde{q}) - \tau_\mu(q)}{\tilde{q} - q} - \epsilon/3 \\ &= \quad \frac{\tau_\mu(\tilde{q}) - \tau_{\mu,n_k}(\tilde{q}) + \tau_{\mu,n_k}(\tilde{q}) - \tau_{\mu,n_k}(q) + \tau_{\mu,n_k}(q) - \tau_\mu(q)}{\tilde{q} - q} - \epsilon/3 \\ &\ge \quad -\epsilon/3 + \tau'_{\mu,n_k}(q) - \epsilon/3 - \epsilon/3 = \tau'_{\mu,n_k}(q) - \epsilon. \end{aligned}$$

and the proof easily follows. •

We can now prove Theorem 1.2.

Proof of Theorem 1.2. Let ν be the Gibbs-measure defined in Lemma 2.3. Since

$$\tau_{\nu,n}(s) = \tau_{\mu,n}(qs) - s\tau_{\mu,n}(q)$$

we get

$$\tau'_{\nu,n}(1) = q\tau'_{\mu,n}(q) - \tau_{\mu,n}(q).$$

Using the convergence of $\tau_{\mu,n}(q)$ we deduce from Proposition 3.1 that

$$\lim_{n\to\infty} \tau'_{\nu,n}(1) = \lim_{n\to\infty} \left(q\tau'_{\mu,n}(q) - \tau_{\mu,n}(q)\right) = q\tau'_\mu(q) - \tau_\mu(q).$$

Since, ν is also an inhomogeneous Bernoulli product, we deduce from remark 2.2 that $\tau'_\nu(1)$ exists and

$$\dim \nu = \operatorname{Dim} \nu = -\tau'_\nu(1) = -q\tau'_\mu(q) + \tau_\mu(q).$$

On the other hand, for $I \in \mathcal{F}_n$, we have

$$\frac{\log \nu(I)}{\log |I|} = q \frac{\log \mu(I)}{\log |I|} + \tau_{\mu,n}(q).$$

Since

$$\lim_{n\to\infty} \frac{\log \nu(I_n(x))}{\log |I_n(x)|} = \dim \nu = \mathrm{Dim}\, \nu \ ; \nu\text{-a.s.}$$

we obtain that $\lim_{n\to\infty} \dfrac{\log \mu(I_n(x))}{\log |I_n(x)|} = -\tau_\mu'(q)$, ν-a.s. We conclude that

$$\dim (E_\alpha \cap F_\alpha) \geq \dim \nu = \tau_\mu^*(\alpha).$$

The opposite inequality being always valid, the proof is done. •

We end this section with a few comments about Theorem 1.2.

As mentionned in the introduction of the paper, the validity of the multifractal formalism is often easier to obtain for the values of α that can be written $\alpha = -\tau'(q)$. However, the following example shows that there exist inhomogeneous Bernoulli products that do not satisfy the multifractal formalism even at their differentiability points $\alpha = -\tau'(q)$. Thus, the convergence of the sequence $(\tau_n(q))$ converges is necessary for the validity of the multifractal formalism in our context.

To see that, let $(n_k)_{k\geq 1}$ be a sequence of integers such that

$$n_1 = 1, \quad n_k < n_{k+1} \quad \text{and} \quad \lim_{k\to+\infty} \frac{n_{k+1}}{n_k} = +\infty,$$

and consider the inhomogeneous Bernoulli product μ given by the sequence (p_n) such that

$$p_i = p \quad \text{if} \quad n_{2n-1} \leq i < n_{2n} \quad \text{and} \quad p_i = \tilde{p} \quad \text{if} \quad n_{2n} \leq i < n_{2n+1},$$

with $0 < p < \tilde{p} < 1/2$.

The calculation of the function τ is classical. By observing that

$$\mu(I_{\epsilon_1\ldots\epsilon_n 0})^q + \mu(I_{\epsilon_1\ldots\epsilon_n 1})^q = [(p_{n+1}^q + (1-p_{n+1})^q]\mu(I_{\epsilon_1\ldots\epsilon_n})^q,$$

we easily deduce that

$$\sum_{I\in\mathcal{F}_n} \mu(I)^q = \prod_{k=1}^n [p_k^q + (1-p_k)^q].$$

Then, if N_n is the number of integer $k \leq n$ such that $p_k = p$, we have

$$\tau_n(q) = \frac{N_n}{n} \log_2(p^q + (1-p)^q) + (1 - \frac{N_n}{n}) \log_2(\tilde{p}^q + (1-\tilde{p})^q).$$

Since that $\liminf_n \frac{N_n}{n} = 0$ and $\limsup_n \frac{N_n}{n} = 1$, we get

$$\tau(q) = \sup(\log_2(p^q + (1-p)^q), \log_2(\tilde{p}^q + (1-\tilde{p})^q).$$

So, except for $q = 0$ and $q = 1$, $\tau'(q)$ exists. Moreover,

$$\forall I \in \mathcal{F}_n, \quad \mu(I) \geq \mu(I_{00\cdots 0}) = p^{N_n}\tilde{p}^{n-N_n}.$$

Thus,

$$\forall I \in \mathcal{F}_n, \quad -\frac{\log_2(\mu(I))}{n} \leq \frac{N_n}{n}(-\log_2 p) + (1 - \frac{N_n}{n})(-\log_2 \tilde{p}),$$

and we have

$$\forall I \in \mathcal{F}_n, \quad \liminf_n -\frac{\log_2(\mu(I))}{n} \leq \inf(-\log_2 p, -\log_2 \tilde{p}) = -\log_2 \tilde{p}.$$

Finally, if $-\log_2 \tilde{p} < \alpha = -\tau'(q) < -\log_2 p$, we have $E_{-\tau'(q)} = \emptyset$ and the multifractal formalism is not satisfied for a such α.

Moreover, this example shows that the function τ may be not differentiable at the positive values of q. Therefore, the situation differs from this one obtained by Heurteaux [Heu98] for quasi-Bernoulli measure for which τ is differentiable on $\mathbb{R}$. It also differs from this one obtained by Testud in [Tes06b] for weak quasi-Bernoulli measure for which the phase transitions only appears for $q < 0$.

In fact, the function τ of an inhomogeneous Bernoulli product may be very irregular. This is the object following section.

4 Proof of Theorem 1.3

From now, we denote par $\tau(p, .)$ the τ function of the homogeneous Bernoulli product of parameter p. Moreover, whenever we use the notation p_i for a weight in $(0, 1)$ we will also note $\tau_i = \tau(p_i, .)$.

Before the proof of Theorem 1.3, we present a few lemmas.

Lemma 4.1 *For any $p_1 < p_2 < p_3$ in $(0, 1/2)$ consider the functions $\tau_1 = \tau(p_1, .), \tau_2 = \tau(p_2, .)$ and $\tau_3 = \tau(p_3, .)$. We have that $\dfrac{\tau_1 - \tau_2}{\tau_2 - \tau_3}$ is decreasing on $(1, +\infty)$.*

Although the proof only uses elementary calculus, it is a litte bit "tricky" and cannot be omitted.

Proof of Lemma 4.1 Taking into account the trivial equality

$$\tau(p',q)-\tau(p'',q)=\int_{p''}^{p'}\frac{\partial\tau}{\partial p}(p,q)dp$$

we only need to show that if $p' < p''$ then $\frac{\partial\tau}{\partial p}(p',q) : \frac{\partial\tau}{\partial p}(p'',q)$ is decreasing on $q\in(1,\infty)$. We get

$$\begin{aligned}\frac{\partial\tau}{\partial p}(p',q):\frac{\partial\tau}{\partial p}(p'',q) &= \frac{1}{p'}\frac{1-(-1+1/p')^{q-1}}{1+(-1+1/p')^{q}}:\frac{1}{p''}\frac{1-(-1+1/p'')^{q-1}}{1+(-1+1/p'')^{q}}\\ &= p''\frac{1-{s_1}^{q-1}}{1+{s_1}^{q}}:p'\frac{1-{s_2}^{q-1}}{1+{s_2}^{q}}\end{aligned}$$

where $s_1=-1+1/p'>1$ and $s_2=-1+1/p''>1$.

If we set $f(s,q)=\ln\frac{1-s^{q-1}}{1+s^q}$, with $s,q>1$, it is sufficient to prove that $\frac{\partial f}{\partial s}f(s,q)$ is decreasing in q. We calculate

$$\frac{\partial f}{\partial s}f(s,q)=\frac{(q-1)s^{q-2}}{s^{q-1}-1}-\frac{qs^{q-1}}{s^q+1}.$$

By multiplying by s, we need to show that $\frac{(q-1)s^{q-1}}{s^{q-1}-1}-\frac{qs^q}{s^q+1}$ is decreasing which is equivalent to $q-1+\frac{q-1}{s^{q-1}-1}-q+\frac{q}{s^q+1}$ being decreasing.

Put $Q=q-1$; it remains to show that $\frac{q-1}{s^{q-1}-1}+\frac{q}{s^q+1}=\frac{Q}{s^Q-1}+\frac{Q}{s^{Q+1}+1}+\frac{1}{s^{Q+1}+1}$ decreases in $Q>0$. The last term being decreasing it suffices to show that $\frac{Q}{s^Q-1}+\frac{Q}{s^{Q+1}+1}$ is doing the same. By taking derivatives we need to show that

$$s^{Q+1}(s^Q-1-s^Q\ln s^Q)+s^Q-1-\ln s^Q\le 0.$$

Since, $(s^Q-1-s^Q\ln s^Q)<0$, it suffices to show that

$$s^Q(s^Q-1-s^Q\ln s^Q)+s^Q-1-\ln s^Q=s^{2Q}-s^{2Q}\ln s^Q-\ln s^Q-1=g(s^Q)\le 0$$

where $g(x)=x^2-x^2\ln x-\ln x-1$. Moreover, the sign of $g'(x)=x-x\ln x^2-1/x$ is the same of the sign of $x^2-x^2\ln x^2-1$ if $x>1$. Since, $y-1\le y\ln y$ for $y>1$, we deduce that g is decreasing on $(1,+\infty)$. By observing than $g(1)=0$, we obtain that $\frac{\partial f}{\partial s}f(s,q)$ is decreasing on $(1,+\infty)$ and the Lemma 4.1 is proved. •

Lemma 4.2 *Take $\tau = \lambda\tau(p_1,.) + (1-\lambda)\tau(p_2,.)$ with $0 < p_1 < p_2 < 1/2$ and $\lambda \in (0,1)$. For $p_0 \in (0,1/2)$ one of the following occurs :*

1. *either $\tau(q) \neq \tau(p_0,q)$, for all $q > 1$,*
2. *either there exists $q_0 > 1$ such that $\tau(q) > \tau(p_0,q)$ for $1 < q < q_0$ and $\tau(q) < \tau(p,q)$ for $q > q_0$. In this case q_0 is then the unique point of $(1,+\infty)$ for which $\tau(q) = \tau(p_0,q)$.*

Proof of Lemma 4.2. Let us first remark that τ and $\tau(p_0,.)$ can coincide at one point only if $p_0 \in (p_1,p_2)$. Moreover, $\tau(q) = \tau(p_0,q)$ implies

$$\frac{\tau(p_1,q) - \tau(p_0,q)}{\tau(p_0,q) - \tau(p_2,q)} = \frac{1-\lambda}{\lambda}.$$

By lemma 4.1 this can only occur once. The lemma 4.2 easily follows on the decreasing property of the ratio. •

Lemma 4.3 *Take $\lambda_1, \lambda_2 \in (0,1)$ such that $\lambda_1 + \lambda_2 = 1$, $1 < p_1 < p_2 < 1/2$ and set $\tau = \lambda_1\tau_1 + \lambda_2\tau_2$. Fix $1 < q_1 < q_2 < +\infty$ and consider $p_1 < p_4 < p_2 < p_5 < 1/2$ such that $\tau(p_4,q_1) = \tau(q_1)$. Then there is a unique convex combination $\tilde{\tau}$ of τ_1, τ_4 and τ_5 such that*

$$\tilde{\tau}(q_1) = \tau(q_1) \text{ and } \tilde{\tau}(q_2) = \tau(q_2).$$

Furthermore, for $i = 1,2$, we have $\tau'(q_i) \neq \tilde{\tau}'(q_i)$ and $\tau(q) \neq \tilde{\tau}(q)$ if $1 < q \neq q_i$.

Proof of Lemma 4.3.

First note that it is easy to see that there exists $p_4 \in (p_1,p_2)$ such that $\tau(p_4,q_1) = \tau(q_1)$.

It then suffices to show that the linear system

$$\left\{\begin{array}{llllll} \lambda_3\tau_1(q_1) & +\lambda_4\tau_4(q_1) & +\lambda_5\tau_5(q_1) & = & \tau(q_1) \\ \lambda_3\tau_1(q_2) & +\lambda_4\tau_4(q_2) & +\lambda_5\tau_5(q_2) & = & \tau(q_2) \\ \lambda_3 & +\lambda_4 & +\lambda_5 & = & 1 \end{array}\right. \qquad (\mathbf{S})$$

has a unique positive solution $(\lambda_3, \lambda_4, \lambda_5)$. The existence of a unique solution is easy to verify. Let us show that this solution is positive.

First note that $\lambda_4 \neq 1$. Indeed, if $\lambda_4 = 1$, since $\tau(q_1) = \tau_4(q_1)$, we have $\lambda_3(\tau_1(q_1) - \tau_5(q_1)) = 0$. Thus, $\lambda_3 = \lambda_5 = 0$ and $\tau(q_2) = \tau_4(q_2)$ which is not possible by Lemma 4.2.

Therefore, since $\tau(q_1) = \tau_4(q_1)$, the first equation of the system gives that

$$\frac{\lambda_3}{\lambda_3+\lambda_5}\tau_1(q_1)+\frac{\lambda_5}{\lambda_3+\lambda_5}\tau_5(q_1)=\lambda_1\tau_1(q_1)+\lambda_2\tau_2(q_1)\in(\tau_5(q_1),\tau_1(q_1)) \tag{5}$$

This implies that $\frac{\lambda_3}{\lambda_3+\lambda_5}\in(0,1)$. We deduce that $\lambda_3\lambda_5>0$. Moreover, since $\tau_5<\tau_2$, we also have $\frac{\lambda_3}{\lambda_3+\lambda_5}>\lambda_1$.

Let us show that λ_3 and λ_5 are positive. Otherwise, by the above remark, we have $\lambda_3<0$, $\lambda_5<0$ and $\lambda_4>0$. By the system (S) we have

$$\tau_4(q)=\frac{\lambda_1-\lambda_3}{\lambda_4}\tau_1(q)+\frac{\lambda_2}{\lambda_4}\tau_2(q)-\frac{\lambda_5}{\lambda_4}\tau_5(q)$$

at the points $q=q_1$ and $q=q_2$. We then obtain that

$$\frac{\lambda_1-\lambda_3}{\lambda_4}\frac{\tau_1-\tau_4}{\tau_4-\tau_2}(q)=\frac{\lambda_2}{\lambda_4}-\frac{\lambda_5}{\lambda_4}\frac{\tau_4-\tau_5}{\tau_4-\tau_2}(q)$$

for $q=q_1$ and $q=q_2$. Since $p_1<p_4<p_2$, by Lemma 4.1 the function $\frac{\tau_1-\tau_4}{\tau_4-\tau_2}$ is decreasing. On the other hand, since $p_4<p_2<p_5$, Lemma 4.1 implies that the function $\frac{\tau_4-\tau_5}{\tau_4-\tau_2}=1+\frac{\tau_2-\tau_5}{\tau_4-\tau_2}$ is increasing. Thus, these functions cannot coincide at two points so we conclude that λ_3 and λ_5 are positive.

Let us now prove that $\lambda_4>0$. By (5) we have

$$\frac{\lambda_3}{\lambda_3+\lambda_5}\tau_1(q_1)+\frac{\lambda_5}{\lambda_3+\lambda_5}\tau_5(q_1)=\lambda_1\tau_1(q_1)+\lambda_2\tau_2(q_1)$$

which gives that

$$\lambda_2\tau_2(q_1)=\left(\frac{\lambda_3}{\lambda_3+\lambda_5}-\lambda_1\right)\tau_1(q_1)+\frac{\lambda_5}{\lambda_3+\lambda_5}\tau_5(q_1).$$

Using Lemma 4.1, for $q>q_1$ we get

$$\lambda_2\tau_2(q)>\left(\frac{\lambda_3}{\lambda_3+\lambda_5}-\lambda_1\right)\tau_1(q)+\frac{\lambda_5}{\lambda_3+\lambda_5}\tau_5(q)$$

and

$$\lambda_1\tau_1(q)+\lambda_2\tau_2(q)>\frac{\lambda_3}{\lambda_3+\lambda_5}\tau_1(q)+\frac{\lambda_5}{\lambda_3+\lambda_5}\tau_5(q).$$

In particular, for $q=q_2$ we find that

$$\lambda_3\tau_1(q_2)+\lambda_5\tau_5(q_2)+\lambda_4\tau(q_2)<\tau(q_2)=\lambda_3\tau_1(q_2)+\lambda_4\tau_4(q_2)+\lambda_5\tau_5(q_2)$$

and we deduce that

$$\lambda_4 \tau(q_2) < \lambda_4 \tau_4(q_2).$$

Since $\tau(q_1) = \tau_4(q_1)$, it follows from Lemma 4.1 that $\lambda_4 > 0$.

The last assertion follows directly from the independancy of the vector families

$$\left\{ \begin{pmatrix} \tau_1(q_1) \\ \tau_4(q_1) \\ \tau_5(q_1) \end{pmatrix}, \begin{pmatrix} \tau_1(q_2) \\ \tau_4(q_2) \\ \tau_5(q_2) \end{pmatrix}, \begin{pmatrix} \tau_1'(q_i) \\ \tau_4'(q_i) \\ \tau_5'(q_i) \end{pmatrix} \right\}$$

and

$$\left\{ \begin{pmatrix} \tau_1(q_1) \\ \tau_4(q_1) \\ \tau_5(q_1) \end{pmatrix}, \begin{pmatrix} \tau_1(q_2) \\ \tau_4(q_2) \\ \tau_5(q_2) \end{pmatrix}, \begin{pmatrix} \tau_1(q) \\ \tau_4(q) \\ \tau_5(q) \end{pmatrix} \right\},$$

which can be easily established. •

Remark 4.4 In the proof of Lemma 4.3 it is clear that when p_5 is close to p_2, the solution of the system (**S**) converges to $(\lambda_1, 0, \lambda_2)$ and $\tilde{\tau}$ converges to τ.

The following result generalizes Lemma 4.3 for any convex combination of functions $\tau(p_i, .)$.

Lemma 4.5 *Let τ be a convex combination of functions $\tau(p_i, .)$ where $0 < p_i \leq 1/2$, $i = 1, ..., n \geq 2$. For any $1 < q_1 < q_2 < \infty$ there exists another convex combination $\tilde{\tau}$ of functions $\tau(p_j', .)$ such that*

- *$\tilde{\tau}(q_i) = \tau(q_i)$ and $\tilde{\tau}'(q_i) \neq \tau'(q_i)$, $i = 1, 2$,*
- *for $q \notin \{q_1, q_2\}$, $\tilde{\tau}(q) \neq \tau(q)$.*

Proof of Lemma 4.5.

The case $n = 2$ is given by Lemma 4.3. The case $n > 2$ is easy to derive. Suppose $\tau = \sum_{k=1}^{n} \lambda_k \tau(p_k, .)$ and let $\tau_1 = \tau(p_1, .)$ and $\tau_2 = \tau(p_2, .)$ be the first two functions of the convex combination. By Lemma 4.3 there exists a convex combination $\hat{\tau}$ of three $\tau(p, .)$ functions such that

1. $\frac{1}{\lambda_1+\lambda_2} (\lambda_1 \tau_1(q_i) + \lambda_2 \tau_2(q_i)) = \hat{\tau}(q_i)$, for $i = 1, 2$
2. $\frac{1}{\lambda_1+\lambda_2} (\lambda_1 \tau_1'(q_i) + \lambda_2 \tau_2'(q_i)) \neq \hat{\tau}'(q_i)$ for $i = 1, 2$.

The function $\tilde{\tau} = (\lambda_1+\lambda_2)\hat{\tau} + \sum_{k=3}^{n} \lambda_k \tau(p_k, .)$ satisfies then the conclusion of Lemma 4.5. •

The following lemma is easy and the proof is left to the reader.

Lemma 4.6 *For any $p_1, ..., p_n$ and any convex combination τ of $\tau(p_1, .), ..., \tau(p_n, .)$ there exists an inhomogeneous Bernoulli measure μ whose multifractal spectrum equals τ.*

We can now prove Theorem 1.3.

Proof of Theorem 1.3

In fact, and in order to avoid technicalities, we only prove the following easier version of Theorem 1.3 and then indicate the changes needed to extend the proof in the general case.

Theorem 4.7 *There exists an inhomogeneous Bernoulli product μ such that the spectrum τ of μ has an infinite set of the phase transitions. Moreover, this set has a finite point of accumulation.*

Proof The strategy of the demonstration is the following : we first find inhomogeneous Bernoulli products that are not derivable at a finite number of predefined points and we construct the measure μ using Cantor's diagonal argument.

Fix $(q_n)_{n\geq 1}$ a sequence of real numbers nested in the sense that $q_1 < ... < q_{2n-1} < q_{2n+1} < q_{2n+2} < q_{2n} < ... < q_2$ for all $n \geq 1$ and $\cap_n (q_{2n+1}, q_{2n+2}) = \{q_0\}$. In particular, $\lim q_n = q_0$. Let $p_1, p_2 \in (0, 1/2)$ and consider $\tau_1 = \frac{1}{2}\tau(p_1, .) + \frac{1}{2}\tau(p_2, .)$. By Lemma 4.6 we can construct a Bernoulli product μ_1 of spectrum τ_1. Then, Lemma 4.5 implies the existence of a convex combination τ_2 of $\tau(p_i, .)$'s functions, such that

$$\tau_1(q_i) = \tau_2(q_i) \text{ , for } i = 1, 2 \text{ and } \tau_1'(q_i) \neq \tau_2'(q_i).$$

We can define a measure μ_2 of spectrum τ_2. Using μ_1 and μ_2, we can construct a measure ν_2 of spectrum $\rho_2 = \max\{\tau_1, \tau_2\}$. To do that, we take a sequence of integers $(\ell_k)_k$ such that $\frac{\ell_{k+1}}{\sum_1^k \ell_i} \to \infty$. On dyadique intervals of length between $2^{-\ell_{2k}}$ and $2^{-\ell_{2k+1}}$ we apply the weight distribution of μ_1 and on dyadique intervalles of length between $2^{-\ell_{2k+1}}$ and $2^{-\ell_{2k+2}}$ we apply the weight distribution of μ_2, where $k \in \mathbb{N}$. It is easy to verify that the resulting inhomogeneous measure ν_2 has spectrum $\rho_2 = \max\{\tau_1, \tau_2\}$. The spectrum of ν_2 is not differentiable at q_1 and q_2.

We proceed by induction to construct a measure ν_n which has a non differentiable spectrum for points $q_1, \cdots, q_{2n-2}$. Suppose the measures $\nu_1 = \mu_1$, μ_2, ν_2,..., μ_n, ν_n constructed and denote by $\rho_n = \max\{\tau_1, ...\tau_n\}$ where τ_i is the spectrum of the measure μ_i, $i \in \{1..., n\}$. Let us construct μ_{n+1} and ν_{n+1}.

One of the following two cases hold :

Case 1 Lemma 4.5 provides a function τ_{n+1} satisfying :

1. $\tau_{n+1}(q_{2n-i}) = \rho_n(q_{2n-i})$ for $i = 0, 1$ and $\tau_{n+1}(q) \neq \rho_n(q)$ if $q \notin \{q_{2n-1}, q_{2n}\}$,
2. $\tau'_{n+1}(q_{2n-1}) > \rho'_n(q_{2n-1})$, $\tau'_{n+1}(q_{2n}) < \rho'_n(q_{2n})$

Therefore we have $\tau_{n+1} > \rho_n$ on (q_{2n-1}, q_{2n}) and $\tau_{n+1} < \rho_n$ on $(1, \infty) \setminus [q_{2n-1}, q_{2n}]$. Let μ_{n+1} be the inhomogeneous Bernoulli measure of spectrum τ_{n+1}. To define the measure ν_{n+1} we use the previous procedure convenably adapted : Take $(\ell_k)_k$ a sequence of integers such that $\dfrac{\ell_{k+1}}{\sum_1^k \ell_i} \to \infty$. On dyadique intervals of length between $2^{-\ell_{(n+1)k+i}}$ and $2^{-\ell_{(n+1)k+i+1}}$ apply the weight distribution of μ_i, where $i = 1, ..., n+1$ and $k \in \mathbb{N}$. It is easy to verify that the resulting inhomogeneous measure ν_{n+1} has spectrum $\rho_{n+1} = \max\{\tau_1, ...\tau_{n+1}\}$ on $(1, \infty)$. Remark that this spectrum equals τ_{n+1} on $[q_{2n-1}, q_{2n}]$ and $\rho_n = \max\{\tau_1, ...\tau_n\}$ elsewhere on $[1, \infty)$. Clearly, in this case, the function $\rho_{n+1} = \max(\tau, \tau_{n+1})$ is not differentiable at $q_1, \cdots, q_{2n}$.

Case 2 Lemma 4.5 provides for all choices of $p_5 > p_2$ a function τ_{n+1} satisfying :

1. $\tau_{n+1}(q_{2n-i}) = \rho_n(q_{2n-i})$, for $i = 0, 1$ and $\tau_{n+1}(q) \neq \rho_n(q)$ if $q \notin \{q_{2n-1}, q_{2n}\}$,
2. $\tau'_{n+1}(q_{2n-1}) < \rho'_n(q_{2n-1})$, $\tau'_{n+1}(q_{2n}) > \rho'_n(q_{2n})$

In this case,

$$\tau_{n+1} < \rho_n \text{ on } (q_{2n-1}, q_{2n}) \text{ and } \tau_{n+1} > \rho_n \text{ on } (q_{2n-3}, q_{2n-1}) \cup (q_{2n}, q_{2n-2}).$$

The function $\tilde{\rho}_{n+1} = \max(\rho_n, \tau_{n+1})$ is not differentiable at q_{2n-1} and q_{2n} but we lose the phase transitions q_{2n-3}, q_{2n-2} and we don't know what happens for the other phase transitions $q_1, \cdots, q_{2n-4}$.
To avoid this problem we use remark 4.4. From this, when p_5 converges to p_2, τ_{n+1} converges to the convex combination T of $\tau(p_i, .)$ functions which is equal to ρ_n on (q_{2n-3}, q_{2n-2}). Since T differs from ρ_n on (q_{2n-5}, q_{2n-3}) and (q_{2n-2}, q_{2n-4}), we can choose p_5 sufficiently close to p_2 such that

$$\tau_{n+1}\left(\frac{q_{2n-3} + q_{2n-5}}{2}\right) < \rho_n\left(\frac{q_{2n-3} + q_{2n-5}}{2}\right)$$

and

$$\tau_{n+1}\left(\frac{q_{2n-2} + q_{2n-4}}{2}\right) < \rho_n\left(\frac{q_{2n-2} + q_{2n-4}}{2}\right).$$

We deduce that there exist $q_{2n-5} < q' < q_{2n-3}$ and $q_{2n-2} < q'' < q_{2n-4}$ such that $\tau_{n+1} = \rho_n$ at q' and q'' and $\tau_{n+1} < \rho_n$ on (q_1, q') and (q'', q_2).

The modified family of $\tilde{q}_i$'s defined by

$$\tilde{q}_i = \begin{cases} q_i & \text{if } i \notin \{2n-3, 2n-2\} \\ q' & \text{if } i = 2n-3 \\ q'' & \text{if } i = 2n-2 \end{cases}$$

have the same properties as the initial q_i's. Moreover, $\rho_{n+1} = \max(\rho_n, \tau_{n+1})$ is not differentiable at points $q_1, \cdots, q_{2n}$. We proceed as above for the construction of the measures μ_{n+1} and ν_{n+1} which have spectra τ_{n+1} and ρ_{n+1} respectively.

To end the proof we use Cantor's diagonal argument : take $(\ell_k)_k$ as before and define the measure ν by applying the weight distribution of ν_k on dyadique intervals of length between $2^{-\ell_k}$ and $2^{-\ell_{k+1}}$. The spectrum of the measure ν equals then $\tau = \sup_{n\in\mathbb{N}} \rho_n = \sup_{n\in\mathbb{N}} \tau_n$. By construction, the set of non-derivability points of the function τ is infinite and has q_0 as accumulation point. •

Remark 4.8 The second case of the proof of Theorem 4.7 seems to be inexistent (in our numerical simulations) but we have not been able to prove that only the first case arises.

Let us now give some hints concerning the proof of Theorem 1.3.

Fix $(q_n)_n$ a sequence of real numbers, dense in $[1, \infty)$ and nested in the sense that $q_{2n+1} < q_{2n+2}$ and $\{q_1, ..., q_{2n}\} \cap [q_{2n+1} - \frac{1}{2^n}, q_{2n+2} + \frac{1}{2^n}] = \emptyset$ for all $n \geq 0$. We can then follow the proof of Theorem 4.7 until case 2, the first case being carried out exactly in the same way.

The second case has to be slightly modified. The technical, but not difficult, part is to ensure that the modified q_i's still form a dense subset of $[1, \infty)$ and that the difference of the left and right derivative at the q_i's does not go to 0. To do that we take p_5 sufficiently close to p_2 (in the construction of τ_{n+1}) to have :

– $|q_i - \tilde{q}_i| < \frac{1}{2^n} \inf_{1\leq j<j'\leq 2n+2} |q_j - q_{j'}|$
– $|\delta_n(q_i) - \delta_{n+1}(\tilde{q}_i)| < \frac{1}{2^n}\delta_n(q_i)$,

where $\delta_n(q_i)$ stands for the difference between the right and left derivative at q_i of $\sup_{1\leq k\leq n} \tau_k$. The proof of Theorem 1.3 is then completed in the same way as above. •

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Publications de la partie 3

Harmonic measure of some sets of Cantor type

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Abstract. We show that for some compact sets $\mathbb{K} \subset \mathbb{R}^2$ of Cantor type, the harmonic measure is supported by a set whose Hausdorff dimension is strictly smaller than the dimension of $\mathbb{K}$.

Resumé. On démontre que pour certains compacts $\mathbb{K}$ du plan (de type ensembles de Cantor), la mesure harmonique du complémentaire est portée par un sous-ensemble de $\mathbb{K}$ de dimension strictement inférieure à celle de $\mathbb{K}$.

1. Introduction.

In [M-V] Makarov and Volberg show that the Hausdorff dimension of the harmonic measure of the complement of a particular kind of Cantor sets is strictly smaller than the dimension of the set, i.e. that there exists a subset of the Cantor set of full harmonic measure but of strictly smaller dimension. Furthermore, Volberg in [V1], [V2] has extended this result to cover a large class of Cantor sets on the real axis. In [C], Carleson has shown that technics and tools of the Ergodic theory could be used to study the harmonic measure of "classical" Cantor sets and [M-V], [V1], [V2], [Z] are some of the results that strongly rely on this idea. Carleson prove that the dimension of the harmonic measure of these Cantor sets in the plane is always strictly smaller than 1. A similar but more general result is proved in [J-W], under the assumption of the "capacity density condition".

The purpose of this work is to prove the inequality between the dimension of the harmonic measure and the dimension of the set for a larger class of Cantor sets in the plane (or the space) without using Ergodic theory. We have been motivated by an idea of Bourgain which has appeared in [B], and we are also making use of some lemmas and results of the works mentionned above. For more information regarding the Hausdorff dimension and measure of Cantor type sets [Be], [F] are some of the possible references.

This paper is organized in five sections: first we present the main theorem and we introduce some notation. In the second section we prove a number of lemmas and in the third we prove the theorem. Two examples are given in the fourth section and finally we make some remarks and we investigate the possibilities of the method.

Let $\{a_j\}$ be a sequence of real numbers such that there exist two constants $\underline{A}, \overline{A}$, $0 < \underline{A} \leq \overline{A} < \dfrac{1}{2}$ with $\overline{A} \geq a_i \geq \underline{A}$, $\forall i \in \mathbb{N}$. We construct a Cantor set $\mathbb{K}$ in the following way: we replace the square $[0,1]^2$ with four equal squares of side-length a_1 situated in the four corners, and each one of them with four new ones of side-length a_1a_2 and so on, see Fig.1. We denote $\tilde{I}^n_{i_1\dots i_n}$, where $i_j \in \{1,2,3,4\}$ for $1 \le j \le n$, the 4^n squares of the nth generation constructed in this way with the enumeration shown in the figure and the usual condition that $\tilde{I}^n_{i_1\dots i_n}$ is the "father" of the sets $\tilde{I}^{n+1}_{i_1\dots i_n i_{n+1}}$, $i_{n+1} \in \{1,2,3,4\}$. It is clear that $\overline{A} \ge \dfrac{\mathrm{diam}\tilde{I}^{n+1}_{i_1\dots i_n i}}{\mathrm{diam}\tilde{I}^n_{i_1\dots i_n}} = a_{n+1} \ge \underline{A}$, $i = 1, ..., 4$. We will denote $I^n_{i_1\dots i_n}$ the intersection of $\tilde{I}^n_{i_1\dots i_n}$ with $I\!\!K$. We will say that a set $F \subset I\!\!R^2 \setminus I\!\!K$ is of full harmonic measure for the domaine $I\!\!R^2 \setminus I\!\!K$ if $\omega(F) = 1$, where ω is the harmonic measure of $I\!\!R^2 \setminus I\!\!K$ (see Notation 1.1).

Theorem 1.0. *For the Cantor set $I\!\!K$ there exists a subset F of $I\!\!K$ of full harmonic measure such that $dim(F) < dim(I\!\!K)$.*

Notation 1.1.

If there is a constant c independent of the parameters α, β such that $\frac{1}{c}\alpha \le \beta \le c\alpha$ we will write $\alpha \sim \beta$ (in what follows the symbols c and C will be used to denote the constants). Suppose that Ω is a domain in $I\!\!R^2$ and that $F \subset \partial\Omega$. For $x \in \Omega$ we denote $\omega(x, F, \Omega)$ the harmonic measure of F in Ω evaluated at x. We denote $\omega(F, \Omega)$ the harmonic measure of F in Ω evaluated at infinity. If $\Omega = I\!\!R^2 \setminus I\!\!K$ we will write $\omega(F)$ instead of $\omega(F, \Omega)$. For a square F we note $l(F)$ it's side-length and finally h_ρ is the ρ-dimensional Hausdorff measure.

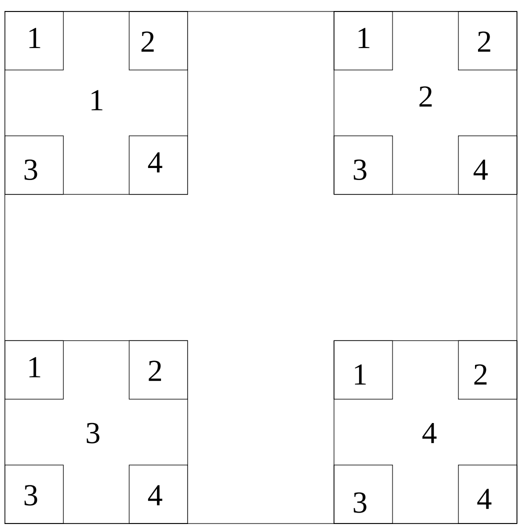

Fig. 1

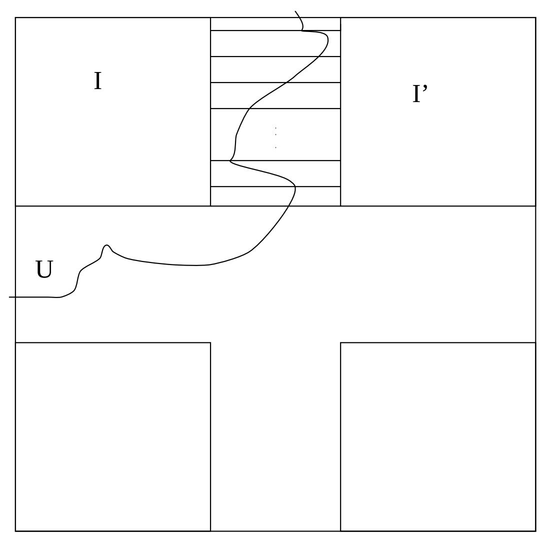

Fig. 2

2. Preparatory lemmas.

The proof of the theorem will be based on a number of lemmas some of which are already well known.

We first remark that there exists a constant $c_0 > 1$ depending only on $\underline{A}$,$\overline{A}$ such that, if

$c_0\tilde{I}^n_{i_1...i_n}$ denotes the square of side-length c_0-times the side-length of $\tilde{I}^n_{i_1...i_n}$ with the same center, then $c_0\tilde{I}^n_{i_1...i_n} \cap I\!\!K = I^n_{i_1...i_n}$. For this c_0 (not depending on $i_1...i_n$) we have the following classical lemma:

Lemma 2.1. *There exists a $\delta > 0$ not depending neither on n, nor on the choice of $i_1...i_n$ such that for all $x \in \frac{1+c_0}{2}\tilde{I}^n_{i_1...i_n}$*

$$\omega(x, I^n_{i_1...i_n}, I\!\!R^2 \setminus I\!\!K) > \delta. \tag{1}$$

Remark 2.2.

If δ_n is the side-length of the square $I^n_{i_1...i_n}$ the Green's function G of the square $c_0\tilde{I}^n_{i_1...i_n}$ satisfies

$$\frac{1}{2\pi}\log\frac{C^{-1}\delta_n}{\mid x-y\mid} \le G(x,y) \le \frac{1}{2\pi}\log\frac{C\delta_n}{\mid x-y\mid}$$

for $x, y \in \frac{1+c_0}{2}\tilde{I}^n_{i_1...i_n}$ where the constant C depends only on c_0. Furthermore we have

$$\begin{aligned}\omega(x, I^n_{i_1...i_n}, c_0\tilde{I}^n_{i_1...i_n} \setminus I\!\!K) &= {}^{c_0\tilde{I}^n_{i_1...i_n}}\mathbf{R}_1^{I^n_{i_1...i_n}}(x)\\ &= G\mu(x)\end{aligned}$$

where ${}^{\Omega}\mathbf{R}^F_1$ the capacitary potential of the set F in the domain Ω and μ is the capacitary measure of $I^n_{i_1...i_n}$ in $c_0\tilde{I}^n_{i_1...i_n}$, $\parallel\mu\parallel = \mathrm{cap}_{c_0\tilde{I}^n_{i_1...i_n}}(I^n_{i_1...i_n})$.

Proof of lemma 2.1.

We first show that there exists a constant $c_1 > 0$ such that for $x \in \frac{1+c_0}{2}\tilde{I}^n_{i_1...i_n}$

$$\omega(x, I^n_{i_1...i_n}, c_0\tilde{I}^n_{i_1...i_n} \setminus I\!\!K) > c_1. \tag{2}$$

If μ is the probability measure on $I\!\!K$ charging every square of the nth generation with mass 4^{-n}, let $\mu_n = 4^n\mu\big|_{I^n_{i_1...i_n}}$ the restriction of the measure μ on the square $I^n_{i_1...i_n}$ renormalized.

Let us calculate the potential of μ_n for $y \in I^n_{i_1...i_n}$

$$\begin{aligned}G\mu_n(y) &\le 4^n\sum_{\kappa>n} 3\cdot 4^{-\kappa}\log\Big(\frac{C\prod_{i=1}^{n}a_i}{\prod_{i=1}^{\kappa}a_i}\Big)\\ &= 3\cdot 4^n\sum_{\kappa>n}4^{-\kappa}\log\big(C\prod_{i=n+1}^{\kappa}a_i^{-1}\big)\\ &= 3\sum_{\kappa=1}^{\infty}4^{-\kappa}\log\big(C\prod_{i=1}^{\kappa}a_{i+n}^{-1}\big)\\ &\le \tilde{C}(\underline{A}, \overline{A}) < \infty.\end{aligned}$$

The same reasoning provides a constant $c_2 > 0$ such that

$$\frac{1}{c_2} \le G\mu_n(y) \le c_2 \quad , \quad \forall y \in I^n_{i_1 \dots i_n}.$$

By the maximum principle we get

$$\omega(x, I^n_{i_1\dots i_n}, c_0\tilde{I}^n_{i_1\dots i_n} \setminus I\!\!K) \sim G\mu_n(x) \quad , \quad \text{for} \quad x \in c_0 \tilde{I}^n_{i_1\dots i_n}. \tag{3}$$

We can easily see that

$$G\mu_n(x) \ge c_3 \quad , \quad \text{for} \quad x \in \partial\Big\{\frac{1+c_0}{2}\tilde{I}^n_{i_1\dots i_n}\Big\}.$$

On the other hand the harmonic measure is non-decreasing as a function of the domain, hence

$$\omega(x, I^n_{i_1\dots i_n}, c_0\tilde{I}^n_{i_1\dots i_n} \setminus I\!\!K) \le \omega(x, I^n_{i_1\dots i_n}, I\!\!R^2 \setminus I\!\!K)$$

and the lemma is proved .

Lemma 2.3. *There exists a $\delta > 0$ not depending on n, such that for the squares of the nth generation $I^n_{1\dots 11}$ and $I^n_{1\dots 14}$ we have*

$$\omega(I^n_{1\dots 11}) > (1+\delta)\omega(I^n_{1\dots 14}). \tag{4}$$

Remark 2.4.

This lemma has been proved in [M-V] in the case of standard planar Cantor sets. The proof given below is similar.

We will make repeated use of the following well known formula (see for instance [Br]): If $\Omega \subset \tilde{\Omega}$ are two domains and if $F \subset \partial\Omega \cap \partial\tilde{\Omega}$ then the harmonic measures of the domains, ω and $\tilde{\omega}$ are associated in the following way

$$\omega(x, F) = \tilde{\omega}(x, F) - \int_{\partial\Omega \cap \tilde{\Omega}} \tilde{\omega}(y)\omega(x, dy).$$

Proof of lemma 2.3.

To begin with, let us point out that the symmetry of the set implies

$$\omega(I^n_{1\dots 11}, I\!\!R^2 \setminus I^{n-1}_{1\dots 1}) = \omega(I^n_{1\dots 14}, I\!\!R^2 \setminus I^{n-1}_{1\dots 1}). \tag{5}$$

For the same reason if x lies on the $I^{n-1}_{1\dots 1}$ square's diagonal separating $I^n_{1\dots 11}$ and $I^n_{1\dots 14}$ we have

$$\omega(x, I^n_{1\dots 11}, I\!\!R^2 \setminus I^{n-1}_{1\dots 1}) = \omega(x, I^n_{1\dots 14}, I\!\!R^2 \setminus I^{n-1}_{1\dots 1}). \tag{6}$$

Let $\mathbb{H}^-$ be the half-plane limited by the line containing this diagonal, such that the square $I^n_{1\dots 14}$ is contained in $\mathbb{H}^-$. Using (2) , the monotony of the harmonic measure, and Harnack's inequalities, one can verify the existence of a constant $c_4 > 0$ such that

$$\omega(x, I^n_{1\dots 14}, \mathbb{H}^- \setminus I^{n-1}_{1\dots 1}) \geq c_4 \;\; , \;\; \forall x \in I^{n-1}_{1\dots 4}. \tag{7}$$

By the maximum principle and (6) we obtain

$$\omega(x, I^n_{1\dots 14}, \mathbb{H}^- \setminus I^{n-1}_{1\dots 1}) = \omega(x, I^n_{1\dots 14}, \mathbb{R}^2 \setminus I^{n-1}_{1\dots 1}) - \omega(x, I^n_{1\dots 11}, \mathbb{R}^2 \setminus I^{n-1}_{1\dots 1}) \;\; , \;\; \forall x \in \mathbb{H}^-.$$

Combining with (7)

$$\omega(x, I^n_{1\dots 14}, \mathbb{R}^2 \setminus I^{n-1}_{1\dots 1}) - \omega(x, I^n_{1\dots 11}, \mathbb{R}^2 \setminus I^{n-1}_{1\dots 1}) \geq c_4 \;\; , \;\; \forall x \in I^{n-1}_{1\dots 4} \tag{8}$$

and (5),(8) imply that

$$\begin{aligned} &\omega(I^n_{1\dots 11}, \mathbb{R}^2 \setminus \mathbb{K}) - \omega(I^n_{1\dots 14}, \mathbb{R}^2 \setminus \mathbb{K}) = \\ &= \int_{K \setminus I^{n-1}_{1\dots 1}} \Big(\omega(y, I^n_{1\dots 14}, \mathbb{R}^2 \setminus I^{n-1}_{1\dots 1}) - \omega(y, I^n_{1\dots 11}, \mathbb{R}^2 \setminus I^{n-1}_{1\dots 1}) \Big) \omega(dy, \mathbb{R}^2 \setminus \mathbb{K}) \\ &\geq c_4 \omega(I^{n-1}_{1\dots 4}, \mathbb{R}^2 \setminus \mathbb{K}). \end{aligned} \tag{9}$$

A final use of Harnack's principle and of (7) gives a constant $c_5 > 0$ not depending on n, verifying

$$\omega(I^{n-1}_{1\dots 4}) \geq c_5 \; \omega(I^n_{1\dots 14}).$$

Hence, (9) turns to

$$\omega(I^n_{1\dots 11}) - \omega(I^n_{1\dots 14}) \geq c_4 \; c_5 \; \omega(I^n_{1\dots 14})$$

and the lemma is proved.

Lemma 2.5. ([C], [M-V]) *Let Ω be a domain containing ∞ and let $A_1 \subset B_1 \subset A_2 \subset B_2 \subset \dots \subset A_n \subset B_n \subset \Omega$ be conformal discs such that the annuli $B_i \setminus A_i$ are contained in Ω, for $1 \leq i \leq n$. If the modules of the annuli are uniformly bounded away from zero and if $\infty \in \Omega \setminus B_n$ then, for all pairs of positive harmonic functions u, v vanishing on $\partial\Omega \setminus A_1$ and for all $x \in \Omega \setminus B_n$ we have*

$$\left| \frac{u(x)}{v(x)} : \frac{u(\infty)}{v(\infty)} - 1 \right| \leq K q^n \tag{10}$$

where $q < 1$ and K are two constants that depend only on the lower bound of the modules of the annuli.

Lemma 2.6. *There exists a $N_0 = N_0(\delta, \underline{A}, \overline{A})$ large enough such that for all $n \in \mathbb{N}$ and all squares $I^n_{i_1\dots i_n}$,*

$$\omega(I^{n+N_0}_{i_1\dots i_n 11\dots 1}) > (1 + \frac{\delta}{2}) \omega(I^{n+N_0}_{i_1\dots i_n 11\dots 4}), \tag{11}$$

where δ is the positive constant defined in lemma 2.3.

Proof of Lemma 2.6.

Let us show first the following estimation for $x \in \dfrac{1+c_0}{2}\tilde{I}^n_{i_1\ldots i_n}$:

$$\omega(x, I^{n+N_0}_{i_1\ldots i_n 11\ldots 1}, c_0\tilde{I}^n_{i_1\ldots i_n} \setminus I^n_{i_1\ldots i_n}) \sim \omega(x, I^{n+N_0}_{i_1\ldots i_n 11\ldots 1}, \mathbb{R}^2 \setminus \mathbb{K}). \tag{12a}$$

For $N \in \mathbb{N}$ we choose x such that

$$\omega(x, I^{n+N}_{i_1\ldots i_n 11\ldots 1}, \mathbb{R}^2 \setminus \mathbb{K}) = \sup\Big\{\omega(y, I^{n+N}_{i_1\ldots i_n 11\ldots 1}, \mathbb{R}^2 \setminus \mathbb{K})\,;\ y \in \partial\{\frac{1+c_0}{2}\tilde{I}^n_{i_1\ldots i_n}\}\Big\}.$$

Then,

$$\begin{aligned}
&\omega(x, I^{n+N}_{i_1\ldots i_n 11\ldots 1}, \mathbb{R}^2 \setminus \mathbb{K}) \geq \omega(x, I^{n+N}_{i_1\ldots i_n 11\ldots 1}, c_0\tilde{I}^n_{i_1\ldots i_n} \setminus I^n_{i_1\ldots i_n})\\
&\geq \omega(x, I^{n+N}_{i_1\ldots i_n 11\ldots 1}, \mathbb{R}^2 \setminus \mathbb{K})\\
&- \int_{\partial\{c_0\tilde{I}^n_{i_1\ldots i_n}\}} \omega(y, I^{n+N}_{i_1\ldots i_n 11\ldots 1}, \mathbb{R}^2 \setminus \mathbb{K})\, \omega(x, dy, c_0\tilde{I}^n_{i_1\ldots i_n} \setminus I^n_{i_1\ldots i_n})\\
&\geq \omega(x, I^{n+N}_{i_1\ldots i_n 11\ldots 1}, \mathbb{R}^2 \setminus \mathbb{K})\\
&- \Big(1 - \omega(x, I^n_{i_1\ldots i_n}, c_0\tilde{I}^n_{i_1\ldots i_n} \setminus I^n_{i_1\ldots i_n})\Big)\omega(x, I^{n+N}_{i_1\ldots i_n 11\ldots 1}, \mathbb{R}^2 \setminus \mathbb{K})\\
&\geq c_1\omega(x, I^{n+N}_{i_1\ldots i_n 11\ldots 1}, \mathbb{R}^2 \setminus \mathbb{K})
\end{aligned}$$

because of (2).

Then (12a) follows on using again Harnack's inequalities.

We have, of course the same estimate for $I^{n+N_0}_{i_1\ldots i_n 11\ldots 4}$:

$$\omega(x, I^{n+N_0}_{i_1\ldots i_n 11\ldots 4}, c_0\tilde{I}^n_{i_1\ldots i_n} \setminus I^n_{i_1\ldots i_n}) \sim \omega(x, I^{n+N_0}_{i_1\ldots i_n 11\ldots 4}, \mathbb{R}^2 \setminus \mathbb{K}). \tag{12b}$$

To simplify the notation in what follows we will write $\omega_1(x)\Big(\tilde{\omega}_1(x)\Big)$ and $\omega_4(x)\Big(\tilde{\omega}_4(x)\Big)$ instead of

$$\omega(x, I^{n+N_0}_{i_1\ldots i_n 11\ldots 1}, \mathbb{R}^2 \setminus \mathbb{K})\Big(\omega(x, I^{n+N_0}_{i_1\ldots i_n 11\ldots 1}, c_0\tilde{I}^n_{i_1\ldots i_n} \setminus I^n_{i_1\ldots i_n})\Big)$$

and

$$\omega(x, I^{n+N_0}_{i_1\ldots i_n 11\ldots 4}, \mathbb{R}^2 \setminus \mathbb{K})\Big(\omega(x, I^{n+N_0}_{i_1\ldots i_n 11\ldots 4}, c_0\tilde{I}^n_{i_1\ldots i_n} \setminus I^n_{i_1\ldots i_n})\Big)$$

respectively.

By the relation (10) in lemma 2.5 we get that for $z \in \partial\Big\{\dfrac{1+c_0}{2}\tilde{I}^n_{i_1\ldots i_n}\Big\}$

$$\frac{\omega_1(\infty)}{\omega_4(\infty)} \sim_{q^{N_0}} \frac{\omega_1(z)}{\omega_4(z)} \sim_{q^{N_0}} \frac{\omega_1(y)}{\omega_4(y)}\ ,\quad \forall y \notin \mathbb{K} \cup c_0\tilde{I}^n_{i_1\ldots i_n}. \tag{13}$$

From (13) it follows that

$$\begin{aligned}
&\left|\frac{\tilde\omega_1(z)}{\tilde\omega_4(z)} : \frac{\omega_1(\infty)}{\omega_4(\infty)} - 1\right| \sim \left|\frac{\tilde\omega_1(z)}{\tilde\omega_4(z)} : \frac{\omega_1(z)}{\omega_4(z)} - 1\right| = \frac{\omega_4(z)}{\tilde\omega_4(z)}\left|\frac{\tilde\omega_1(z)}{\omega_1(z)} - \frac{\tilde\omega_4(z)}{\omega_4(z)} - \right| \\
&= \frac{\omega_4(z)}{\tilde\omega_4(z)} \int_{\partial\left\{c_0 \tilde I^n_{i_1\dots i_n}\right\}} \left|\frac{\omega_4(y)}{\omega_4(z)} - \frac{\omega_1(y)}{\omega_1(z)}\right| \omega(z, dy, c_0 \tilde I^n_{i_1\dots i_n} \setminus I^n_{i_1\dots i_n}) \leq \\
&\leq \frac{1}{c_1} \int_{\partial\left\{c_0 \tilde I^n_{i_1\dots i_n}\right\}} \frac{\omega_4(y)}{\omega_4(z)} \left|\frac{\omega_1(y)}{\omega_1(z)} : \frac{\omega_4(y)}{\omega_4(z)} - 1\right| \omega(z, dy, c_0 \tilde I^n_{i_1\dots i_n} \setminus I^n_{i_1\dots i_n}) \\
&\leq C q^{N_0}.
\end{aligned} \tag{14}$$

If we take $i_1 = ... = i_n = 1$, lemma 2.3 implies $\dfrac{\omega_1(\infty)}{\omega_4(\infty)} > 1 + \delta$. Then (14) shows that there exists a N_0 large enough such that $\dfrac{\tilde\omega_1(z)}{\tilde\omega_4(z)} > 1 + \dfrac{3}{4}\delta$. On the other hand, $\dfrac{\tilde\omega_1(z)}{\tilde\omega_4(z)}$ does not depend on the choice of $i_1, ..., i_n$. It follows that $\dfrac{\omega_1(\infty)}{\omega_4(\infty)} > 1 + \dfrac{\delta}{2}$ for all the possible choices of $i_1, ..., i_n$.

Lemma 2.7. *There exists a $N_1 \in \mathbb{N}$ independent of n and of $i_1 ... i_n$ such that for all the squares $I^n_{i_1\dots i_n}$ there is a square $J_m = I^{n+N_1}_{i_1\dots i_n\dots i_{n+N_1}} \subset I^n_{i_1\dots i_n}$ of the $(n+N_1)$th generation such that*

$$\omega\big(J_m\big) < \frac{1}{4} \frac{\omega\big(I^n_{i_1\dots i_n}\big)}{4^{N_1}}.$$

(In fact, $\dfrac{1}{4}$ could be replaced by any constant $\epsilon > 0$.)

Proof of Lemma 2.7.

Choose a square $I^n_{i_1\dots i_n}$. According to the preceeding lemma there exists an $\alpha < 1$ independent of the choice of $i_1, ..., i_n$ and a $J_1 = I^{n+N_0}_{i_1\dots i_n \dots i_{n+N_0}}$ such that

$$\omega\big(J_1\big) < \alpha \frac{\omega\big(I^n_{i_1\dots i_n}\big)}{4^{N_0}}.$$

Similarly there exists a $J_2 = I^{n+2N_0}_{i_1\dots i_n\dots i_{n+2N_0}} \subset J_1$ with

$$\omega\big(J_2\big) < \alpha \frac{\omega\big(J_1\big)}{4^{N_0}} < \alpha^2 \frac{\omega\big(I^n_{i_1\dots i_n}\big)}{4^{2N_0}}$$

and after k steps, we obtain a square J_k verifying

$$\omega\big(J_k\big) < \alpha^k \frac{\omega\big(I^n_{i_1\dots i_n}\big)}{4^{kN_0}}.$$

To finish the proof take $k = m$ so that $\alpha^k < 1/4$ and let $N_1 = kN_0$.

3. Proof of Theorem 1.0.

The theory of martingales provides a well known technique to prove the inequality between the dimensions of the two measures by using lemma 2.7. If we note S_n the sum of $I_{i_1} + ... + I_{i_1 ... i_n}$ one may show that for ϵ and t appropriately chosen the sequence $exp(\epsilon n - tS_n)$ is a positive supermartingale for the harmonic measure and apply the large deviations law to conclude. However we propose here a different path which does not involve Probabilistic tools and is inspired by [B].

We introduce some more notation. For $n \in \mathbb{N}$ we will denote $\mathcal{E}_n$ the collection of squares $\left\{ I^n_{i_1 ... i_n}; i_j = 1, ..., 4, j = 1, ..., n \right\}$ and for $I \in \mathcal{E}_n$, $\mathcal{E}_{n+s}(I)$ will represent those squares $J \in \mathcal{E}_{n+s}$ that are contained in I.

It can be shown (see for instance lemma 2 of [Be]) that if ρ is the Hausdorff dimension of $\mathbb{K}$, then

$$\rho = \sup\{s > 0;\ \liminf_{n \to \infty} 4^n \prod_{i=1}^{n} a_i^s = \infty\} = \inf\{s > 0;\ \liminf_{n \to \infty} 4^n \prod_{i=1}^{n} a_i^s = 0\}.$$

simply because in order to obtain the Hausdorff dimension of the Cantor set $\mathbb{K}$ it suffices to consider coverings of $\mathbb{K}$ with the squares of construction $I^n_{i_1 ... i_n}$. However, the ρ-Hausdorff measure of the Cantor sets considered here could be infinite.

It easily follows that for $\epsilon > 0$ there exists a strictly increasing sequence of integers $\{n_j\}_{j=1}^{\infty}$ such that

$$4^{n_j} \prod_{i=1}^{n_j} a_i^{\rho+\epsilon} > 4^{n_{j+1}} \prod_{i=1}^{n_{j+1}} a_i^{\rho+\epsilon}. \tag{15}$$

We will also assume that $n_{j+1} - n_j > 2N_1$.

Lemma 2.8. *There exists a $\beta < 1$ such that for $\epsilon > 0$ and $I \in \mathcal{E}_{n_j}$ the following inequality holds*

$$\sum_{J \in \mathcal{E}_{n_{j+1}}(I)} \omega(J)^{\frac{1}{2}} l(J)^{\frac{\rho+\epsilon}{2}} \leq \beta^{n_{j+1} - n_j}\ \omega(I)^{\frac{1}{2}}\ l(I)^{\frac{\rho+\epsilon}{2}}. \tag{16}$$

where n_j is the sequence corresponding to ϵ given by (15).

Proof of Lemma 2.8.

Let us start by showing that there is a $\tilde{\beta}$ such that $I \in \mathcal{E}_n$

$$\sum_{J \in \mathcal{E}_{n+N_1}(I)} \omega(J)^{\frac{1}{2}} \left(\frac{1}{4}\right)^{\frac{n+N_1}{2}} \leq \tilde{\beta} \omega(I)^{\frac{1}{2}} \left(\frac{1}{4}\right)^{\frac{n}{2}}. \tag{17}$$

Take $J_m \in \mathcal{E}_{n+N_1}(I)$ to be the square provided by lemma 2.7, i.e. a square such that $\omega\big(J_m\big) < \dfrac{1}{4}\dfrac{\omega\big(I\big)}{4^{N_1}}$. We have

$$\omega\big(J_m\big)^{\frac{1}{2}}\big(\frac{1}{4}\big)^{\frac{n+N_1}{2}} \leq \frac{1}{2}\frac{1}{4^{N_1}}\big(\frac{1}{4}\big)^{\frac{n}{2}}\omega\big(I\big)^{\frac{1}{2}}$$

$$\sum_{J\in\mathcal{E}_{n+N_1}(I),J\neq J_m}\omega(J)^{\frac{1}{2}}\big(\frac{1}{4}\big)^{\frac{n+N_1}{2}} \leq \omega\big(I\big)^{\frac{1}{2}}\big(4^{N_1}-1\big)^{\frac{1}{2}}\big(\frac{1}{4}\big)^{\frac{n+N_1}{2}}$$

by the Cauchy-Schwarz inequalities. Summing up we get

$$\sum_{J\in\mathcal{E}_{n+N_1}(I)}\omega(J)^{\frac{1}{2}}\big(\frac{1}{4}\big)^{\frac{n+N_1}{2}} \leq \omega\big(I\big)^{\frac{1}{2}}\big(\frac{1}{4}\big)^{\frac{n}{2}}\left(\frac{1}{2}\frac{1}{4^{N_1}}+\Big(\frac{4^{N_1}-1}{4^{N_1}}\Big)^{\frac{1}{2}}\right) \tag{18}$$

and we may let $\tilde{\beta} = \dfrac{1}{2}\dfrac{1}{4^{N_1}}+\Big(\dfrac{4^{N_1}-1}{4^{N_1}}\Big)^{\frac{1}{2}} < 1.$

Choose $\epsilon > 0$ and let $\{n_j\}$ be a corresponding sequence given by (15). Then by (17)

$$\sum_{J\in\mathcal{E}_{n_{j+1}}(I)}\omega(J)^{\frac{1}{2}}l(J)^{\frac{\rho+\epsilon}{2}} \leq 4^{\frac{n_j}{2}}\tilde{\beta}\left(\prod_{i=1}^{n_j}a_i^{\frac{\rho+\epsilon}{2}}\right)\sum_{J\in\mathcal{E}_{n_{j+1}-N_1}(I)}\omega(J)^{\frac{1}{2}}\left(\frac{1}{4^{n_{j+1}-N_1}}\right)^{\frac{1}{2}}.$$

We repeat the procedure and we apply the Cauchy-Schwarz inequalities. We then get

$$\sum_{J\in\mathcal{E}_{n_{j+1}}(I)}\omega(J)^{\frac{1}{2}}l(J)^{\frac{\rho+\epsilon}{2}} \leq 4^{\frac{n_j}{2}}\ \tilde{\beta}^{\frac{n_{j+1}-n_j}{2N_1}}\left(\prod_{i=1}^{n_j}a_i^{\frac{\rho+\epsilon}{2}}\right)\left(\frac{\omega\big(I\big)}{4^{n_j}}\right)^{\frac{1}{2}}.$$

The existence of β is obvious now. For instance, one may take $\beta = \tilde{\beta}^{\frac{1}{2N_1}}$.

Proof of Theorem 1.0.

Let $\mathcal{L}_j = \Big\{J\in\mathcal{E}_{n_j} \mid \omega(J) > l(J)^{\rho-\epsilon}\Big\}$ and $\mathcal{L}'_j = \mathcal{E}_{n_j}\setminus\mathcal{L}_j$, where $\epsilon > 0$ is to be chosen later and let $\{n_j\}$ be a sequence corresponding to ϵ as above. It is clear that

$$\sum_{J\in\mathcal{L}_j} l(J)^{\rho-\epsilon} < \sum_{J\in\mathcal{L}_j}\omega(J) \leq 1. \tag{19}$$

But, we can also estimate

$$\begin{aligned}\sum_{J\notin\mathcal{L}_j}\omega(J) = \sum_{J\notin\mathcal{L}_j}\omega(J)^{\frac{1}{2}}\omega(J)^{\frac{1}{2}} &\leq \sum_{J\in\mathcal{E}_{n_j}}\omega(J)^{\frac{1}{2}}l(J)^{\frac{\rho+\epsilon}{2}-\epsilon}\\ &\leq \prod_{i=1}^{n_j}a_i^{-\epsilon}\ \beta^{n_j-n_{j-1}}\sum_{J\in\mathcal{E}_{n_{j-1}}}\omega(J)^{\frac{1}{2}}l(J)^{\frac{\rho+\epsilon}{2}}\end{aligned}$$

because of (16). By iterating the procedure we get

$$\sum_{J \notin \mathcal{L}_j} \omega(J) \leq \beta^{n_j} \prod_{i=1}^{n_j} a_i^{-\epsilon} \leq \beta^{n_j} \underline{A}^{-\epsilon n_j}.$$

Let $\epsilon > 0$ be such that $\beta < \underline{A}^{\epsilon}$. It is then immediate from the above that

$$\lim_{j \to \infty} \sum_{J \notin \mathcal{L}_j} \omega(J) = 0. \tag{20}$$

Clearly, (19) and (20) allow us to construct a subset of $I\!\!K$ of Hausdorff dimension $< \rho$ but of full harmonic measure and the proof is completed.

4. A Counterexample.

We state the following simple result:

Proposition 4.0. *For a Cantor set $I\!\!K$ as described in the Introduction, the harmonic measure ω of it's complement is "monodimensional", i.e. there is a dimension σ (the dimension of the harmonic measure) such that there exist a subset $F \subset I\!\!K$ of Hausdorff dimension σ with $\omega(F) = 1$, and for every set $F' \subset I\!\!K$ of dimension smaller than σ, $\omega(F') = 0$.*

The proof given below applies to all self-similar Cantor sets and therefore the proposition remains valid even for "general" Cantor sets.

Proof of Proposition 4.0.

Suppose that the proposition is false. Then, there is a dimension σ and a real number $0 < \alpha < 1$ such that

$$\sup\{\omega(F) \ ; \ F \subset I\!\!K \ , \ \dim(F) \leq \sigma\} = \alpha$$

or equivalently, there exist a dimension σ and a $\gamma > 0$ such that

$$\sup\Big\{ \inf_{x \in \frac{1+c_0}{2}[0,1]^2} \omega(x, F, c_0[0,1]^2 \setminus I\!\!K) \ ; \ F \text{ compact} \ , \ F \subset I\!\!K \ , \ \dim(F) \leq \sigma \Big\} = \gamma$$

$$\text{and} \qquad \gamma < \inf_{x \in \frac{1+c_0}{2}[0,1]^2} \omega(x, I\!\!K, c_0[0,1]^2 \setminus I\!\!K),$$

where c_0 is the constant defined in section 2.

For every real number τ , $0 < \tau < 1$ there is a compact set $F \subset I\!\!K$ of Hausdorff dimension σ with

$$\tau\gamma < \inf_{x \in \frac{1+c_0}{2}[0,1]^2} \omega(x, F, c_0[0,1]^2 \setminus I\!\!K) < \frac{1}{\tau}\gamma.$$

Moreover we can find a covering $\mathcal{F} = \{I_j\}_{j\in J}$ of F with squares I_j of the same generation of the construction of $I\!\!K$, satisfying

$$\tau\gamma < \inf_{x\in\frac{1+c_0}{2}[0,1]^2} \omega(x, \cup_{I\in\mathcal{F}} I, c_0[0,1]^2 \setminus I\!\!K) < \frac{1}{\tau}\gamma.$$

There exists at least one $I_j \in \mathcal{F}$ with the following property:

"There is a compact set $F_j \subset I_j \cap I\!\!K$ of Hausdorff dimension σ with

$$\inf_{x\in\frac{1+c_0}{2}I_j} \omega(x, F_j, c_0 I_j \setminus I\!\!K) > c\tau\gamma \inf_{x\in\frac{1+c_0}{2}I_j} \omega(x, I\!\!K, c_0 I_j \setminus I\!\!K)$$

where c is a Harnack constant depending only on $I\!\!K$."

We say then, that F_j is a γ-subset of I_j.

To prove this claim we first remark the existence of at least one I_j satisfying

$$\inf_{x\in\frac{1+c_0}{2}[0,1]^2} \omega(x, F_j, c_0[0,1]^2 \setminus I\!\!K) > \tau\gamma \inf_{x\in\frac{1+c_0}{2}[0,1]^2} \omega(x, I\!\!K \cap I_j, c_0[0,1]^2 \setminus I\!\!K)$$

and then proceed with standard arguments, using the Brelot formula.

Recall that all squares of the same generation of the construction of $I\!\!K$ are identical, and therefore the preceeding property is valid for any square of the generation of I_j, i.e. every such square has a γ-subset. Let $\tilde{\mathcal{F}}$ be the collection of all squares of the same generation with I_j that do not belong to $\mathcal{F}$, and let S be the union of F with the γ-subsets of the squares in $\tilde{\mathcal{F}}$. Thus S is a subset of $I\!\!K$ of Hausdorff dimension σ. By the above it is clear that

$$\inf_{x\in\frac{1+c_0}{2}[0,1]^2} \omega(x, S, c_0[0,1]^2 \setminus I\!\!K) > \tau\gamma + c\tau\gamma\big(\inf_{x\in\frac{1+c_0}{2}[0,1]^2} \omega(x, I\!\!K, c_0[0,1]^2 \setminus I\!\!K) - \frac{1}{\tau}\gamma\big),$$

which is greater than γ if τ is close enough to one; since γ is taken to be the maximal value of harmonic measure for subsets of $I\!\!K$ of Hausdorff dimension equal to σ we have reached a contradiction. The proof is now complete.

We will now construct a Cantor set $I\!\!K'$ as in the Introduction,except that here we replace a square J of the kth generation, $k \geq 1$, by four equal squares, $J_1, ..., J_4$ whose size depends not only on the generation k but on the square J also; we still require $\underline{A} \leq \dfrac{l(J_i)}{l(J)} \leq \overline{A}$ with $0 < \underline{A} \leq \overline{A} < 1/2$. We will show that for an appropriate choice of the sizes of the squares *the Hausdorff dimension of $I\!\!K'$ will be equal to the dimension of it's harmonic measure.* The idea of the construction was suggested to us by a remark of A. Ancona. Let us begin with the standard planar Cantor set $I\!\!K_{1/4}$ of dimension 1, i.e. a Cantor set as defined in the Introduction with $\underline{A} = \overline{A} = 1/4$. Let D be the dimension of it's harmonic measure; if F is a compact subset of $I\!\!K_{1/4}$ such that $\omega(F) > 1/2$, it follows from Proposition 4.0 that it's dimension will be at least D. We may therefore find such a subset F of $I\!\!K_{1/4}$ of Hausdorff dimension D. We then construct the desired Cantor set in the following way: In each generation we replace every square J that do not intersect F by squares of size 4^{-M} times the size of J, where M is a fixed integer with $M > 1/D$, and every square J' that intersects F is replaced by four squares of size $1/4$ times the size of J'. Let $\mathbb{K}'$ be the Cantor set constructed in this way. Observe that $\mathbb{K}' \subset \mathbb{K}_{1/4}$ by construction and that $\dim \mathbb{K}' = D$ because of the choice of M. It is clear (by the monotonicity of the harmonic measure as a function of the domain) that the dimension of the harmonic measure of $\mathbb{K}'$ is also D, and the construction is complete.

We should remark here that the preceeding process gives us Cantor sets whose Hausdorff dimension is equal to their harmonic measure dimension for every possible value of the dimension of the harmonic measure of a Cantor set as described in the Introduction. Also, a result of [J-W] implies that we cannot have $\dim \omega = 1$ for Cantor sets of this type. It is therefore natural to ask if we can have dimensions arbitrarily close to one. The following proposition answers the question.

Proposition 4.1. *For the self-similar Cantor set $\mathbb{K}_\delta$, $0 < \delta \leq \frac{1}{4}$, as defined in the introduction with $\frac{1}{2} - \delta = \underline{A}$ and $\underline{A} = \overline{A}$, the dimension of the harmonic measure $\dim \omega$ is greater than $1 - C\delta$, for some constant $C > 0$.*

This proposition as well as the proof given below is due to professor A. Ancona (compare with [M-V], pages 15-22, 28) .

Proof of Proposition 4.1.

We will need some more notation. Let $\mathbb{K}_n$ be the nth approximation of $\mathbb{K}_\delta$ by squares of the nth generation, let g_n be the Green function of the complement of $\mathbb{K}_n$ and $\mathcal{C}_n$ it's critical points. We shall rely on the following formula :

$$\dim \omega = 1 - \frac{\lim_{n\to\infty} \frac{1}{n} \sum_{\mathcal{C}_n} g_n(c)}{\chi_\mu},$$

where $\chi_\mu = \log(\frac{2}{1-2\delta})$ and the critical points in the sum are counted with their multiplicity.

This formula is a simple variant of the Carleson formula given in [M-V], page 15 (see also [C]); here we consider the sum over the critical points of g_n instead of those of the Green's function of the complement of $\mathbb{K}_\delta$.

It remains to prove that the limit in the previous formula is $O(\delta)$ as δ tends to 0.

We extend g_n on $\mathbb{K}_n$ by the value 0 and consider the critical domains of g_n, i.e. any region U which is a connected component of $\{g_n < \beta\}$ for some $\beta > 0$ and with a critical point $c \in \partial U$. Let $\mathcal{U}$ be the collection of all critical domains.

Note that if $U \in \mathcal{U}$ and if $\tau = \max\{g_n(z); z \in \mathcal{C}_n \cap U\}$, the number of critical domains $U' \subset U$ associated to τ is exactly equal to the number of critical points $z \in U$ with $g_n(z) = \tau$ (counted with their multiplicity) plus one.

To each $U \in \mathcal{U}$ we attach a square $I = I_U$ of some stage k of the construction of the Cantor set , $k \leq n$, with the following property:

(PI) We have $I \cap \mathbb{K}_n \subset U$ and if $\tilde{I}$ denote the "father" of I then there exists a square $I' \subset \tilde{I}$ of the kth generation such that $I' \cap \mathbb{K}_n \cap U = \emptyset$ and I and I' lie on the same side of $\tilde{I}$.

The existence of I_U is easily checked. For instance one may take for $\tilde{I}$ a minimal square such that $\tilde{I} \cap \mathbb{K}_n \cap U^c \neq \emptyset$ and $\tilde{I} \cap \mathbb{K}_n \cap U \neq \emptyset$, and then easily verify the existence an $I \subset \tilde{I}$ with the property (PI).

We now proceed with the following simple algorithm which leeds to the construction of a subcollection $\mathcal{U}_0 \subset \mathcal{U}$, (the "nice" domains) and to the choice of some square $c_U \subset I_U$ of the nth generation (the last generation for $\mathbb{K}_n$), for every $U \in \mathcal{U}_0$.

Each domain $U \in \mathcal{U}$ which is maximal is "nice" and we choose the square c_U arbitrarily in I_U. For $U \in \mathcal{U}$, if the construction has been achieved for all $U' \supset U$, $U \neq U'$ then we decide that $U \notin \mathcal{U}_0$ if there exists $c_{U'} \subset U$ for some $U' \supset U$, $U \neq U'$. Otherwise we say that $U \in \mathcal{U}_0$ and we associate to it some square $c_U \subset I_U$ of the nth generation.

At the end of the procedure every critical domain $U \in \mathcal{U}$ contains exactly one $c_{U'}$ for some U' , $U \subset U'$, and for $U, U' \in \mathcal{U}_0$ we have $I_U = I_{U'}$ if and only if $U = U'$.

Hence we have

$$\frac{1}{n} \sum_{\mathcal{C}_n} g_n(c) = \frac{1}{n} \Big(\sum_{U \in \mathcal{U}_0} g_{_U} - g_{max} \Big)$$

where $g_{_U}$ is the value of g_n on ∂U and g_{max} is the maximal critical value of g_n.

If $U \in \mathcal{U}_0$, let $I = I_U$ be the square attached to it , $\tilde{I}$ it's "father", and I' as in (PI) (see Fig. 2). There are at least $s = [\frac{1}{4\delta}]$ parallel segments, $l_1, ..., l_s$ joining points of $\mathbb{K}_n \cap I$ with points of $\mathbb{K}_n \cap I'$, the distance between any two segments being $\geq \delta \ l(I)$. Necessarily, ∂U cuts all these segments and therefore $\sup\{g_n(t) \ ; \ t \in l_i\} \geq g_{_U}$, $1 \leq i \leq s$.

For every $l_i, i = 1..., s$ let z_i^1, z_i^2 be the endpoints of l_i, $z_i^1 \in \mathbb{K}_n \cap I$, $z_i^2 \in \mathbb{K}_n \cap I'$. It is clear that the set $B(z_i^1, \delta l(I)/2) \cap \mathbb{K}_n \cap I$ has capacity $\geq C_0 > 0$ in the domain $B(z^1, \delta l(I))$, with C_0 independent of $\delta \in [\frac{1}{4}, \frac{1}{2})$. By standard arguments, it follows that

$$g_n(t) \leq C \ \omega(B(z_i^1, \delta l(I)/2), \mathbb{R}^2 \setminus K_n)$$

on the segment l_i with a constant C indepent of δ.

The above finally yields

$$\frac{1}{4\delta} g_{_U} \leq \sum_i \sup\{g_n(t) \ ; \ t \in l_i\} \leq C \ \omega(I, \mathbb{R}^2 \setminus K_n).$$

Summing up we find

$$\frac{1}{n} \sum_{\mathcal{C}_n} g_n(c) \leq \frac{1}{n} \sum_{U \in \mathcal{U}_0} g_{_U} \leq C\delta \frac{1}{n} \sum_{I \in \mathcal{F}_n} \omega(I, \mathbb{R}^2 \setminus \mathbb{K}_n) \leq C\delta \ ,$$

where $\mathcal{F}_n$ is the collection of all squares of some stage k , $k \leq n$, of the construction of $\mathbb{K}_\delta$. The proof of the proposition is complete.

5. Conclusion - Further remarks.

It is clear that the method we developped in sections 2,3 does not apply only to the Cantor sets described above but also to other Cantor sets, for example those indicated in Fig.3. The proof can also be applied to some Cantor sets in higher dimension.

For a general Cantor set $\mathbb{K} \subset \mathbb{R}^d$, a sufficient condition to conclude that $\dim(\omega) < \rho = \dim(\mathbb{K})$ is the following: if $I^n_{i_1 \dots i_n}$ is a square of the nth generation and if $I^{n+1}_{i_1 \dots 1}, ..., I^{n+1}_{i_1 \dots s}$ are the squares of the next generation contained in $I^n_{i_1 \dots i_n}$ then there exist $0 < \alpha < 1, 1 \leq \tau \leq s$, and constants $a^n_j > 0$ such that

$$\omega(I^{n+1}_{i_1 \dots i_n \tau}) < \alpha \frac{\operatorname{diam}\big(I^{n+1}_{i_1 \dots i_n \tau}\big)^\rho}{\sum_{j=1}^s \operatorname{diam}\big(I^{n+1}_{i_1 \dots i_n j}\big)^\rho} \omega(I^n_{i_1 \dots i_n}), \tag{$*$}$$

and

$$\operatorname{diam}\big(I^{n+1}_{i_1 \dots i_n j}\big) = a^n_j \operatorname{diam}\big(I^n_{i_1 \dots i_n}\big), \quad 2^{-d} < \underline{A} \leq a^n_j \leq \overline{A} < 1, \quad \forall j \in \{1, ..., s\}$$

where a^n_j depends only on j, n but not on the square $I^n_{i_1 \dots i_n}$ and $\underline{A}, \overline{A}$ are two constants not depending on n. Both lemmas 2.7 and 2.8 can be applied to prove a formula similar to (16) and the theorem's proof may be completed in the same way.

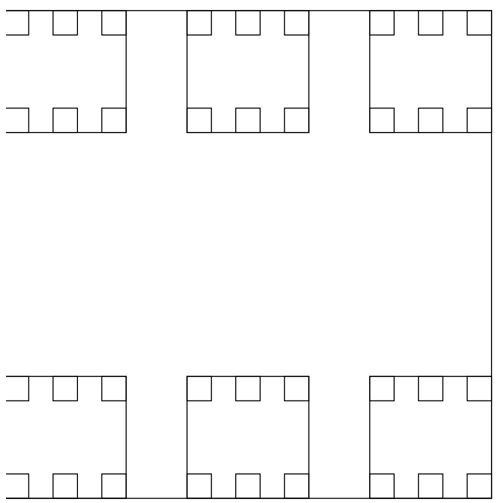

Fig. 3a

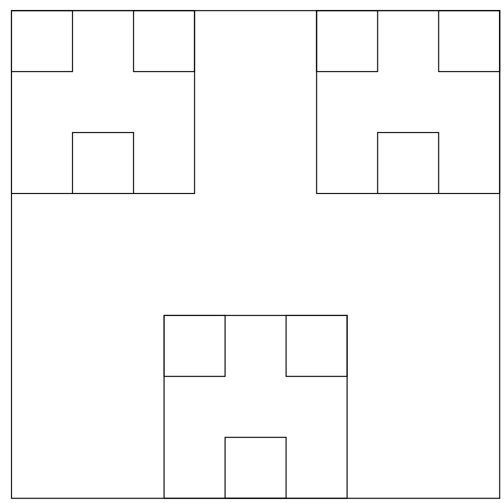

Fig. 3b

In general $(*)$ seems hard to check; however under certains assumptions of symmetry on the Cantor set $\mathbb{K}$ one may verify it by proving some lemmas similars to those presented above. Even though the method presented here seems rather general, we haven't been able to get rid of these assumptions of symmetry, and the proof of lemma 2.3 strongly depends on them.

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A continuity property of the dimension of the harmonic measure of Cantor sets under perturbations

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Abstract: *We investigate the behaviour of the dimension of harmonic measure of the complementary of Cantor sets as a function of parameters determining these sets, and we establish continuity results.*

1 Introduction

The purpose of this work is to study the dimension of the harmonic measure of the complementary of (not necessarily self-similar) Cantor sets as a function of parameters assigned to these sets, and to establish some continuity properties. We develop our method on a particular kind of Cantor sets in the plane for convenience, even though the proof can be applied to all "self-similar" Cantor sets in $\mathbb{R}^n$, $n \geq 2$ (see theorem 1.2).

A 4-corner Cantor set will be a compact set constructed in the following way: let $\underline{A}, \overline{A}$ be two constants with $0 < \underline{A} \leq \overline{A} < \frac{1}{2}$ and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers with $\underline{A} \leq a_n \leq \overline{A}$ for all $n \in \mathbb{N}$. We replace the square $[0,1]^2$ by four squares of sidelength a_1 situated in the four corners of $[0,1]^2$. Each of these squares is then replaced by four squares of sidelength $a_1 a_2$ situated in its four corners. At the nth stage of the construction every square of the $(n-1)$th generation will be replaced by four squares of sidelength $a_1 ... a_n$ situated in its four corners (see figure 1). Let $\mathbb{K}$ be the Cantor set constructed by repeating the procedure.

Let us recall that for a probability measure μ on $\mathbb{R}^n$ the dimension $\dim \mu$ of μ is the smallest Hausdorff dimension of sets of measure 1. Carleson [7] has shown that for self-similar 4-corner Cantor sets (the sequence $(a_n)_{n\in\mathbb{N}}$ is constant) the dimension of the harmonic measure of their complementary is strictly smaller than 1. His proof, involving ergodic theory techniques, was improved by Makarov and Volberg [12] who showed that the dimension of the harmonic measure of any self-similar 4-corner Cantor set is strictly smaller than the dimension of the Cantor set. Finally, Volberg ([14], [15]) extended these results to a class of dynamic Cantor

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[0]Key words: Harmonic measure, Cantor sets, fractals, Hausdorff dimension, entropie

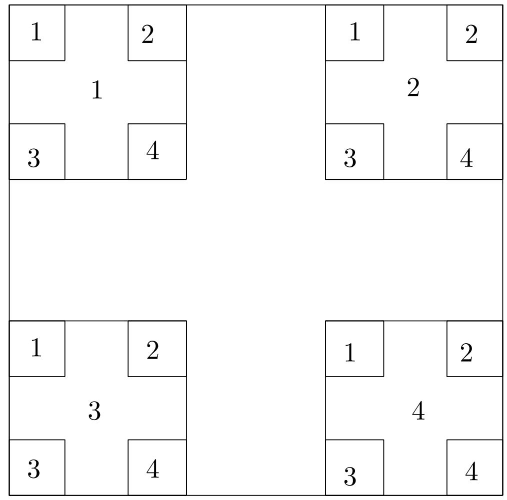

Figure 1: A 4-corner Cantor set and its enumeration

repellers. Other comparisons of harmonic and maximal measures for dynamical systems are proposed in [2], [11].

In [3] it is shown that the dimension of the harmonic measure of the complementary of 4-corner Cantor sets is strictly smaller than the Hausdorff dimension of the Cantor set, even when the sequence $(a_n)_{n\in\mathbb{N}}$ is not constant. In [4] we prove that small perturbations of the sidelength of the squares of the construction of $\mathbb{K}$ do not alterate this property.

It is therefore natural to ask whether the dimension of harmonic measure is continuous as a function of the sequence $(a_n)_{n\in\mathbb{N}}$ with respect to the "sup" norm. We show that constant sequences are continuity "points" of this function. A general continuity statement seems much more difficult to prove as we will point out in section 4.

Theorem 1.1 *Let $\mathbb{K}$ be the 4-corners Cantor set associated to a constant sequence $a_n = a$. Let $\mathbb{K}_{(a'_n)}$ be another Cantor set of the 4-corners type associated to the (not necessarily constant) sequence $(a'_n)_{n\in\mathbb{N}}$ and let ω and ω' be the harmonic measures of $\mathbb{K}$ and $\mathbb{K}_{(a'_n)}$ respectively. Then for all $\epsilon > 0$ there exists a $\delta = \delta(a,\epsilon) > 0$ such that if $|a'_n - a| < \delta$ for all $n \in \mathbb{N}$ then $|\dim\omega - \dim\omega'| < \epsilon$.*

This result is also valid for general self-similar Cantor sets: Let D be an open simply connected bounded set in the plane and let $p_1, ..., p_k$ be k affine functions. Let $p_i(D) = D_i$ for $i = 1, ..., k$ and suppose that the sets D_i are open simply connected subsets of D with disjoint closures (see figure 2). A self-similar Cantor set $\mathbb{K}$ will be the compact set defined by

$$\mathbb{K} = \bigcap_{n\in\mathbb{N}} \bigcup_{i_1,...,i_n} p_{i_1} \circ ... \circ p_{i_n}(D).$$

The following known result (which can also be proved with classical techniques of the thermodynamical formalism) can be obtained using the method presented in this paper:

Theorem 1.2 *Let $p_1, ..., p_k$ be k affine functions and $\mathbb{K}$ be the self-similar Cantor set associated to these functions. Take $\mathbb{K}'$ to be a self similar Cantor set associated to the functions $p'_1, ..., p'_k$. Then for all $\epsilon > 0$ there exists a $\delta = \delta(p_1, ..., p_k, \epsilon) > 0$ such that if $||p_i - p'_i||_\infty < \delta$ for all $i = 1, ..., k$ then $|\dim \omega - \dim \omega'| < \epsilon$, where ω and ω' are the harmonic measures of $\mathbb{R}^2 \setminus \mathbb{K}$ and $\mathbb{R}^2 \setminus \mathbb{K}'$ respectively.*

The following sections are entirely devoted to the proof of theorem 1.1.

2 Preliminary results

In this section we establish some estimates on the harmonic measure of a Cantor set under perturbation, and recall some known results on the harmonic measure of Cantor-type sets. We also introduce the tools needed, such as the Hausdorff dimension and the entropy of a probability measure on a Cantor set.

Notation 2.1 Let $\mathbb{K}$ be a 4-corner Cantor set as described in the introduction. We enumerate $\mathbb{K}$ by identifying it to the abstract Cantor set $\{1, ..., 4\}^{\mathbb{N}}$. We denote $I_{i_1 \dots i_n}$, where $i_j \in \{1, 2, 3, 4\}$ for $1 \leq j \leq n$, the 4^n squares of the n-th generation of the construction of $\mathbb{K}$ with the enumeration shown in the figure 1 and the usual condition that $I_{i_1 \dots i_n}$ is the "father" of the sets $I_{i_1 \dots i_n i}$, $i \in \{1, 2, 3, 4\}$. It is clear that $\overline{A} \geq \dfrac{\mathrm{diam} I_{i_1 \dots i_n i}}{\mathrm{diam} I_{i_1 \dots i_n}} = a_{n+1} \geq \underline{A}$, $\; i = 1, ..., 4$.

The collection of the squares of the n-th generation of the constuction of $\mathbb{K}$ will be $\mathcal{F}_n = \{I_{i_1 \dots i_n} \, ; \, i_1, ..., i_n = 1, ..., 4\}$, for $n \in \mathbb{N}$. For a square $I \in \mathcal{F}_n$ we note $P_k(I)$ the unique square of the $(n-k)$-th generation containing I; in particular we note $\widehat{I} = P_1(I)$ the "father" of I. If $I = I_{i_1 \dots i_k} \in \mathcal{F}_k$ and $J = I_{j_1 \dots j_n} \in \mathcal{F}_n$ we will note $IJ = I_{i_1 \dots i_k j_1 \dots j_n} \in \mathcal{F}_{n+k}$. Finally, for $x \in \mathbb{K}$ and $n \in \mathbb{N}$ let $I_n(x)$ be the unique square of $\mathcal{F}_n$ containing x.

For a domain Ω, a point $x \in \Omega$ and a Borel set $F \subset \mathbb{R}^2$ we denote by $\omega(x, F, \Omega)$ the harmonic measure of $F \cap \partial\Omega$ (for the domain Ω) assigned to the point x. Clearly, F carries no measure if it does not intersect $\partial\Omega$. If Ω is not specified it will be $\mathbb{R}^2 \setminus \mathbb{K}$ and if x is the point at infinity we will simply note $\omega(F)$. Finally, for a Borel set $E \subset \mathbb{R}^2$ we note $\dim E$ the Hausdorff dimension of the set E.

2.1 Dimension of measures and entropy

In this section we recall some known results on the dimensions of measures (see also [13]).

Definition 2.2 *For a probability measure μ in $\mathbb{R}^n$ we note $\dim \mu$ the dimension of μ*

$$\dim \mu = \inf\{\dim E \; ; \; E \textit{ measurable} \; , \; \mu(E) = 1\}.$$

We say that the measure μ is monodimensional if $\mu(E) = 0$ for all measurable sets E of Hausdorff dimension $\dim E < \dim \mu$.

One can prove that (see for instance [8], [4]) if μ is monodimensional then

$$\dim \mu = \liminf_{r \to 0} \frac{\log \mu B(x, r)}{\log r} \; , \; \mu\text{-almost everywhere.} \tag{1}$$

If the probability measure μ is supported by a 4-corner Cantor set, the balls $B(x,r)$ can be replaced by the squares of the construction of the Cantor set (see [6]):

$$\dim \mu = \liminf_{n\to\infty} \frac{\log \mu(I_n(x))}{\log l(I_n(x))} \ , \ \mu\text{-almost everywhere,} \tag{2}$$

where $l(I_n(x))$ is the sidelength of the square $I_n(x)$ and $\underline{A}^n \le l(I_n) \le \overline{A}^n$.

Remark 2.3 If μ is an arbitrary (not necessarily monodimensional) probability measure we get

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \le \dim \mu \quad \mu\text{-almost everywhere.} \tag{3}$$

Moreover $\dim \mu = \mathrm{supess}_\mu \liminf_{r\to 0} \dfrac{\log \mu(B(x,r))}{\log r}$.

Definition 2.4 *The entropy of a probability measure μ supported by a Cantor set, $h(\mu)$, is defined*

$$h(\mu) = \lim_{n\to\infty} \frac{1}{n} \sum_{I\in\mathcal{F}_n} |\log \mu(I)|\mu(I),$$

whenever this limit exists.

For more information on entropy of measures see for instance [16].

For a self-similar Cantor set and an invariant ergodic measure μ on the Cantor set one gets $\dim \mu = \dfrac{h(\mu)}{\chi(\mu)}$, where $\chi(\mu)$ is the Lyapounov exponent of μ. If $\mathbb{K}$ is a 4-corners self-similar Cantor set (i.e. $a_n = a$ for all $n \in \mathbb{N}$), then for all invariant ergodic probability measures μ on $\mathbb{K}$ we have $\chi(\mu) = |\log a|$ (see also [12]).

2.2 Estimating perturbations of the harmonic measure

Suppose that the 4-corner Cantor set $\mathbb{K}$ is associated to the sequence $(a_n)_{n\in\mathbb{N}}$ and let $\mathbb{K}'$ be another Cantor set associated to the sequence $(a'_n)_{n\in\mathbb{N}}$. Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be the collections of squares associated to $\mathbb{K}$ and $(\mathcal{F}'_n)_{n\in\mathbb{N}}$ those associated to $\mathbb{K}'$. For $I \in \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ and $I' \in \bigcup_{n\in\mathbb{N}} \mathcal{F}'{}_n$ we will write $I \overset{\text{cod}}{\sim} I'$ if I and I' have the same enumeration (with respect to the identification to the abstract Cantor set $\{1,2,3,4\}^{\mathbb{N}}$).

If ω is the harmonic measure of $\mathbb{R}^2 \setminus \mathbb{K}$ and ω' the harmonic measure of $\mathbb{R}^2 \setminus \mathbb{K}'$ we have the following theorem.

Theorem 2.5 *For all $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$\sup_{n\in\mathbb{N}} |a_n - a'_n| < \delta \Rightarrow \left| \frac{\omega(I)}{\omega(\widehat{I})} : \frac{\omega'(I')}{\omega'(\widehat{I'})} - 1 \right| < \epsilon, \tag{4}$$

for all $I \in \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ and $I' \in \bigcup_{n\in\mathbb{N}} \mathcal{F}'{}_n$ with $I \overset{\text{cod}}{\sim} I'$.

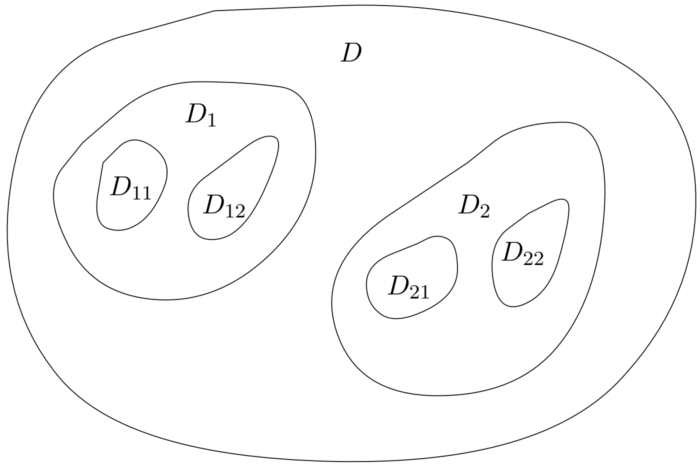

Figure 2: A general Cantor type set

The remaining of this section is devoted to the proof of theorem 2.5. Let us first remark that there exists a constant $c_0 = c_0(\underline{A}, \overline{A}) > 1$ such that for all squares $I = I_{i_1,\dots i_n} \in \mathcal{F}_n$ of the construction of $\mathbb{K}$ (or of $\mathbb{K}'$) we have $(c_0 I) \cap \mathbb{K} = I \cap \mathbb{K}$ (and $(c_0 I) \cap \mathbb{K}' = I \cap \mathbb{K}'$).

We will use the following result which is a variant of a strong Harnack's principle.

Lemma 2.6 *([7], [12]) Let Ω be a domain containing ∞ and let $A_1 \subset B_1 \subset A_2 \subset B_2 \subset \dots \subset A_n \subset B_n$ be conformal discs such that the annuli $B_i \setminus A_i$ are contained in Ω, for $1 \le i \le n$. If the moduli of the annuli are uniformly bounded away from zero and if $\infty \in \Omega \setminus B_n$ then, for all pairs of positive harmonic functions u, v vanishing on $\partial\Omega \setminus A_1$ and for all $x \in \Omega \setminus B_n$ we have*

$$\left| \frac{u(x)}{v(x)} : \frac{u(\infty)}{v(\infty)} - 1 \right| \le K q^n \tag{5}$$

where $q < 1$ and K are two constants that depend only on the lower bound of the moduli of the annuli.

We use the previous result to establish the following lemma. Both of them are closely related to the Boundary Harnack principle (see [1]).

Lemma 2.7 *Let $\epsilon > 0$. Under the conditions of theorem 2.5 there exists a $k_0 = k_0(\underline{A}, \overline{A}) > 0$ such that for all $k \ge k_0$ and all squares I of the construction of $\mathbb{K}$, if $Q = c_0 P_k(I)$ then*

$$\left| \frac{\omega(x, I, Q \setminus \mathbb{K})}{\omega(x, \widehat{I}, Q \setminus \mathbb{K})} : \frac{\omega(I)}{\omega(\widehat{I})} - 1 \right| < \epsilon \text{ for all } x \in \partial \left\{ \frac{1 + c_0}{2} \right\} P_k(I). \tag{6}$$

The result applies also to the Cantor set $\mathbb{K}'$.

Proof By lemma 2.6, if $k_0 = k_0(\underline{A}, \overline{A})$ is big enough (such that $K q^{k_0} < \epsilon$, where K, q are the constants given by the lemma) then

$$\left|\frac{\omega(x,I)}{\omega(x,\widehat{I})} : \frac{\omega(I)}{\omega(\widehat{I})} - 1\right| < \epsilon \text{ , for } x \notin \left\{\frac{1+c_0}{2}\right\} P_k(I) \tag{7}$$

Let $A = \dfrac{\omega(I)}{\omega(\widehat{I})}$. We have

$$\omega(x, I, Q \setminus \mathbb{K}) = \omega(x, I) - \int_{\partial Q} \omega(z, I)\omega(x, dz, Q \setminus \mathbb{K}),$$

for $x \in \partial\left\{\frac{1+c_0}{2}\right\} P_k(I)$.

By the equation (7),

$$A\omega(x,\widehat{I}) - \epsilon A\omega(x,\widehat{I}) \leq \omega(x,I) \leq A\omega(x,\widehat{I}) + \epsilon A\omega(x,\widehat{I}).$$

We get

$$\begin{aligned}
\omega(x, I, Q\setminus\mathbb{K}) &\leq A\omega(x,\widehat{I}) + \epsilon A\omega(x,\widehat{I}) - \int_{\partial Q}\Big(A\omega(z,\widehat{I}) - \epsilon A\omega(z,\widehat{I})\Big)\omega(x,dz,Q\setminus\mathbb{K}) \\
&= A\omega(x,\widehat{I}) - \int_{\partial Q} A\omega(z,\widehat{I})\omega(x,dz,Q\setminus\mathbb{K}) + \\
&+ \epsilon\Big(A\omega(x,\widehat{I}) + \int_{\partial Q} A\omega(z,\widehat{I})\omega(x,dz,Q\setminus\mathbb{K})\Big) \\
&= A\omega(x,\widehat{I},Q\setminus\mathbb{K}) + \epsilon\Big(A\omega(x,\widehat{I}) + \int_{\partial Q} A\omega(z,\widehat{I})\omega(x,dz,Q\setminus\mathbb{K})\Big) \qquad (8)
\end{aligned}$$

Therefore,

$$\frac{\omega(x,I,Q\setminus\mathbb{K})}{\omega(x,\widehat{I},Q\setminus\mathbb{K})} \leq A + \epsilon A \frac{\omega(x,\widehat{I}) + \int_{\partial Q}\omega(z,\widehat{I})\omega(x,dz,Q\setminus\mathbb{K})}{\omega(x,\widehat{I},Q\setminus\mathbb{K})} \tag{9}$$

It suffices now to show that the quantity

$$\frac{\omega(x,\widehat{I}) + \int_{\partial Q}\omega(z,\widehat{I})\omega(x,dz,Q\setminus\mathbb{K})}{\omega(x,\widehat{I},Q\setminus\mathbb{K})}$$

is smaller that a given constant. Take $x_0 \in \partial\left\{\frac{1+c_0}{2}\right\} P_k(I)$ such that

$$\omega(x_0,\widehat{I}) = \max\left\{\omega(x,\widehat{I})\ ;\ x \notin \left\{\frac{1+c_0}{2}\right\} P_k(I)\right\}.$$

We then have

$$\begin{aligned}
\omega(x_0,\widehat{I},Q\setminus\mathbb{K}) &= \omega(x_0,\widehat{I}) - \int_{\partial Q}\omega(z,\widehat{I})\omega(x_0,dz,Q\setminus\mathbb{K}) \\
&\geq \omega(x_0,\widehat{I}) - \int_{\partial Q}\omega(x_0,\widehat{I})\omega(x_0,dz,Q\setminus\mathbb{K}) \\
&= \omega(x_0,\widehat{I})(1 - \omega(x_0,\partial Q,Q\setminus\mathbb{K}))
\end{aligned}$$

By standard techniques one can verify (see [3]) that $1-\omega(x_0,\partial Q,Q\setminus\mathbb{K})$ is greater that a constant $c>0$ depending only on $\underline{A},\overline{A}$.

By using Harnack's principle we get

$$1-\omega(x,\partial Q,Q\setminus\mathbb{K})\geq c \text{ , for all } x\in\partial\left\{\frac{1+c_0}{2}\right\}P_k(I),$$

for a new constant $c>0$.

Hence $\dfrac{\omega(x,\widehat{I})+\int_{\partial Q}\omega(z,\widehat{I})\omega(x,dz,Q\setminus\mathbb{K})}{\omega(x,\widehat{I},Q\setminus\mathbb{K})}\leq\dfrac{2}{c}$ and therefore, by relation (9),

$$\frac{\omega(x,I,Q\setminus\mathbb{K})}{\omega(x,\widehat{I},Q\setminus\mathbb{K})}\leq A(1+\frac{2}{c}\epsilon) \tag{10}$$

On the other hand $A=\dfrac{\omega(I)}{\omega(\widehat{I})}$; we obtain

$$\frac{\omega(x,I,Q\setminus\mathbb{K})}{\omega(x,\widehat{I},Q\setminus\mathbb{K})}:\frac{\omega(I)}{\omega(\widehat{I})}-1<\frac{2}{c}\epsilon \text{ , for all } x\in\partial\left\{\frac{1+c_0}{2}\right\}P_k(I),$$

The left hand inequality can be established in the same way and the proof is complete. •

Lemma 2.8 *Let $Q_1\subset Q_2\subset Q_3\subset\ldots\subset Q_n$ be squares verifying that the moduli of the annuli $Q_j\setminus Q_{j-1}$ are greater than $1/c$ and smaller than $c>1$. Let $S\subset\frac{1}{c}Q_1$ be the intersection of a Cantor set $\mathbb{K}$ as above with Q_1, and suppose that the annuli do not intersect $\mathbb{K}$. Then , there exist two constants $C>0$ and $\delta>0$ depending only on c, $\overline{A}$ and $\underline{A}$ such that for all $x\in\left\{\frac{2}{1+c}\right\}Q_1$*

$$\omega(x,S,Q_n\setminus S)>1-C\prod_{k=1}^{n}\left(1-\frac{\delta}{k}\right) \tag{11}$$

Proof We can assume that $Q_1=[0,1]^2$. Let us recall that there is a constant $c_4=c_4(\underline{A},\overline{A})$ such that for $x\in\partial\left\{\frac{2}{1+c}\right\}Q_j$ we have $\omega(x,S,Q_j\setminus S)\geq c_4\dfrac{1}{\log(\mathrm{diam}Q_j)}$ (see for instance [3]). Let ω_j be the harmonic measures of the domains $Q_j\setminus S$, $j=1,\ldots,n$. Take $x_0\in\partial\left\{\frac{2}{1+c}Q_1\right\}$.

We have

$$\begin{aligned}\omega_{j-1}(x_0,S) &= \omega_j(x_0,S)-\int_{\partial Q_{j-1}}\omega_j(z,S)\omega_{j-1}(x_0,dz)\\ &\leq \omega_j(x_0,S)-c_4\frac{1}{\log(\mathrm{diam}Q_j)}\omega_{j-1}(x_0,\partial Q_{j-1})\end{aligned}$$

Now by the lower bound of the annuli $Q_j\setminus Q_{j-1}$ we get that $\dfrac{1}{\log(\mathrm{diam}Q_j)}\geq j^{-1}C_0$, where $C_0>1$ is a constant. We get

$$\omega_{j-1}(x_0,\partial Q_{j-1})(1-\frac{c_4}{jC_0})\geq\omega_j(x_0,\partial Q_j).$$

To finish the proof, it suffices to recall that $\omega_j(x_0,S) = 1-\omega_j(x_0,\partial Q_j)$ and to take $\delta = \frac{c_4}{C_0}$ and $C = \max\left\{\omega_1(x,S)\ ;\quad x\in\left\{\frac{1+\alpha}{2}\right\}Q_1\right\}$. •

Lemma 2.9 *For $J\in\bigcup_{n\in\mathbb{N}}\mathcal{F}_n$, $J'\in\bigcup_{n\in\mathbb{N}}\mathcal{F}'_n$ and $s>0$ we define*

$$\begin{aligned} U_s(J) &= \{x\in(c_0J)\setminus\mathbb{K} \text{ such that } \omega(x,J,(c_0J)\setminus\mathbb{K})>1-s\} \\ U'_s(J') &= \left\{x\in(c_0J')\setminus\mathbb{K}' \text{ such that } \omega(x,J',(c_0J')\setminus\mathbb{K}')>1-s\right\} \end{aligned}$$

There is an increasing function $\eta>0$ depending only on $\underline{A},\overline{A}$, such that

$$\begin{gathered} \operatorname{dist}(J\cap\mathbb{K},\partial U_s(J)) > \eta(s)l(J) \\ \operatorname{dist}(J'\cap\mathbb{K}',\partial U'_s(J')) > \eta(s)l(J') \end{gathered} \tag{12}$$

Proof By dilating the square J we can assume that $c_0J=[0,1]^2=Q_0$ (idem for the square J'). Let $J_n\subset J$ be a square such that $P_n(J_n)=J$. The squares $Q_1=c_0P_1(J_n)\subset Q_2=c_0P_2(J_n)\subset\ldots\subset Q_n=c_0P_n(J_n)=Q_0$ satisfy the conditions of lemma 2.8, by construction. Hence there exists a $\delta=\delta(\underline{A},\overline{A})>0$ and a constant $C>0$ such that for all $x\in\frac{c_0+1}{2}J_n$

$$\omega(x,J_n,Q_n\setminus\{J_n\cap\mathbb{K}\}) > 1-C\prod_{k=1}^{n}\left(1-\frac{\delta}{k}\right).$$

Using the maximum principle, we can easily verify that for all $x\in\frac{c_0+1}{2}J_n$

$$1-\omega(x,\partial Q_0,Q_0\setminus\{J\cap\mathbb{K}\}) \geq 1-\omega(x,\partial Q_0,Q_0\setminus\{J_n\cap\mathbb{K}\})$$

Therefore,

$$\omega(x,J,Q_0\setminus\{J\cap\mathbb{K}\}) \geq \omega(x,J_n,Q_0\setminus\{J_n\cap\mathbb{K}\}) \geq 1-C\prod_{k=1}^{n}\left(1-\frac{\delta}{k}\right) \tag{13}$$

The square J_n has been chosen arbitrarily, hence the last equation gives

$$\operatorname{dist}(x,J\cap\mathbb{K}) < \frac{(c_0-1)\underline{A}^n}{4} \Rightarrow \omega(x,J,Q_0\setminus\{J\cap\mathbb{K}\}) \geq 1-C\prod_{k=1}^{n}\left(1-\frac{\delta}{k}\right).$$

We can now choose $n=n(\underline{A},\overline{A},s)$ such that $C\prod_{k=1}^{n}\left(1-\frac{\delta}{k}\right)<s$. •

Proof of theorem 2.5 Fix $\epsilon>0$. Let I be a square of the construction of the Cantor set $\mathbb{K}$ and I' a square of the construction of $\mathbb{K}'$, with $I\overset{\text{cod}}{\sim}I'$. We note $Q=c_0P_k(I)$. By translating and dilating the Cantor set $\mathbb{K}'$ we can assume that $c_0P_k(I')=Q$ (recall that the harmonic measure is invariant under affine maps).

By lemma 2.7 it suffices to compare $\dfrac{\omega(x,I,Q\setminus\mathbb{K})}{\omega(x,\widehat{I},Q\setminus\mathbb{K})}$ and $\dfrac{\omega(x,I',Q\setminus\mathbb{K}')}{\omega(x,\widehat{I'},Q\setminus\mathbb{K}')}$, if k is taken sufficiently large. Let us fix $k\in\mathbb{N}$. For $J\in\bigcup_{n\in\mathbb{N}}\mathcal{F}_n$, $J'\in\bigcup_{n\in\mathbb{N}}\mathcal{F}'_n$ with $J\cup J'\subset Q$ and for $0<s<1$, put

$$\begin{aligned}\Omega_s(J) &= \{x\in Q\setminus\mathbb{K} \text{ such that } \omega(x,J,Q\setminus\mathbb{K})>1-s\}\\ \Omega'_s(J') &= \big\{x\in Q\setminus\mathbb{K}' \text{ such that } \omega(x,J',Q\setminus\mathbb{K}')>1-s\big\}\end{aligned}\tag{14}$$

Since the harmonic measure is increasing as a function of the domain we have

$$U_s(J)\subset\Omega_s(J) \text{ and } U'_s(J')\subset\Omega'_s(J').$$

Therefore,

$$\operatorname{dist}(J\cap\mathbb{K},\partial\Omega_s(J))>\eta(s)l(J) \text{ and } \operatorname{dist}(J'\cap\mathbb{K}',\partial\Omega'_s(J'))>\eta(s)l(J'),\tag{15}$$

for all squares $J\overset{\text{cod}}{\sim}J'$.

Now let s be a positive constant which will be precised later. By the equations (15) we can choose $\delta=\delta(s,k)>0$ such that

$$\sup_{n\in\mathbb{N}}|a'_n-a_n|<\delta\Rightarrow I\cap\mathbb{K}\subset\Omega'_s(I') \text{ and } I'\cap\mathbb{K}'\subset\Omega_s(I).\tag{16}$$

The idea is that if the sequences $(a_n)_{n\in\mathbb{N}}$ and $(a'_n)_{n\in\mathbb{N}}$ are close enough, then any two squares of the k-th generation of the constructions (with the same encoding) will be close with regard to the Hausdorff metric (where k is a fixed positive integer).

By the formulas (16) it follows on applying the maximum principle

$$|\omega(\zeta,J,Q\setminus\mathbb{K})-\omega(\zeta,J',Q\setminus\mathbb{K}')|<s, \text{ for all } \zeta\in\partial\left\{\frac{1+c_0}{2}\right\}P_k(I).$$

On the other hand, Harnack's principle gives a constant $c_3=c_3(k)>0$ such that

$$\min\big\{\omega(x,J,Q\setminus\mathbb{K}),\omega(x,J',Q\setminus\mathbb{K}')\big\}>c_3\quad,\quad\forall x\in\partial\left\{\frac{1+c_0}{2}\right\}P_k(I).\tag{17}$$

for all squares J of the construction of $\mathbb{K}$ and all squares J' of the construction of $\mathbb{K}'$ of the same generation with I such that $J\cup J'\subset Q$.

Take $s>0$ verifying $\dfrac{s}{c_3}<\epsilon^2/10$. Then, for all squares J and J' of the same generation with I such that $J\overset{\text{cod}}{\sim}J'\subset Q$, we get

$$\left|\frac{\omega(\zeta,J,Q\setminus\mathbb{K})}{\omega(\zeta,J',Q\setminus\mathbb{K}')}-1\right|<\epsilon^2/10\ ,\quad\forall\zeta\in\partial\left\{\frac{1+c_0}{2}\right\}P_k(I).\tag{18}$$

We deduce

$$\left|\frac{\omega(\zeta,I,Q\setminus\mathbb{K})}{\omega(\zeta,\widehat{I},Q\setminus\mathbb{K})}:\frac{\omega(\zeta,I',Q\setminus\mathbb{K}')}{\omega(\zeta,\widehat{I'},Q\setminus\mathbb{K}')}-1\right|<\epsilon\ ,\quad\forall\zeta\in\partial\left\{\frac{1+c_0}{2}\right\}P_k(I)\tag{19}$$

which completes the proof. $\bullet$

2.3 Some estimates on harmonic measure when $(a_n)_{n\in\mathbb{N}}$ is constant

Throughout this section $\mathbb{K}$ is a 4-corners self similar Cantor set associated to a constant sequence $(a_n)_{n\in\mathbb{N}}$ ($a_n = a$ for all $n \in \mathbb{N}$, $0 < a < 1/2$) and ω will be the harmonic measure of $\mathbb{K}$. The following lemma is a corollary of lemma 2.6.

Lemma 2.10 *([7], [12]) For every $I \in \mathcal{F}_n$, $J \in \mathcal{F}_k$ and every $L \in \mathcal{F}_m$, $n, k, m \in \mathbb{N}$*

$$\left| \frac{\omega(IJL)}{\omega(IJ)} : \frac{\omega(JL)}{\omega(J)} - 1 \right| < C\, q^k \tag{20}$$

where the constants $C > 0$ and $q \in (0,1)$, depend only on a.

Using a slightly weaker version this lemma, Carleson ([7]) shows that for the self-similar Cantor set $\mathbb{K}$ there exists an invariant ergodique measure μ and a constant $C > 0$ such that $\frac{1}{C}\omega \le \mu \le C\omega$. Therefore,

$$\dim \omega = \dim \mu = \frac{h(\mu)}{|\log a|} = \lim_{n\to\infty} \frac{1}{n|\log a|} \sum_{I\in\mathcal{F}_n} |\log \omega(I)| \omega(I) = \lim_{n\to\infty} \frac{\log \omega(I_n(x))}{n \log a} \ ,$$

for ω-a.e. $x \in \mathbb{K}$.

With the same notation as before we have the following technical but essential lemma:

Lemma 2.11 *Take $\epsilon > 0$. There exists $p_0 \in \mathbb{N}$ big enough such that if $p \ge p_0$ then*

$$\left| \frac{1}{p} \sum_{J\in\mathcal{F}_p} \frac{\omega(IJ)}{\omega(I)} \left| \log \left(\frac{\omega(IJ)}{\omega(I)} \right) \right| - h(\mu) \right| < \epsilon \tag{21}$$

for all $I \in \ \cup\{\mathcal{F}_n \,,\, n \in \mathbb{N}\}$.

Proof Take $\epsilon > 0$ and $p \in \mathbb{N}$. We write $p = p_1 + p_2$, with p, p_1, p_2 to be chosen later. We get

$$\begin{aligned} \sum_{J\in\mathcal{F}_p} \frac{\omega(IJ)}{\omega(I)} \log\left(\frac{\omega(IJ)}{\omega(I)}\right) &= \sum_{J_1\in\mathcal{F}_{p_1}} \sum_{J_2\in\mathcal{F}_{p_2}} \frac{\omega(IJ_1J_2)}{\omega(I)} \log\left(\frac{\omega(IJ_1J_2)}{\omega(I)}\right) \\ = \sum_{J_1\in\mathcal{F}_{p_1}} \sum_{J_2\in\mathcal{F}_{p_2}} \frac{\omega(IJ_1J_2)}{\omega(I)} \log\left(\frac{\omega(IJ_1J_2)}{\omega(IJ_1)}\right) &+ \sum_{J_1\in\mathcal{F}_{p_1}} \sum_{J_2\in\mathcal{F}_{p_2}} \frac{\omega(IJ_1J_2)}{\omega(I)} \log\left(\frac{\omega(IJ_1)}{\omega(I)}\right). \end{aligned}$$

Let us note

$$A = \sum_{J_1\in\mathcal{F}_{p_1}} \sum_{J_2\in\mathcal{F}_{p_2}} \frac{\omega(IJ_1J_2)}{\omega(I)} \log\left(\frac{\omega(IJ_1J_2)}{\omega(IJ_1)}\right) \text{ and } B = \sum_{J_1\in\mathcal{F}_{p_1}} \sum_{J_2\in\mathcal{F}_{p_2}} \frac{\omega(IJ_1J_2)}{\omega(I)} \log\left(\frac{\omega(IJ_1)}{\omega(I)}\right).$$

Let us recall that there exists a constant $c_2 > 0$ such that $\omega(I_n(x)) \ge c_2\, \omega(I_{n-1}(x))$ for all $x \in \mathbb{K}$ and all $n \in \mathbb{N}$. It follows that $|B| \le -\log(c_2^{p_1})$. We show that if p_2 is big enough, then $-\dfrac{1}{p}A$ will be close to $h(\mu)$ for all I.

By the Shannon-McMillan theorem we get that for p_1 fixed

$$\lim_{n\to\infty} \frac{1}{n} \left|\log\left(\frac{\omega(I_n(x))}{\omega(I_{p_1}(x))}\right)\right| = h(\mu) \text{ for } \omega\text{-almost every } x \in \mathbb{K}. \tag{22}$$

By the dominated convergence theorem we get that there exists $N_0 = N_0(p_1) \in \mathbb{N}$ big enough such that for all $p_2 \geq N_0$ and all $J_1 \in \mathcal{F}_{p_1}$

$$\left|\frac{1}{p_2} \sum_{J_2\in\mathcal{F}_{p_2}} \left|\log\left(\frac{\omega(J_1J_2)}{\omega(J_1)}\right)\right| \frac{\omega(J_1J_2)}{\omega(J_1)} - h(\mu)\right| < \epsilon. \tag{23}$$

By lemma 2.10 we have

$$\left|\frac{\omega(IJ_1J_2)}{\omega(IJ_1)} : \frac{\omega(J_1J_2)}{\omega(J_1)} - 1\right| < C\, q^{p_1} \text{ with } q < 1. \tag{24}$$

Choose p_1 big enough to have $C\, q^{p_1} < \epsilon$ and take $p_2 \geq N_0(p_1)$ in a way that (23) remains valid. Then,

$$\begin{aligned}
-h(\mu) - \frac{1}{p}A &= -h(\mu) - \frac{1}{p} \sum_{J_1\in\mathcal{F}_{p_1}} \sum_{J_2\in\mathcal{F}_{p_2}} \frac{\omega(IJ_1J_2)}{\omega(I)} \log\left(\frac{\omega(IJ_1J_2)}{\omega(IJ_1)}\right) = \\
&= -h(\mu) - \frac{1}{p} \sum_{J_1\in\mathcal{F}_{p_1}} \frac{\omega(IJ_1)}{\omega(I)} \sum_{J_2\in\mathcal{F}_{p_2}} \frac{\omega(IJ_1J_2)}{\omega(IJ_1)} \log\left(\frac{\omega(IJ_1J_2)}{\omega(IJ_1)}\right) \leq \\
&\leq -h(\mu) + (h(\mu)+2\epsilon)\frac{p_2}{p} \sum_{J_1\in\mathcal{F}_{p_1}} \frac{\omega(IJ_1)}{\omega(I)} = \\
&= -h(\mu) + \frac{p_2}{p}(h(\mu)+2\epsilon).
\end{aligned} \tag{25}$$

It suffices now to modify the choice of p_2 by taking, if necessary, p_2 even greater so that $\frac{p_2}{p} = \frac{p_2}{p_1+p_2} > 1-\epsilon$. The lower bound is obtained in the same manner.

To estimate B remark that

$$\left|\frac{1}{p}B\right| \leq -\frac{p_1}{p} \log c_2.$$

By the choice of p_2 we have $\frac{p_1}{p} < \epsilon$ and therefore

$$\left|\frac{1}{p}B\right| \leq -\epsilon \log c_2.$$

The quantities c_2 and $h(\mu)$ not depending on p_1 or p_2, we have shown that

$$\left|\frac{1}{p} \sum_{J\in\mathcal{F}_p} \frac{\omega(IJ)}{\omega(I)} \left|\log\left(\frac{\omega(IJ)}{\omega(I)}\right)\right| - h(\mu)\right| < \epsilon, \tag{26}$$

if p is large enough. •

3 Proof of theorem 1.1

Take $\epsilon > 0$. By theorem 2.5 there is a $\delta > 0$ such that

$$\Big(|a_n - a| < \delta \ , \ \forall n \in \mathbb{N}\Big) \Longrightarrow \left| \frac{\omega(I)}{\omega(\widehat{I})} : \frac{\omega'(I')}{\omega'(\widehat{I}')} - 1 \right| < \epsilon \tag{27}$$

for all $I \in \bigcup\limits_{n\in\mathbb{N}} \mathcal{F}_n$ and $I' \in \bigcup\limits_{n\in\mathbb{N}} \mathcal{F}'_n$ with $I' \overset{\text{cod}}{\sim} I$.

By lemma 2.11 we can find an integer p big enough for the inequality (21) to be valid for all $I \in \bigcup\limits_{n\in\mathbb{N}} \mathcal{F}_n$.

By (27), if $\delta > 0$ is small one gets

$$\left| \frac{\omega(I)}{\omega(\widehat{I})} : \frac{\omega'(I')}{\omega'(\widehat{I}')} - 1 \right| < \epsilon', \tag{28}$$

where $\epsilon' > 0$ is small enough to have $(1+\epsilon')^p < 1 + \epsilon^{10}$.

We obtain

$$\left| \frac{\omega'(I'J')}{\omega'(I')} : \frac{\omega(IJ)}{\omega(I)} - 1 \right| < \epsilon^{10},$$

for all $I \in \bigcup\limits_{n\in\mathbb{N}} \mathcal{F}_n$ with $I' \overset{\text{cod}}{\sim} I$ and all $J \in \mathcal{F}_p$ with $J' \overset{\text{cod}}{\sim} J$.

In the same way as before, for $\delta < a/2$ there is a constant $c_3 = c_3(a) > 0$ such that for all $I' \in \mathcal{F}'_n$ and all $I \in \mathcal{F}_n$, $n \in \mathbb{N}$

$$\omega(I) \geq c_3^n \ \text{ and } \ \omega'(I') \geq c_3^n.$$

Then, if we take $\epsilon < \inf\{\log^{-1} c_3, 2^{-1}\}$,we get that for all $I \in \bigcup\limits_{n\in\mathbb{N}} \mathcal{F}_n$, $I' \overset{\text{cod}}{\sim} I$

$$\begin{aligned}
&\frac{1}{p} \sum_{J'\in\mathcal{F}'_p} \frac{\omega'(I'J')}{\omega'(I')} \left| \log\left(\frac{\omega'(I'J')}{\omega'(I')}\right)\right| \leq (1+\epsilon^{10}) \sum_{\substack{J\in\mathcal{F}_p \\ J \overset{\text{cod}}{\sim} J'}} \frac{\omega(IJ)}{\omega(I)} \left| \log\left(\frac{\omega'(I'J')}{\omega'(I')}\right)\right| \\
&\leq \frac{1}{p}(1+\epsilon^{10})\Big(\sum_{J\in\mathcal{F}_p} \frac{\omega(IJ)}{\omega(I)} \left| \log\left(\frac{\omega(IJ)}{\omega(I)}\right)\right| + \sum_{J\in\mathcal{F}_p} \frac{\omega(IJ)}{\omega(I)} \left|\log\left(1-\epsilon^{10}\right)\right| \Big) \\
&\leq \frac{1}{p} \sum_{J\in\mathcal{F}_p} \frac{\omega(IJ)}{\omega(I)} \left| \log\left(\frac{\omega(IJ)}{\omega(I)}\right)\right| + \frac{1}{p}\epsilon^{10} \sum_{J\in\mathcal{F}_p} \frac{\omega(IJ)}{\omega(I)} \left| \log\left(\frac{\omega(IJ)}{\omega(I)}\right)\right| + \\
&+\frac{1}{p}(1+\epsilon^{10}) \left|\log\left(1-\epsilon^{10}\right)\right| \\
&\leq \frac{1}{p} \sum_{J\in\mathcal{F}_p} \frac{\omega(IJ)}{\omega(I)} \left| \log\left(\frac{\omega(IJ)}{\omega(I)}\right)\right| + \frac{\epsilon^{10}}{p} |\log c_3| + \frac{1}{p}(1+\epsilon^{10}) \left|\log(1-\epsilon^{10})\right|.
\end{aligned} \tag{29}$$

In the same way we obtain

$$\frac{1}{p}\sum_{J'\in\mathcal{F}'_p}\frac{\omega'(I'J')}{\omega'(I')}\left|\log\left(\frac{\omega'(I'J')}{\omega'(I')}\right)\right|\geq$$
$$\geq\frac{1}{p}\sum_{J\in\mathcal{F}_p}\frac{\omega(IJ)}{\omega(I)}\left|\log\left(\frac{\omega(IJ)}{\omega(I)}\right)\right|-\frac{\epsilon^{10}}{p}|\log c_3|-\frac{1}{p}(1+\epsilon^{10})\left|\log(1-\epsilon^{10})\right|. \tag{30}$$

We combine the equations (26), (29) and (30) to get

$$\left|\frac{1}{p}\sum_{J'\in\mathcal{F}'_p}\frac{\omega'(I'J')}{\omega'(I')}\left|\log\left(\frac{\omega'(I'J')}{\omega'(I')}\right)\right|-h(\mu)\right|<\epsilon+\frac{\epsilon^{10}}{p}|\log c_3|+\frac{(1+\epsilon^{10})\left|\log(1-\epsilon^{10})\right|}{p} \tag{31}$$

and therefore

$$\left|\frac{1}{p}\sum_{J'\in\mathcal{F}'_p}\frac{\omega'(I'J')}{\omega'(I')}\left|\log\left(\frac{\omega'(I'J')}{\omega'(I')}\right)\right|-h(\mu)\right|<3\epsilon. \tag{32}$$

Let us now show that for ω'-almost all $x\in\mathbb{K}'$

$$\left|\liminf_{n\to\infty}\frac{\log\omega'(I'_n(x))}{n}+h(\mu)\right|<3\epsilon \tag{33}$$

With this relation and with the equation (2) we will finish the proof.

Consider the sequence of random variables $(X_n)_{n\in\mathbb{N}}$ defined on $\mathbb{K}'$ in the following way:

For $x\in\mathbb{K}'$ we put

$$X_n(x)=-\frac{1}{p}\log\left(\frac{\omega'(I'_{np}(x))}{\omega'(I'_{(n-1)p}(x))}\right). \tag{34}$$

We will make use of the following known version of the theorem of large numbers (see for instance [9]).

Lemma 3.1 *Let X_n be a sequence of uniformly bounded real random variables on a probability space $(\mathbb{X},\mathcal{B},P)$ and let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be an increasing sequence of σ-subalgebra of $\mathbb{B}$ such that X_n is measurable with respect to $\mathcal{F}_n$ for all $n\in\mathbb{N}$. Then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(X_k-\mathbb{E}(X_k|\mathcal{F}_{k-1}))=0\ \ P\text{-almost surely} \tag{35}$$

Consider the sequence of σ-algebras $(\mathcal{R}'_n)_{n\in\mathbb{N}}$ where $\mathcal{R}'_n$ is generated by $\mathcal{F}'_{np}$. The hypothesis of lemma 3.1 can be easily verified to hold for the sequence of random variables $(X_n)_{n\in\mathbb{N}}$ and the sequence of σ-algebras $(\mathcal{R}'_n)_{n\in\mathbb{N}}$.

We get

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[X_k-\mathbb{E}_{\omega'}(X_k|\mathcal{R}'_{k-1})\right]=0\ \ \omega'\text{-almost everywhere}. \tag{36}$$

On the other hand, on $I' \in \bigcup_{n\in\mathbb{N}} \mathcal{R}'_n$,

$$\mathbb{E}_{\omega'}(X_n|\mathcal{R}'_{n-1}) = \frac{1}{p}\sum_{J'\in\mathcal{F}'_p} \frac{\omega'(I'J')}{\omega'(I')}\left|\log\left(\frac{\omega'(I'J')}{\omega'(I')}\right)\right|$$

By relation (32) we obtain

$$|\mathbb{E}_{\omega'}(X_n|\mathcal{R}'_{n-1}) - h(\mu)| < 3\epsilon,$$

for all $n \in \mathbb{N}$. Hence,

$$h(\mu) - 3\epsilon < \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_k < h(\mu) + 3\epsilon \ \ \omega'\text{-almost everywhere.}$$

One can easily verify that,

$$\frac{1}{n}\sum_{k=1}^{n} X_k = \frac{1}{np}\log\omega'(I'_{np}(x)). \tag{37}$$

Therefore,

$$\left|\liminf_{n\to\infty}\frac{1}{n}\log\omega'(I'_n(x)) + h(\mu)\right| < 3\epsilon \ \ \omega'\text{-almost everywhere.} \tag{38}$$

We use this relation together with relation (2) to achieve the proof. The measure ω is monodimensional (as well as ω', see [3]). Remark that for $n \in \mathbb{N}$, an all $I \in \mathcal{F}_n$ the sidelength of the square I est a^n. On the other hand, for each $I' \in \mathcal{F}'_n$ the sidelength of the square I' is bounded by $(a-\delta)^n$ and $(a+\delta)^n$.

By relation (2)

$$\dim\omega = \liminf_{n\to\infty}\frac{\log\omega(I_n(x))}{\log l(I_n(x))} \ \ \omega\text{-almost everywhere on } \mathbb{K}.$$

By the remark 2.3 and the doubling property of the measure ω'

$$\dim\omega' = \text{supess}_{\omega'}\liminf_{n\to\infty}\frac{\omega'(I'_n(x))}{\log l(I'_n(x))} = \text{supess}_{\omega'}\liminf_{n\to\infty}\frac{\omega'(I'_{np}(x))}{\log l(I'_{np}(x))}.$$

Using the Shannon-McMilan's theorem and the fact that $l(I) = a^n$ for all $I \in \mathcal{F}_n$, we get

$$\dim\omega = \frac{h(\mu)}{|\log a|}.$$

On the other hand the relation (38) gives

$$\dim\omega' \in \left]\frac{h(\mu)-3\epsilon}{|\log(a-\delta)|}, \frac{h(\mu)+3\epsilon}{|\log(a+\delta)|}\right[$$

It suffices to choose δ even smaller to have

$$\dim\omega' \in \left]\dim\omega - 5\epsilon, \dim\omega + 5\epsilon\right[.$$

which completes the proof. •

Remark 3.2 Let us point out that we did not need any regularity conditions on the harmonic measure ω' during the proof of theorem 1.1.

Theorem 1.2 is proved with the same arguments: we use the formula $\dim \omega = \dfrac{h(\mu)}{\chi(\mu)}$, valid for the harmonic measure of all self-similar Cantor sets. The entropy $h(\mu)$ is controlled in the same way as in section 3 and it is easy to verify that the Lyapounov exponent varies continuously.

4 Consequences and remarks

A. Ancona showed that the dimension of the harmonic measure of a 4-corners Cantor set, assigned to a constant sequence, converges to 1 as the value of the sequence tends to $\frac{1}{2}$ (see [3]). With this and the previously presented results one can easily prove the following.

Corollary 4.1 *If ω_α is the harmonic measure of the Cantor set assigned to the constant sequence $a_n = \alpha$, then the function $f(\alpha) = \dim \omega_\alpha$ is continuous and takes all values between 0 and 1 as α varies in $]0, \frac{1}{2}[$.*

In [3] we show that for every possible value of the dimension of harmonic measure of a 4-corners Cantor set, there is a Cantor-type set $\mathbb{K}$ (without symmetry properties) such that the dimension of the harmonic measure of the complementary of $\mathbb{K}$ equals $\dim \mathbb{K}$. We deduce

Corollary 4.2 *For all $0 < \alpha < 1$ there is a Cantor-type set $\mathbb{K}$ such that the dimension of the harmonic measure of $\mathbb{R}^2 \setminus \mathbb{K}$ equals $\dim \mathbb{K} = \alpha$.*

We should point out that the proof strongly depends on the invariance properties of harmonic measures of self-similar Cantor sets. Therefore, we have not been able to prove the continuity in the neighborhoods of Cantor sets associated to non-constant sequences $(a_n)_{n\in\mathbb{N}}$.

It is natural to ask whether the relation (4) suffices to conclude that the dimensions of two measures ω and ω' (not necessarily harmonic) are close. This is not the case. There are counterexemples (see [4]) even when the measures are doubling on $(\mathcal{F}_n)_{n\in\mathbb{N}}$ and monodimensional.

The equality between the Hausdorff dimension and the entropy of the harmonic measure of self-similar Cantor sets plays a crucial role in the proof of theorems 1.1 and 1.2. In a more general case, for a monodimensional probability measure μ supported by a Cantor set $\mathbb{K}$, let us define its lower and higher information dimensions (we are using the notation introduced previously):

$$h_*(\mu) = \liminf_{n\to\infty} \frac{1}{n} \sum_{I\in\mathcal{F}_n} |\log \mu(I)|\mu(I) \quad , \quad h^*(\mu) = \limsup_{n\to\infty} \frac{1}{n} \sum_{I\in\mathcal{F}_n} |\log \mu(I)|\mu(I).$$

In order to simplify, suppose that $\mathbb{K}$ is associated to the constant sequence $a_n = e^{-1}$ and therefore $\chi(\mu) = 1$.

We can introduce the random variables X_n, $n \in \mathbb{N}$ as in the equation (34). By Fatou's lemma and relation (2) we get

$$\dim \mu = \mathbb{E}_\mu \left\{ \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} X_k \right\} \leq \liminf_{n\to\infty} \frac{1}{n} \mathbb{E}_\mu \sum_{k=1}^{n} X_k,$$

which gives the known formula (see [10]) $\dim \mu \leq h_*(\mu)$. Some necessary and sufficient conditions in order to have equality are given in [5]. However, we have not been able to prove this equality for the harmonic measure of the Cantor sets of our context.

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Dimension of the harmonic measure of non-homogeneous Cantor sets

Athanasios BATAKIS

Abstract

We prove that the dimension of the harmonic measure of the complementary of a translation-invariant type of Cantor sets as a continuous function of the parameters determining these sets. This results extend a previous one of the author and do not use ergodic theoretic tools, not applicable to our case.

1 Introduction

The purpose of this work is to complement the study of the dimension of the harmonic measure of the complementary of (not necessarily self-similar) Cantor sets as a function of parameters assigned to these sets. In a previous work [5] we have proved that the parameters assigned to self-similar Cantor sets are continuity points for this function. A new method allows us to treat the continuity over the entire family of parameters determining these translation invariant Cantor sets. We restrain ourselves to sets in the plane for convenience, even though the proof can be applied to all "translation-invariant" Cantor sets in $\mathbb{R}^n$, $n \geq 2$.

Let us start by recalling the definition of the Hausdorff dimension of a measure; we will use the notation $\dim_{\mathcal{H}}$ for the Hausdorff dimension of sets.

Definition 1.1 *If μ is a measure on $\mathbb{K}$, we will denote by $\dim_*(\mu)$ the lower Hausdorff dimension of μ:*

$$\dim_* \mu = \inf\{\dim_{\mathcal{H}} E \; ; \; E \subset \mathbb{K} \textit{ and } \mu(E) > 0\}$$

and by $\dim^*(\mu)$ the upper Hausdorff dimension of μ:

$$\dim^* \mu = \inf\{\dim_{\mathcal{H}} E \; ; \; E \subset \mathbb{K} \textit{ and } \mu(\mathbb{K} \setminus E) = 0\}.$$

If, for a measure μ on $\mathbb{K}$, we have $\dim^*(\mu) = \dim_*(\mu)$ then we note this common value $\dim(\mu)$. In the latter case the measure is called exact.

For convenience and in order to fix ideas we consider a particular case of translation invariant Cantor sets; we study 4-corner Cantor sets constructed in the following way (see

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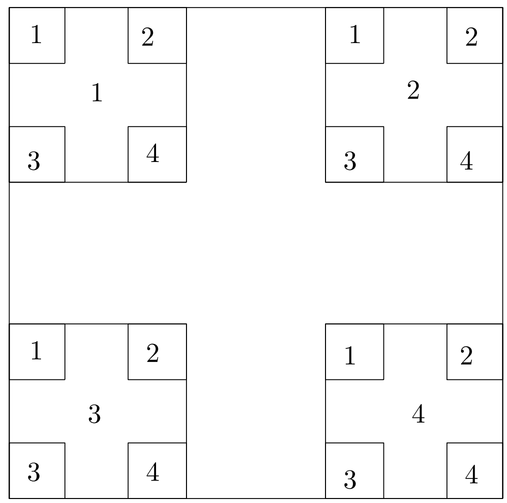

Figure 1: A 4-corner Cantor set and its enumeration

also [3]): let $\underline{A}, \overline{A}$ be two constants with $0 < \underline{A} \leq \overline{A} < \frac{1}{2}$ and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers with $\underline{A} \leq a_n \leq \overline{A}$ for all $n \in \mathbb{N}$.

We replace the square $[0,1]^2$ by four squares of sidelength a_1 situated in the four corners of $[0,1]^2$. Each of these squares is then replaced by four squares of sidelength $a_1 a_2$ situated in its four corners. At the nth stage of the construction every square of the $(n-1)$th generation will be replaced by four squares of sidelength $a_1 ... a_n$ situated in its four corners (see figure 1). Let $\mathbb{K}$ be the Cantor set constructed by repeating the procedure.

Recall that the harmonic measure of a domain is supported by its boundary and can be seen as the distribution of the exit points of Brownian motion starting at some (any) point of the domain (for more details see [13], [16] and [11]). Carleson [12] has shown that for self-similar 4-corner Cantor sets (the sequence $(a_n)_{n\in\mathbb{N}}$ is constant) the dimension of the harmonic measure of their complementary is strictly smaller than 1. His proof, involving ergodic theory techniques, was improved by Makarov and Volberg [20] who showed that the dimension of the harmonic measure of any self-similar 4-corner Cantor set is strictly smaller than the dimension of the Cantor set. Volberg ([23], [24]) extended these results to a class of dynamic Cantor repellers. Other comparisons of harmonic and maximal measures for dynamical systems are proposed in [2], [18], [22]. More recently, a multifractal study of harmonic measure on simply connected domains and on Julia sets of polynomial mappings is carried out in [19], [10].

In [3] it is shown that the dimension of the harmonic measure of the complementary of 4-corner Cantor sets is strictly smaller than the Hausdorff dimension of the Cantor set, even when the sequence $(a_n)_{n\in\mathbb{N}}$ is not constant. In [4] we prove that small perturbations of the sidelength of the squares of the construction of $\mathbb{K}$ do not alterate this property. This last result can also be seen as an immediate consequence of the following theorem.

Theorem 1.2 *Let $\mathbb{K} = \mathbb{K}_{(a_n)}$ be the 4-corners Cantor set associated to a sequence $(a_n)_{n\in\mathbb{N}}$ and $\mathbb{K}' = \mathbb{K}_{(a'_n)}$ a second Cantor set of the same type associated to the sequence $(a'_n)_{n\in\mathbb{N}}$. Let*

ω and ω' be the harmonic measures of $\mathbb{R}^2\setminus\mathbb{K}$ and $\mathbb{R}^2\setminus\mathbb{K}'$ respectively. Then for all $\epsilon>0$ there exists a $\delta=\delta(\epsilon,\overline{A},\underline{A})>0$ such that if $|a'_n-a_n|<\delta$ for all $n\in\mathbb{N}$ then $|\dim\omega-\dim\omega'|<\epsilon$.

When the sequence $(a_n)_{n\in\mathbb{N}}$ is constant the partial result is already established in [5] using ergodic theoretic tools, which are not applicable in the general case.

Remark 1.3 Let $\mathcal{D}:\ell^\infty([\overline{A},\underline{A}])\to[0,1]$ be the function that assigns to a sequence $(a_n)_{n\in\mathbb{N}}\subset[\overline{A},\underline{A}]$ the dimension of harmonic measure of the Cantor set associated to $(a_n)_{n\in\mathbb{N}}$. By refining the estimations in the demonstration of the theorem, we can even show that $\mathcal{D}$ is a Lipschitz continuous function. The proof of this statement is very technical but straightforward and therefore ommited.

In particular this refinement implies that if $\sum_{n\in\mathbb{N}}|a_n-a'_n|<\infty$ then the harmonique measures of the corresponding Cantor sets are of the same dimension, and even equivalent when we report them onto the abstract Cantor set $\{1,2,3,4\}^{\mathbb{N}}$.

2 Notations and Preliminary results

In this section we establish some estimates on the harmonic measure of a Cantor set under perturbation, and recall some known results on the harmonic measure of Cantor-type sets. We also introduce the tools needed, such as the Hausdorff dimension and the entropy of a probability measure on a Cantor set.

Let $\mathbb{K}$ be a 4-corner Cantor set as described in the introduction. We enumerate $\mathbb{K}$ by identifying it to the abstract Cantor set $\{1,...,4\}^{\mathbb{N}}$. We denote $I_{i_1\ldots i_n}$, where $i_j\in\{1,2,3,4\}$ for $1\le j\le n$, the 4^n squares of the n-th generation of the construction of $\mathbb{K}$ with the enumeration shown in the figure 1 and the usual condition that $I_{i_1\ldots i_n}$ is the "father" of the sets $I_{i_1\ldots i_n i}$, $i\in\{1,2,3,4\}$. It is clear that $\overline{A}\ge\dfrac{\mathrm{diam}I_{i_1\ldots i_n i}}{\mathrm{diam}I_{i_1\ldots i_n}}=a_{n+1}\ge\underline{A}$, $i=1,...,4$.

The collection of the squares of the n-th generation of the constuction of $\mathbb{K}$ will be $\mathcal{F}_n=\{I_{i_1\ldots i_n}\,;\,i_1,...,i_n=1,...,4\}$, for $n\in\mathbb{N}$. For a square $I\in\mathcal{F}_n$ we note $\widehat{I}$ the "father" of I, i.e. the unique square of $\mathcal{F}_{n-1}$ containing I. If $I=I_{i_1\ldots i_k}\in\mathcal{F}_k$ and $J=I_{j_1\ldots j_n}\in\mathcal{F}_n$ we will note $IJ=I_{i_1\ldots i_k j_1\ldots j_n}\in\mathcal{F}_{n+k}$. Finally, for $x\in\mathbb{K}$ and $n\in\mathbb{N}$ let $I_n(x)$ be the unique square of $\mathcal{F}_n$ containing x.

For a domain Ω, a point $x\in\Omega$ and a Borel set $F\subset\mathbb{R}^2$ we denote by $\omega(x,F,\Omega)$ the harmonic measure of $F\cap\partial\Omega$ (for the domain Ω) assigned to the point x. Clearly, F carries no measure if it does not intersect $\partial\Omega$. If Ω is not specified it will be $\mathbb{R}^2\setminus\mathbb{K}$ and if x is the point at infinity we will simply denote $\omega(F)$. Finally, for a Borel set $E\subset\mathbb{R}^2$ we note $\dim E$ the Hausdorff dimension of the set E.

2.1 Dimension of measures

In this section we recall some known results on the dimensions of measures (see also [21], [9], [14], [25]). One can prove (see for instance [14], [17], [7]) that if μ is exact, i.e. if

$\dim_* \mu = \dim^* \mu$, then

$$\dim \mu = \liminf_{r \to 0} \frac{\log \mu B(x,r)}{\log r} \ , \ \mu\text{-almost everywhere.} \tag{1}$$

If the probability measure μ is supported by a 4-corner Cantor set, the balls $B(x,r)$ can be replaced by the squares of the construction of the Cantor set (see [8], [4]):

$$\dim \mu = \liminf_{n \to \infty} \frac{\log \mu(I_n(x))}{\log l(I_n(x))} \ , \ \mu\text{-almost everywhere,} \tag{2}$$

where $l(I_n(x))$ is the sidelength of the square $I_n(x)$ and $\underline{A}^n \leq l(I_n) \leq \overline{A}^n$.

Remark 2.1 If μ is an arbitrary (not necessarily exact) probability measure we get

$$\dim_* \mu \leq \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \leq \dim^* \mu \quad \mu\text{-almost everywhere.} \tag{3}$$

Moreover $\dim^* \mu = \operatorname{supess}_\mu \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}$ and $\dim_* \mu = \operatorname{infess}_\mu \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}$.

Some results of the following section are stated without demonstration since they are already proved in [3] and in [5].

2.2 Estimating perturbations of the harmonic measure

Suppose that the 4-corner Cantor set $\mathbb{K}$ is associated to the sequence $(a_n)_{n \in \mathbb{N}}$ and let $\mathbb{K}'$ be another Cantor set associated to the sequence $(a'_n)_{n \in \mathbb{N}}$. Let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be the collections of squares associated to $\mathbb{K}$ and $(\mathcal{F}'_n)_{n \in \mathbb{N}}$ those associated to $\mathbb{K}'$.

For $I \in \mathcal{F}_n$ and $I' \in \mathcal{F}'_m$ we will write $I \overset{\text{cod}}{\sim} I'$ if $n = m$ and if I and I' have the same encoding (with respect to the identification to the abstract Cantor set $\{1,2,3,4\}^{\mathbb{N}}$).

Finally, if $I \subset \mathbb{R}^2$ is a square of sidelength ℓ and c is a positive number we note $c\dot{I}$ the square of sidelength $c\ell$ having the same barycenter as I. If ω is the harmonic measure of $\mathbb{R}^2 \setminus \mathbb{K}$ and ω' the harmonic measure of $\mathbb{R}^2 \setminus \mathbb{K}'$ we have established the following theorem.

Theorem 2.2 *(cf. [5]) For all $\epsilon > 0$ there exists a $\delta = \delta(\epsilon, \overline{A}, \underline{A}) > 0$ such that*

$$\sup_{n \in \mathbb{N}} |a_n - a'_n| < \delta \Rightarrow \left| \frac{\omega(I)}{\omega(\widehat{I})} : \frac{\omega'(I')}{\omega'(\widehat{I'})} - 1 \right| < \epsilon, \tag{4}$$

for all $I \in \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ and $I' \in \bigcup_{n \in \mathbb{N}} \mathcal{F}'_n$ with $I \overset{\text{cod}}{\sim} I'$.

We will also need the following estimations of the harmonic measure of cylinders (see also [12] and [20] for a version adapted to self-similar sets). The proof uses the ideas already explored in [5].

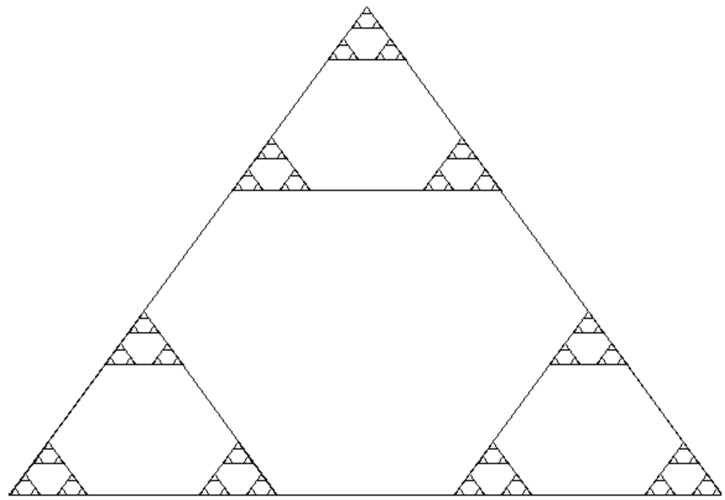

Figure 2: A different type of translation-invariant Cantor set

Lemma 2.3 *For every $I, I' \in \mathcal{F}_n$, $J \in \mathcal{F}_k$ and every $L \in \mathcal{F}_m$, $n, k, m \in \mathbb{N}$*

$$\left| \frac{\omega(IJL)}{\omega(IJ)} : \frac{\omega(I'JL)}{\omega(I'J)} - 1 \right| < C\, q^k \tag{5}$$

where the constants $C > 0$ and $q \in (0,1)$, depend only on $\underline{A}, \overline{A}$.

Let us give the proof of this statement.

Proof of Lemma 2.3 To begin with we need the following Harnack principle (see also [4], [1]).

Lemma 2.4 *(cf. [12], [20]) Let Ω be a domain containing ∞ and let $A_1 \subset B_1 \subset A_2 \subset B_2 \subset \ldots \subset A_n \subset B_n$ be conformal discs such that the annuli $B_i \setminus A_i$ are contained in Ω, for $1 \le i \le n$. If the moduli of the annuli are uniformly bounded away from zero and if $\infty \in \Omega \setminus B_n$ then, for all pairs of positive harmonic functions u, v vanishing on $\partial\Omega \setminus A_1$ and for all $x \in \Omega \setminus B_n$ we have*

$$\left| \frac{u(x)}{v(x)} : \frac{u(\infty)}{v(\infty)} - 1 \right| \le K q^n \tag{6}$$

where $q < 1$ and K are two constants that depend only on the lower bound of the moduli of the annuli.

We use this result to prove the following:

Lemma 2.5 *There are constants $K > 0$ and $0 < q < 1$ depending only on $\underline{A}, \overline{A}$ such that for all $i, j, k \in \mathbb{N}$ and for all squares $I \in \mathcal{F}_i, J \in \mathcal{F}_j, K \in \mathcal{F}_k$ of the construction of $\mathbb{K}$, if*

$Q = c_0 \cdot I$,

$$\left| \frac{\omega(x, IJK, Q \setminus \mathbb{K})}{\omega(x, IJ, Q \setminus \mathbb{K})} : \frac{\omega(IJK)}{\omega(IJ)} - 1 \right| < Kq^j \textit{ for all } x \in \partial \left\{ \frac{1+c_0}{2} \cdot I \right\}. \tag{7}$$

The result applies also to the Cantor set $\mathbb{K}'$.

Proof By Lemma 2.4,

$$\left| \frac{\omega(x, IJK)}{\omega(x, IJ)} : \frac{\omega(IJK)}{\omega(IJ)} - 1 \right| < Kq^j \text{ , for } x \notin \frac{1+c_0}{2} \cdot I \tag{8}$$

Let $A = \dfrac{\omega(IJK)}{\omega(IJ)}$. We have

$$\omega(x, IJK, Q \setminus \mathbb{K}) = \omega(x, IJK) - \int_{\partial Q} \omega(z, IJK)\omega(x, dz, Q \setminus \mathbb{K}),$$

for $x \in \partial \left\{ \frac{1+c_0}{2} \cdot I \right\}$.

By the equation (8),

$$A\omega(x, IJ) - Kq^j A\omega(x, IJ) \le \omega(x, IJK) \le A\omega(x, IJ) + Kq^j A\omega(x, IJ).$$

We get

$$\begin{aligned}
\omega(x, IJK, Q \setminus \mathbb{K}) &\le A\omega(x, IJ) + Kq^j A\omega(x, IJ) - \\
&- \int_{\partial Q} \Big(A\omega(z, IJ) - Kq^j A\omega(z, IJ) \Big) \omega(x, dz, Q \setminus \mathbb{K}) \\
&= A\omega(x, IJ) - \int_{\partial Q} A\omega(z, IJ)\omega(x, dz, Q \setminus \mathbb{K}) + \\
&+ Kq^j \Big(A\omega(x, IJ) + \int_{\partial Q} A\omega(z, IJ)\omega(x, dz, Q \setminus \mathbb{K}) \Big) \\
&= A\omega(x, IJ, Q \setminus \mathbb{K}) + Kq^j \Big(A\omega(x, IJ) + \\
&+ \int_{\partial Q} A\omega(z, IJ)\omega(x, dz, Q \setminus \mathbb{K}) \Big)
\end{aligned} \tag{9}$$

Therefore,

$$\frac{\omega(x, IJK, Q \setminus \mathbb{K})}{\omega(x, IJ, Q \setminus \mathbb{K})} \le A + Kq^j A \frac{\omega(x, IJ) + \int_{\partial Q} \omega(z, IJ)\omega(x, dz, Q \setminus \mathbb{K})}{\omega(x, IJ, Q \setminus \mathbb{K})} \tag{10}$$

It suffices now to show that the quantity

$$\frac{\omega(x, IJ) + \int_{\partial Q} \omega(z, IJ)\omega(x, dz, Q \setminus \mathbb{K})}{\omega(x, IJ, Q \setminus \mathbb{K})}$$

is smaller that a given constant. Take $x_0 \in \partial\left\{\frac{1+c_0}{2} \cdot I\right\}$ such that

$$\omega(x_0, IJ) = \max\left\{\omega(x, IJ)\ ;\ x \notin \left\{\frac{1+c_0}{2} \cdot I\right\}\right\}.$$

Using the maximum principle we get

$$\begin{aligned}
\omega(x_0, IJ, Q \setminus \mathbb{K}) &= \omega(x_0, IJ) - \int_{\partial Q} \omega(z, IJ)\omega(x_0, dz, Q \setminus \mathbb{K}) \\
&\geq \omega(x_0, IJ) - \int_{\partial Q} \omega(x_0, IJ)\omega(x_0, dz, Q \setminus \mathbb{K}) \\
&= \omega(x_0, IJ)(1 - \omega(x_0, \partial Q, Q \setminus \mathbb{K}))
\end{aligned}$$

By standard capacitary techniques one can verify (see [3]) that $1 - \omega(x_0, \partial Q, Q \setminus \mathbb{K})$ is greater that a constant $c > 0$ depending only on $\underline{A}, \overline{A}$.

By using Harnack's principle we get

$$1 - \omega(x, \partial Q, Q \setminus \mathbb{K}) \geq c \text{ , for all } x \in \partial\left\{\frac{1+c_0}{2} \cdot I\right\},$$

for a new constant $c > 0$.

Hence, $\dfrac{\omega(x, IJ) + \int_{\partial Q} \omega(z, IJ)\omega(x, dz, Q \setminus \mathbb{K})}{\omega(x, IJ, Q \setminus \mathbb{K})} \leq \dfrac{2}{c}$ and therefore, by relation (10),

$$\frac{\omega(x, IJK, Q \setminus \mathbb{K})}{\omega(x, IJ, Q \setminus \mathbb{K})} \leq A(1 + \frac{2}{c}Kq^j) \tag{11}$$

On the other hand $A = \dfrac{\omega(IJK)}{\omega(IJ)}$; we obtain

$$\frac{\omega(x, IJK, Q \setminus \mathbb{K})}{\omega(x, IJ, Q \setminus \mathbb{K})} : \frac{\omega(IJK)}{\omega(IJ)} - 1 < \frac{2}{c}Kq^j \text{ , for all } x \in \partial\left\{\frac{1+c_0}{2} \cdot I\right\},$$

The left hand inequality and hence the Lemma 2.5 is established in the same way . •

It is now evident that

$$\frac{\omega(x, IJK, Q \setminus \mathbb{K})}{\omega(x, IJ, Q \setminus \mathbb{K})} = \frac{\omega(x, I'JK, Q' \setminus \mathbb{K})}{\omega(x, I'J, Q' \setminus \mathbb{K})},$$

for any square $I' \in \mathcal{F}_n$, where $Q' = c_0 \cdot I'$. The proof of Lemma 2.3 is complete. •

Corollary 2.6 *There is a constant $\tilde{C} > 1$ such that for any $n, k \in \mathbb{N}$, all $I, I' \in \mathcal{F}_n$ and every $J \in \mathcal{F}_k$ we have*

$$\frac{\omega(IJ)}{\omega(I)} \leq \tilde{C}\frac{\omega(I'J)}{\omega(I')} \tag{12}$$

where the constant $\tilde{C} > 0$ depends only on $\underline{A}, \overline{A}$.

The proof of the corollary is an easy application of Lemma 2.3.

3 Proof of the main result.

This section is dedicated to the proof of theorem 1.2. We will make use of the following known version of the theorem of large numbers (see for instance [15]).

Lemma 3.1 *Let X_n be a sequence of uniformly bounded real random variables on a probability space $(\mathbb{X}, \mathcal{B}, P)$ and let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be an increasing sequence of σ-subalgebra of $\mathbb{B}$ such that X_n is measurable with respect to $\mathcal{F}_n$ for all $n \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (X_k - \mathbb{E}(X_k|\mathcal{F}_{k-1})) = 0 \ \ P\text{-almost surely} \tag{13}$$

The following elementary Lemma is also useful; the proof is left to the reader.

Lemma 3.2 *Let $\alpha_1, ..., \alpha_n$ be real numbers such that $\displaystyle\sum_{i=1}^{n} \alpha_i = 0$. Then, for any choise of real values $h_1, ..., h_n$, we have*

$$\begin{aligned} |\textstyle\sum_{i=1}^{n} \alpha_i h_i| \ &\leq \ \max\left\{ \sum_{\{i\ ;\ \alpha_i>0\}} \alpha_i\ ,\ -\sum_{\{i\ ;\ \alpha_i<0\}} \alpha_i \right\} \left(\max_{1\leq i\leq n} h_i - \min_{1\leq i\leq n} h_i \right) \\ &= \ \sum_{\{i\ ;\ \alpha_i>0\}} \alpha_i \left(\max_{1\leq i\leq n} h_i - \min_{1\leq i\leq n} h_i \right). \end{aligned}$$

Proof of theorem 1.2 For $p \in \mathbb{N}$ consider the sequence of σ-algebras $(\mathcal{R}_n)_{n\in\mathbb{N}}$ where $\mathcal{R}_n$ is generated by $\mathcal{F}_{np}$.

The hypothesis of Lemma 3.1 can be easily verified to hold for the sequence of random variables $(X_n^p)_{n\in\mathbb{N}}$ given by $X_n^p(x) = \dfrac{1}{p}\left|\log\left(\dfrac{\omega\left(I_{np}(x)\right)}{\omega\left(I_{(n-1)p}(x)\right)}\right)\right|$ and the sequence of σ-algebras $(\mathcal{R}_n)_{n\in\mathbb{N}}$.

We get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Big[X_k^p - \mathbb{E}_\omega(X_k^p|\mathcal{R}_{k-1}) \Big] = 0 \ \ \omega\text{-almost everywhere.} \tag{14}$$

On the other hand, on $I \in \mathcal{R}_{n-1}$, $n \in \mathbb{N}$,

$$\mathbb{E}_\omega(X_n^p|\mathcal{R}_{n-1}) = \frac{1}{p} \sum_{J\in\mathcal{F}_p} \frac{\omega(IJ)}{\omega(I)} \left|\log\left(\frac{\omega(IJ)}{\omega(I)}\right)\right|.$$

We show that this quantity is almost constant on $x \in \mathbb{K}$ if p is taken sufficiently large.

Take $\epsilon > 0$, $n \in \mathbb{N}$ and $I \in \mathcal{F}_n$. For $j, k \in \mathbb{N}$ we have

$$\sum_{J\in\mathcal{F}_{j+k}} \frac{\omega(IJ)}{\omega(I)} \log\left(\frac{\omega(IJ)}{\omega(I)}\right) = \sum_{J\in\mathcal{F}_j} \sum_{K\in\mathcal{F}_k} \frac{\omega(IJK)}{\omega(I)} \log\left(\frac{\omega(IJK)}{\omega(I)}\right) =$$

$$= \sum_{J\in\mathcal{F}_j}\sum_{K\in\mathcal{F}_k}\frac{\omega(IJK)}{\omega(IJ)}\log\left(\frac{\omega(IJK)}{\omega(IJ)}\right)\frac{\omega(IJ)}{\omega(I)}+\sum_{J\in\mathcal{F}_j}\frac{\omega(IJ)}{\omega(I)}\log\left(\frac{\omega(IJ)}{\omega(I)}\right) \tag{15}$$

For $L\in\mathcal{F}_n$ and $k,n,j\in\mathbb{N}$ we note

$$h_k(L)=-\frac{1}{k}\sum_{K\in\mathcal{F}_k}\frac{\omega(LK)}{\omega(L)}\log\left(\frac{\omega(LK)}{\omega(L)}\right).$$

In particular, we put

$$h_k(IJ)=-\frac{1}{k}\sum_{K\in\mathcal{F}_k}\frac{\omega(IJK)}{\omega(IJ)}\log\left(\frac{\omega(IJK)}{\omega(IJ)}\right) \text{ and } \Delta_j^k(I)=\max_{J\in\mathcal{F}_j}h_k(IJ)-\min_{J\in\mathcal{F}_j}h_k(IJ).$$

We will use the following lemma.

Lemma 3.3 *For all $\epsilon>0$, if $j,k\in\mathbb{N}$ are big enough (depending only on $\underline{A}$ and $\overline{A}$) then for all $n\in\mathbb{N}$ and $I\in\mathcal{F}_n$ we have $\Delta_j^k(I)<\epsilon$.*

We first proceed with the proof of this sub-Lemma.

Proof of Lemma 3.3 We can rewrite formula (15):

$$(j+k)h_{j+k}(I)=\sum_{J\in\mathcal{F}_j}k\frac{\omega(IJ)}{\omega(I)}h_k(IJ)+jh_j(I) \tag{16}$$

By applying formula (16) to a cylinder $I=I_1I_2$ with $I_1\in\mathcal{F}_{i_1}$ and $I_2\in\mathcal{F}_{i_2}$ we have

$$(j+k)h_{j+k}(I_1I_2)=\sum_{J\in\mathcal{F}_j}k\frac{\omega(I_1I_2J)}{\omega(I_1I_2)}h_k(I_1I_2J)+jh_j(I_1I_2)$$

Now take j big enough to have $Kq^j<\epsilon$ and afterwards choose k in order to have that $\dfrac{j}{k+j}<\epsilon$. Remark that, by Lemma 2.3, $h_k(I_1I_2'J)-h_k(I_1I_2J)<2\epsilon$. We have

$$\begin{aligned}
\Delta_{i_2}^{k+j}(I_1) &= \max_{I_2\in\mathcal{F}_{i_2}}h_{k+j}(I_1I_2)-\min_{I_2\in\mathcal{F}_{i_2}}h_{k+j}(I_1I_2)\\
&\leq 5\epsilon+\max_{I_2,I_2'\in\mathcal{F}_{i_2}}\left[\sum_{J\in\mathcal{F}_j}\frac{\omega(I_1I_2J)}{\omega(I_1I_2)}h_k(I_1I_2J)-\sum_{J\in\mathcal{F}_j}\frac{\omega(I_1I_2'J)}{\omega(I_1I_2')}h_k(I_1I_2'J)\right]\\
&\leq 10\epsilon+\max_{I_2,I_2'\in\mathcal{F}_{i_2}}\sum_{J\in\mathcal{F}_j}\left(\frac{\omega(I_1I_2J)}{\omega(I_1I_2)}-\frac{\omega(I_1I_2'J)}{\omega(I_1I_2')}\right)h_k(I_1I_2J).
\end{aligned} \tag{17}$$

We can now apply Lemma 3.2 to get

$$\Delta_{i_2}^{k+j}(I_1)\leq 10\epsilon+\max_{I_2,I_2'\in\mathcal{F}_{i_2}}\sum_{J\in S_{i_1}^j(I_2,I_2')}\left(\frac{\omega(I_1I_2J)}{\omega(I_1I_2)}-\frac{\omega(I_1I_2'J)}{\omega(I_1I_2')}\right)\Delta_j^k(I_1I_2) \tag{18}$$

where $\mathcal{S}^j_{I_1}(I_2,I'_2)$ is the set of cylinders $J \in \mathcal{F}_j$ such that $\dfrac{\omega(I_1I_2J)}{\omega(I_1I_2)} > \dfrac{\omega(I_1I'_2J)}{\omega(I_1I'_2)}$.

The following lemma is easy to prove:

Lemma 3.4 *There is a constant $0<\zeta<1$ such that for all $i_1, i_2, j \in \mathbb{N}$, all $I_1 \in \mathcal{F}_{i_1}$ and all $I_2, I'_2 \in \mathcal{F}_{i_2}$ we have*

$$\sum_{J\in S^j_{I_1}(I_2,I'_2)} \left(\frac{\omega(I_1I_2J)}{\omega(I_1I_2)} - \frac{\omega(I_1I'_2J)}{\omega(I_1I'_2)} \right) < \zeta$$

where $\mathcal{S}^j_{I_1}(I_2,I'_2)$ is the set of cylinders $J \in \mathcal{F}_j$ such that $\dfrac{\omega(I_1I_2J)}{\omega(I_1I_2)} > \dfrac{\omega(I_1I'_2J)}{\omega(I_1I'_2)}$.

The proof follows from the translation invariance of ω (corollary 2.6).

Proof of Lemma 3.4 Remark that by Lemma 2.3, if $C = \dfrac{K}{1-q}$, we have $\dfrac{\omega(IJ)}{\omega(I'J)} \le C$ for all $I, I' \in \mathcal{F}_n$, $n \in \mathbb{N}$, and all $J \in \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$. Hence,

$$\sum_{J\in S^j_{I_1}(I_2,I'_2)} \left(\frac{\omega(I_1I_2J)}{\omega(I_1I_2)} - \frac{\omega(I_1I'_2J)}{\omega(I_1I'_2)} \right) \le \left(1-\frac{1}{C}\right) \sum_{J\in S^j_{I_1}(I_2,I'_2)} \frac{\omega(I_1I_2J)}{\omega(I_1I_2)} \le 1-\frac{1}{C},$$

which is the Lemma conclusion for $\zeta = 1-\dfrac{1}{C}$. •

By applying Lemma 3.4 to relation (18) we conclude that there is $0<\zeta<1$ depending only on $\underline{A}, \overline{A}$ such that

$$\Delta^{k+j}_{i_2}(I_1) \le 10\epsilon + \zeta\Delta^k_j(I_1I_2) \tag{19}$$

By repeating the same reasoning if we write $k = k_1 + k_2$ we can establish the inequalities

$$\Delta^{k_1+k_2}_j(I_1I_2) < 10\epsilon + \zeta\Delta^{k_2}_{k_1}(I_1I_2J). \tag{20}$$

Either $\Delta^{k_2}_{k_1}(I_1I_2J) < \frac{10\epsilon}{1-\zeta}$ and the proof is complete, or $\Delta^{k_2}_{k_1}(I_1I_2J) > \frac{10\epsilon}{1-\zeta}$ and we obtain

$$\Delta^{k+j}_{i_2}(I_1) - \frac{10\epsilon}{1-\zeta} \le \zeta\left(\Delta^{k_2}_{k_1}(I_1I_2J) - \frac{10\epsilon}{1-\zeta}\right) \le \zeta^2\left(\Delta^{k_2}_{k_1}(I_1I_2J) - \frac{10\epsilon}{1-\zeta}\right).$$

The sequence $\Delta^m_l(I)$ being uniformly bounded we get, by decomposing again k_2 and repeating the procedure, that if k is big enough, $\Delta^{k+j}_{i_2} < \dfrac{20\epsilon}{1-\zeta}$.

The real constant ζ depending only on $\underline{A}$, $\overline{A}$, the proof is complete. •

We now apply Lemma 3.1 to an adapted filtration $\mathcal{R}_n$: By the previous Lemma we can choose j, k such that $\Delta_j^k(I) < \epsilon$. By formula (16), for this choice of j and k and for all $n \in \mathbb{N}$ there are constants c_n such that

$$\left| \frac{1}{k+j} \mathbb{E} \left\{ \sum_{\ell=1}^{k+j} X_{n+\ell}^1 \Big| \mathcal{F}_n \right\} - c_n \right| < \epsilon.$$

By Lemma 3.1 and the relation (14) following it we then deduce (for $p = k + j$)

$$\left| \liminf_{n\to\infty} \frac{1}{n(k+j)} \sum_{\ell=1}^{n(k+j)} X_\ell^1 - \liminf_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^{n} c_\ell \right| < \epsilon \ , \ \omega \text{ - a.e. on } \mathbb{K}$$

This implies that

$$\left| \liminf_{n\to\infty} \left| \frac{\log \omega(I_{n(k+j)}(x))}{\log \left(\prod_{i=1}^{n(k+j)} a_i \right)} \right| - \liminf_{n\to\infty} \left| \frac{k+j}{\log \left(\prod_{i=1}^{n(k+j)} a_i \right)} \sum_{\ell=1}^{n} c_\ell \right| \right| < \epsilon \ , \ \omega \text{ - a.e. on } \mathbb{K}.$$

On the other hand, once we have fixed k, j we can use Lemma 2.2 to choose δ in a way that, for all $n \in \mathbb{N}$,

$$|c_n - c'_n| < \epsilon, \text{ and } \frac{1}{n} \left| \log \left(\prod_{i=1}^{n} a_i \right) - \log \left(\prod_{i=1}^{n} a'_i \right) \right| < \epsilon,$$

where c'_n is the same sequence associated to the harmonique measure ω'. We can finally use relation (2) to conclude that $|\dim \omega - \dim \omega'| < 4\epsilon$. •

4 Consequences and remarks

It is implicitely proved that the harmonic measure of the sets $\mathbb{K}$ studied here satisfy the relationship $\dim_* \omega = h_*^{\mathbb{K}}(\omega)$, where

$$h_*^{\mathbb{K}}(\omega) = \liminf_{n\to\infty} \frac{1}{\log \prod_{i=1}^{n} a_i} \sum_{I \in \mathcal{F}_n} \log \omega(I) \omega(I),$$

and $(a_n)_{n\in\mathbb{N}}$ is the construction sequence associated to $\mathbb{K}$. This fact is a consequence of the space invariance of ω and is a key factor in the proof of our results.

It is natural to ask whether the relation (4) suffices to conclude that the dimensions of two measures ω and ω' (not necessarily harmonic) are close. This is not the case. There are counterexemples (see [4]) even when the measures are doubling on $(\mathcal{F}_n)_{n\in\mathbb{N}}$ and exact (cf [6]).

Even though the equality between the Hausdorff dimension and the entropy of the harmonic measure of the complementary of Cantor sets plays a crucial role in the proof of theorem 1.2, it is in general not sufficient to establhish continuity results : the measures constructed μ and ν in the example of [6] are exact and satisfy the relation

$$\left| \frac{\mu(I)}{\mu(\widehat{I})} : \frac{\mu(I)}{\mu(\widehat{I})} - 1 \right| < \delta,$$

with δ as small as we want. Nevertheless, $|\dim \mu - \dim \nu| \geq \dfrac{1}{2}$.

To establish the result claimed in remark 1.3 we first need a fine precision of inequalities in theorem 2.2 and secondly we need to quantify the dependance on ϵ of the choice of k in Lemma 3.3. In fact, we can find suitable k's that are bounded by $-C\log(\epsilon)$, where C is a positive constant depending on $\underline{A}$, $\overline{A}$, which is sufficient in order to prove the claim.

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On the exit distribution of partially reflected brownian motion in planar domains

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Abstract: *We show that the dimension of the exit distribution of planar partially reflected Brownian motion can be arbitrarily close to 2.*

1 Introduction

Let Ω be a domain in $\mathbb{R}^2$. It is well known (see [Mak85], [JW88]) that the exit distribution of Brownian motion in Ω is carried by a borel subset of the boundary of dimension at most 1 (equal to one for simply connected domains). We are interested in the minimal dimension of sets carrying the exit distribution of partially reflected Brownian motion.

The problem is posed as follows. Consider an (ϵ, δ) domain Ω , take $F \subset \Omega$ a closed subset of the boundary of Ω and consider Brownian Motion inside Ω absorbed by F and reflected on $\partial\Omega \setminus F$ (for definitions of the (ϵ, δ) domains and of reflected brownian motion see section 2). Note $\mathcal{R}_t$ the above process and τ_F the (first) hitting time of F by $\mathcal{R}_t$. In general, τ_F may not be finite or may be finite but of infinite expectation (see also the so called "trap domains" [BCM06]).

We prove the following theorem

Theorem 1.1 *For all $\eta > 0$ there exist a domain Ω (that can be taken simply connected) and $F \subset \partial\Omega$ such that $\mathbb{P}_x(\tau_F < \infty) = 1$ and for all $x \in \Omega$ and for all $A \subset F$ of dimension $\dim A < 2 - \eta$ we have,*

$$\mathbb{P}_x(\mathcal{R}_{\tau_F} \in A) = 0$$

In particular this answers a question of B. Sapoval concerning Brownian motion as we will point out at the end of the paper: Consider a domain Ω and let A be a subset of the boundary of (standard) harmonic measure equal to 1. If we change A into reflecting boundary, is the dimension of the exit distribution for this new diffusion still less than 1?

Our proof can not be generalized in higner dimensions; the distribution of harmonic measure in dimesion greater than 3 is considered by Bourgain [Bou87]. For more information about progress in harmonic measure see also the recent review of C. Bishop concerning

[GM05] and the references therein. Let us also point out that the case of p-harmonic measure has been studied by Lewis, Nystrm and Poggi-Corradini [LVV05].

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2 Definitions of the main objects

The following definition is due to P. Jones [Jon81].

Definition 2.1 *We say that a (not necessarily simply connected) domain Ω is an (ϵ,δ)-domain or locally uniform if there exist constants ϵ and δ such that for all $x, y \in \Omega$ with $|x-y| < \delta$ there is a (rectifiable) curve γ joining x and y satisfying*

1. $\epsilon\ell(\gamma) \leq |x-y|$
2. $\epsilon \min\{|x-z|, |y-z|\} \leq \operatorname{dist}(z, \partial\Omega)$

The (ϵ,δ)-domains satisfy the so called $W^{1,2}$- extension property, cf [Jon81]: if we note $W^{1,2}(\Omega) = \{f \in L^2(\Omega)\ ;\ \nabla f \in L^2(\Omega)\}$ with the usual Sobolev norm $||f||_{1,2} = ||f||_2 + ||\nabla f||_2$, we assume that there is a bounded linear operator $T : W^{1,2}(\Omega) \to W^{1,2}(\mathbb{R})$ extending the identity of $W^{1,2}(\Omega)$.

For $f, g \in W^{1,2}(\Omega)$ define

$$\mathcal{E}(f,g) = \int_\Omega < \nabla f, \nabla g > dx,$$

and

$$\mathcal{E}_1(f,g) = \mathcal{E}(f,g) + \int_\Omega fg dx.$$

The Dirichlet form $(\mathcal{E}, W^{1,2}(\Omega))$ is said to be regular on $\overline{\Omega}$ if $W^{1,2}(\Omega) \cap C(\overline{\Omega})$ is dense both in $(W^{1,2}(\Omega), \mathcal{E}_1^{\frac{1}{2}})$ and in $(C(\overline{\Omega}), ||.||_\infty)$. Clearly, if Ω is a (ϵ,δ)-domain the Dirichlet form $(\mathcal{E}, W^{1,2}(\Omega))$ is regular on $\overline{\Omega}$.

Following [Che93], [BCR04] we can now define the "reflected" Brownian motion. If Ω in an (ϵ,δ)-domain, there is a strong Markov process $\mathcal{R}$ associated, with continuous sample paths. Furthermore, we can construct a family of distributions $(\mathcal{R}_t^x)_t$ for this process starting at every $x \in \overline{\Omega}$ (for further detail see also [FOT94]).

Take F a closed subset of $\partial\Omega$ and consider τ_F the hitting time of F for the process $\mathcal{R}_t^x$. Now if we suppose that $\mathbb{E}_x[\tau_F] < +\infty$ for at least one $x \in \Omega$, we get that for any $f \in C(F)$, the function

$$u : x \mapsto \mathbb{E}_x \left[f(\mathcal{R}_{\tau_F}) \right]$$

is bounded harmonic in Ω and takes the value f at all regular points of F.

If we suppose that $\partial\Omega \backslash F$ is smooth then u is the solution to the mixed Dirichlet-Neumann problem

$$\begin{cases} u \text{ harmonic in } \Omega \\ \dfrac{\partial u}{\partial \eta} = 0 \text{ on } \partial\Omega \setminus F \\ u = f \text{ on } F \end{cases}, \tag{1}$$

where η denotes the normal vector to the boundary $\partial\Omega$.

Remark 2.2 We denote by $C_F(\overline{\Omega})$ the set of continuous functions on $\overline{\Omega}$ vanishing on F. Suppose that $W^{1,2}(\Omega) \cap C_F(\overline{\Omega})$ is dense in $C_F(\overline{\Omega}), ||.||_\infty)$. We can then define the stochastic process $\mathcal{R}_t^F$ associated. This process agrees with the previous one for all (ϵ, δ)-domains (see also [AB10]).

Let $\omega_.$ denote the harmonic measure of this diffusion, ie. for $x \in \Omega$ and $A \subset \partial\Omega$,

$$\omega_x(A) = \mathbb{P}_x(\mathcal{R}_{\tau_F} \in A).$$

Remark that, from (1), for $A \subset \partial\Omega$ measurable, the function $x \mapsto \omega_x(A)$ is positive harmonic in Ω, tending to 1 on A, to 0 on $F \setminus A$ and of nul normal derivative on $\partial\Omega \setminus F$.

In the following we keep this same notation.

3 Preliminary lemmas and remarks

Let $E \subset \mathbb{R}^2$ be any set and, for every covering $\mathcal{V}_\delta(E)$ of E with discs of radius less than δ, let $H_\alpha\left(\mathcal{V}_\delta(E)\right) = \sum_{B \in \mathcal{V}_\delta} \operatorname{diam}(B)^\alpha$. Consider

$$\mathcal{H}_\alpha^\delta(E) = \inf_{\mathcal{V}_\delta(E)} H_\alpha\left(\mathcal{V}_\delta(E)\right) \text{ and } \mathcal{H}_\alpha(E) = \lim_{\delta \to 0} \mathcal{H}_\alpha^\delta(E)$$

Then, there exists an $\alpha_0 \geq 0$ such that $\mathcal{H}_\alpha(E) = 0$ for all $\alpha > \alpha_0$ and $\mathcal{H}_\alpha(E) = \infty$ for all $0 \leq \alpha < \alpha_0$. This α_0 is denoted $\dim_{\mathcal{H}}(E)$, the Hausdorff dimension of E.

For a Borel measure μ we define the Hausdorff dimension of μ as

$$\dim_{\mathcal{H}}(\mu) = \inf\{\dim_{\mathcal{H}}(E) \; ; \; \mu(E) > 0\}$$

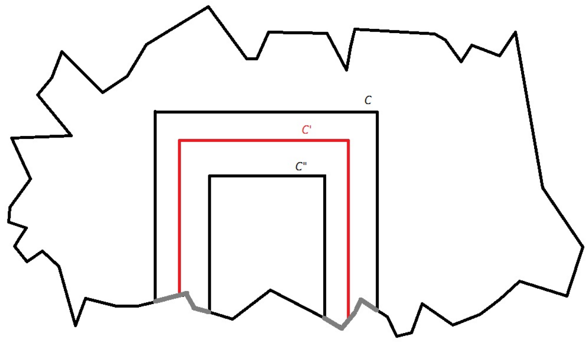

Figure 1: *Boundary Harnack Principle.*

In particular, let μ be the harmonic measure $\omega_.$ defined above. Using the fact that, for any $A \subset F$, $x \mapsto \omega_x(A)$ is harmonic we get that $\dim_{\mathcal{H}}(\omega_x)$ does not depend on the choice of $x \in \Omega$ and will be therefore denoted by $\dim_{\mathcal{H}}(\omega)$.

In this paper we are interested in the dimension of harmonic measure for partially reflected Brownian motion in domains in $\mathbb{R}^2$. Theorem 1.1 can now be reformulated in the following terms:

"for all $\eta > 0$, there exists a uniform planar domain Ω and a closed set $F \subset \partial\Omega$ such that if $\omega_.$ is the harmonic measure for partially reflected Brownian motion (ie. reflected on $\partial\Omega \setminus F$, absorbed on F) we have $\dim_{\mathcal{H}}(\omega) > 2 - \eta$*."*

Clearly, $\dim_{\mathcal{H}}(\omega) \leq \dim_{\mathcal{H}}(F)$. Therefore the boundary of Ω will be of Hausdorff dimension $\geq 2 - \eta$.

3.1 Potential theoretic lemmas

By "adapted cylinder" $\mathcal{C}$ to a graph Γ of a Lipschitz function f we understand the intersection of a vertical revolution cylinder of finite height centered on Γ with the $\Gamma^+ = \{(x, y)\ ;\ y > f(x)\}$. We also ask the ratio (height):(revolution radius) of $\mathcal{C}$ to be greater than 2 times $||f||_L$, la lipschitz norm of f.

We recall the boundary Harnack principle for reflected Brownian motion (see [BH91], [Anc90]). We say that D is a Lipschitz domain if it is a Jordan domain and if the boundary is locally the graph of a Lipschitz function (with uniform lipschitz norm).

Let D be a Lipschitz domain, u and v be positive harmonic functions on D with vanishing normal derivatives on the graph between the adapted cylinder (to a graph-component of the boundary) $\mathcal{C}$ and the "sub"-adapted cylinder $\mathcal{C}''$ of the same center and revolution axis but of ℓ times the size, $\ell < 1$ (see figure 1).

Proposition 3.1 *If $\mathcal{C}'$ is the "middle" cylinder of the same center and revolution axis but of $\frac{1+\ell}{2}$ times the size of $\mathcal{C}$. Then for all $x \in \partial\mathcal{C}' \cap V$*

$$\frac{v(x)}{u(x)} \sim \frac{v(P)}{u(P)},$$

where P is the intersection point of the revolution axis of the cylinder $\mathcal{C}'$ and of its boundary.

The multiplicative constants in the equivalence relation depend on the ratio (revolution radius):(height) of $\mathcal{C}$, on ℓ, on the Lipschitz norm of the boundary and on the dimension of the space n (here $n = 2$) see also [Anc78].

We also need a Dirichlet-Neumann version of the maximum principle.

Proposition 3.2 *Let D be a planar domain , Γ a continuous subset of the boundary of D, graph of a Lipschitz function, and u a function harmonic in D such that $\liminf_{y \to x} u(y) \geq 0$ for all $x \in \partial D \setminus \Gamma$ and $\frac{\partial u}{\partial \eta} = 0$ on Γ, where η denotes the normal vector on Γ. Then $u \geq 0$ on D.*

This is a consequence of the unicity of solutions (see for instance [Hör94]) and the probabilistic description of these same solutions of the mixed Dirichlet-Neumann problem, described above.

3.2 Subsidiary results

We will use the following result due to Benjamini, Chen and Rohde.

Theorem 3.3 *(Theorem 5.1 of [BCR04]) Let Ω be a locally uniform bounded planar domain. Then, $\dim_{\mathcal{H}}(R([0,\infty)) \cap \partial\Omega) = \dim_{\mathcal{H}}(\partial\Omega)$, $\mathbb{P}_x$-almost surely, for all $x \in \overline{\Omega}$.*

In particular, under the assumptions of the theorem, if $F \subset \Omega$ is a closed set such that $\dim_{\mathcal{H}}(\partial\Omega \setminus F) < \dim_{\mathcal{H}}(F)$ we have

$$\mathbb{P}_x(\tau_F < +\infty) = 1 \tag{2}$$

for all $x \in \overline{\Omega}$.

Proposition 3.4 *Under the same assumptions, formula (2) implies $\mathbb{E}_x[\tau_F] < +\infty$, for all $x \in \overline{\Omega}$.*

To prove this proposition we recall a result of Burdzy, Chen and Marshall.

Theorem 3.5 *([BCM06]) Let Ω be any bounded locally uniform domain and $\mathbb{B}$ a closed ball in Ω. If we note $\tau_{\mathbb{B}}$ the hitting time of $\mathbb{B}$ by $\mathcal{R}_t$ then $\sup_{x\in\overline{\Omega}} \mathbb{E}_x[\tau_{\mathbb{B}}] < \infty$.*

Proof of Proposition 3.4. Let z be a point in Ω and $\mathbb{B}_z \subset \Omega$ a closed disc centered at z. For $x \in \Omega$ and any $s > 0$,

$$\mathbb{E}_x[\tau_F] \leq \sum_{n\in\mathbb{N}} s\mathbb{P}_x(\tau_F \geq ns).$$

By formula (2) for all $N \in \mathbb{N}$ there exists s sufficiently big such that $\mathbb{P}_x\left(\tau_F > \frac{s}{2}\right) < \frac{1}{N}$ and $\mathbb{P}_z\left(\tau_F > \frac{s}{2}\right) < \frac{1}{N}$. Furthermore we can choose $s > 2\sup_{x\in\overline{\Omega}} \mathbb{E}_x[\tau_{\mathbb{B}_z}]$. We get that, for $n > 1$,

$$\mathbb{P}_x(\tau_F \geq ns) = \mathbb{P}_x(\tau_F \geq ns | \tau_F \geq (n-1)s)\mathbb{P}_x(\tau_F \geq (n-1)s)$$

We can bound $\mathbb{P}_x(\tau_F \geq ns|\tau_F \geq (n-1)s) \leq \sup_{y\in\Omega} \mathbb{P}_y(\tau_F > s)$. On the other hand $\mathbb{P}_y(\tau_F > s) \leq \mathbb{P}_y(\tau_{\mathbb{B}_z} > s/2) + \mathbb{P}_y\left(\tau_{\mathbb{B}_z} < s/2\ ,\ \mathcal{R}_{[\tau_{\mathbb{B}_z},s]} \cap F = \emptyset\right)$.

Using the Markov property of $\mathcal{R}$,

$$\mathbb{P}_y\left(\tau_{\mathbb{B}_z} < s/2\ ,\ \mathcal{R}_{[\tau_{\mathbb{B}_z},s]} \cap F = \emptyset\right) \leq \mathbb{P}_y(\tau_{\mathbb{B}_z} < s/2) \sup_{v\in\mathbb{B}_z} \mathbb{P}_v(\tau_F > s/2).$$

Using parabolic Harnack principle (see [BCM06]) we get that there is a constant $c > 1$ such that

$$\sup_{v\in\mathbb{B}_z} \mathbb{P}_v\left(\tau_F > \frac{s}{2}\right) \leq c\mathbb{P}_z\left(\tau_F > \frac{s}{2}\right) < c/N.$$

We also have

$$\mathbb{P}_y(\tau_{\mathbb{B}_z} < s/2) < \frac{2\sup_{x\in\overline{\Omega}} \mathbb{E}_x[\tau_{\mathbb{B}_z}]}{s}$$

therefore, for s big enough,

$$\mathbb{P}_x(\tau_F \geq ns|\tau_F \geq (n-1)s) < 1/2$$

By induction we get $\mathbb{P}_x(\tau_F \geq ns) \leq \left(\frac{1}{2}\right)^n$ and hence $\mathbb{E}_x[\tau_F] < +\infty$. •

In fact we have proved that $\sup_{x\in\Omega} \mathbb{E}_x[\tau_F] < +\infty$.

4 Proof of theorem 1.1

Even though our proof can be carried out using only simply connected domains we have chosen to present a totally disconneted example: the constructions appear better and the lemmas get easier to write.

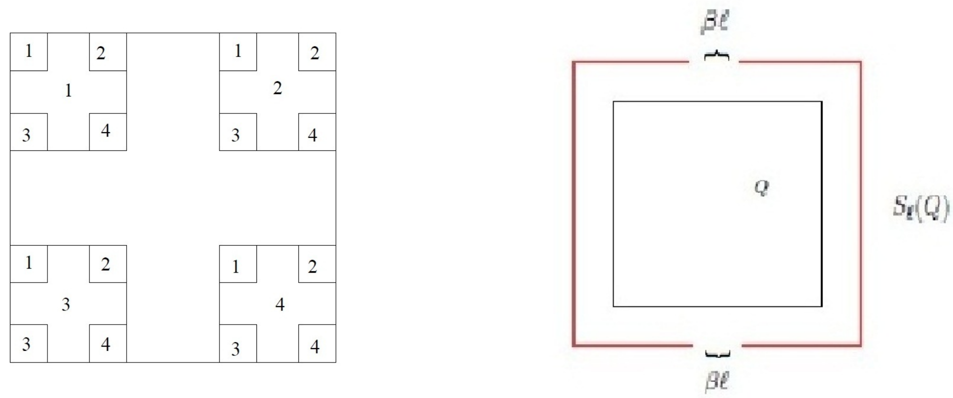

Figure 2: *A.* 4*-corner Cantor set and its encoding. B. The squares* $S_\ell(Q)$.

4.1 Construction of the domain

We construct, for $\alpha \in (0, \frac{1}{2})$ a 4-corner Cantor set (fig 2.A) in the following way. We start with the square $Q = [-\frac{1}{2}, \frac{1}{2}]^2$ that we replace by four squares of sidelength α situated at the four corners of Q. We name these squares $Q_1, ..., Q_4$. We replace then each Q_i , $i = 1, ...4$ by four smaller squares of sidelength α^2 situated at the corners of Q_i. We note these squares of the second generation Q_{ij}, where $j = 1, ..., 4$ and so on. Let us denote $\mathbb{K}$ the Cantor set constructed in this way. We endowe $\mathbb{K}$ with the natural encoding identifying it to the abstract Cantor set $\{1, ..., 4\}^{\mathbb{N}}$.

Observe that $\dim_{\mathcal{H}}(\mathbb{K}) = \left|\frac{\log 4}{\log \alpha}\right|$ and hence for α close to $\frac{1}{2}$ the dimension of the Cantor set is close to 2.

The set $\mathbb{K}$ will be the absorbing part of the boundary of Ω. Let us know construct the reflecting part. First of all, in order to ensure boundedness let us consider a ball $\mathbb{B}_0$, centered at 0 of radius, say, 10^6. The domain Ω will be a subset of $\mathbb{B}_0 \setminus \mathbb{K}$.

Let Q be a square of sidelength ρ centered at (x^*, y^*) and for $0 < \beta < 10^{-2}\rho$ and $\ell > 1$ consider the "unfinished" squares

$$S_{\beta,\ell}(Q) = \{(x, y) \in \mathbb{R}^2 \ ; \ |x - x^*| = |y - y^*| = \rho\ell \text{ and } x \notin (x^* - \beta\ell/2, x^* + \beta\ell/2)\}$$

(see figure 2.B). Finally consider the blown-up version of $S_{\beta,\ell}(Q)$ (see figure 3):

$$L_{\beta,\ell}(Q) = \{z \in \mathbb{R}^2 \ ; \ \ \text{dist}(z, S_{\beta,\ell}(Q)) \leq 10^{-6}\beta\rho\} \tag{3}$$

Note that, if ℓ is less than $\frac{1}{2\alpha}$, for any Q and Q' squares of the construction of the Cantor set $L_{\beta,\ell}(Q) \cap L_{\beta,\ell}(Q') = \emptyset$. Consider the union of the Cantor set $\mathbb{K}$ with

$$M_{\beta,\ell}(\mathbb{K}) = \bigcup_{n\in\mathbb{N}} \bigcup_{i_1,\dots i_n} \left(L_{\beta,\ell}\left(Q_{i_1,\dots i_n}\right)\right)$$

The domain Ω is defined as the complementary of this union within the ball $\mathbb{B}_0$ of radius 10^6:

$$\Omega = \mathbb{B}_0 \setminus \left(\mathbb{K} \cup M_{\beta,\ell}(\mathbb{K})\right).$$

Remark 4.1 For $\ell > 1$ and $\beta > 0$ fixed the domain Ω is clearly a bounded uniform domain. Therefore we can construct partially reflected Brownian motion $\mathcal{R}_t$ in Ω with the partition of the boundary of Ω into an absorbing part of the boundary $F = \mathbb{K}$ and a reflecting part $\partial\Omega \setminus F$.

Note also that the Hausdorff dimension of $\partial\Omega$ equals the Hausdorff dimension of $\mathbb{K}$ if $\dim_{\mathcal{H}} \mathbb{K} > 1$ (ie. if $\alpha > \frac{1}{4}$). This is because $\partial\Omega \setminus K$ consists of a countable union of rectifiable arcs and is therefore of Hausdorff dimension 1.

We will show that for $\alpha \in (0, \frac{1}{2})$ and every $\epsilon > 0$ there exists $\ell < \frac{1}{2\alpha}$ and β close to 0 such that the domain Ω, constructed in the previous way, satisfy $\dim_{\mathcal{H}} \omega > (1-\epsilon) \dim_{\mathcal{H}} \mathbb{K}$.

4.2 Preparatory lemmas

Let $Q = Q_{i_1,\dots,i_n}$ be a square of the construction of $\mathbb{K}$ of sidelength ρ and let (x^*, y^*) be it's center. Let $F_{\beta,\ell}(Q) = L_{\beta,\ell}(Q) \cup C_{\beta,\ell}(Q)$ where

$$\begin{aligned} C_{\beta,\ell}(Q) &= \{(x,y)\ ;\ y > y^* + \rho\ell \text{ and } ||(x-x^*, y-y^*-\rho\ell)|| = (\frac{1}{2\alpha} - \ell)\rho\} \\ &\quad \bigcup \{(x,y)\ ;\ y < y^* - \rho\ell \text{ and } ||(x-x^*, y-y^*+\rho\ell)|| = (\frac{1}{2\alpha} - \ell)\rho\}, \end{aligned}$$

see figure 3.

Let $\tilde{D}$ be the bounded component of the complementary of $F_{\beta,\ell}(Q)$ and $D = \Omega \cap \tilde{D}$ (as in figure 3).

We consider reflected Brownian motion ${}^D\mathcal{R}$ in D and we note τ^D the hitting time of $\mathbb{K} \cup C_{\beta,\ell}(Q)$ by ${}^D\mathcal{R}$. It follows on the previous discussion that for all $x \in D$, $\mathbb{E}_x \tau^D < \infty$ and, furthermore, $\mathbb{P}_x\left({}^D\mathcal{R}_{\tau^D} \in \mathbb{K}\right) > 0$. To prove this last claim one can also use the arguments of relation (8) below, applied to the domaine D and to the diffusion ${}^D\mathcal{R}$ respectively.

Remark 4.2 Consider $Q_1 = Q_{i_1,\dots,i_n a}, Q_2 = Q_{i_1,\dots,i_n b}$ (with $a, b = 1,..,4$) two sub-cubes of $Q = Q_{i_1,\dots,i_n}$. By symmetry we get that, if $x_Q = (x^*, y^*)$ is the center of Q,

$$\mathbb{P}_{x_Q}\left({}^D\mathcal{R}_{\tau^D} \in \mathbb{K} \cap Q_1\right) = \mathbb{P}_{x_Q}\left({}^D\mathcal{R}_{\tau^D} \in \mathbb{K} \cap Q_2\right).$$

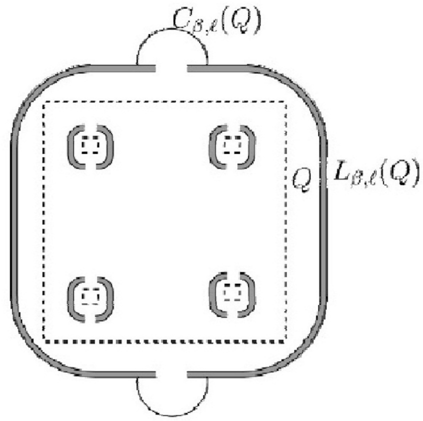

Figure 3: $F_{\beta,\ell}(Q)$

It follows, using Harnack's principle, that for all $\epsilon > 0$ there exists $r = r_\epsilon > 0$ (depending only on ϵ, ℓ, α but not on β) such that

$$||x - x_Q|| < r \Longrightarrow \mathbb{P}_{x_Q}\left({}^D\mathcal{R}_{\tau^D} \in \mathbb{K} \cap Q_1\right) \leq (1+\epsilon)\mathbb{P}_{x_Q}\left({}^D\mathcal{R}_{\tau^D} \in \mathbb{K} \cap Q_2\right). \quad (4)$$

Let us now prove that for β small enough, the harmonic functions (measures) $U_i(.) = \mathbb{P}_.\left({}^D\mathcal{R}_{\tau^D} \in \mathbb{K} \cap Q_i\right)$, $i = 1, 2$, satisfy inequality (4) in the subdomain D' of D :

$$D' = D \setminus \left(\mathbb{B}\left((x^*, y^* + \rho\ell), (\frac{1}{2\alpha} - \ell)\rho\right) \cup \mathbb{B}\left((x^*, y^* - \rho\ell), (\frac{1}{2\alpha} - \ell)\rho\right)) \cup \bigcup_{i=1\ldots4} \ell Q_i \right). \quad (5)$$

Remark that the closure of D' is a compact subset of $D \cup L_{\beta,\ell}(Q)$.

Lemma 4.3 *For all $\epsilon > 0$ there exists $\beta_0 > 0$ such that for all $\beta < \beta_0$ and all $x \in D'$*

$$\mathbb{P}_x\left(\exists t < \tau^D \,;\, {}^D\mathcal{R}_t \in \mathbb{B}(x_Q, r_\epsilon)\right) > 1 - \epsilon$$

Proof We introduce an auxilliary subdomain D'' of D, $D'' = D \setminus \overline{B}(x_Q, r_\epsilon)$. Consider, in D'' the harmonic function ζ satisfying the mixed Dirichlet-Neumann boundary conditions $\zeta = 0$ on $B(x_Q, r_\epsilon)$, $\zeta = 1$ on $\mathbb{K} \cup C_{b,\ell}(Q)$ and $\frac{\partial\zeta}{\partial\eta} = 0$ elsewhere on $\partial D''$.

It is immediate that, since r_ϵ does not depend on β, ζ tends to 0 when β goes to 0. By the maximum principle 3.2,

$$1 - \zeta(x) < \mathbb{P}_x\left(\exists t < \tau^D \,;\, {}^D\mathcal{R}_t \in \mathbb{B}(x_Q, r_\epsilon)\right).$$

On the other hand, for every x there is an β_0 such that $\zeta(x) < \epsilon$ for all $\beta < \beta_0$ and by Harnack's principle this inequality can be taken uniform in $\overline{D'}$. $\bullet$

Keeping the same notation we also have:

Lemma 4.4 *Let $Q_i = Q_{i_1,\ldots,i_n i}$, $i = 1, \ldots 4$, be a sub-cube of $Q = Q_{i_1,\ldots,i_n}$. Then, there is a constant $C > 0$ depending only on α, ℓ such that for all $x \in C_{\beta,\ell}(Q_i)$*

$$\mathbb{P}_x(\exists \ 0 < t_1 < t_2 < \tau^D \ ; {}^D\mathcal{R}_{t_1} \in B(x_Q, r_\epsilon) \ , \ {}^D\mathcal{R}_{t_2} \in C_{\beta,\ell}(Q_i)) \geq C$$

The proof of this lemma is standard and hence omitted.

4.3 Harmonic measure estimates

As before, let $Q_1 = Q_{i_1,\ldots,i_n a}, Q_2 = Q_{i_1,\ldots,i_n b}$ be two sub-cubes of a given cube $Q = Q_{i_1,\ldots,i_n}$ of the construction of $\mathbb{K}$.

Take U_1 and U_2 to be the harmonic functions previously defined in D, ie. satisfying the mixed Dirichlet-Neumann boundary conditions :

$$\begin{cases} U_1 = 1 \text{ on } \mathbb{K} \cap Q_1 \\ U_1 = 0 \text{ on } (\mathbb{K} \cap Q_2) \cup C_{\beta,\ell}(Q) \\ \frac{\partial U_1}{\partial \eta} = 0 \text{ elsewhere on } \partial D \end{cases} \text{ and } \begin{cases} U_2 = 1 \text{ on } \mathbb{K} \cap Q_2 \\ U_2 = 0 \text{ on } (\mathbb{K} \cap Q_1) \cup C_{\beta,\ell}(Q) \\ \frac{\partial U_2}{\partial \eta} = 0 \text{ elsewhere on } \partial D \end{cases} \tag{6}$$

Thus, $U_i(.) = \mathbb{P}_. \left({}^D\mathcal{R}_{\tau^D} \in \mathbb{K} \cap Q_i \right)$, $i = 1, 2$.

Lemma 4.5 *For every $\epsilon > 0$ there existe $\beta_0 > 0$ such that for all $0 < \beta < \beta_0$ and all $x \in D'$,*

$$U_1(x) < (1 + \epsilon) U_2(x).$$

Proof The proof relies on lemma 4.3. By Harnack's principle there exists $C > 0$ such that $U_i(x) \geq C U_i(x_Q)$, for all $x \in D'$. On the other hand,

$$\begin{aligned} U_i(x) &= \mathbb{P}_. \left({}^D\mathcal{R}_{\tau^D} \in \mathbb{K} \cap Q_i \right) = \\ &= \mathbb{P}_. \left({}^D\mathcal{R}_{\tau^D} \in \mathbb{K} \cap Q_i \ , \ {}^D\mathcal{R}_{[0,\tau^D]} \cap \mathbb{B}(x_Q, r_\epsilon) = \emptyset \right) + \\ &+ \mathbb{P}_. \left({}^D\mathcal{R}_{\tau^D} \in \mathbb{K} \cap Q_i \ , \ {}^D\mathcal{R}_{[0,\tau^D]} \cap \mathbb{B}(x_Q, r_\epsilon) \neq \emptyset \right) \end{aligned}$$

Lemmas 4.3 and 4.4 imply that

$$\mathbb{P}_. \left({}^D\mathcal{R}_{\tau^D} \in Q_i \ , {}^D\mathcal{R}_{[0,\tau^D]} \cap \mathbb{B}(x_Q, r_\epsilon) = \emptyset \right) \leq \epsilon \mathbb{P}_. \left({}^D\mathcal{R}_{\tau^D} \in Q_i \ , {}^D\mathcal{R}_{[0,\tau^D]} \cap \mathbb{B}(x_Q, r_\epsilon) \neq \emptyset \right)$$

On the other hand, by the Markov property,

$$\mathbb{P}_. \left({}^D\mathcal{R}_{\tau^D} \in Q_i \ , {}^D\mathcal{R}_{[0,\tau^D]} \cap \mathbb{B}(x_Q, r_\epsilon) \neq \emptyset \right) \leq \sup_{x \in \mathbb{B}(x_Q, r_\epsilon)} \mathbb{P}_x \left({}^D\mathcal{R}_{\tau^D} \in Q_i \right).$$

Therefore, using once more Harnack's inequality

$$U_i(x) \leq (1 + c\epsilon) U_i(x_Q),$$

and the lemma's claim follows using symmetry. •

Consider now the functions V_1 and V_2 that solve the following mixed Dirichlet-Neumann problem in Ω.

$$\left\{\begin{array}{l} V_1 = 0 \text{ on } \mathbb{K}\cap Q_1 \\ V_1 = U_1 \text{ on } \mathbb{K}\setminus Q_1 \\ \frac{\partial V_1}{\partial \eta} = \frac{\partial U_1}{\partial \eta} \text{ on } \partial\Omega\setminus\mathbb{K} \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} V_2 = 0 \text{ on } \mathbb{K}\cap Q_2 \\ V_2 = V_1 \text{ on } \mathbb{K}\setminus Q_2 \\ \frac{\partial V_2}{\partial \eta} = \frac{\partial U_2}{\partial \eta} \text{ on } \partial\Omega\setminus\mathbb{K} \end{array}\right. \tag{7}$$

4.4 Proof of theorem

We need to show that for β small enough $V_i \leq (1+\epsilon)V_j$, for $i,j = 1,2$. Since $V_i(x) = \mathbb{P}_x(\mathcal{R}_{\tau_\mathbb{K}} \in \mathbb{K}\cap Q_i)$, this inequality clearly implies that the harmonic measure for partially reflected Brownian motion ω satisfies

$$(1+\epsilon)^{-n}4^{-n} \leq \omega(Q_{i_1,\ldots,i_n}) \leq (1+\epsilon)^n 4^{-n},$$

for all n and all indices $i_1, \ldots i_n \in \{1, \ldots, 4\}$ and hence the claim.

We note

$$C'_{\beta,\ell} = D \cap \partial\left(\mathbb{B}\left((x^*, y^*+\rho\ell), (\frac{1}{2\alpha}-\ell)\rho\right)\cup\mathbb{B}\left((x^*, y^*-\rho\ell), (\frac{1}{2\alpha}-\ell)\rho\right)\right)$$

For $n\in\mathbb{N}$, consider the increasing sequences of hitting times

$$T_n = \inf\{t\ ;\ \exists t_1 < s_1 \ldots < t_{n-1} < s_{n-1} < t \text{ s.t. } \mathcal{R}_{t_i}\in C'_{\beta,\ell}\ ,\ \mathcal{R}_{s_i}\in C_{\beta,\ell}\}$$

and

$$S_n = \inf\{s\ ;\ \exists t_1 < s_1 \ldots < t_n < s \text{ s.t. } \mathcal{R}_{t_i}\in C'_{\beta,\ell}\ ,\ \mathcal{R}_{s_i}\in C_{\beta,\ell}\}$$

with the convention $T_n = \infty$ ($S_n = \infty$) if the corresponding set is empty.

$$\begin{aligned} V_i(x) &= \mathbb{P}_x(\mathcal{R}_{\tau_\mathbb{K}} \in Q\cap\mathbb{K}\ ,\ \mathcal{R}_{\tau_\mathbb{K}} \in \mathbb{K}\cap Q_i) \\ &= \sum_n \mathbb{P}_x\left(0 < T_1 < \ldots < T_n < \tau_\mathbb{K} < \infty\ ,\ S_n = \infty\ ,\ \mathcal{R}_{\tau_\mathbb{K}}\in\mathbb{K}\cap Q_i\right) \\ &= \sum_n \mathbb{P}_x\left(0 < T_1 < \ldots < T_n < \infty\right)\mathbb{E}_x\mathbb{P}_{R_{T_n}}\left(S_1 = \infty\ ,\ \mathcal{R}_{\tau_\mathbb{K}}\in\mathbb{K}\cap Q_i\right) \end{aligned} \tag{8}$$

where the last equality is derived by Markov's property. By lemma 4.5, for $i,j = 1,2$ and all $z\in C'_{\beta,\ell}$,

$$\mathbb{P}_z\left(S_1 = \infty\ ,\ \tau_\mathbb{K}\in\mathbb{K}\cap Q_i\right) \leq (1+\epsilon)\mathbb{P}_z\left(S_1 = \infty\ ,\ \mathcal{R}_{\tau_\mathbb{K}}\in\mathbb{K}\cap Q_j\right).$$

Implementing this inequality in (8) we get

$$V_i(x) \leq \sum_n \mathbb{P}_x\left(0 < T_1 < \ldots < T_n < \infty\right)\mathbb{E}_x\mathbb{P}_{R_{T_n}}\left(S_1 = \infty\ ,\ \mathcal{R}_{\tau_\mathbb{K}}\in\mathbb{K}\cap Q_j\right)$$

and hence $V_i \leq (1+\epsilon)V_j$, which completes the proof.

Comments-Further Remarks: If the boundary of the domain Ω is entirely absorbing, ie. for the Laplace equation with Dirichlet boundary conditions, then harmonic measure is carried by $\partial\Omega \setminus \mathbb{K}$. This is not difficult to see. In fact, using previous notation, to get to $\mathbb{K}$ Brownian motion has to go through an infinity of conformal annuli of the type $L\beta, \ell(Q) \setminus Q$. But, at every passage of this type, there is a -bounded from below probability- to hit $L\beta, \ell(Q)$. Hence the probability that brownian motion hits $\mathbb{K}$ is 0. Moreover, harmonic measure will be carried by a union of curves of finite length.

This answers the question of B. Sapoval mentionned in the introduction.

Another way to study the passage between Dirichlet boundary condition to the Neumann boundary condition through the mixed Dirichlet-Neumann is through a random approach. This the object of a previous work [BLZ11] that should be completed in a forthcoming artcile [BZ10].

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Publications de la partie 4

On Brownian flights

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Abstract : Let K be a compact subset of $\mathbb{R}^n$. We choose at random with uniform law a point at distanceε of K and start a Brownian motion (BM) from this point. We study the probability that this BM hits K for the first time at a distance $\geq r$ from the starting point.

Keywords : Brownian Motion, Minkowski dimension, Harmonic measure, quasi-conformal maps, John domains.

1 Introduction and motivation.

Porous materials, concentrated colloidal suspensions or physiological organs such that lung or kidney are systems developping large specific surfaces with a rich variety of shapes that influence the diffusive dynamics of Brownian particles. A typical example is the diffusion of water molecules in a colloidal suspension. NMR relaxation allows to measure the statistics of the flights of these molecules over long colloidal shapes such as proteins or DNA chains. It is thus tempting to rely these statistics to the geometry of the molecules, the goal being to probe shapes using this method. An ideal (but far-reaching) objective would be to make up a DNA-test for example using NMR relaxation. This program has been developped in [GKL$^+$06] where various kind of simulations or experiments have shown remarkable commun properties.
All the simulations measure the statistics of the same random phenomenon: an irregular curve or surface is implemented, consisting of a union of a large (but finite) number of equal affine pieces. Such a piece is chosen at random with uniform distribution and a random walker is started at some small but fixed distance from this piece inside the complement of the surface. One is interested in the law of the variable $X =$ the length of the flight, i.e. the distance between the starting point and the first hitting point on the surface of the random walker. Whatever shape the surface shows, the experiment shows the same behaviour

$$\mathbb{P}\left(X > r\right) \sim_\infty r^{d_e - d - 2} \tag{1}$$

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where d_e is the dimension of the ambiant space and d is the Minkowski dimension of the surface.
A first mathematical explanation of this behavior was outlined in [GKL$^+$06]. The main purpose of the present paper is to give first a rigorous statement of this result and to prove it with minimal assumptions so that all the cases of the simulations are covered. This will be the content of the second paragraph.
The third section concerns an alternative approach of the result in a special but important case: a quasiconformal perturbation of the line in 2D. The paper being dedicated to Fred Gehring is one of the reasons of this section, but not the only one: indeed this alternative proof is neat and instructive.
There is a case of particular importance for physics and particularly polymer physics. It is the case of a curve being a self-avoiding walk, since this has been shown to be a good model for polymers. In 2D there are two evidences that the result should remain in this case : first intensive simulations performed by P.Levitz and secondly some computations by Duplantier using conformal field theory. We present in the last section these simulations and sho how, despite the fact that this case is not covered by results of section 3, one can use results about SLE to prove the result in this situation.
We would like to conclude this introduction with two remarks:
1) The result may be surprising since it involves only the Minkowski dimension while the experiment suggests that harmonic measure is involved (and it is!) and consequently that the result should depend on some multifractal property of the harmonic measure. This is not the case because of the law of the choice of the starting point. A completely different behavior would occur if instead of uniform law we had chosen harmonic measure and this case is extremely interesting since essentially it models the second flight: this case will be considered in a forecoming paper.
2) The paper [GKL$^+$06] has been written while the second author was hosted by the laboratory of physics of condensed matter at Ecole Polytechnique whose members he thanks for their warm hospitality. This paper is an example of a successfull pluridisciplinar research and we would like to emphasize the fact that firstly problems coming from physics (especially polymer physics) are extremely rich mathematically speaking and that, secondly, modern Function Theory as we herited from great mathematicians as Fred Gehring is an extremely efficient tool to attack these challenging problems.
Our goal will be achieved if the reader gets convinced of this last statement after having read this paper.

2 Geometric considerations

For convenience all domains in the next two sections of this paper will be assumed to have compact boundary. An estimate like (1) cannot be true for every domain. Some geometric conditions are needed : one of them is that Minkowski dimension of the boundary exists, i.e. that $d = \lim_{\epsilon \to 0} \frac{\log \# N_\epsilon}{-\log \epsilon}$ exists, where N_ϵ is the minimal number of cubes of size ϵ needed to cover the boundary.

As we shall see later Whitney decomposition is a central tool in our proof; a second condition we have to impose is that the number of cubes of the Whitney decomposition at distance r must also be comparable to r^{-d}. A sufficient condition for this is given by the first property of NTA domains ([JK82]) called the "corkscrew" condition:

Définition 2.1 *We say that a domain Ω satisfies the "corkscrew" condition if there exists $r_0 > 0$ and a constant $c > 0$ such that for all $r < r_0$ and any $x \in \partial\Omega$ there exists $y \in \Omega$ such that $cr < \mathrm{dist}(x, y) < r$ and $\mathrm{dist}(y, \partial\Omega) > cr$.*

Proposition 2.1 *Under the corkscrew condition the number of Whitney cubes with size that intersect the level surface $\Gamma_r = \{x \in \Omega\,;\, \mathrm{dist}(x, \partial\Omega) = r\}$ is comparable to the minimal of cubes of size r needed to cover the boundary.*

The straightforward proof is left to the reader.

3 The case of open sets in $\mathbb{R}^n$, $n \geq 3$

Although our approach can be easily adapted to open sets in the plane, we present it for $\mathbb{R}^n$, $n \geq 3$ for two reasons: First, Green function and related formulas being different we try to avoid writing everything twice. Secondly, a different approach is proposed in section 4 for open sets in the complex plane, using the quasi-conformal theory.

Notation 3.1 Given an open set Ω, a point $x \in \Omega$ and a set $A \subset \bar{\Omega}$ we denote $\mathbb{P}(x \hookrightarrow_\Omega A)$ the probability that Brownian motion started at x touches A before leaving Ω. The Green function of Ω will be denoted G_Ω and let G_n be the Green function of $\mathbb{R}^n$. The ball of center x and radius r is denoted by $\mathbb{B}(x, r)$ and the distance from x to $\partial\Omega$ is denoted by d_x. We say that two quantities A, B are "comparable" (we denote $A \sim B$) if the exists a constant c such that $\frac{1}{c}A \leq B \leq cA$. The harmonic measure of a set F (usually a subset of $\partial\Omega$) at a point $x \in \Omega$ is denoted by $\omega_\Omega(x, F)$ or $\omega(x, F, \Omega)$.

3.1 Green estimates

Proposition 3.1 *Let Ω be a domain in $\mathbb{R}^n$, $n > 2$ and $x, y \in \Omega$. There exists a universal constant C depending only on n such that for all $\ell \leq \frac{1}{2}$*

$$\mathbb{P}\Big(x \hookrightarrow_\Omega \mathbb{B}(y, \ell d_y)\Big) \leq C \left(\frac{d_y}{d_x}\right)^{n-2} \mathbb{P}\Big(y \hookrightarrow_\Omega \mathbb{B}(x, \ell d_x)\Big)$$

Proof First, notice that by the maximum principle, $G_{\mathbb{B}(x,d_x)}(x,z) \leq G_\Omega(x,z) \leq G_n(x,z)$ for all $z \in \mathbb{B}(x, d_x/2)$. It is then easy to check that there exists a constant $c = c(n)$ such that $c^{-1}G_n(x,z) \leq G_\Omega(x,z) \leq G_n(x,z)$ for all $z \in \mathbb{B}(x, \ell d_x)$.

Therefore the function that assigns $s \mapsto (\ell d_y)^{n-2}G_n(s,y)$ is harmonic in $\Omega \backslash \mathbb{B}(y, \ell d_y)$, tends to 0 at $\partial\Omega$ and takes values between c^{-1} and 1 on $\partial\mathbb{B}(y, \ell d_y)$. The probability $\mathbb{P}\Big(x \hookrightarrow_\Omega \mathbb{B}(y, \ell d_y)\Big)$ is then comparable to $(\ell d_y)^{n-2}G_n(x,y)$. In a similar way $\mathbb{P}\Big(y \hookrightarrow_\Omega \mathbb{B}(x, \ell d_x)\Big)$ is equivalent to $(\ell d_x)^{n-2}G_n(y,x)$ and the proof is complete since G_n is symmetric. •

We suppose from now on that the domain Ω is uniformly "fat", i.e. that there exists a constant $c > 0$ such that for any $x \in \Omega$ and $r \leq 1$, we have

$$\mathrm{cap}_{\mathbb{B}(x,2r)}\Big(\mathbb{B}(x,r) \cap \partial\Omega\Big) \geq c\, \mathrm{cap}_{\mathbb{B}(x,2r)}\Big(\mathbb{B}(x,r)\Big) \tag{2}$$

In particular, this implies that there exists a uniform lower bound $L > 0$ of the probability that Brownian motion started at x hits the boundary of Ω before leaving $\mathbb{B}(x, 2d_x)$ (see [Anc86], lemma 5). This notion (also called "uniform capacity density condition") has previously been introduced in various contexts, cf. [JW88], [Anc86], [HKM93].

3.2 Main Results

We consider a Whitney decomposition of Ω in **dyadic** cubes Q satisfying $c_1|Q| \leq d(Q, \partial\Omega) \leq c_2|Q|$, where $c_1 < 1 < c_2$ are positive constants (powers of 2, for convenience) depending on n. For $t > 0$ we note $\mathcal{Q}_t$ the subcollection of cubes of the Whitney decomposition that intersect the level surface $\Gamma_t = \{x \in \Omega \,;\, d_x = t\}$.

Theorem 3.2 *Take $\varepsilon < r$, fix a cube $Q_r \in \mathcal{Q}_r$ and consider, for every $Q \in \mathcal{Q}_\varepsilon$, its center $x_Q \in Q$. Then $\sum_{Q \in \mathcal{Q}_\varepsilon} P\Big(x_Q \hookrightarrow_\Omega Q_r\Big)$ is equivalent to $\left(\frac{r}{\varepsilon}\right)^{n-2}$.*

Proof According to proposition 3.1 we have

$$\frac{1}{C}\sum_{Q \in \mathcal{Q}_\varepsilon} P\Big(x_{Q_r} \hookrightarrow_\Omega Q\Big) \leq \left(\frac{\varepsilon}{r}\right)^{n-2} \sum_{Q \in \mathcal{Q}_\varepsilon} P\Big(x_Q \hookrightarrow_\Omega Q_r\Big) \leq C \sum_{Q \in \mathcal{Q}_\varepsilon} P\Big(x_{Q_r} \hookrightarrow_\Omega Q\Big),$$

where x_{Q_r} is the center of the cube Q_r. We now show that $\sum_{Q \in \mathcal{Q}_\varepsilon} P\Big(x_{Q_r} \hookrightarrow_\Omega Q\Big)$ is equivalent to the harmonic measure $\omega_\Omega(x_{Q_r}, \partial\Omega)$ of $\partial\Omega$ at x_{Q_r} (in Ω) which equals 1. For this purpose we will use hypothesis (2) together with an easy control of multiple coverings.

Take any $Q \in \mathcal{Q}_\varepsilon$ and consider the cube $3c_2Q$ of same center but of $3c_2$ times the sidelength of Q (c_2 being the constant of the Whitney decomposition). By condition (2) the probability for Brownian motion started anywhere in Q to exit Ω before exiting $3c_2Q$ is bounded below by a positive constant c. Hence, the harmonic measure $\omega_\Omega(x_{Q_r}, 3c_2Q \cap \partial\Omega)$ of $3c_2Q \cap \partial\Omega$ at x_{Q_r} (in Ω) is greater than $cP\Big(x_{Q_r} \hookrightarrow_\Omega Q\Big)$. Summing over all cubes $Q \in \mathcal{Q}_\varepsilon$ we get

$$c \sum_{Q \in \mathcal{Q}_\varepsilon} P\Big(x_{Q_r} \hookrightarrow_\Omega Q\Big) \leq \sum_{Q \in \mathcal{Q}_\varepsilon} \omega_\Omega\Big(x_{Q_r}, 3c_2Q\Big)$$

On the other hand, every $x \in \partial\Omega$ can only belong to a finite number of cubes $3c_2Q$; this proves that $\sum_{Q \in \mathcal{Q}_\varepsilon} \omega_\Omega\Big(x_{Q_r}, 3c_2Q\Big)$ is comparable to the harmonic measure of the boundary $\partial\Omega$ of Ω and therefore $\sum_{Q \in \mathcal{Q}_\varepsilon} P\Big(x_{Q_r} \hookrightarrow_\Omega Q\Big)$ has an upper bound depending only on n and on condition's (2) constant. The lower bound is trivial by the "fatness" condition. $\bullet$

Theorem 3.3 *Choose Q at random with uniform law in $\mathcal{Q}_\varepsilon$. The probability for a Brownian motion started at any point x of Q to hit Γ_r before exiting Ω is comparable to* $\dfrac{\#\mathcal{Q}_r}{\#\mathcal{Q}_\varepsilon}\left(\dfrac{r}{\varepsilon}\right)^{n-2}$.

Proof First observe that by the Harnack principle we can choose $x = x_Q$ the center of the cube Q. Since the cube Q is arbitrarily chosen there are $\#\mathcal{Q}_\varepsilon$ possible choices.

We consider now the cubes of $\mathcal{Q}_r$ and we define S_r as the part of the boundary of $\cup_{Q \in \mathcal{Q}_r} Q$ separating the set $\Gamma_r = \{x \in \Omega \text{ such that } d_x = r\}$ and $\partial\Omega$. We say that a cube in $\mathcal{Q}_r$ has a seashore if part of its boundary is also part of S_r, and , in this case, this access to the "sea" is at least a square whose diameter is $\geq$ to a constant depending only on n times the size of the cube. We then consider the open set U consisting in the union of the components of $\Omega \setminus \bigcup_{Q \in \mathcal{Q}_r} \overline{Q}$. We denote by $\tilde{S}_r$ the boundary of this open set: we are interested in the probability that Brownian motion started at any point x of Q hits $\tilde{S}_r$ before exiting Ω. Denote by V the component of U containing Q. Let O be a cube in $\mathcal{Q}_r$ having a seashore. Each one of its sides touching $\tilde{S}_r$ contains a dyadic square R of $c_1/8$ times the size of O, c_1 being the constant in Whitney decomposition, such that $R \subset \tilde{S}_r$. Let O_L be a cube of the same center as O but $1 + c_1/2$ its size.We consider the dyadique cube R' contained in O_L of size $c_1/8$ times the size of O vertically above R and at distance $c_1/8$ from O .

We use the Boundary Harnack Principle to prove that the probability that Brownian motion started at x_Q leave V through R is comparable to $P\Big(x_Q \hookrightarrow_V R'\Big)$.

By "adapted cylinder" to a graph of a Lipschitz function we understand a vertical revolution cylinder of finite height centered on the graph. Let us remind the Boundary Harnack Principle : Let u and v be positive harmonic functions on a Lipschitz domain vanishing on the graph between the adapted cylinder (to a graph-component of the boundary) $\mathcal{C}$ and the "sub"-adapted cylinder $\tilde{\mathcal{C}}$ of the same center and revolution axis but of ℓ times the size, $\ell < 1$. If $\mathcal{C}'$ is the "middle" cylinder of the same center and revolution axis but of $\frac{1+\ell}{2}$ times the size of $\mathcal{C}$. Then for all $x \in \partial\mathcal{C}' \cap V$

$$\frac{v(x)}{u(x)} \sim \frac{v(P)}{u(P)},$$

where P is the intersection point of the revolution axis of the cylinder $\mathcal{C}'$ and of its boundary. The multiplicative constants in the equivalence relation depend on the ratio (revolution radius):(height) of $\mathcal{C}$, on ℓ, on the local Lipschitz norm of the boundary and on the dimension of the space n, see [Anc78].

Remark that O_L only touches the neighbouring cubes of R. Clearly, $O_L \cap V$ is a Lipschitz domain (its boundary is composed of a finite union of squares). We can find a finite number of adapted cylinders $\mathcal{C}_i$ that do not touch the cubes R' such that the "sub"-adapted cylinders $\tilde{\mathcal{C}}_i$ of half their size cover the boundary of $\partial R \cap \partial V$. Furthermore, we can ask these cylinders to touch the boundary of the cube O_l of the same center as O but of $1 + c_1/100$ the size (see figure 1). Note that, since there is a finite number of configurations of the neighborhood of O in the Whitney decomposition, the number of adapted cylinders needed to this covering is bounded by a uniform constant κ depending only on n. We consider the harmonic functions $u(x) = \omega(x, R', V \setminus \bigcup_{\text{"wet" sides of } O} R')$ and $v_i(x) = \omega(x, \tilde{\mathcal{C}}_i \cap \partial O_L, V)$ in the domain $V' = V \setminus \bigcup_{\text{"wet" sides of } O} R'$. Clearly,

$$\omega(x, \partial O \cap \partial V, V') \leq \sum_i v_i(x) \leq \kappa\omega(x, \partial O \cap \partial V, V'). \tag{3}$$

We can apply the Boundary Harnack Principle to every one of these cylinders $\mathcal{C}_i$ for the functions u and v_i. We get that for all i and all $x \in \partial\mathcal{C}_i \cap V$

$$\frac{v_i(x)}{u(x)} \sim \frac{v_i(P_i)}{u(P_i)}, \tag{4}$$

where P_i is the intersection point of the revolution axis of the cylinder and of it's boundary. The boundary of our domain is the inner boundary of the cubes of the Whitney decomposition; hence we may restrain ourselves to a finite number of configurations of the "adapted cylinders" (up to a contractions-dilatations that do not affect constants) and therefore the multiplicative constants in the equivalence relation will be finite in number and so uniformly bounded away from 0 and infinity. Since the distance of $\text{dist}(P_i, R')$ is equivalent to the distance $\text{dist}(P_i, \partial R)$ we can prove by standard arguments of harmonic analysis that $u(P_i)$ is bounded below by a constant depending only on dimension n.

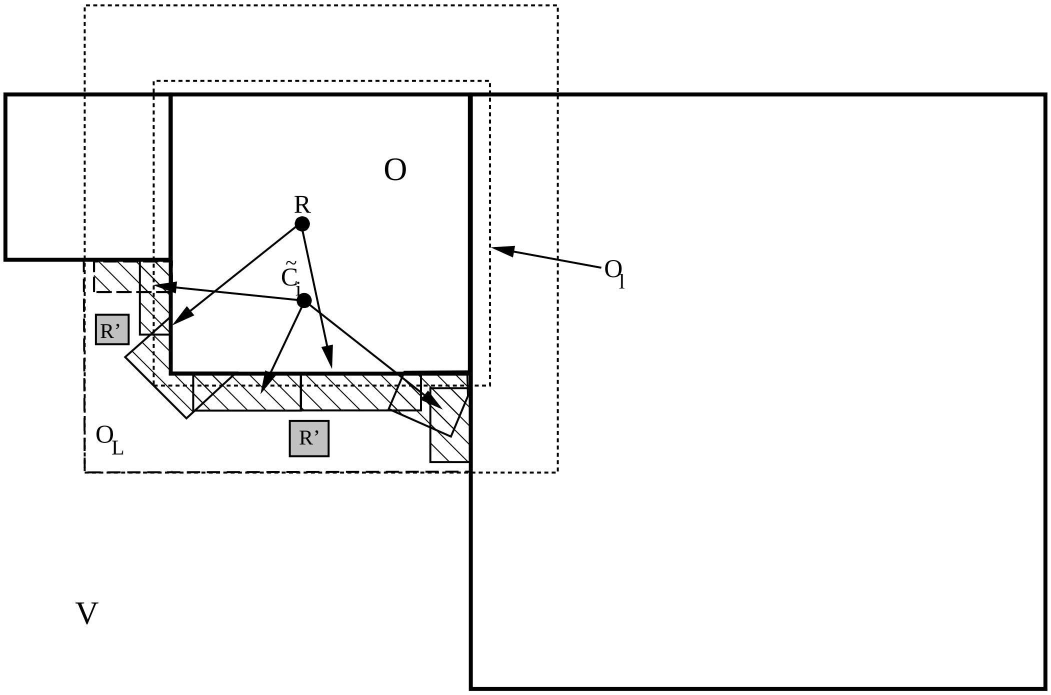

Figure 1: ***A configuration in*** $\mathbb{R}^2$***.***

Using the Harnack principle and the fact that all adapted cylinders intersect ∂R_l we get the existence of a constant c depending only on n such that $v_i(P_k) \leq c v_i(P_j)$ for all i, j, k.

After summing over i, taking in account equations (3) and (4) and using standard Harnack inequalities we get

$$\omega(x, \partial R \cap \partial V', V') \leq \omega(x, R', V') \leq \kappa' \omega(x, \partial R \cap \partial V', V')$$

for all $x \in \partial R_l \cup \cup_i \partial \mathcal{C}_i \cap V$. By the maximum principle we thus obtain

$$\omega(x, \partial R \cap \partial V', V') \leq \omega(x, R', V') \leq \kappa' \omega(x, \partial R \cap \partial V', V')$$

for all $x \in Q \in \mathcal{Q}_\varepsilon$.

Theorem 3.2 applies to the domain V' where $\mathcal{Q}_r$ is replaced by the collection R' coming from all “wet”sides of all cubes with a seashore. We obtain that

$$\sum_{Q \in \mathcal{Q}_\varepsilon, Q \subset V'} P\Big(x_Q \hookrightarrow_{V'} R'\Big) \sim \left(\frac{r}{\varepsilon}\right)^{n-2} \omega_{V'}(x_{R'}, \partial\Omega \cap V')$$

a quantity bounded from below by some $\alpha > 0$ by the fatness condition. The proof is completed by suming over all R' and V's and by noticing that up to a multiplicative factor depending only on the dimension, the number of cubes with a seashore is comparable to the number of cubes in $\mathcal{Q}_r$. •

Theorem 3.4 *Choose Q at random with uniform law in $\mathcal{Q}_\varepsilon$. The probability for a Brownian motion started at any point x of Q to exit Ω at distance greater than r from the starting point is comparable to* $\dfrac{\#\mathcal{Q}_r}{\#\mathcal{Q}_\varepsilon}\left(\dfrac{r}{\varepsilon}\right)^{n-2}$.

Proof It depends on a comparison between the probability of "cruising along the coast" $\partial\Omega$ and the probability to move at distance r before coming back to the coast. The second is comparable to $\dfrac{\#\mathcal{Q}_r}{\#\mathcal{Q}_\varepsilon}\left(\dfrac{r}{\varepsilon}\right)^{n-2}$ according theorem 3.3 while the first is exponentially small.

Take $s > 0$ and $x_Q \in Q \in \mathcal{Q}_\varepsilon$. Consider the annuli centered at x of inner radii ℓs and outer radii $(\ell+1)s$ where $\ell = 0, ..., \left[\frac{r}{s}\right]$. Brownian motion started at x and moving at distance r from x before exiting Ω must go through all these annuli. The probability of going through such an annulus while staying at distance at most $\frac{s}{4}$ from the boundary is bounded by a $p_0 \in (0,1)$ by the "fatness" hypothesis. To see this take any point y in the middle of the annulus (i.e. at distance $\frac{\ell+1}{2}s$ from x) and consider the ball of center y and radius $\frac{s}{2}$. If $d_y < \frac{s}{4}$, the probability to exit the ball without touching $\partial\Omega$ is uniformly bounded away from 1 by the "fatness" hypothesis. This probability being greater than the probability of going through the annulus we have the statement. By the independence of the "crossing annulli" events we get that the probability that Brownian motion goes through all the annulli is smaller that $p_0^{\left[\frac{r}{s}\right]}$.

Let us now prove the following statement: "there exist $0 < c_1, c_2 < 1$ positive constants depending only on dimension and on the constant L of the fatness condition such that for any $x \in \Omega$ there exist disjoint sets $K_1, K_2 \subset \partial\Omega \cap \mathbb{B}(x, 2d_x)$ verifying $\mathrm{dist}(K_1, K_2) > c_1 d_x$ and $\omega_\Omega(x, K_i) > c_2$ for $i = 1, 2$. Once more this is a consequence of the "fatness" property. Cut the sphere $\partial\mathbb{B}(x, 2d_x)$ in small equal normal polygons (spherical triangles in $\mathbb{R}^3$) and consider the intersections L_i of the cones of summit x and basis these polygons with the set $\mathbb{B}(x, 2d_x) \setminus \mathbb{B}(x, d_x)$ (see figure 2). Clearly, we can choose the polygons small enough (independently of x and d_x) to have that $\omega(x, L_i, \mathbb{B}(x, 2d_x) \setminus L_i) < L/2^n$, where L is the constant in the fatness condition. It is now clear that $\omega(x, L_i \cap \partial\Omega, \mathbb{B}(x, 2d_x) \cap \Omega) < L/100$ and therefore there are two non-neighboring L_i's having harmonic measure greater than $L/2^n \# L_i$, which proves the statement.

The previous statement implies that once we reached distance r from the boundary of the domain the Brownian motion will revisit the boundary at distance comparable to r from the starting point with probability greater than c_2. Putting together all the above we get the theorem. •

As a corollary we get the following.

Theorem 3.5 *If Ω satisfies*

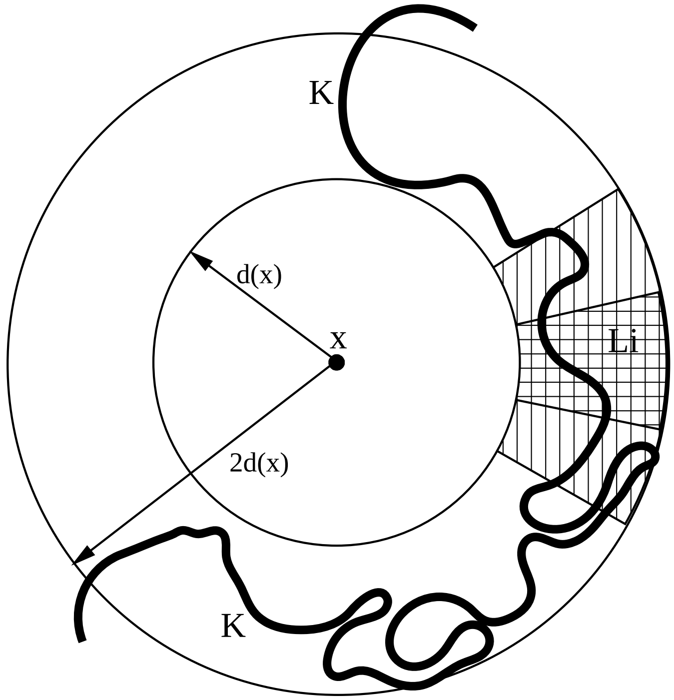

Figure 2: ***Note that*** $d(x)$ ***is the distance of*** x ***from the boundary*** $K = \partial\Omega$***.***

1. *The corkscrew condition*
2. *The fatness condition*
3. *and if $\partial\Omega$ has a Minkowski content,*

then for every $\eta > 0$ there exists a constant $c_{\eta,n} > 0$ such that for all $r > \varepsilon > 0$ if we choose Q at random with uniform law in $\mathcal{Q}_\varepsilon$, the probability that a Brownian motion started at any point x of Q hits for the first time $\partial\Omega$ at distance greater than r from the starting point $\mathbb{P}\left(X > r\right)$ verifies

$$\frac{1}{c_{\eta,n}} \left(\frac{r}{\varepsilon}\right)^{n-d-2+\eta} \leq \mathbb{P}\left(X > r\right) \leq c_{\eta,n} \left(\frac{r}{\varepsilon}\right)^{n-d-2-\eta}.$$

4 An alternative approach in the quasicircle perturbative case.

We present here a simple 2D case for which the proof of the main result is particularly simple using conformal mapping which preserves Brownian trajectories. The curve we consider will be a quasiconformal perturbation of the line. More precisely we consider a domain $\Omega = \varphi(\mathbb{R}^2_+)$ where $\varphi : \mathbb{R}^2_+ \to \mathbb{C}$ is holomorphic and such that

$$\sup_{x+iy\in\mathbb{R}^2_+} y|\frac{\varphi''(x+iy)}{\varphi'(x+iy)}| < 1/2. \tag{5}$$

It is known that under this hypothesis φ is injective and has a quasiconformal extention to the whole plane. In particular $\Gamma = \varphi(\mathbb{R})$ is a quasicircle close to a line. We will moreover assume that Γ has a Minkowski dimension which we denote by d. By Koebe theorem, the quantity $y|\varphi'(x+iy)|$ is uniformly comparable to the distance from $\varphi(x+iy)$ to Γ. For $\alpha > 0$ we then define $L_\alpha = \{x+iy \in \mathbb{R}^2_+\,;\, y|\varphi'(x+iy)| = \alpha\}$ and $\varphi(L_\alpha)$ will serve as a substitute for the level set $\{\zeta \in \Omega\,;\, d(\zeta,\Gamma) = \alpha\}$.

Lemma 4.1 *If φ satisfies (5) then L_α is the graph of a Lipschitz function $f_\alpha : \mathbb{R} \to \mathbb{R}$.*

Proof We apply the implicit function theorem to the function

$$F(x,y) = y\varphi'(x+iy)\overline{\varphi'}(x+iy) - \alpha^2.$$

The computation gives

$$\frac{\partial F}{\partial x} = 2y|\varphi'(x+iy)|^2\mathcal{R}(\psi(x+iy)), \frac{\partial F}{\partial y} = |\varphi'(x+iy)|^2(1+2y\mathcal{F}(\psi(x+iy))$$

where

$$\psi(z) = \frac{\varphi''(z)}{\varphi'(z)}.$$

The result follows because $\frac{\partial F}{\partial y} > 0$ and by (5).
We now consider a portion of the curve with diameter 1 and $0 < \varepsilon < r$. We divide the portion of Ω between Γ and L_ε into pieces of diameter $\sim \varepsilon$. The preimages of these pieces are rough squares of sidelength $f_\varepsilon(x_j)$ where x_j is the left-hand point of the intersection with $\mathbb{R}$. By the results of the preceeding paragraph it suffices to compute the probability that a Brownian motion started at $\varphi(x_j + if_\varepsilon(x_j))$ will hit $\varphi(L_r)$ before returning to Γ. By conformal invariance, this probability is comparable to $f_\varepsilon(x_j)/f_r(x_j)$ a quantity which is equivalent by Koebe to

$$\frac{r}{\varepsilon}\frac{|\varphi'(x_j + if_r(x_j)|}{|\varphi'(x_j + if_\varepsilon(x_j)|}.$$

On the other hand, by quasiconformality,

$$(x_{j+1} - x_j)|\varphi'(x_j + if_\varepsilon(x_j)| \sim \varepsilon.$$

Combining all the estimates we see that the probability we look for is comparable to

$$\varepsilon^d\left(\frac{r}{\varepsilon}\right)\sum_j \frac{|\varphi'(x_j + if_r(x_j))|}{|\varphi'(x_j + if_\varepsilon(x_j)|} \sim \varepsilon^d\left(\frac{r}{\varepsilon}\right)\sum_j \frac{(x_{j+1} - x_j)}{\varepsilon}|\varphi'(x_j + if_r(x_j))|$$

and we see a Riemann sum appearing: we finally get as an estimate for the probability we seek

$$\frac{\varepsilon^d}{r}\text{length}(\varphi(L_r)) \sim \frac{\varepsilon^d}{r}\frac{r}{r^d} \sim \left(\frac{r}{\varepsilon}\right)^d,$$

which is precisely what we wanted. •

The preceeding proof has been presented because it is particurlarly simple, but of course the result is not optimal. The results of paragraph 2 remain true in dimension 2 and the proof requires only minor changes. In particular the result is true for all quasicircles; the difference with the case presented is that the topology of the level sets of the function distance to the boundary is more complicated in general.

5 The self-avoiding walks case

Self-avoiding walks (S.A.W.) (see [MS93] and [dG79] for a definition) serve as a good model for polymers in physics. On the other hand it is strongly believed that in 2D self-avoiding walks is the same as $SLE_{8/3}$ (see [RS05] for a definition of SLE_κ). This conjecture, highly plausible, is comforted by the adequation between the computed dimension, which is $\dfrac{4}{3}$ and the proved dimension for SLE_κ curves , $1+\frac{\kappa}{8}$ (Beffara, [Bef06]). The following simulations can be seen as a new way of probing the adequation between SAW's and $SLE_{8/3}$:
In order to check that the statistics of flights over a self avoiding walk follow the expected law with $d=\frac{4}{3}$, we have performed extended computer simulations. We have generated a set of self-avoiding walks on a square lattice using an implementation of the pivot algorithm described by Kennedy [Ken02] (see also [Ken05]). The number of steps of the self avoiding walk is fixed at 10^5. Two S.A.W. are shown in Fig. 3.

We have checked by a box counting method that the mass fractal dimension of these curves are numerically founded around 1.33 ± 0.005. These values are very close to the expected value 4/3. We have performed an on-lattice simulation analyzing the first passage statistics of a random walk starting in the close vicinity of the SAW and going back for the first time nearby the SAW. The numerical computations were performed on several configurations of SAW, using a statistical analysis over more than 210^9 flights. Two probability density functions were computed. First, the probability density $\psi(n)$ that a flight has a total length equals to n. Second, the probability distribution of displacements $\theta(r)$. In order to limit edge effects, we have selected flights starting and ending on the same side of the S.A.W.. As shown Fig. 4 and Fig. 5, we found that $\psi(n) \propto n^{-\alpha}$ and $\theta(r) \propto r^{-\beta}$.

It was shown in [GKL$^+$06] that for a boundary of fractal dimension d embedded in an Euclidian space of Euclidean dimension d_e, we should have

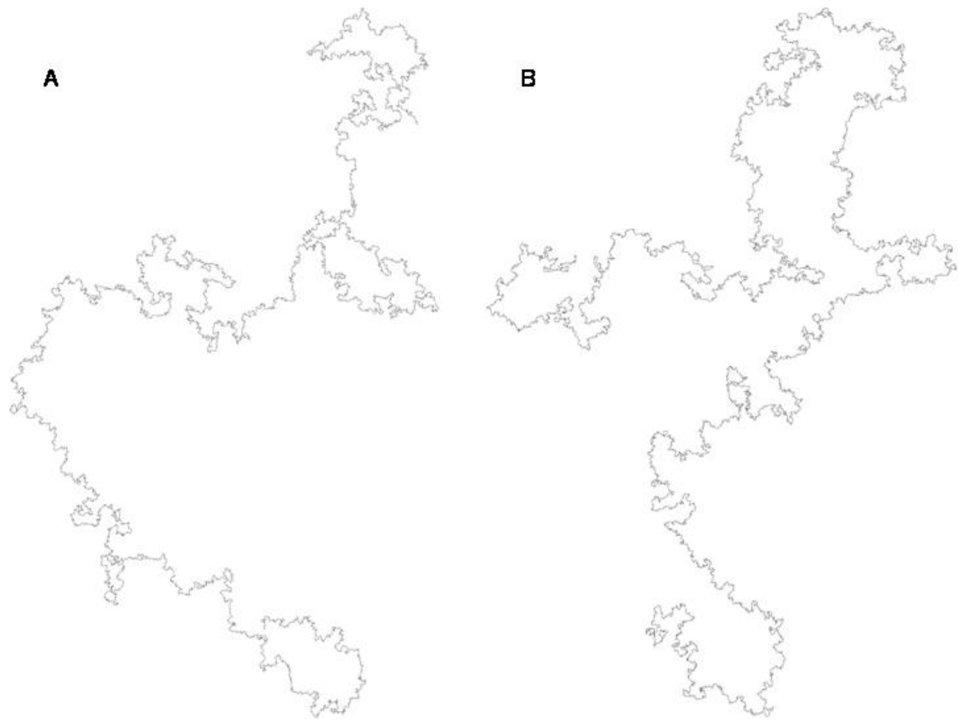

Figure 3: ***Two examples of self avoiding walk in $2D$ generated by the pivot algorithm [Ken02]. The number of steps in each SAW is fixed at 10^5.***

$$\alpha = \frac{d - d_e + 4}{2}. \tag{6}$$

We are now in position to prove this estimate rigorously; this result is contained a paper to come.

Moreover as $\theta(r) = -dP(r)/dr$, we get

$$\beta = d - d_e + 3. \tag{7}$$

For $d = 4/3$, we expect to have $\alpha = 10/6$ and $\beta = 7/3$. As shown in Figs. 4and 5, numerical results provide a very good approximation of these above predictions.

After these convincing simulations, let us prove rigourously these asymptotics for SLE_κ curves. First, combining results of Rohde-Schramm [RS05] and Beffara [Bef06], we see that the limsup in the definition of Minkowski upper-dimension for SLE_κ is actually a limit, allowing asymptotic values for all values of r. Secondly, we observed in the last section that the result follows from the understanding of the number of Whitney cubes of given order. By a nice result of Bishop [Bis96] , it follows immediately that we get the right estimate if we allow the starting point to be chosen on both sides of the curve, which is actually the case in the above simulations.
If we always start from the same side, then not only Bishop's result does not apply but

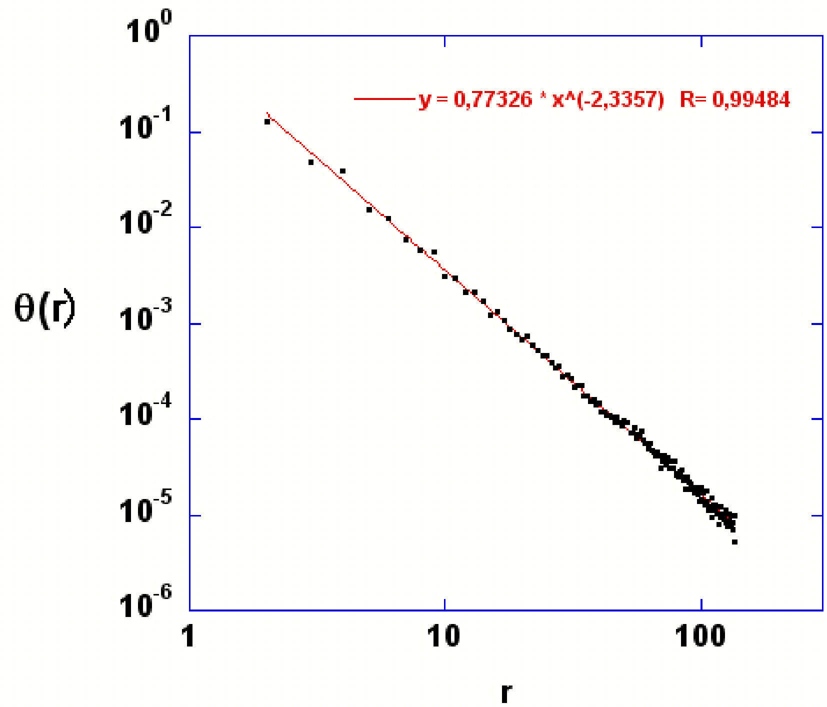

Figure 4: ***Evolution of the probability density $\theta(r)$ that a particle starting from close vicinity of SAW, returns to the SAW, for the first time, after an end to end displacement found between r and $r + dr$. The numerical estimation of the exponent β is very close to $7/3$.***

neither does it follow from the previous discussion because the corresponding domains do not have the corkscrew property. But, as Rohde and Schramm have proved, these domains are Hölder, meanning that the Riemman mapping from the upper half-plane onto them is Hölder continuous. This condition implies a weaker form of the the corkscrew condition which is sufficient to ensure the possibility to compute the Minkowski dimension of the SLE_κ curve via the counting of Whitney cubes. As we have seen this is enough to prove the main result about statistics of flights.

The case of self affine curves is a little more delicate and will be treated in a forthcoming paper.

It is to be noticed that, using quantum gravity arguments, Duplantier also obtained the right exponents for all SLE_κ(cf. [Dup04])

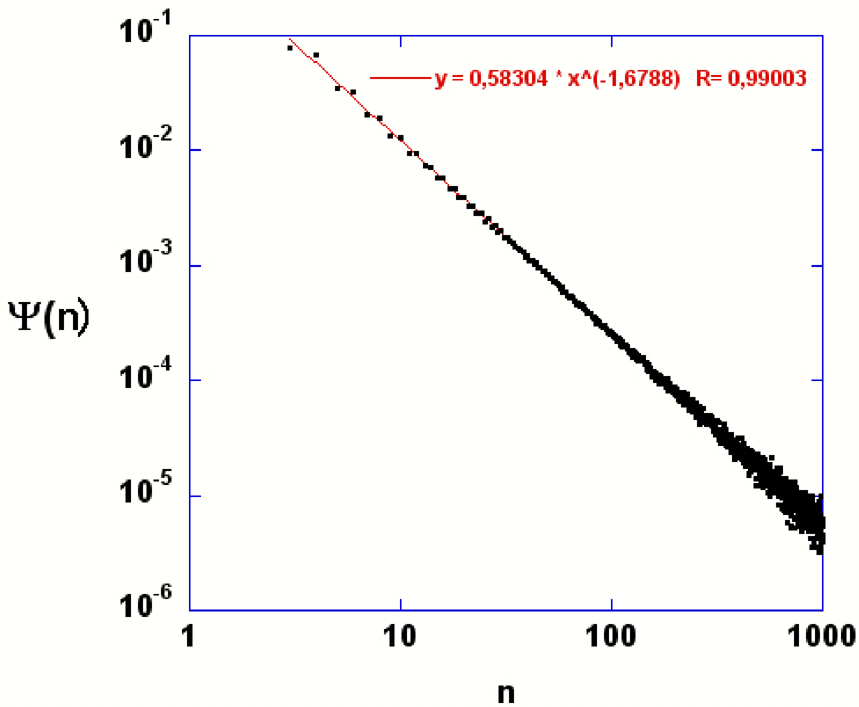

Figure 5: ***Evolution of the probability density*** $\psi(n)$ ***that the particle starting from a close vicinity of SAW and returning for the first time to this SAW, has a total length displacement between*** n ***and*** $n+1$***. The numerical estimation of the exponant*** α ***is very close to*** $10/6$***.***

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On the time schedule of Brownian Flights

Athanasios BATAKIS and Michel ZINSMEISTER

Abstract: *We are interested in the statistics of the duration of Brownian diffusions started at distance ϵ from the boundary of a given domain and stopped when they hit back this boundary.*

1 Introduction

The motivation of the following work has its origin in experimental physics. Some long molecules are solvable in a liquid (for instance imogolite in water or DNA in lithium) and the molecules forming the liquid show an intermittent dynamics, alternating diffusion in the bulb and adsorption on the long molecules. For the physicist's point of view, it is very important to have as precise as possible knowledge of the statistics of these brownian flights.

In [GKL$^+$06] a connection is established between the statistics of the long flight lengths and the geometry of the long molecules (more precisely their Minkowski dimension). This connection has been made rigorous in [BLZ09]. These two papers concern almost exclusively lengths. How does one check experimentally the results? A very powerful tool for that is relaxation methods in nuclear magnetic resonance (see [DPP$^+$08]): but this method only allows to compute (the statistics of the) duration of long flights. Some heuristic link between time and length was derived in [GKL$^+$06], [DPP$^+$08]. The aim of this paper is to make this heuristics rigorous.

2 Geometric Backgound

In the sequel, Ω will always denote a domain in $\mathbb{R}^d$ with compact boundary. The crucial tool we need to use is the notion of Whintey cubes. We thus recall the

Proposition 2.1 *(cf. [Gra08], p. 463) Given any non-empty open proper subset Ω of $\mathbb{R}^d$, there exists a family of closed dyadic cubes $\{Q_j\}_j$ such that*

- $\bigcup_j Q_j = \Omega$ *and the cubes Q_j's have disjoint interiors*
- $\sqrt{d}\ell(Q_j) \leq \ dist(Q_j, \partial\Omega) \leq 4\sqrt{d}\ell(Q_j)$
- *if Q_j and Q_k touch then $\ell(Q_j) \leq 4\ell(Q_k)$*

- *for a given Whitney cube Q_j there are at most 12^d Whitney cubes Q_k's that touch Q_j.*

In this statement, $\ell(Q)$ stands for the side-length of the cube Q and, for $\lambda > 0$, λQ is the cube of the same center and of sidelength $\lambda\ell(Q)$. For $k \in \mathbb{Z}$, we denote by $\mathcal{Q}_k$, the collection of Whitney cubes Q_j with $\ell(Q_j) = 2^k$. We also recall the definition of the Minkowski sausage: for $r > 0$,

$$M_r = \{x \in \Omega\ ;\ \ \mathrm{dist}(x, \partial\Omega) \le r\}$$

and

$$\Gamma_r = \{x \in \Omega\ ;\ \ \mathrm{dist}(x, \partial\Omega) = r\}$$

We then define $\mathcal{S}_r$ as the collection of Whitney cubes intersecting Γ_r. Notice that $\mathcal{S}_r$ is a finite set.

Definition 2.1 *Let $\varepsilon > 0$. We will call Brownian flight the random process $F_t, t \ge 0$ consisting in picking at random with equiprobability one of the dyadic Whitney cubes of $\mathcal{S}_\varepsilon$ and starting from the center of the cube a Brownian motion B_t killed once it reaches $\partial\Omega$. We denote by $\tau_\Omega = \inf\{t\ ;\ F_t \notin \Omega\}$ the lifetime of this process.*

We are interested in the asymptotics of $\mathbb{P}(\tau_\Omega > t)$ as t grows, but this needs some explanation:

It is well known that, if Ω is bounded, this quantity decreases exponentially, as $t \to \infty$, as $e^{\lambda t}$, where λ is the first eigenvalue of the Laplacian. We define

$$R_\Omega = \min\left\{1, \sup_{x\in\Omega}\ \mathrm{dist}(x, \partial\Omega)\right\}.$$

Our aim is to evaluate $\mathbb{P}(\tau_\Omega > t)$ in the interval $\epsilon^2 \le t \le R_\Omega^2$, and this independently of ε. In fact, the estimate we are looking for is thus an estimate with respect to ε rather than for "pure" t.

We study first a simple example, since we will need the partial result anyhow. Let $[0, a]$ be a real segment and take $x \in (0, a)$. The probability that brownian motion started at x has not exit the interval $(0, a)$ by time t, $\mathbb{P}(\tau_x > t)$, is given by the following equivalent formulas (see [Fel71], pg. 342)

$$\mathbb{P}(\tau_x > t) = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{t}}^{-x/\sqrt{t}} \exp\left(-\frac{1}{2}y^2\right) dy + \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} (-1)^k \int_{(ka-x)\sqrt{t}}^{(ka+x)\sqrt{t}} \exp\left(-\frac{1}{2}y^2\right) dy$$

and

$$\mathbb{P}(\tau_x > t) = \frac{4}{\pi} \sum_{k=1}^{\infty} \exp\left(-\frac{(2n+1)^2\pi^2}{2a^2} t\right) \sin\frac{(2n+1)\pi x}{a}.$$

By symmetry we can assume $x \le \frac{a}{2}$. If $\frac{a}{\sqrt{t}}$ is not too big, say $\frac{a}{\sqrt{t}} < 1/2$, we have an easy estimate of $\mathbb{P}(\tau_x > t)$ using the first formula : $\mathbb{P}(\tau_x > t) \sim \frac{x}{\sqrt{t}}$. On the other hand, for $\frac{a}{\sqrt{t}}$ not too small we get, by the second formula, that $\mathbb{P}(\tau_x > t) \leq \exp\left(-\frac{\pi^2}{2a^2}t\right)$. Hence, that there is a change of regime, the decay of $\mathbb{P}(\tau_x > t)$ with t going from polynomial to exponential and the quantity $\frac{x}{\sqrt{t}}$ is relevant for small times.

In a higher dimensional context, let Q be the cube centered at 0 and of side r. By the preceeding remark it follows that, T denoting the exit time from Q of the Brownian motion starting at 0, we have

$$\mathbb{P}(T > t) \leq c\left(\frac{r}{\sqrt{t}}\right)^d \tag{1}$$

where c depends only on d. Here, $\varepsilon = r \simeq R_\Omega$ and the opposite inequality thus holds for $\varepsilon^2 \leq t \leq R_\Omega^2$.

If the Brownian motion is started at distance ε from the boundary then the exit time is essentially the exit time from a half space and we thus get

$$\mathbb{P}(T > t) \leq c\left(\frac{\varepsilon}{\sqrt{t}}\right) = c\left(\frac{\varepsilon}{\sqrt{t}}\right)^{d-1+2-d}, \tag{2}$$

and, as we will see, the opposite inequality is valid for t not too big.

Our goal is to extend (2) to general domains with rough boundary. In order to describe the domain of validity of our result let us recall a few definitions. Let K be a compact subset of $\mathbb{R}^d$. For $j \geq 0$ let N_j be the number of dyadic cubes of the j-th generation (i.e of size 2^{-j}) that intersect K.

Definition 2.2 *The Minkowski dimension of K is*

$$d_M(K) = \limsup_{j\to\infty} \frac{\log_2(N_j)}{j}$$

Returning to our situation, we can define similarly the Whitney dimension of $\partial\Omega$ as

$$d_W = d_W(\partial\Omega) = \limsup_{j\to\infty} \frac{\log_2(W_j)}{j}, \tag{3}$$

where W_j is the number of elements of $\mathcal{Q}_j$.

Under very mild conditions (see [Bis96], [JK82], [BLZ09]) these two dimensions coincide. If the boundary of Ω has some self similarity we can moreover say that there is a constant $c > 0$ such that

$$\frac{1}{c}\varepsilon^{d_M} \leq \#\mathcal{S}_\varepsilon \leq c\varepsilon^{d_M}, \tag{4}$$

for all $\varepsilon \leq R_\Omega$, where $d_M = d_M(\partial\Omega)$.

In our main theorem we will assume that our domain Ω satisfies (4).

We also suppose that the domain Ω satisfies so-called Δ-regularity condition (see also [JW88], [Anc86], [HKM93]): there exists $L > 0$ such that for all $x \in \Omega$, if $d_x = \operatorname{dist}(x, \partial\Omega) < R_\Omega$ then

$$\omega^x_{\mathbb{B}(x,2d_x)\cap\Omega}(\partial\Omega) \geq L, \tag{5}$$

where $\omega^x_{\mathbb{B}(x,2d_x)\cap\Omega}$ is the distribution law of the hitting point of Brownian motion starting at x and killed when reaching the boundary of $\mathbb{B}(x, 2d_x) \cap \Omega$). This is a very mild condition (satisfied, for instance, by all domains in $\mathbb{R}^2$ with non-trivial connected boundary) that appears frequently in related literature in various forms (for instance "uniform capacity condition" or Hardy inequality).

3 Time and length estimates for Brownian flights

We can now state the main result. Let Ω be a bounded domain in $\mathbb{R}^d$ satisfying (4) and (5). If τ_Ω denotes the life-time of a Brownian flight F_t with parameter ε we have

Theorem 3.1 *There exists $c > 0$ depending only on constants in (4), (5) (and in particular not on ε) such that*

$$\frac{1}{c}\left(\frac{\varepsilon}{\sqrt{t}}\right)^{d_M+2-d} \leq \mathbb{P}(\tau_\Omega > t) \tag{6}$$

and

$$\mathbb{P}(\tau_\Omega > t) \leq c\left(\frac{\varepsilon}{\sqrt{t}}\right)^{d_M+2-d}\left|\log\left(\frac{\varepsilon}{\sqrt{t}}\right)\right|^{2d}, \tag{7}$$

for all $\varepsilon^2 < t < R_\Omega^2$.

This theorem has a "cousin" theorem, which was proved in [BLZ09].

Theorem 3.2 *Let $\varepsilon < r < R_\Omega$. The probability that the hitting point of F is at distance greater than r from the starting point x is comparable to*

$$\left(\frac{\#\mathcal{S}_r}{\#\mathcal{S}_\varepsilon}\right)^{d_M}\left(\frac{r}{\varepsilon}\right)^{d-2} \tag{8}$$

Notice that we do not assume (4) for this theorem. If we do, we have

$$\left(\frac{\#\mathcal{S}_r}{\#\mathcal{S}_\varepsilon}\right)^{d_M}\left(\frac{r}{\varepsilon}\right)^{d-2} \sim \left(\frac{r}{\varepsilon}\right)^{d_M-(d-2)} \tag{9}$$

Notice that the quantity on the left of (9) is the same as the one in (6) where we have replaced r by $\sqrt{t}$ which is coherent with standard behaviour of Brownian motion.

4 Proof of the theorem 3.1

For $s > 0$ we denote by β_s the total time spent by Brownian flight F_t in the Minkowski sausage $\{x \in \Omega\ ;\ \ \mathrm{dist}(x, \partial\Omega) \leq s\}$ and $\delta\beta_s = \beta_s - \beta_{s/2}$.

We define analogue quantities more adapted to the Whitney decomposition ; namely, $\delta\tilde{\beta}_{2^k}$ will denote the time spent by F_t inside $\tilde{M}_k = \bigcup\{Q\ ;\ Q \in \mathcal{Q}_k\}$.

If Q is a Whintney cube we define the "vicinity" of Q as

$$\tilde{Q} = Q \cup \bigcup\{Q'\ ;\ Q' \in \mathcal{E}\},$$

where

$$\mathcal{E} = \{Q' \in \bigcup_k \mathcal{Q}_k\ ;\ \lambda Q \cap Q' \neq \emptyset \text{ and } \lambda Q' \cap Q \neq \emptyset\},$$

and $\lambda = 8\sqrt{d}$ satisfy that for all Whitney cubes

$$\lambda Q \supset \mathbb{B}(x_Q, 2\ \mathrm{dist}(x_Q, \partial\Omega)),$$

x_Q being the center of Q.

We may now start the proof and we begin with

4.1 The upper bound

We separate the event $\{\tau_\Omega > t\}$ by the partition $\{\tau_\Omega = \beta_{\sqrt{t}}\}$ and $\{\tau_\Omega > \beta_{\sqrt{t}}\}$, that is whether the process goes or does not go at distance $\sqrt{t}$ from the boundary. From theorem 3.2 we get

$$\mathbb{P}(\tau_\Omega > t \text{ and } \tau_\Omega > \beta_{\sqrt{t}}) \leq \mathbb{P}(\tau_\Omega > \beta_{\sqrt{t}}) \leq c\left(\frac{\varepsilon}{\sqrt{t}}\right)^{d_M+2-d}.$$

In order to estimate the term $\mathbb{P}(\tau_\Omega = \beta_{\sqrt{t}} > t)$ we begin by writing

$$\tau_\Omega = \sum_{k=-\infty}^{\log_2\sqrt{t}} \delta\beta_{2^k},$$

and thus, putting $k_0 = \log_2 \epsilon$ and $k_1 = \log_2\sqrt{t}$,

$$\mathbb{P}\left(\sum_{k=-\infty}^{\log_2\sqrt{t}} \delta\beta_{2^k} > t\right) \leq \sum_{k=k_0+1}^{k_1} \mathbb{P}\left(\delta\beta_{2^k} > \frac{t}{2(k_1-k_0)}\right) + \mathbb{P}\left(\beta_{2^{k_0}} > \frac{t}{2}\right) \tag{10}$$

We now invoke the following lemma whose proof is post poned to the next section.

Lemma 4.1 *There exists a number k^* depending only on d and constants C , $0 < p < 1$ depending only on d and L such that for all $t > 0$, $k \in \mathbb{Z}$ and $N \in \mathbb{N}$ we have*

$$\mathbb{P}(\delta\beta_{2^k} > t) \leq Cp^N +$$
$$\mathbb{P}\left(\exists Q \in \mathcal{Q}_k \cup ... \cup \mathcal{Q}_{k-k^*}; \exists\, 0 < s_1 < s_2 < t \textit{ with } F_{[s_1,s_2]} \subset \tilde{Q} \textit{ and } s_2 - s_1 > t/N\right)$$

Using this lemma, we get

$$\mathbb{P}\left(\delta\beta_{2^j} > \frac{t}{2(k_1 - k_0)}\right) \leq cp^N +$$
$$\mathbb{P}\left(\exists Q \in \mathcal{Q}_j \cup ... \cup \mathcal{Q}_{j-k^*}\ \exists s_1 < s_2 < \tau_\Omega\ ;\ F_{[s_1,s_2]} \subset \tilde{Q}\ ,\ s_2 - s_1 > \frac{t}{2(k_1 - k_0)N}\right)$$

By (1), (9) and the strong Markov property of Brownian motion we then have :

$$\mathbb{P}\left(\delta\beta_{2^j} > \frac{t}{2(k_1 - k_0)}\right) \leq cp^N +$$
$$\mathbb{P}\left(\exists Q \in \mathcal{Q}_j \cup ... \cup \mathcal{Q}_{j-k^*}\ \exists s_1 \tau_\Omega\ ;\ F_{s_1} \in \tilde{Q}\right) \times$$
$$\mathbb{P}\left(\exists s_2 > s_1\ ,\ F_{[s_1,s_2]} \subset \tilde{Q} \text{ and } s_2 - s_1 > \frac{t}{2(k_1 - k_0)N}\right) \leq$$
$$cp^N + c\left(\frac{2^j}{\sqrt{\frac{t}{2(k_1-k_0)N}}}\right)^d \left(\frac{\varepsilon}{2^j}\right)^{d_M+2-d}$$

Suming up the first term in (10) we get

$$\sum_{k=k_0+1}^{k_1} \mathbb{P}\left(\delta\beta_{2^k} > \frac{t}{2(k_1 - k_0)}\right) \leq c(k_1 - k_0)p^N + c\sum_{k=k_0+1}^{k_1}\left(\frac{2^k}{\sqrt{\frac{t}{2(k_1-k_0)N}}}\right)^d \left(\frac{\varepsilon}{2^k}\right)^{d_M+2-d}$$
$$\leq c(k_1 - k_0)p^N + c(2(k_1 - k_0)N)^d \varepsilon^{d_M+d-2}\left(\frac{1}{\sqrt{t}}\right)^d \sum_{k=k_0+1}^{k_1} 2^{2d-d_M+2}$$
$$\leq c(k_1 - k_0)p^N + c(2(k_1 - k_0)N)^d \varepsilon^{d_M+d-2}\left(\frac{1}{\sqrt{t}}\right)^d 2^{k_1(2d-d_M+2)}$$
$$\leq cp^N \log_2\left(\frac{\sqrt{t}}{\varepsilon}\right) + c\left(\log_2\left(\frac{\sqrt{t}}{\varepsilon}\right)N\right)^d \left(\frac{\varepsilon}{\sqrt{t}}\right)^{d_M+2-d}$$

Take $N \simeq (d_M + 2 - d)\log_p\left(\frac{\varepsilon}{\sqrt{t}}\right)$ to obtain

$$\sum_{k=k_0+1}^{k_1} \mathbb{P}\left(\delta\beta_{2^k} > \frac{t}{2(k_1 - k_0)}\right) \leq c\left(\log\left(\frac{\varepsilon}{\sqrt{t}}\right)\right)^{2d}\left(\frac{\varepsilon}{\sqrt{t}}\right)^{d_M+2-d} \qquad (11)$$

To bound the second term $\mathbb{P}\left(\beta_{2^{k_0}} > \frac{t}{2}\right)$ of the sum (10) we need a lemma of the same nature as lemma 4.1. The proof of this lemma is also post poned to the next section.

Lemma 4.2 *Let $\mathcal{R}_{k_0}$ be the collection of all dyadic cubes of sidelength 2^{k_0} intersecting $\partial\Omega$. There exist constants C , $0 < p < 1$ depending only on d and L such that for all $t > 0$, $k_0 \in \mathbb{Z}$ and $N \in \mathbb{N}$*

$$\mathbb{P}(\delta\beta_{2^{k_0}} > t) \leq Cp^N +$$
$$\mathbb{P}\left(\exists Q \in \mathcal{R}_{k_0}; \exists\, 0 < s_1 < s_2 < t \text{ with } F_{[s_1,s_2]} \subset NQ \text{ and } s_2 - s_1 > t/N\right)$$

We therefore deduce

$$\mathbb{P}(\delta\beta_{2^{k_0}} > \frac{t}{2}) \leq cp^N + \left(\frac{2^{k_0}N}{\sqrt{t}}\right)^d.$$

We minimize on $N \simeq \log_p(\varepsilon)$ to get

$$\mathbb{P}(\delta\beta_{2^{k_0}} > \frac{t}{2}) \leq c\left(\log(\varepsilon)\frac{\varepsilon}{\sqrt{t}}\right)^d.$$

Combining this last inequality and (11) we get the upper bound, since $d_M + d - 2 \leq d$.

4.2 The Lower Bound

Following the same reasoning for $k_1 = [\log_2 \sqrt{t}] + 1$ we get

$$\mathbb{P}(\tau_\Omega > t) \geq \mathbb{P}(\exists s_1 > 0 \text{ s.t. } F_{s_1} \in \bigcup_{Q \in \mathcal{Q}_{k_1}} Q \text{ and } \exists s_2 > s_1 + t \text{ s.t. } F_{[s_1,s_2]} \in \bigcup_{Q \in \mathcal{Q}_{k_1}} 2Q)$$

Using strong Markov property the this probability can be written as the product of $\mathbb{P}(\exists s_1 > 0 \text{ s.t. } F_{s_1} \in \mathcal{Q}_{k_1})$ with $\mathbb{P}(\exists s_2 > s_1 + t \text{ s.t. } F_{[s_1,s_2]} \in \bigcup_{Q\in\mathcal{Q}_{k_1}} 2Q)$. The second term of the product is greater than the probability that Brownian motion exits a cube of size $2^{k_1+1} \simeq \sqrt{t}$ at time greater that t which is bounded below by a positive constant depending only on d. The first one is simply the probability that Brownian flight gets to $\mathcal{Q}_{k_1}$ which is equivalent to $\left(\frac{\epsilon}{\sqrt{t}}\right)^{d_M+2-d}$ and the proof is complete.

5 Proofs of lemmas

Let un first deal with lemma 4.1. The proof of 4.2 is quite similar and will hence be abridged.

5.1 Proof of lemma 4.1

We need the following

Lemma 5.1 *Under the Δ-regularity hypothesis the probability that BM touches more than N Whitney cubes of a given size decreases as Cp^N, with $0 < p < 1$, C a positive constant.*

The proof of the lemma relies on an annuli reasoning.

Proof Le $(B_t)_{t>0}$ be Brownian motion started at any point $x \in \Omega$ and choose $k \in \mathbb{Z}$. Choose any $Q \in \mathcal{Q}_k$ and let λQ be the cube of the same center but λ times the side-length $\ell(Q)$ of Q. By the definition of Whitney cubes, there is a $\lambda = 8\sqrt{d}$ depending only on d such that

$$\frac{\lambda}{2}\ell(Q) \leq \text{ dist}(Q, \partial\Omega) \leq 2\lambda\ell(Q).$$

Suppose that there exists $t_0 > 0$ such that $B_{t_0} \in Q$. By the Δ-regularity condition (5), the probability that there exists $t_1 > t_0$ with $B_{[t_0,t_1]} \subset \Omega$ and $B_{t_1} \notin \lambda Q$ is bounded above by $p < 1$ depending only on L, λ:

$$\mathbb{P}\left(\exists t_1 > t_0\ ;\ B_{[t_0,t_1]} \subset \Omega \text{ and } B_{t_1} \notin \lambda Q \middle| \exists t_0 > 0\ ;\ B_{t_0} \in Q\right) < p \tag{12}$$

On the other hand, the number of Whitney cubes of $\mathcal{Q}_k$ lying inside λQ is bounded by a constant $c_1 = c_1(d)$. The probability that there exists a Whitney cube $Q_1 \in \mathcal{Q}_k$ outside λQ that is visited by Brownian motion is hence bounded above by $p < 1$.

We study probability that there exist Whitney cubes $Q_1, ..., Q_m \in \mathcal{Q}_k$ such that $Q_1 \cap \lambda Q = Q_2 \cap \lambda Q_1 = ... = Q_m \cap \lambda Q_{m-1} = \emptyset$ all visited by Brownian motion. It is sufficient to prove that this probability decays exponentially with m.

By the strong Markov property the probability that there exists $t_m > t_{m-1} > ... > t_0$ such that $B_{t_0} \in Q$, $B_{t_1} \in Q_1$ ... , $B_{t_m} \in Q_m$ is given by

$$\begin{aligned}
&\mathbb{P}\left(\exists\, t_m > t_{m-1} > ... > t_0\ ;\ \text{and } Q, ...Q_m \text{ as above such that } B_{t_m} \in Q_m, ..., B_{t_0} \in Q\right)\\
&= \mathbb{P}\left(\exists\, t_m > t_{m-1}\ ;\ B_{t_m} \in Q_m | \exists\, t_{m-1} > ... > t_0\ B_{t_{m-1}} \in Q_{m-1}, ..., B_{t_0} \in Q\right) \cdot\\
&\mathbb{P}\left(\exists\, t_{m-1} > ... > t_0\ B_{t_{m-1}} \in Q_{m-1}, ..., B_{t_0} \in Q\right)\\
&= \mathbb{P}\left(\exists\, t_m > t_{m-1}\ ;\ B_{t_m} \in Q_m \text{ with } Q_m \cap \lambda Q_{m-1} = \emptyset | \exists\, t_{m-1}\ ; B_{t_{m-1}} \in Q_{m-1}\right) \cdot\\
&\mathbb{P}\left(\exists\, t_{m-1} > ... > t_0\ B_{t_{m-1}} \in Q_{m-1}, ..., B_{t_0} \in Q\right)
\end{aligned}$$

Now, by (12),

$$\mathbb{P}\left(\exists\, t_m > t_{m-1}\ ;\ B_{t_m} \in Q_m \text{ with } Q_m \cap \lambda Q_{m-1} = \emptyset | \exists\, t_{m-1}\ ; B_{t_{m-1}} \in Q_{m-1}\right) < p.$$

By induction we get that

$$\mathbb{P}\left(\exists\, t_m > t_{m-1} > ... > t_0\ ;\ \text{and } Q, ...Q_m \text{ as above such that } B_{t_m} \in Q_m, ..., B_{t_0} \in Q\right) < p^m$$

and hence the lemma. $\bullet$

Recall that for a given dyadic Whitney cube Q we have defined the vicinity $\tilde{Q}$ of Q as the union of all Whitney cubes Q' verifying

$$Q' \cap \lambda Q \neq \emptyset \text{ and } Q \cap \lambda Q' \neq \emptyset.$$

We can easily check that there are less than $(100\sqrt{d})^d$ such cubes Q' of size at most $\ell(Q)/12$ (the constants are not optimal). We say that the k-level layers are visited at least n times if there exist $t_0 < s_1 < t_1 < ... < s_n < t_n$ satisfying

$$B_{t_j} \in \bigcup_{Q \in \mathcal{S}_{2^k}} Q \text{ and } B_{s_j} \notin \bigcup_{Q \in \mathcal{S}_{2^k}} \tilde{Q},$$

for all $j = 1, ..., n$. For any $k \in \mathbb{Z}$ note

$$\nu_k = \sup\{n \in \mathbb{N}\ ;\ \text{the } k\text{-level layers are visited at least } n \text{ times}\}$$

Lemma 5.2 *There exists $0 < p < 1$ and a positive constant C such that, given $k \in \mathbb{Z}$, for all $n \in \mathbb{N}$*

$$\mathbb{P}(\nu_k > n) \leq p^n \mathbb{P}\left(\exists t_0 > 0 \textit{ and } Q \in \mathcal{S}_{2^k}\ ;\ B_{t_0} \in Q\right).$$

Proof The arguments as similar as in lemma 5.1. We only need to prove that $\mathbb{P}(\nu_k > 1) < p$ and apply strong Markov property. We have

$$\begin{aligned}
\mathbb{P}(\nu_k > 1) &\leq \mathbb{P}\left(\exists 0 < t_0 < s_1 < t_1\ ,\ Q \in \mathcal{S}_{2^k}\ ;\ B_{t_0} \in Q\ ,\ B_{s_1} \notin \bigcup_{Q \in \mathcal{S}_{2^k}} \tilde{Q}\ ,\ B_{t_1} \in \bigcup_{Q \in \mathcal{S}_{2^k}} Q\right) \\
&= \mathbb{P}\left(\exists t_1 > s_1 > t_0\ ;\ B_{s_1} \notin \bigcup_{Q \in \mathcal{S}_{2^k}} \tilde{Q}\ ,\ B_{t_1} \in \bigcup_{Q \in \mathcal{S}_{2^k}} Q \middle| \exists t_0 > 0\ ;\ B_{t_0} \in Q \in \mathcal{S}_{2^k}\right) \\
&\times\ \mathbb{P}\left(\exists t_0 > 0 \text{ and } Q \in \mathcal{S}_{2^k}\ ;\ B_{t_0} \in Q\right)
\end{aligned}$$

To abbreviate formulas we note $\mathbb{P}_c(.) = \mathbb{P}\left(.|\exists t_0 > 0\ ;\ B_{t_0} \in Q \in \mathcal{S}_{2^k}\right)$. With this notation,

$$\mathbb{P}_c\left(\exists t_1 > s_1 > t_0\ ;\ B_{s_1} \notin \bigcup_{Q \in \mathcal{S}_{2^k}} \tilde{Q}\ ,\ B_{t_1} \in \bigcup_{Q \in \mathcal{S}_{2^k}} Q)\right) =$$

$$\mathbb{P}_c\left(\exists t_1 > s_1\ ;\ B_{t_1} \in \bigcup_{Q \in \mathcal{S}_{2^k}} Q \middle| A\right) \mathbb{P}_c(A) + \mathbb{P}_c\left(\exists t_1 > s_1\ ;\ B_{t_1} \in \bigcup_{Q \in \mathcal{S}_{2^k}} Q \middle| B\right) \mathbb{P}_c(B)$$

where

$$A = \left\{\exists s_1 > t_0\ ;\ B_{s_1} \notin \bigcup_{Q \in \mathcal{S}_{2^k}} \lambda Q\right\} \text{ and}$$

$$B = \left\{ \exists s_1 > t_0 \; ; \; B_{s_1} \in Q' \; , \; \lambda Q' \cap \bigcup_{Q \in \mathcal{S}_{2^k}} Q = \emptyset \; , \; B_{s_1} \in \bigcup_{Q \in \mathcal{S}_{2^k}} \lambda Q \right\}$$

form a partition of the event $\left\{ \exists s_1 > t_0 \; ; \; B_{s_1} \notin \bigcup_{Q \in \mathcal{S}_{2^k}} \tilde{Q} \right\}$.

By (12), $\mathbb{P}_c(A) \leq p$. Similarly, by the strong Markov property of Brownian motion and by (12),

$$\mathbb{P}_c \left(\exists t_1 > s_1 \; ; \; B_{t_1} \in \bigcup_{Q \in \mathcal{S}_{2^k}} Q \, \Big| B \right) = \mathbb{P} \left(\exists t_1 > s_1 \; ; \; B_{t_1} \in \bigcup_{Q \in \mathcal{S}_{2^k}} Q \, \Big| B_{s_1} \in Q' \right) \leq p.$$

We deduce that

$$\mathbb{P}_c \left(\exists t_1 > s_1 > t_0 \; ; \; B_{s_1} \notin \bigcup_{Q \in \mathcal{S}_{2^k}} \tilde{Q} \; , \; B_{t_1} \in \bigcup_{Q \in \mathcal{S}_{2^k}} Q) \right) \leq \mathbb{P}_c(A) + p(1 - \mathbb{P}_c(A))$$

The function $t \mapsto t + p(1-t)$ being increasing on $[0, p]$ we get

$$\mathbb{P}_c \left(\exists t_1 > s_1 > t_0 \; ; \; B_{s_1} \notin \bigcup_{Q \in \mathcal{S}_{2^k}} \tilde{Q} \; , \; B_{t_1} \in \bigcup_{Q \in \mathcal{S}_{2^k}} Q) \right) \leq 2p - p^2 < 1$$

and the lemma is proven. •

Remark that by definition of dyadic Whitney cubes there exist k^* depending only on the dimension of the space ($k^* = [\log_2(8\sqrt{d})] + 3$ will do) such that for all $k \in \mathbb{Z}$,

$$\{x \in \Omega \; ; \; 2^{k-1} \leq \; \mathrm{dist}(x, \partial\Omega) \leq 2^k\} \subset \bigcup_{j=k-k^*}^{k} \bigcup_{Q \in \mathcal{Q}_j} Q. \tag{13}$$

Proof of lemma 4.1 We clearly have

$$\mathbb{P}(\delta\beta_{2^k} > t) \leq \mathbb{P}(\delta\beta_{2^k} > t \; , \; \nu_k + ... + \nu_{k-k^*} > N) + \mathbb{P}(\delta\beta_{2^k} > t \; , \; \nu_k + ... + \nu_{k-k^*} \leq N) \tag{14}$$

Given $t > 0$, $k \in \mathbb{Z}$, by lemma 5.2 we have, for all N,

$$\mathbb{P}(\delta\beta_{2^k} > t \; , \; \nu_k + ... + \nu_{k-k^*} > N) < k^* p^{\frac{1}{k^*} N} \mathbb{P}\left(\exists t_0 > 0 \text{ and } Q \in \mathcal{S}_{2^{k-k^*}} \; ; \; B_{t_0} \in Q\right) < c\tilde{p}^N$$

Let us estimate the second term of the sum (14). By (13) and the definition of ν_k we get

$$\mathbb{P}(\delta\beta_{2^k} > t \; , \; \nu_k + ... + \nu_{k-k^*} \leq N)$$

$$\leq \mathbb{P}(\exists l \leq N \; , \; Q_1, ..., Q_l \in \bigcup_{j=k-k^*}^{k} \mathcal{Q}_j \; ; \; Q_s \cap \tilde{Q}_{s-1} = \emptyset \; , \; \forall s = 2, ..., l \text{ and}$$

$$\exists t_1 < s_1 \leq t_2 < s_2 < ... \leq t_l < s_l \; ; \; B_{[t_i, s_i]} \subset \tilde{Q}_s \; \forall s = 1, ..., l \text{ and } \sum_{i=1}^{l} s_i - t_i > t).$$

Since $l \leq N$ we get that $\mathbb{P}(\delta\beta_{2^k} > t \ , \ \nu_k + ... + \nu_{k-k^*} \leq N)$ is bounded above by

$$\mathbb{P}\left(\exists Q \in \mathcal{Q}_k \cup ... \cup \mathcal{Q}_{k-k^*}; \exists\, 0 < t_{i_0} < s_{i_0} \text{ with } B_{[t_{i_0}, s_{i_0}]} \subset \tilde{Q} \text{ and } s_{i_0} - t_{i_0} > t/N\right),$$

which completes the proof. $\bullet$

5.2 Proof of lemma 4.2

The ideas are the same but we will work with cubes touching the boundary instead of Whitney cubes.

Lemma 5.3 *Under the Δ-regularity hypothesis, the probability that BM started at distance $\varepsilon < r$ from the boundary gets at distance greater than R from the starting point without leaving the Minkowski sausage $M_r = \{x \in \Omega \ ; \ \ dist(x, \partial\Omega) \leq r\}$ is bounded above by $cp^{R/r}$ where $c > 0$ and $0 < p < 1$ are constants (depending only on L of the Δ-regularity hypothesis and on d).*

The proof, similar to the one of lemma 5.1, is therefore abridged.

Proof Let $x \in M_r$ and consider the annuli centered at x of inner radii $4\ell r$ and outer radii $4(\ell+1)r$ where $\ell = 0, ..., \left[\frac{R}{4r}\right]$. Brownian motion started at x and moving at distance R from x before exiting Ω must go through all these annuli. The probability of going through such an annulus while staying at distance at most r from the boundary is bounded by a $p_0 \in (0,1)$ by the Δ-regularity hypothesis. To see this take any point y in the middle of the annulus (i.e. at distance $\frac{6\ell+6}{r}$ from x) and consider the ball of center y and radius $2r$. If $d_y < r$, the probability to exit the ball without touching $\partial\Omega$ is uniformly bounded away from 1 by the same hypothesis. This probability being greater than the probability of going through the annulus we have the statement. By the independence of the "crossing annuli" events we get that the probability that Brownian motion goes through all the annulli is smaller that $cp_0^{\left[\frac{R}{4r}\right]} \sim c\tilde{p}^{\left[\frac{R}{r}\right]}$. $\bullet$

We say that the Minkowki sausage M_r is visited by the Brownian motion at least k times if there exist $t_0 < s_1 < t_1 < ... < s_n < t_n < \tau_\Omega$ satisfying $B_{t_i} \in M_r$ for all $i = 0, ..., n$ and $B_{s_i} \notin M_{4r}$.

In a similar way with ν_k we define ξ_k as the

$$\xi_r = \sup\{n \in \mathbb{N} \ ; \ M_r \text{ has been visited at least } k \text{ times}\}.$$

Lemma 5.4 *There exists $0 < p < 1$ depening only on the Δ-regularity's L such that, given $r > 0$, for all $n \in \mathbb{N}$*

$$\mathbb{P}(\xi_r > n) \leq p^n.$$

Proof The proof of this lemma is a straightforward application of the Δ-regularity condition. It suffices to show that there exists $0 < p < 1$ such that

$$\mathbb{P}((\exists t_1 > s_1 > t_0 \ ; \ B_{t_0} \in M_r \ , \ B_{s_1} \notin M_{4r} \ , \ B_{t_1} \in M_r) < p,$$

and then apply the Markov property. Remark that, the probability

$$\mathbb{P}((\exists t_1 > s_1 > t_0 \ ; \ B_{t_0} \in M_r \ , \ B_{s_1} \notin M_{4r})$$

is smaller than the probability that brownian motion started at t_0 exits a ball of radius $2r > 2 \ \mathrm{dist}(B_{t_0}, \partial\Omega)$ without hitting $\partial\Omega$. By the Δ-regularity this last probability is bounded by a constant $p < 1$. $\bullet$

Proof of lemma 4.2 As before we get :

$$\mathbb{P}(\delta\beta_{2^{k_0}} > t) = \mathbb{P}(\delta\beta_{2^{k_0}} > t \ , \ \xi_{2^{k_0}} > N) + \mathbb{P}(\delta\beta_{2^{k_0}} > t \ , \ \xi_{2^{k_0}} \leq N) \tag{15}$$

By lemma 5.4 we get $\mathbb{P}(\delta\beta_{2^{k_0}} > t \ , \ \xi_{2^{k_0}} > N) \leq p^N$, where $0 < p < 1$, for all $N \in \mathbb{N}$. Let us now deal with the second term of the sum.

$$\mathbb{P}(\delta\beta_{2^{k_0}} > t \ , \ \xi_{2^{k_0}} \leq N) \leq \mathbb{P}(\exists s_1 < s_2 < \tau_\Omega \ ; \ s_2 - s_1 > t/N \ , \ B_{[s_1,s_2]} \subset M_{4r})$$

Using lemma 5.3 we get that, for $R > 4r$, this probability is bounded by

$$cp^{\frac{R}{r}} + \mathbb{P}\left(\exists s_1 < s_2 < \tau_\Omega \ ; \ s_2 - s_1 > t/N \ , \ B_{[s_1,s_2]} \subset M_{4r} \cap \mathbb{B}(B_{s_1}, R)\right),$$

and the statement of the lemma 4.2 follows on taking $N = \left[\frac{R}{r}\right]$ $\bullet$

6 Further Comments

We should point out that hypothesis (4) in theorem 3.1 can be dropped; in this case, the same reasoning as in subsection 4.2 gives the lower bound

$$\left(\frac{\#\mathcal{S}_{\sqrt{t}}}{\#\mathcal{S}_\varepsilon}\right)^{d_M} \left(\frac{\sqrt{t}}{\varepsilon}\right)^{d-2}.$$

The best upper bound is less evident; nevertheless a slight improvement of the above proofs gives

$$\mathbb{P}(\tau_\Omega > t) \leq c\left(\log\frac{\sqrt{t}}{\epsilon}\right)^{cd} \sup_N \left(2^{-Nd} + \sum_{k=0}^{N}\left(\frac{\#\mathcal{S}_{\sqrt{t}/2^k}}{\#\mathcal{S}_\varepsilon}\right)^{d_M}\left(\frac{2^{-k}\sqrt{t}}{\varepsilon}\right)^{d-2}\right).$$

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Invariant measures for intermittent transport

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Abstract: *We are interested in the existence and properties of limits of invariant measures for Brownian diffusions started at distance ϵ from the boundary of a given domain and stopped when they hit back this boundary, when ϵ goes to* 0*.*

1 Introduction

The motivation of the following work has its origin in experimental physics. Some long molecules are solvable in a liquid (for instance imogolite in water or DNA in lithium) and the molecules forming the liquid show an intermittent dynamics, alternating diffusion in the bulb and adsorption on the long molecules. For the physicist's point of view, it is very important to a knowledge have as precise as possible of the statistics of these brownian flights.

In [GKL$^+$06] a connection is established between the statistics of the long flight lengths and the geometry of the long molecules (more precisely their Minkowski dimension). This connection has been made rigorous in [BLZ11],[BZ10].

Nevertheless, the statistics of Brownian flights are depending on the distribution of the initial starting point. In all previous papers this distribution is taken uniform on the set Γ_ε of the points at distance ε to the boundary. This choice, justified by experimental data, seemed mathematically unfounded. In fact, iteration of Brownian flights seems to have a limite steady state : uniform distribution is stationnary.

The aim of this paper is to rigorously prove this statement (all definitions of objects will be reminded in the following section) .

For a Green domain Ω in $\mathbb{R}^d$ we can define (following [LS84], [BL96]) a random walk on the centers of dyadique Whitney cubes in Ω with time-homogeneous transition probabilities and (discrete) Green function equal a constant times the Green function of Ω. The positive harmonic functions associated to this Markov chain are the traces of positive harmonic functions on the centers of Whitney cubes. The trajectories of the so-defined random walk are called discretized Brownian paths.

We choose any $\varepsilon > 0$ and we consider the collection $\mathcal{S}_\varepsilon$ of all dyadique Whitney cubes intersecting $\Gamma_\varepsilon = \{x \in \Omega\ ;\ \ \mathrm{dist}(x, \partial\Omega) = \varepsilon\}$. Let μ be a (discrete) probability measure on $\mathcal{S}_\varepsilon$ and choose a cube Q with probability $\mu(Q)$.

For every discretized Brownian path ξ^Q started at the center of Q, we consider the last cube $Q' = \xi^Q_{\tau_\varepsilon}$ of $\mathcal{S}_\varepsilon$ visited by the path ξ. The Markov chain being transient this exit time is well defined and is a.s. finite. This defines a function π on the set of discrete probability measures on $\mathcal{S}_\varepsilon$, that assigns to μ a new mesure $\pi(\mu) : \pi\mu(Q') = \mathbb{E}_\mu \xi^Q_{\tau_\varepsilon}$.

Theorem 1.1 *For every $\varepsilon > 0$, there exists a unique probability measure μ_ε such that $\pi(\mu_\varepsilon) = \mu_\varepsilon$. Moreover, there exists a constant γ not depending on ε such that for all $Q \in \mathcal{S}_\varepsilon$*

$$\frac{1}{\gamma}\frac{1}{\#\mathcal{S}_\varepsilon} \leq \mu_\varepsilon \leq \gamma \frac{1}{\#\mathcal{S}_\varepsilon}.$$

Some mild hypothesis on the domain is needed to prove this theorem, and the last exit time must be properly redefined. To carry out the proofs, we will suitably discretize Brownian motion, following [BL96] and [LS84] and apply an adapted version of the Perron-Frobenius theorem to a finite Markov chain.

2 Backgound and Motivation

In the sequel, Ω will always denote a domain in $\mathbb{R}^d$ with compact boundary. The main tool we need to use is the notion of Whitney cubes. We thus recall the

Proposition 2.1 *(cf. [Gra08], p. 463) Given any non-empty open proper subset Ω of $\mathbb{R}^d$, there exists a family $\mathcal{W}$ of closed dyadic cubes $\{Q_j\}_j$ such that*

- $\bigcup_j Q_j = \Omega$ *and the cubes Q_j's have disjoint interiors*
- $\sqrt{d}\ell(Q_j) \leq$ *dist*$(Q_j, \partial\Omega) \leq 4\sqrt{d}\ell(Q_j)$
- *if Q_j and Q_k touch then $\ell(Q_j) \leq 4\ell(Q_k)$*
- *for a given Whitney cube Q_j there are at most 12^d Whitney cubes Q_k's that touch Q_j.*

In this statement, $\ell(Q)$ stands for the side-length of the cube Q and, for $\lambda > 0$, λQ is the cube of the same center and of sidelength $\lambda\ell(Q)$. For $k \in \mathbb{Z}$, we denote by $\mathcal{Q}_k$, the collection of Whitney cubes Q_j with $\ell(Q_j) = 2^k$. We also recall the definition of the Minkowski sausage: for $r > 0$,

$$M_r = \{x \in \Omega\ ;\ \ \mathrm{dist}(x, \partial\Omega) \leq r\}$$

and

$$\Gamma_r = \{x \in \Omega\ ;\ \ \mathrm{dist}(x, \partial\Omega) = r\}$$

We then define $\mathcal{S}_r$ as the collection of Whitney cubes intersecting Γ_r. Notice that $\mathcal{S}_r$ is a finite set.

Definition 2.1 *Let $\varepsilon > 0$. We will call Brownian flight the random process $F_t, t \geq 0$ consisting in picking at random with equiprobability one of the dyadic Whitney cubes of $\mathcal{S}_\varepsilon$ and starting from the center of the cube a Brownian motion g_t killed once it reaches $\partial\Omega$. We denote by $\tau_\Omega = \inf\{t \; ; \; F_t \notin \Omega\}$ the lifetime of this process.*

Definition 2.2 *The Minkowski dimension of K is*

$$d_M(K) = \limsup_{j\to\infty} \frac{\log_2(N_j)}{j}$$

We can define similarly the Whitney dimension of $\partial\Omega$ as

$$d_W = d_W(\partial\Omega) = \limsup_{j\to\infty} \frac{\log_2(W_j)}{j}, \tag{1}$$

where W_j is the number of elements of $\mathcal{Q}_j$.

Under very mild conditions (see [Tri83], [Bis96], [JK82], [BLZ11]) these two dimensions coincide. If the boundary of Ω has some self similarity we can moreover say that there is a constant $c > 0$ such that

$$\frac{1}{c}\varepsilon^{d_M} \leq \#\mathcal{S}_\varepsilon \leq c\varepsilon^{d_M}, \tag{2}$$

for all $\varepsilon \leq R_\Omega$, where $d_M = d_M(\partial\Omega)$.

We also suppose that the domain Ω satisfies so-called Δ-regularity condition (see also [JW88], [Anc86], [HK93]): there exists $L > 0$ such that for all $x \in \Omega$, if $d_x = \operatorname{dist}(x, \partial\Omega) < R_\Omega$ then

$$\omega^x_{\mathbb{B}(x,2d_x)\cap\Omega}(\partial\Omega) \geq L, \tag{3}$$

where $\omega^x_{\mathbb{B}(x,2d_x)\cap\Omega}$ is the distribution law of the hitting point of Brownian motion starting at x and killed when reaching the boundary of $\mathbb{B}(x, 2d_x) \cap \Omega$. This is a very mild condition (satisfied, for instance, by all domains in $\mathbb{R}^2$ with non-trivial connected boundary) that appears frequently in related literature in various forms (for instance "uniform capacity condition" or Hardy inequality).

The following result has been proven in [BLZ11]:

Theorem 2.3 *Let $\varepsilon < r < R_\Omega$. The probability that the hitting point of F is at distance greater than r from the starting point x is comparable to*

$$\left(\frac{\#\mathcal{S}_r}{\#\mathcal{S}_\varepsilon}\right)^{d_M} \left(\frac{r}{\varepsilon}\right)^{d-2} \tag{4}$$

Notice that we do not assume (2) for this theorem. If we do, we have

$$\left(\frac{\#\mathcal{S}_r}{\#\mathcal{S}_\varepsilon}\right)^{d_M} \left(\frac{r}{\varepsilon}\right)^{d-2} \sim \left(\frac{r}{\varepsilon}\right)^{d_M-(d-2)} \tag{5}$$

Aknowledgment: The authors wish to thank Alano Ancona for helpful and enlightning discussions on the discretization of Brownian motion.

3 Discretization of Brownian Motion

We will modify the continuous diffusion process into a discrete one, with the same potential theory. In this section, Ω is a Green domain, $\mathcal{B}_t$ stands for Brownian motion in Ω, τ_Ω is the exit time (for brownian motion) of Ω, ie. the hitting time of $\partial\Omega$.

If G denotes the Green function of the domain $\Omega \subset \mathbb{R}^d$ and Q is a cube in Ω, recall that there exist a constant C such that for all $y \in Q$

$$\log \frac{\ell(Q)}{|x_Q - y|} \leq G(x_Q, y) \leq \log \frac{C\ell(Q)}{|x_Q - y|},$$

C depending on $\Omega \subset \mathbb{R}^2$ and

$$\frac{1}{||x_Q - y||^{d-2}} - \frac{1}{\ell(Q)^{d-2}} \leq G(x, y) \leq \frac{1}{||x_Q - y||^{d-2}},$$

for domains $\Omega \subset \mathbb{R}^d$, $d \geq 3$. Moreover,

$$\log \frac{\ell(Q)}{2|x_Q - y|} \leq G_Q(x_Q, y) \leq \log \frac{2\ell(Q)}{|x_Q - y|},$$

and

$$\frac{1}{||x_Q - y||^{d-2}} - \frac{C}{\ell(Q)^{d-2}} \leq G_Q(x, y) \leq \frac{1}{||x_Q - y||^{d-2}} - \frac{\sqrt{d}}{\ell(Q)^{d-2}},$$

for $d \geq 3$, G_Q being the Green function of the cube Q.

We denote by $\mathcal{N}$ be the collection of the centers of cubes in $\mathcal{W}$ and we consider the complete graph $\mathcal{G}$ associated. Let $x_Q \in \mathcal{N}$ be the center of a Whitney cube $Q \in \mathcal{W}$.

3.1 Planar domains

We consider separately planar domains not (only) because of the recurrence of brownian motion in $\mathbb{R}^2$ but in order to better explain the ideas of the proof.

Let $F_Q(\eta) = \{y \in \Omega\ ;\ G_Q(x_Q, y) \geq \eta\}$. Clearly, $F_Q(\eta)$ is a compact connected set, such that $x_Q \in F_Q$. Furthermore, by the preceeding observations and the definition of Whitney cubes we can deduce that, for η big enough, $F_Q = F_Q(\eta) \subset \mathring{Q}$ and that there is a constant $c_0 < 1$ not depending on Q such that $c_0 Q \subset \mathring{F}_Q$, where cQ will denote the (contracted) cube centered at x_Q but of sidelength $\ell(cQ) = c\ell(Q)$.

The triplet $(\mathcal{N}, \mathbf{F}, \mathbf{W})$, where $\mathbf{F} = \{F_Q\ ;\ Q \in \mathcal{W}\}$, and $\mathbf{W} = \{\mathring{Q}\ ;\ Q \in \mathcal{W}\}$ is a *balanced Lyons-Sullivan data*, defined in [BL96]. For convienience of the reader we remind hereby the principal facts of this paper.

1. The collection $\mathbf{F}$ is recurrent for Brownian motion in Ω, ie.

$$\mathbb{P}_x(\exists t < \tau_\Omega \ ; \ \mathcal{B}_t \in \bigcup_{\mathbf{F}} F_Q) = 1 \quad \text{for all } x \in \Omega.$$

2. $x_Q \in F_Q \subset \mathring{Q}$, for all $Q \in \mathcal{W}$,

3. $F_Q \cap Q' = \emptyset$, for all $Q \neq Q' \in \mathcal{W}$,

4. there exists a constant c such that for all $Q \in \mathcal{W}$, any positive harmonic function h in $\mathring{Q}$ and all $z \in F_Q$ we have

$$\frac{1}{c} h(x_Q) \leq h(z) \leq c h(x_Q)$$

Following [BL96] we define a Markov chain X on $\mathcal{N}$: for $y \in F = \bigcup_{\mathbf{F}} F_Q$ denote by $\phi(y) \in \mathcal{N}$ the center of the unique cube $Q = Q_y \in \mathcal{W}$ containing y. For a path ξ in the space of brownian paths Ξ starting at $y \in F$, let $S_0(\xi)$ be the exit time of ξ from Q_y. Recursively, we define the stopping times R_n and S_n in the following way

- $R_n(\xi) = \inf\{t > S_{n-1}(\xi) \ ; \ \xi(t) \in F\}$
- $S_n(\xi) = \inf\{t > R_{n-1}(\xi) \ ; \ \xi(t) \notin \mathring{Q}_{\xi(R_{n-1}(\xi))}\}$.

Recall that, if V is an open set and for any $x \in V$ we denote by ω_V^x the harmonic measure of V at x. By our hypothesis, there exist C such that for all $Q \in \mathcal{W}$ and all $y \in F_Q$,

$$\frac{1}{C} d\omega_{\mathring{Q}}^{x_Q} \leq d\omega_{\mathring{Q}}^{y} \leq C d\omega_{\mathring{Q}}^{x}.$$

Let now

$$\kappa_n(\xi) = \frac{1}{C} \frac{d\omega_{\mathring{Q}_{\phi(\xi(R_n(\xi)))}}^{\phi(\xi(R_n(\xi)))}(\xi(S_n(\xi)))}{d\omega_{\mathring{Q}_{\phi(\xi(R_n(\xi)))}}^{\xi(R_n(\xi))}(\xi(S_n(\xi)))} \leq 1$$

Using these stopping times Ballmann and Ledrappier consider the probability space

$$(\tilde{\Xi} = \Xi \times [0,1]^{\mathbb{N}}, \tilde{\mathbb{P}}_y = \mathbb{P}_y \otimes \lambda^{\mathbb{N}}),$$

λ being the Lebesgue measure in $[0,1]$. For $(\xi, \alpha) \in \tilde{\Xi}$ define recursively

- $N_0(\xi, \alpha) = 0$
- $N_k(\xi, \alpha) = \inf\{n > N_{k-1}(\xi, \alpha) \ ; \ \alpha_n < \kappa_n(\xi)\}$

One can then define a Markov chain (discrete random walk) X_i on $\mathcal{N}$ the centers of cubes in $\mathcal{W}$ with time homogeneous transition probabilities

$$p_{Q,Q'} = \tilde{\mathbb{P}}_{x_Q}(\xi(N_1(\xi,\alpha)) = x_{Q'}).$$

Let g be the Green function of this Markov chain on $\mathcal{N}$. The Markov chain is hence irreducible and aperiodic.

In [BL96, LS84] it is shown that for all $x = x_Q \in \mathcal{N}$ and all $y \neq x$

$$g(y,x) = \frac{1}{C} \sum_{n\in\mathbb{N}} \mathbb{P}_y(\xi(R_n(\xi)) \in F_Q) \tag{6}$$

and also that

$$G(y,x) = \sum_{n\in\mathbb{N}} \int_{F_Q} G_{\hat{Q}}(z,x)\mathbb{P}_y(\xi(R_n(\xi)) \in dz) \tag{7}$$

By the choice of $F_Q = F_Q(\eta)$ and relations (6) and (7) we deduce that

$$g(x,y) = C\eta G(x,y) \tag{8}$$

and, moreover, that the transition probabilities of the Markov chain $p_{Q,Q'}$ are symmetric in Q, Q', ie. $p_{Q,Q'} = p_{Q',Q}$.

3.2 Domains in higher dimensions

We consider now bounded domains $\Omega \subset \mathbb{R}^d$, $d \geq 3$. In this setting we can not choose the sets $F_Q(\eta)$ in the same way. Such a choice would be in contradiction with the fourth definition property of Lyons-Sullivan data.

We will choose $\eta = \eta(Q)$ proportionnal to the distance of Q to the boundary. To start with, remark that, by the definition of Whitney cubes, if $Q \cap \Gamma_s \neq \emptyset$, then necessarily $Q \cap \Gamma_{s/4\sqrt{d}} = \emptyset$. Let $b = 1/(4\sqrt{d})$. For $Q \in \mathcal{W}$ put $\eta(Q) = b^{n(d-2)}$ if $Q \cap \Gamma_{b^n} \neq 0$ for some n and $\eta(Q) = \ell(Q)^{d-2}$ otherwise.

All previous definitions and properties stay valid except for (8). This equality must now be replaced by the following one : $\forall x \neq y \in \mathcal{N}$ such that for some $n \in \mathbb{N}$ both Q_x, Q_y are in $\mathcal{S}_{b^n}$ (ie. $Q_x \cap \Gamma_{b^n} \neq \emptyset$ and $Q_x \cap \Gamma_{b^n} \neq \emptyset$) we have

$$g(x,y) = Cb^n G(x,y) \tag{9}$$

and transition probabilities $p_{Q,Q'}$ are symmetric under the same conditions.

Potential theory for this Markov chain is equivalent to the potential theory for Brownian motion in Ω: in fact, the positive harmonic functions of the Markov chain are precisely the traces on $\mathcal{N}$ of positive harmonic functions in Ω, [Anc90].

4 An equivalent discrete model for Brownian flights

Let us now modify the initial model to make it "compatible" with discretized Brownian motion. The idea is to adapt the following remark (in fact Perron-Frobenius theorem) : if we replace brownian motion by X_k, a symmetric simple random walk on a graph, say $T = (\mathbb{Z}/n)^d$, we can consider the random process that consists on picking a boundary point x of T with probability distribution μ , starting random walk at this point and consider the first time τ the random walk gets back to the boundary of T. Clearly, the uniform measure ν on the boundary of T is invariant by the process $\nu(y) = \sum_x \nu(x)\mathbb{P}_x(X_\tau = y)$.

Recall that $\mathcal{S}_{2^{-n}}$ is the collection of Whitney cubes intersecting $\Gamma_{2^{-n}}$ (essentially the cubes at distance 2^{-n} to the boundary). Let us also assume, for the moment, that $\partial\Omega$ is bounded, of diameter say 1.

The dynamical system we are interested in is the following. Given a (discrete) probability measure μ on $\mathcal{S}_{2^{-n}}$, choose a cube $Q \in \mathcal{S}_{2^{-n}}$ with probability $\mu(Q)$. Consider the Markov chain $({}^QX_k)$ defined above started at the center of Q, $X_0 = x_Q$. Since Ω is Greenian (random walk and Brownian motion are transient) so is the X_k on $\mathcal{N}$. Therefore, there is, almost surely, a finite time $\tau_n = \sup\{k \geq 0\ ;\ {}^QX_k \in \mathcal{S}_{2^{-n}}\}$, the last exit time of the random walk from the union of cubes in $\mathcal{S}_{2^{-n}}$.

We consider the function π assigning at every μ the exit distribution of ${}^QX_{\tau_n}$. It is now clear that there is a discrete invariant measure for this function, μ_n (we have identified the cubes in $\mathcal{S}_{2^{-n}}$ with their centers $\mathcal{N} \cap \mathcal{S}_{2^{-n}}$).

The same tools used in [BLZ11] can now be used to prove the analogue of theorem 2.3:

Theorem 4.1 *Choose Q at random with uniform law within $\mathcal{S}_{2^{-n}}$. The probability that the distance $||x_Q - {}^QX_{\tau_n}|| > r$ is comparable to*

$$\left(\frac{\#\mathcal{S}_r}{\#\mathcal{S}_{2^{-n}}}\right)^{d_M} (r2^n)^{d-2} \tag{10}$$

Recall that the domain Ω is assumed to verify the Δ-regularity condition (3). Under the same hypothesis we also have the main result 1.1 that can clearly be reformulated in the folowing way :

Theorem 4.2 *There is a constant γ independent of n such that for all $Q \in \mathcal{S}_{2^{-n}}$,*

$$\frac{1}{\gamma\#\mathcal{S}_{2^{-n}}} \leq \mu_n(Q) \leq \frac{\gamma}{\#\mathcal{S}_{2^{-n}}},$$

ie. the measure μ_n is uniformly equivalent to the uniform measure on $\mathcal{S}_{2^{-n}}$.

Moreover, for any measure μ on $\mathcal{S}_{2^{-n}}$ we have that $\lim_k \pi^k(\mu) = \mu_n$.

Proof For $Q, Q' \in \mathcal{S}_{2^{-n}}$, denote by

$$g_{Q,Q'} = g(x_Q, x_{Q'}) = \delta_{x_Q}(x'_Q) + \sum_{k=1}^{\infty} \mathbb{P}_{x_Q}({}^Q X_k = x_{Q'})$$

the mean time the random walk ${}^Q X_k$ started at x_Q spends inside Q'.

It follows on the construction of the random walk that there is a constant $\eta > 0$ such that $g_{Q,Q'} = g_{Q',Q} = \eta G(x_Q, x_{Q'})$. Let us point out here that for domains in higher dimension we need to pay attention that all $Q \in \mathcal{S}_{2^{-n}}$ have the same η.

Let us now consider, for $Q \in \mathcal{S}_{2^{-n}}$ the probability r_n^Q that random walk definitely leaves $\mathcal{S}_{2^{-n}}$ immediately after reaching Q, that is $r_n^Q = \mathbb{P}_Q(\tau_n = 0)$ by the Markov property.

Lemma 4.3 *There exists a constant $c > 0$ such that for all $n \in \mathbb{N}$ and all $Q \in \mathcal{S}_{2^{-n}}$, $r_n^Q \geq c$.*

We assume this lemma for the moment. The random walk being transient on $\mathcal{G}$, using the Markov property we get :

$$\begin{aligned}
1 &= \sum_{k=0}^{\infty} \sum_{Q' \in \mathcal{S}_{2-n}} \mathbb{P}_{x_Q}({}^Q X_k = x_{Q'} \ , \ \tau_n = k) \\
&= \sum_{k=0}^{\infty} \sum_{Q' \in \mathcal{S}_{2-n}} \mathbb{P}_{x_Q}(\tau_n = k \mid {}^Q X_k = x_{Q'}) \mathbb{P}_{x_Q}({}^Q X_k = x_{Q'}) \\
&= \sum_{k=0}^{\infty} \sum_{Q' \in \mathcal{S}_{2-n}} \mathbb{P}_{x_{Q'}}(\tau_n = 0) \mathbb{P}_{x_Q}({}^Q X_k = x_{Q'}) \\
&= \sum_{Q' \in \mathcal{S}_{2-n}} \sum_{k=0}^{\infty} \mathbb{P}_{x_{Q'}}(\tau_n = 0) \mathbb{P}_{x_Q}({}^Q X_k = x_{Q'}) \\
&= \sum_{Q' \in \mathcal{S}_{2-n}} \mathbb{P}_{x_{Q'}}(\tau_n = 0) \sum_{k=0}^{\infty} \mathbb{P}_{x_Q}({}^Q X_k = x_{Q'}) \\
&= \sum_{Q' \in \mathcal{S}_{2-n}} g_{Q,Q'} r_n^{Q'},
\end{aligned}$$

for all $Q \in \mathcal{S}_{2^{-n}}$.

Observe that $g_{Q,Q'} r_n^{Q'}$ is the probability that the random walk, started at Q, leaves $\mathcal{S}_{2^{-n}}$ through Q'. Consider the measure μ_n on $\mathcal{S}_{2^{-n}}$ defined by

$$\mu_n(Q) = \frac{r_n^Q}{\sum_{Q' \in \mathcal{S}_{2-n}} r_n^{Q'}}.$$

Clearly, for any $\tilde{Q} \in \mathcal{S}_{2^{-n}}$

$$\begin{aligned}
\sum_{Q\in\mathcal{S}_{2^{-n}}} \mu_n(Q) b_{Q,\tilde{Q}} r_n^{\tilde{Q}} &= \sum_{Q\in\mathcal{S}_{2^{-n}}} \frac{r_n^Q}{\sum_{Q'\in\mathcal{S}_{2^{-n}}} r_n^{Q'}} b_{Q,\tilde{Q}} r_n^{\tilde{Q}} \\
&= \frac{r_n^{\tilde{Q}}}{\sum_{Q'\in\mathcal{S}_{2^{-n}}} r_n^{Q'}} \sum_{Q\in\mathcal{S}_{2^{-n}}} g_{Q,\tilde{Q}} r_n^Q \\
&= \frac{r_n^{\tilde{Q}}}{\sum_{Q'\in\mathcal{S}_{2^{-n}}} r_n^{Q'}} \sum_{Q\in\mathcal{S}_{2^{-n}}} g_{\tilde{Q},Q} r_n^Q,
\end{aligned}$$

because $g_{\tilde{Q},Q} = g_{Q,\tilde{Q}}$. Since $\sum_{Q\in\mathcal{S}_{2^{-n}}} g_{\tilde{Q},Q} r_n^Q = 1$ we get that μ_n is invariant:

$$\sum_{Q\in\mathcal{S}_{2^{-n}}} \mu_n(Q) g_{Q,\tilde{Q}} r_n^{\tilde{Q}} = \mu_n(\tilde{Q}).$$

By lemma 4.3, for all $Q \in \mathcal{S}_{2^{-n}}$, $c \le r_n^Q \le 1$. Hence, there is a constant $\gamma = \frac{1}{c}$ such that

$$\frac{1}{\gamma \# \mathcal{S}_{2^{-n}}} \le \mu_n(Q) \le \frac{\gamma}{\# \mathcal{S}_{2^{-n}}},$$

which is the first claim of the theorem.

The second claim follows on the fact that $(g_{Q,Q'} r_{Q'})_{Q,Q'\in\mathcal{S}_{2^{-n}}}$ is a stochastic matrix with strictly positive coefficients. •

We now turn to the proof of the lemma which strongly relies on the Δ-regularity hypothesis.

Proof of lemma 4.3 First observe that, by the definition of Whitney cubes, $\forall Q \in \mathcal{W}$ there is a Whitney cube $Q' \subset 8\sqrt{d}Q$ such that $\frac{\ell(Q)}{64\sqrt{d}^2} \le \ell(Q') \le \frac{\ell(Q)}{16\sqrt{d}}$.

Moreover, if $Q \in \mathcal{S}_{2^{-n}}$ and Q' as above, there is a constant $c > 0$ depending only on dimension such that the probability that Brownian motion stating at x_Q hits F at Q' for the first time, $\omega^{x_Q}_{\Omega\setminus F}(F_{Q'})$ is greater than c. Hence, there is a constant $c' > 0$ (depending only on the Lyons-Sullivan data) such that $\mathbb{P}_{x_Q}(X_1 = x_{Q'}) \ge c'$.

On the other hand, it follows on (3) that,

$$\omega^{x_{Q'}}_{\Omega\cap\mathbb{B}(x_{Q'},8\sqrt{d}\ell(Q'))}(\partial\Omega) \ge L.$$

We deduce that there exist c'' such that

$$\mathbb{P}_{x_{Q'}}(X_n \in \mathbb{B}(x_{Q'}, 8\sqrt{d}\ell(Q'))\ ,\ \forall n \in \mathbb{N}) \ge c''.$$

And finally, by Markov's property $r_n^Q \ge c'c''$, which is the claim of the lemma. •

Now let ν be any measure on $\mathcal{S}_{2^{-n}}$. An immediate consequence of standard facts on stochastic matrices is the

Corollary 4.4 *Under the same hypothesis as in theorem 4.2 we have*

$$\lim_{k\to\infty} \pi^k(\nu) = \mu_n.$$

The following result is a corollary of theorem 4.2.

Theorem 4.5 *Suppose that $\partial\Omega$ is an Alhfors s-regular set of finite Hausdorff measure $\mathcal{H}_s$. There is a constant γ such that every weak limite μ of the sequence μ_n satisfies*

$$\frac{1}{\gamma} r^s \leq \mu(\mathbb{B}_r) \leq \gamma r^s,$$

where $\mathbb{B}_r$ is any ball of radius r centered on $\partial\Omega$.

The uniqueness of the weak limit is false in general. Nevertheless, if the boundary is self-similar it is probable that the limit exists. Let us point out that, if $\partial\Omega$ is smooth enough, the above limit exists and is equal to the normalized surface measure. We must also cite here the results of [GS03] in the same vein.

There is a special case of planar domains , small perturbations of the disk by quasiconformal maps with small constant. In these domains we can prove a continuous version of theorem 4.2 but also we can define the mass transport in a diferent way: instead of taking the last exit point of Γ_ε we can consider the hitting point on the boundary and afterwards get back to Γ_ε using internal rays (see also [BLZ11]). This approach is adopted in a forthcoming article.

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