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Abelianization of Subgroups of Reflection Groups and their Braid Groups; an Application to Cohomology

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Abstract. The final result of this article gives the order of the extension

$$1 \longrightarrow P/[P,P] \xrightarrow{j} B/[P,P] \xrightarrow{p} W \longrightarrow 1$$

as an element of the cohomology group $H^2(W, P/[P,P])$ (where B and P stands for the braid group and the pure braid group associated to the complex reflection group W). To obtain this result, we first refine Stanley-Springer's theorem on the abelianization of a reflection group to describe the abelianization of the stabilizer N_H of a hyperplane H. The second step is to describe the abelianization of big subgroups of the braid group B of W. More precisely, we just need a group homomorphism from the inverse image of N_H by p (where $p : B \to W$ is the canonical morphism) but a slight enhancement gives a complete description of the abelianization of $p^{-1}(W')$ where W' is a reflection subgroup of W or the stabilizer of a hyperplane. We also suggest a lifting construction for every element of the centralizer of a reflection in W.

Key words. Reflection Group – Braid Group – Abelianization – Cohomology – Hyperplane Arrangement

1. Introduction

Let us start with setting the framework. Let V be a finite dimensional complex vector space; a *reflection* is a non trivial finite order element s of $\mathrm{GL}(V)$ which pointwise fixes a hyperplane of V, called *the hyperplane of* s. The *line of* s is the one dimensional eigenspace of s associated to the non trivial eigenvalue of s. Let $W \subset \mathrm{GL}(V)$ be a *(complex) reflection group* that is to say a finite group generated by reflections. We denote by $\mathscr{S}$ the set of reflections of W and $\mathscr{H}$ the set of hyperplanes of W :

$$\mathscr{S} = \{s \in W, \quad \mathrm{codim}\ker(s-\mathrm{id}) = 1\} \quad \text{and} \quad \mathscr{H} = \{\ker(s-\mathrm{id}), \quad s \in \mathscr{S}\}\,.$$

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For a reflection group, we denote by $V^{\mathrm{reg}} = V \setminus \cup_{H\in\mathscr{H}} H$ the set of regular vectors. According to a classical result of Steinberg [12, Corollary 1.6], V^{reg} is precisely the set of vectors that no non trivial element of W fixes. Thus, the canonical map $\pi : V^{\mathrm{reg}} \to V^{\mathrm{reg}}/W$ is a Galois covering. So let us fix a base point $x_0 \in V^{\mathrm{reg}}$ and denote by $P = \pi_1(V^{\mathrm{reg}}, x_0)$ and $B = \pi_1(V^{\mathrm{reg}}/W, \pi(x_0))$ the fundamental groups of V^{reg} and its quotient V^{reg}/W, we obtain the short exact sequence

$$1 \longrightarrow P \longrightarrow B \xrightarrow{p} W \longrightarrow 1 \tag{1}$$

The groups B and P are respectively called the *braid group* and the *pure braid group* of W.

The final result of this article (Corollary 1) gives the order of the extension

$$1 \longrightarrow P/[P,P] \xrightarrow{j} B/[P,P] \xrightarrow{p} W \longrightarrow 1$$

as an element of the cohomology group $H^2(W, P/[P,P])$. This order turns out to be the integer $\kappa(W)$ defined by Marin in [9]. As explained in [9], this integer is linked to the periodicity of the monodromy representation of B associated to the action of W on its set of hyperplanes.

To obtain this result, we first describe in section 2 the abelianization of some subgroups of complex reflection groups. Specifically, we study the stabilizer of a hyperplane H which is the same as the centralizer of a reflection of hyperplane H. Contrary to the case of Coxeter groups, this is not a reflection subgroup of the complex reflection group W in general. The first step is to refine Stanley-Springer's theorem [13][14] on the abelianization of a reflection group (see Proposition 1 in Section 2).

The rationale also relies on a good description (Section 3) of abelianization of various types of big subgroups of the braid group B of W (here "big" stands for "containing the pure braid group P"). Though we just need to construct a group homomorphism from the inverse image of the stabilizer of H by p with values in $\mathbb{Q}$ (see Definition 3 and Proposition 3), we give in fact a complete description of the abelianization of $p^{-1}(W')$ with W' a reflection subgroup of W or the stabilizer of a hyperplane (Proposition 2 and Proposition 4). We also suggest a lifting construction for every element of the centralizer of a reflection in W generalizing the construction of the generator of monodromy of [4, p.14] (see Remark 10).

Orbits of hyperplanes and ramification index are gathered in tables in the last section.

We finish the introduction with some notations. We fix a W-invariant hermitian product on V denoted by $\langle\cdot,\cdot\rangle$: orthogonality will always be relative to this particular hermitian product. We denote by $S(V^*)$ the symmetric algebra of the dual of V which is also the polynomial functions on V.

Notation 1 (Around a Hyperplane of W). For $H \in \mathscr{H}$,

– one chooses $\alpha_H \in V^*$ a linear form with kernel H;

- one sets $W_H = \mathrm{Fix}_W(H) = \{g \in W, \quad \forall x \in H, \quad gx = x\}$. This is a cyclic subgroup of W. We denote by e_H its order and by s_H its generator with determinant $\zeta_H = \exp(2i\pi/e_H)$. Except for identity, the elements of W_H are precisely the reflections of W whose hyperplane is H. The reflection s_H is called the *distinguished reflection for H* in W.

For n a positive integer, we denote by $\mathbb{U}_n$ the group of the n^{th} root of unity in $\mathbb{C}$ and by $\mathbb{U}$ the group of unit complex numbers. For a group G, we denote by $[G,G]$ the commutator subgroup of G and by $G^{\mathrm{ab}} = G/[G,G]$ the abelianization of G.

2. Abelianization of Subgroups of Reflection Groups

Stanley-Springer's Theorem (see [13, Theorem 4.3.4][14, Theorem 3.1]) gives an explicit description of the group of linear characters of a reflection group using the conjugacy classes of hyperplanes. Naturally, it applies to all reflection subgroups of a reflection group and in particular to parabolic subgroups thanks to Steinberg's theorem [12, Theorem 1.5]. But since $P/[P,P]$ is the $\mathbb{Z}W$ permutation module defined by the hyperplanes of W, we are interested in the stabilizer of a hyperplane which is not in general a reflection subgroup of W. So we have to go deeper in the study of the stabilizer of a hyperplane.

Before starting our study of the stabilizer of a hyperplane, we write down Stanley-Springer's Theorem because we will use it many times.

Theorem 1 (Stanley-Springer's Theorem). *For every map $n : \mathscr{H} \to \mathbb{N}$ constant on the W-orbits of $\mathscr{H}$, there exists a linear character $\chi : W \to \mathbb{C}^\times$ such that $\chi(s_H) = \det(s_H)^{-n_H}$.*

Moreover, the χ-isotypic component of $S(V^*)$ is a free $S(V^*)^W$-module of rank 1 generated by

$$Q_\chi = \prod_{H\in\mathscr{H}} {\alpha_H}^{n_H}$$

where the n_H are related to χ by the formula above and satisfy the relations $0 \le n_H \le e_H - 1$.

For $H \in \mathscr{H}$, we set

$$N_H = \{w \in W, \quad wH = H\} = \{w \in W, \quad ws_H = s_H w\}$$

the stabilizer of H which is also the centralizer of s_H. We denote by $D = H^\perp$ the line of s_H (or of every reflection of W with hyperplane H) it is the unique N_H-stable line of V such that $H \oplus D = V$ and N_H is also the stabilizer of D (see [3, Proposition 1.19]). We denote the parabolic subgroup associated to D by $C_H = \{w \in W, \quad \forall x \in D, \ wx = x\}$.

Since D is a line stable by every element of N_H and $W_H \subset N_H$, there exists an integer f_H such that $e_H \mid f_H$ and the following sequence is exact

$$1 \longrightarrow C_H \xrightarrow{\ i\ } N_H \xrightarrow{\ r\ } \mathbb{U}_{f_H} \longrightarrow 1 \qquad (2)$$

where i is the natural inclusion and r denote the restriction to D. We define r to be the *natural linear character of* N_H.

Before stating our main result on the abelianization of the stabilizer of a hyperplane, let us start with a straightforward lemma.

Lemma 1 (Abelianization of an exact sequence). *Let us consider the following exact sequence of groups where M is an abelian group.*

$$1 \longrightarrow C \xrightarrow{i} N \xrightarrow{r} M \longrightarrow 1$$

Then the following sequence is exact

$$C^{ab} \xrightarrow{i^{ab}} N^{ab} \xrightarrow{r^{ab}} M \longrightarrow 1$$

Moreover the map i^{ab} is injective if and only if $[N,N]=[C,C]$. When C^{ab} and M are finite, the injectivity of i^{ab} is equivalent to $|C^{ab}||M|=|N^{ab}|$.

We also have the following criterion : the map i^{ab} is injective if and only if the canonical restriction map $\mathrm{Hom}_{gr.}(N,\mathbb{C}^\times) \to \mathrm{Hom}_{gr.}(C,\mathbb{C}^\times)$ is surjective.

Proof. The first injectivity criterion is an easy diagram chasing computation. The second one is trivial. Let us focus on the third one. Since $\mathbb{C}^\times$ is a commutative group, we have the following commutative square whose vertical arrows are isomorphisms

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{gr.}}(N,\mathbb{C}^\times) & \longrightarrow & \mathrm{Hom}_{\mathrm{gr.}}(C,\mathbb{C}^\times) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{gr.}}(N^{\mathrm{ab}},\mathbb{C}^\times) & \xrightarrow{\circ i^{\mathrm{ab}}} & \mathrm{Hom}_{\mathrm{gr.}}(C^{\mathrm{ab}},\mathbb{C}^\times) \end{array}$$

Moreover, since $\mathbb{C}^\times$ is a divisible abelian group, i^{ab} is injective if and only if $\circ i^{\mathrm{ab}}$ is surjective.

Before applying the preceding lemma to the stabilizer of a hyperplane, let us introduce a definition and a classical linear algebra lemma.

Definition 1 (Commuting hyperplanes). *Let $H, H' \in \mathscr{H}$. We say that H and H' commute if s_H and $s_{H'}$ commute.*

We denote by $\mathscr{H}_H$ the set of hyperplanes which commute with H and $\mathscr{H}'_H = \mathscr{H}_H \setminus \{H\}$.

The next lemma on commuting hyperplanes is stated in [3, Lemma 1.7].

Lemma 2. *We have the following equivalent characterizations :*

(i) *the hyperplanes H and H' commute*
(ii) $H = H'$ *or* $D = H^\perp \subset H'$
(iii) *every reflection of W with hyperplane H commutes with every reflection of W with hyperplane H'*

(iv) there exists a reflection of W with hyperplane H which commutes with a reflection of W with hyperplane H'

As a consequence of Lemma 1, we are now able to formulate the following proposition.

Proposition 1 (Abelianization of the stabilizer of a hyperplane). *For a hyperplane $H \in \mathscr{H}$, the following sequence is exact*

$$C_H^{ab} \xrightarrow{i^{ab}} N_H^{ab} \xrightarrow{r^{ab}} \mathbb{U}_{f_H} \longrightarrow 1$$

Moreover, we have the following geometric characterization of the injectivity of i^{ab} : the map i^{ab} is injective if and only if the orbits of the hyperplanes commuting with H under N_H and C_H are the same.

Proof. Steinberg's theorem and Lemma 2 (*ii*) ensure us that C_H is the reflection subgroup of W generated by the $s_{H'}$ for $H' \neq H$ commuting with H. Thanks to Theorem 1, we are able to describe the linear characters of C_H. For every linear character δ of C_H, there exists integers $e_{\mathcal{O}}$ for $\mathcal{O} \in \mathscr{H}'_H/C_H$ such that the polynomial

$$Q_\delta = \prod_{\mathcal{O} \in \mathscr{H}'_H/C_H} \prod_{H' \in \mathcal{O}} {\alpha_{H'}}^{e_{\mathcal{O}}} \in S(V^*)$$

verifies $gQ_\delta = \delta(g)Q_\delta$ for every $g \in C_H$.

Let us assume that the orbits of the hyperplanes commuting with H under N_H and C_H are the same. For every $g \in N_H$ and every $\mathcal{O} \in \mathscr{H}'_H/C_H$, there exists $\lambda_g \in \mathbb{C}^\times$ such that

$$g \prod_{H' \in \mathcal{O}} \alpha_{H'} = \lambda_g \prod_{H' \in \mathcal{O}} \alpha_{H'}$$

So we obtain that, for every $g \in N_H$, there exists $\mu_g \in \mathbb{C}^\times$ such that $gQ_\delta = \mu_g Q_\delta$. Thus every linear character of C_H extends to N_H and Lemma 1 tells us that the map i^{ab} is injective.

Let us assume now that every linear character of C_H extends to N_H. The orbit of H under N_H and C_H is $\{H\}$. So let us consider an orbit $\mathcal{O} \in \mathscr{H}'_H/C_H$. We define

$$Q = \prod_{H' \in \mathcal{O}} \alpha_{H'} \in S(V^*)\,.$$

Thanks to Theorem 1, Q define a linear character χ of C_H: for every $c \in C_H$, there exists $\chi(c) \in \mathbb{C}^\times$ such that $cQ = \chi(c)Q$ for every $c \in C_H$. We then consider the N_H-submodule M of $S(V^*)$ generated by Q. As a vector space, M is generated by the family $(nQ)_{n \in N_H}$. But, since C_H is normal in N_H, we obtain for $c \in C_H$,

$$cnQ = nn^{-1}cnQ = n\chi(n^{-1}cn)Q\,.$$

Since χ extends to N_H, we have $\chi(n^{-1}cn) = \chi(c)$ and then $cnQ = \chi(c)nQ$. Theorem 1 allows us to conclude that $nQ = \lambda_n Q$ for some $\lambda_n \in \mathbb{C}^\times$. Since $S(V^*)$ is a UFD, we obtain that $\mathcal{O}$ is still an orbit under N_H.

Remark 1 (Commuting Orbits). The orbits of the hyperplanes commuting with H under N_H and C_H are the same for every hyperplane H of every complex reflection group except the hyperplanes of the exceptional group G_{25} and the hyperplanes H_i $(1 \leq i \leq r)$ of the group $G(de, e, r)$ when $r = 3$ and e is even (see section 5 for the notations).

In Section 5, we give tables for the various orbits of hyperplanes for the infinite series $G(de, e, r)$. For the exceptional complex reflection groups, we check the injectivity or non-injectivity of i^{ab} using the package CHEVIE of GAP [6][8].

For a hyperplane $H \in \mathscr{H}$, the comparison of e_H and f_H leads to the following definition.

Definition 2 (Ramification at a hyperplane). *We define $d_H = f_H/e_H$ to be the index of ramification of W at the hyperplane H. We say that W is unramified at H if $d_H = 1$.*

We say that an element $w \in N_H$ such that $r(w) = \exp(2i\pi/f_H)$ realizes the ramification.

Remark 2 (The Coxeter Case). When H is an unramified hyperplane, we have $N_H = C_H \times W_H$ which is generated by reflections thanks to Steinberg's theorem [12, Theorem 1.5] and s_H realizes the ramification. Moreover i^{ab} is trivially injective.

In a Coxeter group, every hyperplane is unramified. Indeed, the eigenvalue on the line D of an element of N_H is a finite order element of the field of the real numbers.

Remark 3 (The 2-dimensional Case). When W is a 2-dimensional reflection group, N_H is an abelian group and $i = i^{\mathrm{ab}}$ is injective.

In section 5, we give tables for the values of e_H, f_H and d_H for every hyperplane of every complex reflection groups. From these tables, we obtain the following remarks.

Remark 4 (Unramified $G(de, e, r)$). All the hyperplanes of $G(de, e, r)$ are unramified only when $r = 1$ or when $G(de, e, r)$ is a Coxeter group (that is to say if $d = 2$ and $e = 1$ and $r \geq 2$ (Coxeter group of type B_r) or if $d = 1$ and $e = 2$ and $r \geq 3$ (Coxeter group of type D_r) or if $d = 1$ and $e = 1$ and $r \geq 3$ (Coxeter group of type A_{r-1}) of if $d = 1$ and $r = 2$ (Coxeter group of type $I_2(e)$).

Remark 5 (Unramified exceptional groups). The only non Coxeter groups for which every hyperplane is unramified are G_8, G_{12} and G_{24}.

Remark 6 (Generating Set). Since C_H is the parabolic subgroup associated to D, it is generated by the reflections it contains (this is Steinberg's theorem). Moreover, if $w_H \in N_H$ realizes the ramification. Then, the exact sequence (2) tells us that N_H is generated by w_H and the family of $s_{H'}$ such that $H' \in \mathscr{H}'_H$.

3. Abelianization of Subgroups of Braid Groups

In this section, we describe abelianizations of subgroups of B containing P that is to say of inverse images of subgroups W' of W. Explicitly, we are able to give a complete description of $p^{-1}(W')^{\mathrm{ab}}$ if W' is a reflection subgroup (Proposition 2) or if W' is the stabilizer of a hyperplane under geometrical assumptions on the hyperplane (Proposition 4). We also construct a particular linear character of $p^{-1}(N_H)$ lifting the natural linear character r of N_H which is of importance for the next section (Definition 3).

Our method is similar to the method of [4] for the description of B^{ab} : we integrate along paths invariants polynomial functions. So, we have first to construct invariant polynomial functions and then verify that we have constructed enough of them.

3.1. Subgroup Generated by Reflections

In this subsection, we fix C a subgroup of W generated by reflections. We denote by $\mathscr{H}_C \subset \mathscr{H}$ the set of hyperplanes of C. For $H \in \mathscr{H}_C$, then $C_H = \{c \in C, \quad \forall x \in H,\ cx = x\}$ is a subgroup of W_H and so generated by ${s_H}^{a_H}$ with $a_H \mid e_H$. For $H \in \mathscr{H} \setminus \mathscr{H}_C$, we set $a_H = e_H$. We then obtain $C = \langle {s_H}^{a_H},\ H \in \mathscr{H} \rangle$. For $\mathcal{C} \in \mathscr{H}/C$ a C-class of hyperplanes of W, we denote by $a_{\mathcal{C}}$ the common value of a_H for $H \in \mathcal{C}$.

The aim of this subsection is to give a description of the abelianization of $p^{-1}(C) \subset B$. For this, we follow the method of [4] and we start to exhibit invariants which will be useful to show the freeness of our generating set of $p^{-1}(C)^{\mathrm{ab}}$.

Lemma 3 (An invariant). *We define, for $\mathcal{C} \in \mathscr{H}/C$,*

$$\alpha_{\mathcal{C}} = \prod_{H \in \mathcal{C}} {\alpha_H}^{e_H/a_H} \in S(V^*)\,.$$

Then $\alpha_{\mathcal{C}}$ is invariant under the action of C.

Proof. If $\mathcal{C}$ is a class of hyperplanes of $\mathscr{H}_C$ then this is an easy consequence of Theorem 1.

Assume that $\mathcal{C}$ is not a class of hyperplanes of $\mathscr{H}_C$. Let us choose a reflection s of C and let n_s be the order of s. Since $\mathcal{C}$ is not a class in $\mathscr{H}_C$, the hyperplane of s does not belong to $\mathcal{C}$. We then deduce that the orbits of $\mathcal{C}$ under the action of $\langle s \rangle$ are of two types.

First type : the orbits of $H \in \mathcal{C}$ such that s_H and s commute. Since $ss_Hs^{-1} = s_{s(H)}$, we then deduce that $s(H) = H$. And so the orbit of H under $\langle s \rangle$ is reduced to H. We denote by H_s the hyperplane of s. Since $H \neq H_s$, Lemma 2 tells us that $D = H^\perp \subset H_s$ and so s acts trivially on D which is identified to the line spanned by α_H through the inner product.

Second type : the orbits of $H \in \mathcal{C}$ such that $s_Hs \neq ss_H$. If $s^iH = H$ then s^i and s_H commute and thus, Lemma 2 ensures us that s^i is trivial. We then obtain that the orbit of H under $\langle s \rangle$ has cardinality n_s. So if we denote by Q the following product $\alpha_H\alpha_{sH}\cdots\alpha_{s^{n_s-1}H} = \lambda\alpha_H s\alpha_H \cdots s^{n_s-1}\alpha_H$ with $\lambda \in \mathbb{C}^\times$, we have $sQ = Q$.

We then easily obtain $s\alpha_\mathcal{C} = \alpha_\mathcal{C}$ for every $s \in C$ and so $\alpha_\mathcal{C}$ is invariant under the action of C.

Before stating the main result of the subsection, we recall the notion of "generator of the monodromy around a hyperplane" as defined in [4, p.14]. For $H \in \mathscr{H}$, we define a generator of the monodromy around H to be a path $s_{H,\gamma}$ in V^{reg} which is the composition of three paths. The first path is a path γ going from x_0 to a point x_H which is near H and far from other hyperplanes. To describe the second path, we write $x_H = h + d$ with $h \in H$ and $d \in D = H^\perp$, and the second path is $t \in [0,1] \mapsto h + \exp(2i\pi t/e_H)d$ going from x_H to $s_H(x_H)$. The third path is $s_H(\gamma^{-1})$ going from $s_H(x_H)$ to $s_H(x_0)$. We can now state our abelianization result.

Proposition 2 (Abelianization of subgroups of the braid group). *Let C be a subgroup of W generated by reflections. Then $p^{-1}(C)^{ab}$ is the free abelian group over $\mathscr{H}/C$ the C-classes of hyperplanes of W.*

Explicitly, we have $p^{-1}(C) = \langle {s_{H,\gamma}}^{a_H},\ (H,\gamma)\rangle$ (see [4, Theorem 2.18]). For $\mathcal{C} \in \mathscr{H}/C$, we denote by $(s_\mathcal{C}^{a_\mathcal{C}})^{ab}$ the common value in $p^{-1}(C)^{ab}$ of the ${s_{H,\gamma}}^{a_H}$ for $H \in \mathcal{C}$. Then $p^{-1}(C)^{ab} = \langle (s_\mathcal{C}^{a_\mathcal{C}})^{ab},\quad \mathcal{C} \in \mathscr{H}/C\rangle$. Moreover, for $\mathcal{C} \in \mathscr{H}/C$, there exists a group homomorphism $\varphi_\mathcal{C} : p^{-1}(C) \to \mathbb{Z}$ such that $\varphi_\mathcal{C}((s_\mathcal{C}^{a_\mathcal{C}})^{ab}) = 1$ and $\varphi_\mathcal{C}((s_{\mathcal{C}'}^{a_{\mathcal{C}'}})^{ab}) = 0$ for $\mathcal{C}' \neq \mathcal{C}$.

Proof. First of all, Lemma 2.14.(2) of [4] shows that ${s_{H,\gamma}}^{a_H} = {s_{H,\gamma'}}^{a_H}$ in $p^{-1}(C)^{\text{ab}}$. Now, for $c \in C$, we choose $x \in p^{-1}(C)$ such that $p(x) = c$. We have $xs_{H,\gamma}x^{-1} = s_{cH,x(c\gamma)}$. So ${s_{cH,x(c\gamma)}}^{a_{cH}}$ and ${s_{H,\gamma}}^{a_H}$ are conjugate by an element of $p^{-1}(C)$. So, we have

$$ {s_{cH,x(c\gamma)}}^{a_{cH}} = {s_{H,\gamma}}^{a_H} \in p^{-1}(C)^{\text{ab}}\,. $$

And then $p^{-1}(C)^{\text{ab}} = \langle (s_\mathcal{C}^{a_\mathcal{C}})^{\text{ab}},\quad \mathcal{C} \in \mathscr{H}/C\rangle$.

Let us now show that the family $((s_\mathcal{C}^{a_\mathcal{C}})^{\text{ab}})_{\mathcal{C}\in\mathscr{H}/C}$ is free over $\mathbb{Z}$. We identify $p^{-1}(C)$ with

$$ p^{-1}(C) = \left(\bigsqcup_{c,c'\in C} \pi_1(V^{\text{reg}}, cx_0, c'x_0)\right)/C $$

where $\pi_1(V^{\text{reg}}, cx_0, c'x_0)$ denotes the homotopy classes of paths from $c(x_0)$ to $c'(x_0)$ and the action of C on paths is simply the composition.

Since $\alpha_{\mathcal{C}} : V^{\mathrm{reg}} \to \mathbb{C}^\times$ is C-invariant (Lemma 3), the functoriality of π_1 defines a group homomorphism $\pi_1(\alpha_{\mathcal{C}})$ from $p^{-1}(C)$ to $\pi_1(\mathbb{C}^\times, \alpha_{\mathcal{C}}(x_0))$. Moreover, the map

$$I : \gamma \longmapsto \frac{1}{2i\pi}\int_\gamma \frac{\mathrm{d}z}{z}$$

realizes a group isomorphism between $\pi_1(\mathbb{C}^\times, \alpha_{\mathcal{C}}(x_0))$ and $\mathbb{Z}$. The composition of these two maps defines a group homomorphism. We denote it by $\varphi_{\mathcal{C}}$ and we now want to show that $\varphi_{\mathcal{C}}$ verifies the condition stated in the Proposition.

For $H \in \mathcal{C}$ and $\mathcal{C}' \in \mathscr{H}/C$, let us compute $\varphi_{\mathcal{C}'}({s_{H,\gamma}}^{a_H})$. The path ${s_{H,\gamma}}^{a_H}$ is the composition of three paths. The first one is γ, the third one is ${s_H}^{a_H}(\gamma^{-1})$ and the second one is $\eta : t \in [0,1] \mapsto h + \exp(2i\pi a_H t/e_H)d$.

Since $\alpha_{\mathcal{C}'}$ is C-invariant, when we apply $\pi_1(\alpha_{\mathcal{C}'})$, the first part of the path and the third one are inverse from each other. So when applying I, they do not appear. We thus obtain

$$\varphi_{\mathcal{C}'}({s_{H,\gamma}}^{a_H}) = \frac{1}{2i\pi}\int_{\alpha_{\mathcal{C}'}\circ\eta} \frac{\mathrm{d}z}{z}\,.$$

Using the logarithmic derivative, we obtain

$$\varphi_{\mathcal{C}'}({s_{H,\gamma}}^{a_H}) = \frac{1}{2i\pi}\sum_{H'\in\mathcal{C}'}\frac{e_{H'}}{a_{H'}}\int_0^1 2i\pi\frac{a_H}{e_H}\frac{\exp(2i\pi a_H t/e_H)\alpha_{H'}(d)}{\alpha_{H'}(h+\exp(2i\pi a_H t/e_H)d)}\,\mathrm{d}t$$

To compute this sum, we regroup the terms according to the orbit of H' under $\langle {s_H}^{a_H}\rangle$.

Lemma 2 shows that there are three types of orbits : two types of orbits reduced to one single hyperplane and one other type of orbits corresponding to reflections that do not commute with s_H.

Let us first study the orbits reduced to one single hyperplane. The first type corresponds to the hyperplane H whose term of the sum is 1 and this term appears if and only if $H \in \mathcal{C}'$. The second type corresponds to hyperplanes H' such that $D = H^\perp \subset H'$. The corresponding term of the sum is 0 since $\alpha_{H'}(d) = 0$

Let us now study the non trivial orbits. The orbits of H' under ${s_H}^{a_H}$ is $\{H', \ldots, {s_H}^{a_H(e_H/a_H-1)}(H')\}$. Moreover, since a quotient of the form $\alpha_{H'}(x)/\alpha_{H'}(y)$ does not depend of the linear form with kernel H', we can replace $\alpha_{{s_H}^k H'}$ by $s_H^k\alpha_{H'}$ to obtain

$$\begin{aligned}
&\frac{\exp(2i\pi a_H t/e_H)\alpha_{{s_H}^{-ka_H}H'}(d)}{\alpha_{{s_H}^{-ka_H}H'}(h)+\exp(2i\pi a_H t/e_H)\alpha_{{s_H}^{-ka_H}H'}(d)}\\
&\qquad= \frac{\exp(2i\pi a_H t/e_H){s_H}^{-a_H k}\alpha_{H'}(d)}{{s_H}^{-a_H k}\alpha_{H'}(h)+\exp(2i\pi a_H t/e_H){s_H}^{-a_H k}\alpha_{H'}(d)}\\
&\qquad= \frac{\exp(2i\pi a_H(t+k)/e_H)\alpha_{H'}(d)}{\alpha_{H'}(h)+\exp(2i\pi a_H(t+k)/e_H)\alpha_{H'}(d)}\,.
\end{aligned}$$

Considering the sum over the orbit under $\langle {s_H}^{a_H} \rangle$ of H', we obtain

$$\sum_{k=0}^{e_H/a_H-1} \int_0^1 \frac{\exp(2i\pi a_H(t+k)/e_H)\alpha_{H'}(d)}{\alpha_{H'}(h)+\exp(2i\pi a_H(t+k)/e_H)\alpha_{H'}(d)}\,\mathrm{d}t = \frac{e_H}{a_H}\int_0^1 \frac{\exp(2i\pi t)\alpha_{H'}(d)}{\alpha_{H'}(h)+\exp(2i\pi t)\alpha_{H'}(d)}\,\mathrm{d}t\,.$$

Since x_H is chosen such that $\alpha_{H'}(h) \neq 0$ for $H' \neq H$ and d is small, the last term is 0 as the index of the circle of center 0 and radius $|\alpha_{H'}(d)|$ relatively to the point $-\alpha_{H'}(h)$.

Remark 7 (Extreme cases). The two extreme cases where $C = 1$ and $C = W$ may be found in [4, Prop. 2.2.(2)] and [4, Theorem 2.17.(2)]. In the first case, $p^{-1}(C) = P$ is the pure braid group whose abelianization is the free abelian group over $\mathscr{H}$. In the second case $p^{-1}(C) = B$ is the braid group whose abelianization is the free abelian group over $\mathscr{H}/W$.

Remark 8. The logarithmic derivative shows that for every $\gamma \in p^{-1}(C)$ and $n \in \mathbb{Z}$, we have

$$\int_{{\alpha_{\mathcal{C}}}^n \circ \gamma} \frac{\mathrm{d}z}{z} = n\varphi_{\mathcal{C}}(\gamma)\,.$$

3.2. Stabilizer of a hyperplane

Let us recall the notation of Section 2; we consider $H \in \mathscr{H}$ a hyperplane of the reflection group W. We denote by N_H the stabilizer of H in W and C_H the parabolic subgroup of W associated to the line $D = H^{\perp}$. The set of hyperplanes commuting with H is $\mathscr{H}_H$ (see Definition 1).

3.2.1. A group homomorphism The aim of this paragraph is to construct an "extension" of the natural character of N_H to the group $p^{-1}(N_H)$ which will be useful for the third section. We still follow the method of [4] : we construct an invariant function with values in $\mathbb{C}^\times$ (Lemma 4) and integrate it (Definition 3). To obtain the "extension" properties of the linear character of $p^{-1}(N_H)$ (Proposition 3), we construct a lifting in the braid group of the elements of N_H (Remark 10). This lifting is inspired from the construction of the generator of the monodromy.

Lemma 4 (An invariant function). *The function $\alpha_{N_H} = {\alpha_H}^{f_H} \in S(V^*)$ is invariant under N_H.*

Proof. This is clear since the line spanned by α_H is identified to D through the inner product.

Definition 3 (The group homomorphism). *As in the proof of Proposition 2, we write*

$$p^{-1}(N_H) = \left(\bigsqcup_{n,n' \in N_H} \pi_1(V^{reg}, nx_0, n'x_0) \right) / N_H \, .$$

Since $\alpha_{N_H} : V^{reg} \to \mathbb{C}^\times$ *is* N_H*-invariant (Lemma 4), the functoriality of* π_1 *allows us to define a group homomorphism* $\pi_1(\alpha_{N_H})$ *from* $p^{-1}(N_H)$ *to* $\pi_1(\mathbb{C}^\times, \alpha_{N_H}(x_0))$*. Moreover, the map*

$$I : \gamma \longmapsto \frac{1}{2i\pi} \int_\gamma \frac{dz}{z}$$

realizes a group isomorphism between $\pi_1(\mathbb{C}^\times, \alpha_{N_H}(x_0))$ *and* $\mathbb{Z}$*. The composition of this two maps defines a group homomorphism* $\rho' : p^{-1}(N_H) \to \mathbb{Z}$*. We also define* $\rho = {f_H}^{-1}\rho' : p^{-1}(N_H) \to \mathbb{Q}$*.*

Remark 9 (Center of the braid group of G_{31}*).* In [4, Theorem 2.24], it is shown that the center of the braid group B of an irreducible reflection group W is an infinite cyclic group generated by $\beta : t \mapsto \exp(2i\pi t/|Z(W)|)x_0$ (where $x_0 \in V^{\mathrm{reg}}$ is a base point) for all but six exceptional reflection groups. In his articles [1][2], Bessis proves that the result holds for all reflection groups but the exceptional one G_{31}.

This remark is a first step toward the case of G_{31} : we show that if ZB is an infinite cyclic group, it is generated by β. For this, let us consider $H \in \mathscr{H}$ a hyperplane of G_{31} and ρ' the group homomorphism defined above. Since $ZB \subset p^{-1}(N_H)$, ρ' restricts to a group homomorphism from ZB to $\mathbb{Z}$ such that $\rho'(\beta) = 1$. So if ZB is an infinite cyclic group, it is generated by β.

Remark 10 (The lifting construction). Let us consider $w \in N_H$. We now construct a path $\widetilde{w}$ in V^{reg} starting from x_0 and ending at $w(x_0) : p(\widetilde{w}) = w$. We use the notations of the description of the generators of the monodromy around H : we write $x_H = h + d$ with $h \in H$ and $\alpha_{H'}(h) \neq 0$ for $H' \neq H$ and $d \in D = H^\perp$. Since $w \in N_H$, we have $w(x_H) = h' + \exp(2ik\pi/f_H)d$ with $h' \in H$ and $0 \leq k < f_H$.

The path $\widetilde{w}$ consists into four parts. As in the case of the generators of the monodromy, the first part is a path γ from x_0 to x_H and the fourth path is $w(\gamma^{-1})$ from $w(x_H)$ to $w(x_0)$. Let us now describe the second part and the third part. The second part of $\widetilde{w}$ is the path

$$t \in [0,1] \mapsto h + \exp(2ik\pi t/f_H)d \in V^{\mathrm{reg}} \, .$$

The third part is of the form $t \in [0,1] \mapsto \theta(t) + \exp(2ik\pi/f_H)d$ where $\theta(t)$ is a path in the complex affine line $\mathscr{D}$ generated by h' and h. It is easy to force the third part of $\widetilde{w}$ to stay in V^{reg} since its image is contained in the affine line $\exp(2ik\pi/f_H)d + \mathscr{D}$ which is parallel to the hyperplane H and meets each of the other hyperplanes in a single point : so we just have to avoid a finite number of points in $\mathbb{C}$.

Remark 11 (Generating set). We have seen in Remark 6 that

$$N_H = \langle w_H, s_{H'},\ H' \in \mathscr{H}_H \rangle$$

where $w_H \in N_H$ is a once and for all fixed element realizing the ramification. It is now an easy consequence of Theorem 2.18 of [4] that

$$p^{-1}(N_H) = \langle \widetilde{w_H}, s_{H',\gamma}, s_{H'',\gamma'}{}^{e_{H''}}, \quad H' \in \mathscr{H}_H,\ H'' \in \mathscr{H} \setminus \mathscr{H}_H,\ \gamma,\ \gamma' \rangle\,.$$

It remains to show that the constructed group homomorphism ρ is an "extension" of the natural character of N_H. More precisely, we have the following proposition.

Proposition 3 (The "extension" property). *We have the following commutative square*

$$\begin{array}{ccc} p^{-1}(N_H) & \xrightarrow{\ \rho\ } & \mathbb{Q} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \pi'} \\ N_H & \xrightarrow{\ r\ } & \mathbb{U}_{f_H} \end{array}$$

where $\pi' : x \in \mathbb{Q} \mapsto \exp(2i\pi x)$.

Proof. Using the generating set of N_H given in Remark 11, we only need to show that

(i) $\rho'(\widetilde{w_H}) = 1$
(ii) $\rho'(s_{H,\gamma}) = f_H/e_H$
(iii) $\rho'(s_{H',\gamma}) = 0$ for $H' \in \mathscr{H}'_H = \mathscr{H}_H \setminus \{H\}$
(iv) $\rho'(s_{H',\gamma}{}^{e_{H'}}) = 0$ for $H' \in \mathscr{H} \setminus \mathscr{H}_H$.

As in the proof of Proposition 2, the γ-part of $s_{H,\gamma}$ (resp. $s_{H',\gamma}$ for $H' \in \mathscr{H}'_H$ and $s_{H',\gamma}{}^{e_{H'}}$ for $H' \in \mathscr{H} \setminus \mathscr{H}_H$) does not appear in the computation of ρ'. We thus obtain

$$\rho'(s_{H,\gamma}) = \frac{1}{2i\pi}\int_0^1 f_H \frac{2i\pi}{e_H}\frac{\alpha_H(\exp(2i\pi t/e_H)d)}{\alpha_H(h + d\exp(2i\pi t/e_H))}\mathrm{d}t = f_H/e_H\,.$$

For $H' \in \mathscr{H}'_H$, we set $x_{H'} = h' + d'$ with $h' \in H'$ and $d' \in D' = H'^{\perp}$. We then obtain

$$\rho'(s_{H',\gamma}) = \frac{1}{2i\pi}\int_0^1 f_H \frac{2i\pi}{e_{H'}}\frac{\alpha_H(\exp(2i\pi t/e_{H'})d')}{\alpha_H(h' + d'\exp(2i\pi t/e_{H'}))}\mathrm{d}t = 0$$

since $\alpha_H(d') = 0$ for $H \in \mathscr{H}'_H$. With the same arguments, we obtain for $H' \in \mathscr{H} \setminus \mathscr{H}_H$

$$\rho'(s_{H',\gamma}{}^{e_H}) = \frac{1}{2i\pi}\int_0^1 2i\pi f_H \frac{\exp(2i\pi t)\alpha_H(d')}{\alpha_H(h') + \exp(2i\pi t)\alpha_H(d')}\mathrm{d}t = 0$$

since d' is small and $\alpha_H(h') \neq 0$.

For $\widetilde{w_H}$, neither the first and fourth part are involved in the computation nor the third one. Moreover, as in the computation of $\rho'(s_{H,\gamma})$ the second part of $\widetilde{w_H}$ gives 1.

3.2.2. The Stabilizer Case In this paragraph, we extend the results of Section 2 to the braid group. Namely, since $p : B/[P,P] \to W$ is a surjective homomorphism, the classical isomorphism theorems give the following short exact sequence

$$1 \longrightarrow p^{-1}(C_H) \xrightarrow{j} p^{-1}(N_H) \xrightarrow{rp} \mathbb{U}_{f_H} \longrightarrow 1$$

which gives rise to the following exact sequence (Lemma 1)

$$p^{-1}(C_H)^{\mathrm{ab}} \xrightarrow{j^{\mathrm{ab}}} p^{-1}(N_H)^{\mathrm{ab}} \xrightarrow{(rp)^{\mathrm{ab}}} \mathbb{U}_{f_H} \longrightarrow 1$$

and Proposition 1 extends to the braid group in the following way.

Proposition 4 (Abelianization in the braid group). *If the orbits of the hyperplanes of $\mathscr{H}$ under N_H and C_H are the same, the map j^{ab} is injective.*

Moreover under this hypothesis, $p^{-1}(N_H)^{ab}$ is the free abelian group with basis $\widetilde{w_H}$, $(s_{\mathcal{C}})^{ab}$ for $\mathcal{C} \in \mathscr{H}'_H/C_H$ and $(s_{\mathcal{C}}^{e_{\mathcal{C}}})^{ab}$ for $\mathcal{C} \in (\mathscr{H} \setminus \mathscr{H}_H)/C_H$.

Proof. From Lemma 1, it is enough to show that every linear character of $p^{-1}(C_H)$ with values in $\mathbb{C}^\times$ extends to $p^{-1}(N_H)$. But the group of linear characters of $p^{-1}(C_H)$ is generated by the $\exp(z\varphi_{\mathcal{C}})$ for $z \in \mathbb{C}$ and $\mathcal{C}$ an orbit of $\mathscr{H}$ under C_H. So it suffices to show that $\varphi_{\mathcal{C}}$ extends to $p^{-1}(N_H)$.

Since the orbits of $\mathscr{H}$ under C_H and N_H are the same, then for every $\mathcal{C} \in \mathscr{H}/C_H$, there exists $n \in \mathbb{N}^*$ such that $\alpha_{\mathcal{C}}{}^n$ is invariant under N_H (see Lemma 3 for the definition of $\alpha_{\mathcal{C}}$). Then Remark 8 shows that

$$\psi_{\mathcal{C}} : \gamma \in p^{-1}(N_H) \longmapsto \frac{1}{n} \int_{\alpha_{\mathcal{C}}{}^n \circ \gamma} \frac{\mathrm{d}z}{z} \in \mathbb{Q}$$

is a well defined linear character of $p^{-1}(N_H)$ extending $\varphi_{\mathcal{C}}$.

Proposition 2 applied to C_H ensures us that $p^{-1}(C_H)^{\mathrm{ab}}$ is the free abelian group generated by $(s_{\{H\}}^{e_{\{H\}}})^{\mathrm{ab}}$, $(s_{\mathcal{C}})^{\mathrm{ab}}$ for $\mathcal{C} \in \mathscr{H}'_H/C_H$ and $(s_{\mathcal{C}}^{e_{\mathcal{C}}})^{\mathrm{ab}}$ for $\mathcal{C} \in (\mathscr{H} \setminus \mathscr{H}_H)/C_H$. Moreover, we have $\widetilde{w_H}^{f_H} \in p^{-1}(C_H)$ and, thanks to Remark 8,

$$\varphi_{\{H\}}(\widetilde{w_H}^{f_H}) = \frac{1}{f_H}\rho'(\widetilde{w_H}^{f_H}) = 1$$

We then deduce that the family $\widetilde{w_H}^{f_H}$, $(s_{\mathcal{C}})^{\mathrm{ab}}$ for $\mathcal{C} \in \mathscr{H}'_H/C_H$ and $(s_{\mathcal{C}}^{e_{\mathcal{C}}})^{\mathrm{ab}}$ for $\mathcal{C} \in (\mathscr{H} \setminus \mathscr{H}_H)/C_H$ is a basis for $p^{-1}(C_H)^{\mathrm{ab}}$. The short exact sequence

$$1 \longrightarrow p^{-1}(C_H)^{\mathrm{ab}} \xrightarrow{j^{\mathrm{ab}}} p^{-1}(N_H)^{\mathrm{ab}} \xrightarrow{(rp)^{\mathrm{ab}}} \mathbb{U}_{f_H} \longrightarrow 1$$

gives the result.

Remark 12 (Comparison of orbits). In this remark, we give a list of the hyperplanes for which the orbits of hyperplanes under N_H and C_H are not the same. Of course, we find again in this list the hyperplanes of Remark 1 but we have to add some others.

Let us consider the infinite series (see Section 5 for notations). When $H = H_i$, the orbits under N_H and C_H are always the same except when $r = 3$ and e is even and when $r = 2$ and $e \geq 3$. If $H = H_{i,j,\zeta}$, the orbits under N_H and C_H are the same when de is even and $r \neq 3$ or when $r = 3$ and $e \in \{1, 3\}$ or when $r = 2$ and $d = e = 1$.

For the exceptional types, G_{25} is the only case where the commuting orbits under N_H and C_H are not the same. The only exceptional types where the non commuting orbits under N_H and C_H are not the same are G_4, the second (named after GAP) class of hyperplanes of G_6, the first (named after GAP) class of hyperplanes of G_{13} and the third (named after GAP) class of hyperplanes of G_{15}.

4. An Application to Cohomology

In this section, we apply the preceding constructions and results to obtain a group cohomology result. Specifically, the derived subgroup of P is normal in B, so we obtain the following short exact sequence

$$1 \longrightarrow P/[P,P] \xrightarrow{\ j\ } B/[P,P] \xrightarrow{\ p\ } W \longrightarrow 1 \tag{3}$$

which induces a structure of W-module on P^{ab}. By a classical result on hyperplanes arrangements (see [10] for example), the W-module P^{ab} is nothing else that the permutation module $\mathbb{Z}\mathscr{H}$ and this section describes the extension (3) as an element of $H^2(W, \mathbb{Z}\mathscr{H})$ using methods of low-dimensional cohomology.

The rationale breaks down into three steps and each step consists of a translation of a standard isomorphism between cohomology groups in terms of group extensions.

(*i*) We decompose $\mathscr{H}$ into orbits under $W : \mathscr{H} = \sqcup \mathcal{C}$ and uses the isomorphism

$$H^2(W, \mathbb{Z}\mathscr{H}) = \bigoplus_{\mathcal{C} \in \mathscr{H}/W} H^2(W, \mathbb{Z}\mathcal{C})$$

(*ii*) In each orbit, we set a hyperplane $H_{\mathcal{C}}$ and then $\mathbb{Z}\mathcal{C} = \mathrm{Ind}_{N_{\mathcal{C}}}^{W}(\mathbb{Z})$ where $N_{\mathcal{C}}$ is the stabilizer of $H_{\mathcal{C}}$. Shapiro's lemma (see [5, Proposition III.6.2]) then gives us

$$H^2(W, \mathbb{Z}\mathscr{H}) = \bigoplus_{\mathcal{C} \in \mathscr{H}/W} H^2(N_{\mathcal{C}}, \mathbb{Z})$$

(*iii*) The short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ of $N_{\mathcal{C}}$-modules gives a long exact sequence in cohomology. Since $|N_{\mathcal{C}}|$ is invertible in $\mathbb{Q}$, we have $H^1(N_{\mathcal{C}}, \mathbb{Q}) = H^2(N_{\mathcal{C}}, \mathbb{Q}) = 0$ and so we obtain the isomorphism $H^2(N_{\mathcal{C}}, \mathbb{Z}) = H^1(N_{\mathcal{C}}, \mathbb{Q}/\mathbb{Z})$ and

$$H^2(W, \mathbb{Z}\mathscr{H}) = \bigoplus_{\mathcal{C}\in\mathscr{H}/W} H^1(N_{\mathcal{C}}, \mathbb{Q}/\mathbb{Z}) = \bigoplus_{\mathcal{C}\in\mathscr{H}/W} \mathrm{Hom}_{\mathrm{gr.}}(N_{\mathcal{C}}, \mathbb{Q}/\mathbb{Z}).$$

The results of this section are the following proposition and corollary.

Proposition 5 (Description). *Under the isomorphism*

$$H^2(W, \mathbb{Z}\mathscr{H}) = \bigoplus_{\mathcal{C}\in\mathscr{H}/W} \mathrm{Hom}_{gr.}(N_{\mathcal{C}}, \mathbb{Q}/\mathbb{Z})$$

the extension (3) corresponds to the family $(r_{\mathcal{C}} : N_{\mathcal{C}} \to \mathbb{Q}/\mathbb{Z})_{\mathcal{C}\in\mathscr{H}/W}$ *where* $r_{\mathcal{C}}$ *is the natural linear character of* $N_{\mathcal{C}}$ *(we identify* $\mathbb{U}_{f_{H_{\mathcal{C}}}}$ *with a subgroup of* $\mathbb{Q}/\mathbb{Z}$ *via the exponential map).*

The next corollary is a trivial consequence of Proposition 5 and generalizes a result of Digne [7, 5.1] for the case of Coxeter groups.

Corollary 1 (Order in $H^2(W, \mathbb{Z}\mathscr{H})$). *Since the order of* $r_{\mathcal{C}}$ *is* $f_{H_{\mathcal{C}}}$*, we deduce that the order of the extension (3) is* $\kappa(W) = lcm(f_{H_{\mathcal{C}}}, \mathcal{C} \in \mathscr{H}/W)$ *(this integer* $\kappa(W)$ *was first introduced in [9]).*

The rest of the section is devoted to the proof of Proposition 5 : one subsection for each of the three steps.

4.1. First step : splitting into orbits

The isomorphism

$$H^2(W, \mathbb{Z}\mathscr{H}) = \bigoplus_{\mathcal{C}\in\mathscr{H}/W} H^2(W, \mathbb{Z}\mathcal{C})$$

is simply given by applying the various projections $p_{\mathcal{C}} : \mathbb{Z}\mathscr{H} \to \mathbb{Z}\mathcal{C}$ to a 2-cocycle with values in $\mathbb{Z}\mathscr{H}$ where

$$p_{\mathcal{C}} : \sum_{H\in\mathscr{H}} \lambda_H H \longmapsto \sum_{H\in\mathcal{C}} \lambda_H H .$$

To give a nice expression of the corresponding extensions, we need the following lemma.

Lemma 5 (Extension and direct sum). *Let* G *be a group,* $X = Y \oplus Z$ *a direct sum of* G*-modules and*

$$0 \longrightarrow X \xrightarrow{u} E \xrightarrow{v} G \longrightarrow 1$$

an extension of G *by* X*. We denote by* $q : X \to Y$ *the first projection and* φ *the class of the extension* E *in* $H^2(G, X)$*. The extension associated to* $q(\varphi)$ *is*

$$0 \longrightarrow Y \longrightarrow E/Z \longrightarrow G \longrightarrow 1$$

Proof. Let us denote by $\theta : E \to E/Z$ the natural surjection and $i : Y \to Y \oplus Z$ the natural map. Let us first remark that Z is normal in E since Z is stable by the action of G. Since v is trivial on Z, then it induces a group homomorphism $\widetilde{v} : E/Z \to G$ whose kernel is $X/Z = Y$. Thus the sequence

$$0 \longrightarrow Y \xrightarrow{\theta u i} E/Z \xrightarrow{\widetilde{v}} G \longrightarrow 1 \tag{4}$$

is an exact one.

If $s : G \to E$ is a set-theoretic section of v, then θs is a set-theoretic section of $\widetilde{v}$. The expression of a 2-cocycle associated to an extension in terms of a set-theoretic section gives the result.

For $\mathcal{C} \in \mathscr{H}/W$, we denote by $B_{\mathcal{C}}$ the quotient group

$$B_{\mathcal{C}} = B/\langle [P,P], {s_{H,\gamma}}^{e_H}, \quad H \notin \mathcal{C}\rangle$$

Lemma 5 tells us that the extension (3) is equivalent to the family of extensions

$$0 \longrightarrow \mathbb{Z}\mathcal{C} \xrightarrow{j_{\mathcal{C}}} B_{\mathcal{C}} \xrightarrow{p_{\mathcal{C}}} W \longrightarrow 1 \tag{5}$$

for $\mathcal{C} \in \mathscr{H}/W$.

4.2. Second step : the induction argument

In each orbit $\mathcal{C} \in \mathscr{H}/W$, we choose a hyperplane $H_{\mathcal{C}} \in \mathcal{C}$ and write $\mathbb{Z}\mathcal{C} = \mathrm{Ind}_{N_{\mathcal{C}}}^{W}(\mathbb{Z})$ where $N_{\mathcal{C}} \subset W$ is the stabilizer of $H_{\mathcal{C}}$. Shapiro's isomorphism lemma [5, Proposition III.6.2] shows that $H^2(W, \mathbb{Z}\mathcal{C}) = H^2(N_{\mathcal{C}}, \mathbb{Z})$. Exercise III.8.2 of [5] tells us that in term of 2-cocycles Shapiro's isomorphism is described as follow

$$S : (\varphi : G^2 \to \mathbb{Z}\mathcal{C}) \longmapsto (f_{\mathcal{C}} \circ \varphi : {N_{\mathcal{C}}}^2 \to \mathbb{Z})$$

where $f_{\mathcal{C}} : \mathbb{Z}\mathcal{C} \to \mathbb{Z}$ is the projection onto the $H_{\mathcal{C}}$-component.

Decomposing Shapiro's isomorphism into the following two steps

$$(\varphi : G^2 \to \mathbb{Z}\mathcal{C}) \longmapsto (\varphi : {N_{\mathcal{C}}}^2 \to \mathbb{Z}\mathcal{C}) \longmapsto (f_{\mathcal{C}} \circ \varphi : {N_{\mathcal{C}}}^2 \to \mathbb{Z}),$$

allows us to interpret it in terms of group extensions. Exercice IV.3.1.(a) of [5] gives a description of the first step : the corresponding extension is given by

$$0 \longrightarrow \mathbb{Z}\mathcal{C} \longrightarrow {p_{\mathcal{C}}}^{-1}(N_{\mathcal{C}}) \longrightarrow N_{\mathcal{C}} \longrightarrow 1$$

since ${p_{\mathcal{C}}}^{-1}(N_{\mathcal{C}})$ is the fiber product of $B_{\mathcal{C}}$ and $N_{\mathcal{C}}$ over W. Moreover, since $f_{\mathcal{C}}$ is a split surjection as a $N_{\mathcal{C}}$-module map, Lemma 5 gives us the following extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow {p_{\mathcal{C}}}^{-1}(N_{\mathcal{C}})/\langle {s_{H,\gamma}}^{e_H},\ H \in \mathcal{C} \setminus \{H_{\mathcal{C}}\}\rangle \longrightarrow N_{\mathcal{C}} \longrightarrow 1$$

Finally, the extension (3) is equivalent to the family of extensions

$$0 \longrightarrow \mathbb{Z} \longrightarrow B'_{\mathcal{C}} \xrightarrow{p_{\mathcal{C}}} N_{\mathcal{C}} \longrightarrow 1 \tag{6}$$

where $B'_{\mathcal{C}} = p^{-1}(N_{\mathcal{C}})/\langle [P,P], {s_{H,\gamma}}^{e_H},\ H \neq H_{\mathcal{C}}\rangle$ and $\mathcal{C} \in \mathscr{H}/W$.

4.3. Third step : linear character

For the third step, we use results and notations of Section 2 and Section 3. Let us consider the group homomorphism $\rho_{\mathcal{C}} : p^{-1}(N_{\mathcal{C}}) \to \mathbb{Q}$ of Definition 3. Since it is trivial on $\langle [P,P], {s_{H,\gamma}}^{e_H},\ H \neq H_{\mathcal{C}} \rangle$, it induces a group homomorphism from $B'_{\mathcal{C}}$ to $\mathbb{Q}$ still denoted by $\rho_{\mathcal{C}}$. Moreover, since $\rho_{\mathcal{C}}({s_{H_{\mathcal{C}},\gamma}}^{e_{H_{\mathcal{C}}}}) = 1$, Proposition 3 gives the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle \rho_{\mathcal{C}}} & & \uparrow{\scriptstyle r_{\mathcal{C}}} & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & B'_{\mathcal{C}} & \xrightarrow{p_{\mathcal{C}}} & N_{\mathcal{C}} & \longrightarrow & 1
\end{array}$$

Exercises IV.3.2 and IV.3.3 of [5] tell us precisely that the group homomorphism corresponding to (6) is $r_{\mathcal{C}}$. So the extension (3) is equivalent to the family $(r_{\mathcal{C}})_{\mathcal{C} \in \mathscr{H}/W}$. This concludes the proof of Proposition 5.

5. Tables

5.1. The infinite series

In this subsection, we bring together tables for the orbits of the hyperplanes of $G(de,e,r)$ under the centralizer of a reflection and under the parabolic subgroup associated to the line of the reflection and tables for the values of f_H and the index of ramification. So let us consider the complex reflection group $G(de,e,r)$ acting on $\mathbb{C}^r$ with canonical basis $(e_1,\ldots,e_r)$. The standard point of $\mathbb{C}^r$ is denoted by $(z_1,\ldots,z_r)$.

The hyperplanes of $G(de,e,r)$ are $H_i = \{z_i = 0\}$ for $i \in \{1,\ldots,r\}$ (when $d>1$) and $H_{i,j,\zeta} = \{z_i = \zeta z_j\}$ for $i<j$ and $\zeta \in \mathbb{U}_{de}$ (when $r \geq 2$). They split in general into two conjugacy classes under $G(de,e,r)$ whose representant may be chosen as follow H_1 and $H_{1,2,1}$.

Let us continue with more notations. For every triple of integers d,e,r, we denote by $\pi : G(de,e,r) \to \mathbb{U}$ the following group morphism : for $g \in G(de,e,r)$, $\pi(g)$ is the product of the nonzero coefficients of the monomial matrix g. When e is even, we denote by $e' = e/2$. We denote by $e'' = e/\gcd(e,3)$ and by P the set of elements of $\mathbb{U}_{de}$ with strictly positive imaginary part.

5.1.1. The case of the hyperplane $H_1 = \{z_1 = 0\}$ We then have $d>1$. The stabilizer N of H_1 is described by

$$N = \{(\alpha,g), \quad g \in G(de,1,r-1),\ \alpha \in \mathbb{U}_{de},\ (\pi(g)\alpha)^d = 1\}$$

and the pointwise stabilizer C of $D_1 = {H_1}^{\perp} = \mathbb{C}e_1$ is $C = G(de,e,r-1)$. Table 1 gives the orbits of the hyperplanes under N and C. In Table 1, C.O. stands commuting orbits and N.C.O. stand for non commuting orbits.

5.1.2. The hyperplane $H_{1,2,\exp(2i\pi/de)}$ with $r=2$ We set $\zeta = \exp(2i\pi/de)$. The reflection of $G(de,e,2)$ with hyperplane $H_{1,2,\zeta}$ is

$$s = \begin{bmatrix} & \zeta \\ \zeta^{-1} & \end{bmatrix}$$

The line of s is $D = \mathbb{C}(e_2 - \zeta e_1)$. The centralizer of s is given by

$$N = \left\{ d_\lambda = \begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix}, t_\lambda = \begin{bmatrix} & \lambda\zeta \\ \lambda\zeta^{-1} & \end{bmatrix},\ \lambda^{2d} = 1 \right\}.$$

The eigenvalue of t_λ on D is λ whereas the eigenvalue of d_λ on D is $-\lambda$. So the parabolic subgroup C associated to D is $C = \{\mathrm{id}, -s\}$. The orbits of hyperplane under C and N are the same. The commuting ones are $\{H_{1,2,\zeta}\}$ and $\{H_{1,2,-\zeta}\}$. The non commuting ones are $\{H_1, H_2\}$ and $\{H_{1,2,\mu}, H_{1,2,\zeta^2\mu^{-1}}\}$ for $\mu \in \mathbb{U}_{de} \setminus \{\pm\zeta\}$.

5.1.3. The hyperplane $H_{1,2,1} = \{z_1 = z_2\}$ We then have $r \geq 2$. The reflection of $G(de,e,r)$ with hyperplane $H_{1,2,1}$ is the transposition τ_{12} swapping 1 and 2. Since the elements of $G(de,e,r)$ are monomial matrices, an element of $G(de,e,r)$ commuting with τ_{12} stabilizes the subspace spanned by e_1 and e_2. Thus the stabilizer N of $H_{1,2,1}$ is given by

$$N = \left\{ d_{\lambda,g} = \left[\begin{array}{cc|c} \lambda & & \\ & \lambda & \\ \hline & & g \end{array}\right],\quad t_{\lambda,g} = \left[\begin{array}{cc|c} & \lambda & \\ \lambda & & \\ \hline & & g \end{array}\right], \right.$$
$$\left. \lambda \in \mathbb{U}_{de},\ g \in G(de,1,r-2),\ (\pi(g)\lambda^2)^d = 1 \right\}$$

The line of τ_{12} is $\mathbb{C}(e_1 - e_2)$. So the eigenvalue of $d_{\lambda,g}$ on $\mathbb{C}(e_1 - e_2)$ is λ whereas the eigenvalue of $t_{\lambda,g}$ on $\mathbb{C}(e_1 - e_2)$ is $-\lambda$. Thus, when de is odd, the parabolic subgroup associated to the line $\mathbb{C}(e_1 - e_2)$ is given by

$$C = \left\{ \left[\begin{array}{cc|c} 1 & & \\ & 1 & \\ \hline & & g \end{array}\right],\quad g \in G(de,e,r-2) \right\}$$

and when de is even, the parabolic subgroup associated to the line $\mathbb{C}(e_1 - e_2)$ is given by

$$C = \left\{ \left[\begin{array}{cc|c} 1 & & \\ & 1 & \\ \hline & & g \end{array}\right], \left[\begin{array}{cc|c} & -1 & \\ -1 & & \\ \hline & & g \end{array}\right],\quad g \in G(de,e,r-2) \right\}$$

Table 2 gives the orbits the hyperplanes of $G(de,e,r)$ under N and C. In Table 2, C.O. stands commuting orbits and N.C.O. stand for non commuting orbits.

5.1.4. Value for f_H and the index of ramification The computations of the preceding paragraphs also lead to Table 3 which brings together the values of e_H, f_H and d_H for every class of hyperplanes. In Table 3, we set $\zeta = \exp(2i\pi/de)$.

We obtain the following errata for the proposition 6.1 of [9]. Let us consider $r \geq 2$. For $W = G(de, e, r)$, we have $\kappa(W) = 2de$ if de is odd and $r \geq 3$. We have $\kappa(W) = de$ if ($d \neq 1$ and $r = 2$) or ($r \geq 3$ and de even).

5.2. Exceptional types

With the package CHEVIE of GAP [8][6], we obtain Table 4 for the values of e_H, f_H and f_H/e_H for the hyperplanes of the exceptional reflection groups. In particular, the only non Coxeter groups with only unramified hyperplanes are G_8, G_{12} and G_{24}. Table 4 can also easily be obtained from the table of [9] for the value of $\kappa(W)$. The first and fifth columns stand for the number of the group in the Shephard and Todd classification. We also write instructions which determines, for a given hyperplane H, the orbits of commuting and non commuting hyperplanes under N_H and C_H which is used to obtain the results of Remark 1 and Remark 12.

		N_{H_1}	C_{H_1}
$r \geq 4$	C.O.	H_1 $\{H_i,\ i \geq 2\}$ $\{H_{i,j,\zeta},\ i \neq j \geq 2,\ \zeta \in \mathbb{U}_{de}\}$	H_1 $\{H_i,\ i \geq 2\}$ $\{H_{i,j,\zeta},\ i \neq j \geq 2,\ \zeta \in \mathbb{U}_{de}\}$
	N.C.O.	$\{H_{1,j,\zeta},\ j \geq 2,\ \zeta \in \mathbb{U}_{de}\}$	$\{H_{1,j,\zeta},\ j \geq 2,\ \zeta \in \mathbb{U}_{de}\}$
$r = 3$ and e odd	C.O.	H_1 $\{H_2, H_3\}$ $\{H_{2,3,\zeta},\ \zeta \in \mathbb{U}_{de}\}$	H_1 $\{H_2, H_3\}$ $\{H_{2,3,\zeta},\ \zeta \in \mathbb{U}_{de}\}$
	N.C.O.	$\{H_{1,i,\zeta},\ i = 2,3,\ \zeta \in \mathbb{U}_{de}\}$	$\{H_{1,i,\zeta},\ i = 2,3,\ \zeta \in \mathbb{U}_{de}\}$
$r = 3$ and e even	C.O.	H_1 $\{H_2, H_3\}$ $\{H_{2,3,\zeta},\ \zeta \in \mathbb{U}_{de}\}$	H_1 $\{H_2, H_3\}$ $\{H_{2,3,\zeta},\ \zeta \in c\}$ for $c \in \mathbb{U}_{de}/\mathbb{U}_{de'}$
	N.C.O.	$\{H_{1,i,\zeta},\ i = 2,3,\ \zeta \in \mathbb{U}_{de}\}$	$\{H_{1,i,\zeta},\ i = 2,3,\ \zeta \in \mathbb{U}_{de}\}$
$r = 2$ and e odd	C.O.	H_1 H_2	H_1 H_2
	N.C.O	$\{H_{1,2,\zeta},\ \zeta \in \mathbb{U}_{de}\}$	$\{H_{1,2,\zeta},\ \zeta \in c\}$ for $c \in \mathbb{U}_{de}/\mathbb{U}_d$
$r = 2$ and e even	C.O.	H_1 H_2	H_1 H_2
	N.C.O.	$\{H_{1,2,\zeta},\ \zeta \in c\}$ for $c \in \mathbb{U}_{de}/\mathbb{U}_{de'}$	$\{H_{1,2,\zeta},\ \zeta \in c\}$ for $c \in \mathbb{U}_{de}/\mathbb{U}_d$
$r = 1$	C.O.	H_1	H_1

Table 1: Orbits of hyperplanes under N_{H_1} and C_{H_1}

		$N_{H_{1,2,1}}$	$C_{H_{1,2,1}}$
$r \geq 5$ and de odd	C.O.	$H_{1,2,1}$ $\{H_i,\ i \geq 3\}$ if $d > 1$ $\{H_{i,j,\zeta},\ i \neq j \geq 3,\ \zeta \in \mathbb{U}_{de}\}$	$H_{1,2,1}$ $\{H_i,\ i \geq 3\}$ if $d > 1$ $\{H_{i,j,\zeta},\ i \neq j \geq 3,\ \zeta \in \mathbb{U}_{de}\}$
	N.C.O.	$\{H_1, H_2\}$ if $d > 1$ $\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$ $\{H_{1,j,\zeta}, H_{2,j,\zeta},\ j \geq 3,\ \zeta \in \mathbb{U}_{de}\}$	$\{H_1\}, \{H_2\}$ if $d > 1$ $\{H_{1,2,\zeta}\}$ for $\zeta \in \mathbb{U}_{de}$ $\{H_{1,j,\zeta},\ j \geq 3,\ \zeta \in \mathbb{U}_{de}\}$ $\{H_{2,j,\zeta},\ j \geq 3,\ \zeta \in \mathbb{U}_{de}\}$
$r \geq 5$ and de even	C.O.	$\{H_{1,2,1}\}, \{H_{1,2,-1}\}$ $\{H_i,\ i \geq 3\}$ if $d > 1$ $\{H_{i,j,\zeta},\ i \neq j \geq 3,\ \zeta \in \mathbb{U}_{de}\}$	$\{H_{1,2,1}\}, \{H_{1,2,-1}\}$ $\{H_i,\ i \geq 3\}$ if $d > 1$ $\{H_{i,j,\zeta},\ i \neq j \geq 3,\ \zeta \in \mathbb{U}_{de}\}$
	N.C.O.	$\{H_1, H_2\}$ if $d > 1$ $\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$ $\{H_{1,j,\zeta}, H_{2,j,\zeta},\ j \geq 3,\ \zeta \in \mathbb{U}_{de}\}$	$\{H_1, H_2\}$ if $d > 1$ $\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$ $\{H_{1,j,\zeta}, H_{2,j,\zeta},\ j \geq 3,\ \zeta \in \mathbb{U}_{de}\}$
$r = 4$ and de odd	C.O.	$\{H_{1,2,1}\}; \{H_i,\ i \geq 3\}$ if $d > 1$ $\{H_{i,j,\zeta},\ i \neq j \geq 3,\ \zeta \in \mathbb{U}_{de}\}$	$\{H_{1,2,1}\}; \{H_i,\ i \geq 3\}$ if $d > 1$ $\{H_{i,j,\zeta},\ i \neq j \geq 3,\ \zeta \in \mathbb{U}_{de}\}$
	N.C.O.	$\{H_1, H_2\}$ if $d > 1$ $\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$ $\{H_{1,j,\zeta}, H_{2,j,\zeta},\ j \geq 3,\ \zeta \in \mathbb{U}_{de}\}$	$\{H_1\}, \{H_2\}$ if $d > 1$ $\{H_{1,2,\zeta}\}$ for $\zeta \in \mathbb{U}_{de}$ $\{H_{1,j,\zeta},\ j \geq 3,\ \zeta \in \mathbb{U}_{de}\}$ $\{H_{2,j,\zeta},\ j \geq 3,\ \zeta \in \mathbb{U}_{de}\}$
$r = 4$ and e odd and d even	C.O.	$\{H_{1,2,1}\}, \{H_{1,2,-1}\}$ $\{H_i,\ i \geq 3\}$ if $d > 1$ $\{H_{i,j,\zeta},\ i \neq j \geq 3,\ \zeta \in \mathbb{U}_{de}\}$	$\{H_{1,2,1}\}, \{H_{1,2,-1}\}$ $\{H_i,\ i \geq 3\}$ if $d > 1$ $\{H_{i,j,\zeta},\ i \neq j \geq 3,\ \zeta \in \mathbb{U}_{de}\}$
	N.C.O.	$\{H_1, H_2\}$ if $d > 1$ $\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$ $\{H_{1,j,\zeta}, H_{2,j,\zeta},\ j \geq 3,\ \zeta \in \mathbb{U}_{de}\}$	$\{H_1, H_2\}$ if $d > 1$ $\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$ $\{H_{1,j,\zeta}, H_{2,j,\zeta},\ j \geq 3,\ \zeta \in \mathbb{U}_{de}\}$

Orbits of hyperplanes under the stabiliser of $H_{1,2,1}$ and parabolic subgroup associated to $\mathbb{C}(e_1 - e_2)$

		$N_{H_{1,2,1}}$	$C_{H_{1,2,1}}$
$r = 4$ and e even	C.O.	$\{H_{1,2,1}\}, \{H_{1,2,-1}\}$ $\{H_i,\ i \geq 3\}$ if $d > 1$ $\{H_{i,j,\zeta},\ i \neq j \geq 3,\ \zeta \in \mathbb{U}_{de'}\}$ $\{H_{i,j,\zeta},\ i \neq j \geq 3,\ \zeta \in \mathbb{U}_{de} \setminus \mathbb{U}_{de'}\}$	$\{H_{1,2,1}\}, \{H_{1,2,-1}\}$ $\{H_i,\ i \geq 3\}$ if $d > 1$ $\{H_{i,j,\zeta},\ i \neq j \geq 3,\ \zeta \in \mathbb{U}_{de'}\}$ $\{H_{i,j,\zeta},\ i \neq j \geq 3,\ \zeta \in \mathbb{U}_{de} \setminus \mathbb{U}_{de'}\}$
	N.C.O.	$\{H_1, H_2\}$ if $d > 1$ $\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$ $\{H_{1,j,\zeta}, H_{2,j,\zeta},\ j \geq 3,\ \zeta \in \mathbb{U}_{de}\}$	$\{H_1, H_2\}$ if $d > 1$ $\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$ $\{H_{1,j,\zeta}, H_{2,j,\zeta},\ j \geq 3,\ \zeta \in \mathbb{U}_{de}\}$
$r = 3$ and de odd	C.O.	$\{H_{1,2,1}\}; \{H_3\}$ if $d > 1$	$\{H_{1,2,1}\}; \{H_3\}$ if $d > 1$
	N.C.O.	$\{H_1, H_2\}$ if $d > 1$ $\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$ $\{H_{1,3,\zeta}, H_{2,3,\zeta},\ \zeta \in c\}$ for $c \in \mathbb{U}_{de}/\mathbb{U}_{de''}$	$\{H_1\}, \{H_2\}$ if $d > 1$ $\{H_{1,2,\zeta}\}$ for $\zeta \in \mathbb{U}_{de}$ $U_c = \{H_{1,3,\zeta},\ \zeta \in c\}$ for $c \in \mathbb{U}_{de}/\mathbb{U}_d$ $U'_c = \{H_{2,3,\zeta},\ \zeta \in c\}$ for $c \in \mathbb{U}_{de}/\mathbb{U}_d$
$r = 3$ and de even	C.O.	$\{H_{1,2,1}\}, \{H_{1,2,-1}\}$ H_3 if $d > 1$	$\{H_{1,2,1}\}, \{H_{1,2,-1}\}$ H_3 if $d > 1$
	N.C.O.	$\{H_1, H_2\}$ if $d > 1$ $\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$ $\{H_{1,3,\zeta}, H_{2,3,\zeta}, \zeta \in c\}$ for $c \in \mathbb{U}_{de}/\mathbb{U}_{de''}$	$\{H_1, H_2\}$ if $d > 1$ $\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$ $\{H_{1,3,\zeta}, H_{2,3,-\zeta}, \zeta \in c\}$ for $c \in \mathbb{U}_{de}/\mathbb{U}_d$
$r = 2$ and de odd	C.O.	$H_{1,2,1}$	$H_{1,2,1}$
	N.C.O.	$\{H_1, H_2\}$ if $d > 1$ $\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$	$\{H_1\}, \{H_2\}$ if $d > 1$ $\{H_{1,2,\zeta}\}$ for $\zeta \in \mathbb{U}_{de}$
$r = 2$ and de even	C.O.	$\{H_{1,2,1}\}, \{H_{1,2,-1}\}$	$\{H_{1,2,1}\}, \{H_{1,2,-1}\}$
	N.C.O.	$\{H_1, H_2\}$ if $d > 1$ $\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$	$\{H_1, H_2\}$ if $d > 1$ $\{H_{1,2,\zeta}, H_{1,2,\zeta^{-1}}\}$ for $\zeta \in P$

Table 2: Orbits of hyperplanes under $N_{H_{1,2,1}}$ and $C_{H_{1,2,1}}$

H			f_H	e_H	d_H
$z_1 = 0$	$r = 1$		d	d	1
$z_1 = 0$	$r \geq 2$	$d \neq 1$	de	d	e
$z_2 = z_1$	$r \geq 3$	de odd	$2de$	2	de
$z_2 = z_1$	$r \geq 3$	de even	de	2	$de/2$
$z_2 = z_1$	$r = 2$	e odd and d even	d	2	$d/2$
$z_2 = z_1$	$r = 2$	e even or d odd	$2d$	2	d
$z_2 = \zeta z_1$	$r = 2$	e even	$2d$	2	d

Table 3: Values for the ramification index for $G(de, e, r)$

ST	e_H	f_H	d_H	ST	e_H	f_H	d_H
4	3	6	2	21	2,3	12,12	6,4
5	3,3	6,6	2,2	22	2	4	2
6	2,3	4,12	2,4	23	2	2	1
7	2,3,3	12,12,12	6,4,4	24	2	2	1
8	4	4	1	25	3	6	2
9	2,4	8,8	4,2	26	3,2	6,6	2,3
10	3,4	12,12	4,3	27	2	6	3
11	2,3,4	24,24,24	12,8,6	28	2,2	2,2	1,1
12	2	2	1	29	2	4	2
13	2,2	8,4	4,2	30	2	2	1
14	2,3	6,6	3,2	31	2	4	2
15	2,3,2	12,12,24	6,4,12	32	3	6	2
16	5	10	2	33	2	6	3
17	2,5	20,20	10,4	34	2	6	3
18	3,5	30,30	10,6	35	2	2	1
19	2,3,5	60,60,60	30,20,12	36	2	2	1
20	3	6	2	37	2	2	1

Table 4: Values for the ramification index for the exceptional groups

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