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Exterior algebra structure on relative invariants of reflection groups

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Abstract Let G be a reflection group acting on a vector space V (over a field with zero characteristic). We denote by $S(V^*)$ the coordinate ring of V, by M a finite dimensional G-module and by χ a one-dimensional character of G. In this article, we define an algebra structure on the isotypic component associated to χ of the algebra $S(V^*) \otimes \Lambda(M^*)$. This structure is then used to obtain various generalizations of usual criterions on regularity of integers.

Keywords Reflection group · Relative invariant · Exterior algebra · Regular integer · Hyperplane arrangement

Mathematics Subject Classification (2000) 13A50 · 15A75

1 Introduction

In the first part of this article, we will study the following situation. Let G be a reflection group acting on the vector space V, M be a finite dimensional representation of G and χ be a one-dimensional character of G. Following the ideas of Shepler [18], we construct an exterior algebra structure on the χ-isotypic component of $T^{-1}S(V^*) \otimes \Lambda(M^*)$ for a suitable multiplicative set T of $S(V^*)$ (which we are able to control). This work is in line with the articles [1,9,13,18] which construct algebra structures on the χ-isotypic of the algebra $S(V^*) \otimes \Lambda(M^*)$ under conditions over the restrictions of M and χ to certain subgroups of G. Here, the idea is to transfer the hypotheses on M and χ to conditions on the base ring: we substitute $S(V^*)$ into a bigger ring (a fraction ring of $S(V^*)$) in which some linear forms associated to hyperplanes of G are invertible. The conditions will be held by the "bad"

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hyperplanes that are needed to be invertible. The main results of [1,9,13,18] are exceptional cases of Theorem 1 (see Remark 4). The articles [6–8] explain the situation in prime characteristic.

In the second part of this article, we will give consequences of the exterior algebra structure with links to the notion of regular integers. These consequences are similar to those that can be found in [3,9,10,13,19].

Various types of hyperplanes appear in the first part of the article. The hyperplanes to invert (the multiplicative set T) are chosen following these types. The third part studies these types for concrete reflection groups : the symmetric group, $G(de, e, 2)$, $G(d, 1, r)$ and the exceptional group G_4, G_5 and G_{24}.

Let us begin with some usual definitions and notations.

Definition 1 (Reflection) Let k be a field of characteristic 0 and V a finite dimensional vector space over k. Any $g \in \mathrm{GL}(V)$ so that g is of finite order and $\ker(g-1)$ is a hyperplane of V is called a *reflection*.

Definition 2 (Reflection Groups) Let k be a field of characteristic 0. The pair (G, V) is said to be a *reflection group over* k if V is a finite dimensional vector space over k (we denote by ℓ the dimension of V) and G be a finite subgroup of $\mathrm{GL}(V)$ generated by reflections. It will often be more comfortable to write "let G be a reflection group" omitting the vector space V.

Notation 1 (Reflections and hyperplanes) Let (G, V) be a reflection group. We denote by $\mathscr{S}$ the set of reflections of G and $\mathscr{H}$ the set of hyperplanes of G:

$$\mathscr{S} = \{s \in G, \quad \dim\ker(s - \mathrm{id}) = \dim V - 1\} \quad \text{and} \quad \mathscr{H} = \{\ker(s - \mathrm{id}), \quad s \in \mathscr{S}\}.$$

Notation 2 (Around a Hyperplane) Let (G, V) be a reflection group. For $H \in \mathscr{H}$,

- one chooses $\alpha_H \in V^*$ a linear form with kernel H;
- one sets $G_H = \mathrm{Fix}_G(H) = \{g \in G, \quad \forall x \in H, \quad gx = x\}$. This is a cyclic subgroup of G. We denote by e_H its order and by s_H its generator with determinant $\zeta_H = \exp(2i\pi/e_H)$;
- For any finite dimensional kG-module N and for $j \in [\![0\,,\, e_H - 1]\!]$, we define the integers $n_{j,H}(N)$ as the multiplicity of ${\zeta_H}^j$ as an eigenvalue of s_H acting on N^*. We also define the *total multiplicity of* N by

$$n_H(N) = \sum_{j=0}^{e_H-1} j n_{j,H}(N);$$

  Since the irreducible representations of G_H are the ${\det_V}^{-j}$ for $j \in [\![0\,,\, e_H - 1]\!]$, we have an abstract characterization of the integers $n_{j,H}(N)$:

$$\mathrm{Res}_{G_H}^G(N) = \bigoplus_{j=0}^{e_H-1} n_{j,H}(N){\det_V}^{-j};$$

- let $\chi : G \to k^\times$ be a linear character of G, we denote by k_χ the representation of G with character χ over k and $n_H(\chi)$ for $n_H(k_\chi)$; according to the definition, $n_H(\chi)$ is the unique integer j verifying $0 \leq j < e_H$ and $\chi(s_H) = \det(s_H)^{-j}$. Finally, for any kG-module N, we denote by $N^\chi = \{x \in N, \quad gx = \chi(g)x\}$ the χ-isotypic component of N.

Definition 3 (Polynomial Function associated to a Representation) Let (G, V) be a reflection group and N be a finite dimensional G-module. We set

$$Q_N = \prod_{H\in\mathscr{H}} {\alpha_H}^{n_H(N)} \in S(V^*).$$

When χ is a linear character of G, we set Q_χ rather than Q_{k_χ}, so that

$$Q_\chi = \prod_{H\in\mathscr{H}} {\alpha_H}^{n_H(\chi)} \in S(V^*).$$

2 Construction of the algebra structure

In this section and the next one, we fix a reflection group (G, V) over k, a kG-module M with dimension r and $\chi : G \to k^\times$ a linear character of G. We denote by $\det_M$ (resp. $\det_{M^*}$) the determinant of the representation M (resp. M^*).

Notation 3 Let $\mathscr{B} \subset \mathscr{H}$ be a G-stable subset (which will be the hyperplanes to invert, i.e. the "bad" hyperplanes), we denote by $\mathscr{G} = \mathscr{H}\backslash\mathscr{B}$ and $T = \langle\alpha_H,\quad H \in \mathscr{B}\rangle$ the multiplicative subset of $S(V^*)$ associated to $\mathscr{B}$. We then set $\Omega = T^{-1}S(V^*) \otimes \Lambda(M^*)$ and $\Omega^p = T^{-1}S(V^*) \otimes \Lambda^p(M^*)$ for $p \in [\![0\,,\, r]\!]$. Thus we have

$$\Omega^\chi = \bigoplus_{p=0}^{r} (\Omega^p)^\chi .$$

As said in the introduction, the idea of the proof is to bring the "bad" hyperplanes together in the subset $\mathscr{B}$. Thus in Subsect. 2.1, once M and χ have been chosen, we define various types of hyperplanes, so that we are able to differentiate the behavior of the hyperplanes of $\mathscr{H}$ with respect to M and χ and then choose the hyperplanes to invert. In Subsect. 2.2, we give conditions on $\mathscr{B}$ (see Hypothesis 2) such that we are able to construct an algebra structure on Ω^χ. Finally, in Subsect. 2.3, we refine these conditions on $\mathscr{B}$ (see Hypothesis 3 and Hypothesis 4) and then determine the algebra structure of Ω^χ. Roughly speaking, we obtain results of the following form : if $\mathscr{B}$ is big enough then Ω^χ is an exterior algebra. Moreover, Hypothesis 3 and Hypothesis 4 give a very precise meaning to the expression "$\mathscr{B}$ is enough". In particular, we can always choose $\mathscr{B} = \mathscr{H}$: when we invert all the hyperplanes, we obtain an exterior algebra structure on Ω^χ (see Remark 5).

2.1 Hyperplanes

Let us start to define our types of hyperplanes. To define the notion of a hyperplane which is multiplicity-partitionable with respect to M and χ, we need to introduce a notation, which will also be useful for the next subsection.

Notation 4 For $H \in \mathscr{H}$, we denote by $(j_1, \dots, j_r)$ the multiset of integers such that, for all $i \in [\![1\,,\, r]\!]$, we have $0 \leq j_i \leq e_H - 1$ and the eigenvalues of s_H acting on M^* are the ${\zeta_H}^{j_i}$ for $i \in [\![1\,,\, r]\!]$. More abstractly, the family $(j_1, \dots, j_r)$ is characterized by $0 \leq j_i \leq e_H - 1$ for all i and $\mathrm{Res}^G_{G_H}(M) = {\det_V}^{-j_1} \oplus \dots \oplus {\det_V}^{-j_r}$. The integers j_i are closely related to the integers $n_{j,H}(M)$ (see Notation 2). Precisely, for $j \in [\![0\,,\; e_H - 1]\!]$, $n_{j,H}(M)$ is the number of $i \in [\![1\,,\; r]\!]$ so that $j_i = j$, so that we have

$$\sum_{i=1}^{r} j_i = \sum_{j=0}^{e_H-1} j n_{j,H}(M) = n_H(M).$$

We now define four types of hyperplanes with respect to M or with respect to M and χ. The hyperplanes which are reflection-preserving with respect to M or the hyperplanes which are of a low multiplicity with respect to M and χ will be those that we do not need to invert (see Hypotheses 3 and 4).

Definition 4 (Hyperplane Types with respect to M) Let $H \in \mathscr{H}$; H is said to be

(i) *of a low multiplicity with respect to M* if $n_H(M) < e_H$;
(ii) *reflection-preserving with respect to M* if s_H acts on M as identity or as a reflection;
(iii) *of a low multiplicity with respect to M and χ* if $n_H(M) + n_H(\chi) < e_H$;
(iv) *multiplicity-partitionable with respect to M and χ* if for all partitions of the set $[\![1\,,\; r]\!]$ into two disjoint sets (denoted respectively by I_1 and I_2), we have

$$\sum_{i \in I_1} j_i < e_H - n_H(\chi) \quad \text{or} \quad \sum_{i \in I_2} j_i < e_H - n_H(\chi).$$

The next remark studies the links between the preceding notions of hyperplane types.

Remark 1 (Comparison of the Types of Hyperplane) Let $H \in \mathscr{H}$. We denote by 1 the trivial character on G. Let us show the following properties.

(i) H is reflection-preserving with respect to M if and only if H is of a low multiplicity with respect to M and M^*.
(ii) If H is of a low multiplicity with respect to M and χ then H is of a low multiplicity with respect to M.
(iii) H is reflection-preserving with respect to M if and only if it is multiplicity-partitionable with respect to M and χ for all $\chi \in \mathrm{Hom}_{\mathrm{gr}}(G, k^\times)$.
(iv) If H is of a low multiplicity with respect to M and χ then H is multiplicity-partitionable with respect to M and χ.
(v) H is of a low multiplicity with respect to M if and only if H is of a low multiplicity with respect to M and 1.
(vi) If H is of a low multiplicity with respect to M then H is multiplicity-partitionable with respect to M and 1.

Let us begin with (i). We start to express $n_H(M^*)$ using $n_H(M)$:

$$n_H(M^*) = \sum_{j=1}^{e_H-1} (e_H - j) n_{j,H}(M) = e_H(r - n_{0,H}(M)) - n_H(M).$$

We have $n_H(M) = 0$ if and only if $n_H(M^*) = 0$ if and only if s_H acts trivially on M. We then deduce

$$n_H(M) \le e_H - 1 \quad \text{and} \quad n_H(M^*) \le e_H - 1 \quad \Longleftrightarrow \quad r - n_{0,H}(M) < 2.$$

Since $n_{0,H}(M)$ (resp. $n_{0,H}(M^*)$) is the multiplicity of 1 as an eigenvalue of s_H acting on M (resp. M^*), we obtain $n_{0,H}(M) = n_{0,H}(M^*)$ and the condition $n_{0,H}(M) \in \{r-1, r\}$ can be expressed geometrically as s_H acts trivially on M or acts as a reflection on M. In particular, a hyperplane which is reflection-preserving with respect to M is always of a low multiplicity with respect to M.

Let us show (ii). We have $n_H(\chi) \geq 0$, so a hyperplane of a low multiplicity with respect to M and χ is always of a low multiplicity with respect to M.

Now, let us consider (iii). Let us assume that H is an hyperplane which is reflection-preserving with respect to M. There exists at most one $i_0 \in [\![1\,,\,r]\!]$ so that j_{i_0} is nonzero. When I_1 and I_2 are two disjoint subsets of $[\![1\,,\,r]\!]$, only one of those two sets can contain i_0. Thus, we have

$$\sum_{i \in I_1} j_i = 0 < e_H - n_H(\chi) \quad \text{or} \quad \sum_{i \in I_2} j_i = 0 < e_H - n_H(\chi).$$

Reciprocally, let us assume that H is not reflection-preserving with respect to M. We then deduce that there exists $i_1 \neq i_2$ so that $j_{i_1} \neq 0$ and $j_{i_2} \neq 0$. In addition, we know the existence of a linear character χ of G so that $n_H(\chi) = e_H - 1$ by Stanley's theorem [22]. The disjoint sets $I_1 = \{i_1\}$ and $I_2 = \{i_2\}$ verify

$$\sum_{i \in I_1} j_i \geq 1 = e_H - n_H(\chi) \quad \text{and} \quad \sum_{i \in I_2} j_i \geq 1 = e_H - n_H(\chi).$$

We then deduce that H is not multiplicity-partitionable with respect to M and χ.

Let us show (iv). Let I be a subset of $[\![1\,,\,r]\!]$ so that

$$e_H - n_H(\chi) \leq \sum_{i \in I} j_i.$$

Any such I contains every $i \in [\![1\,,\,r]\!]$ so that $j_i \neq 0$ since

$$\sum_{i \in [\![1\,,\,r]\!]} j_i = n_H(M) \leq e_H - n_H(\chi).$$

Thus, any set I' disjoint of I verifies

$$\sum_{i \in I'} j_i = 0 < e_H - n_H(\chi).$$

Let us show (v) and (vi). Since $n_H(1) = 0$ for all $H \in \mathscr{H}$, a hyperplane is of a low multiplicity with respect to M if and only if it is of a low multiplicity with respect to M and 1. (iv) shows that such a hyperplane is multiplicity-partitionable with respect to M and 1.

Example 1 Let (G, V) be a reflection group with $\dim V = \ell$. We have

$$n_H(V) = n_H(\det_V) = e_H - 1 \quad \text{and} \quad n_H(V^*) = n_H\left(\det_V^{-1}\right) = 1.$$

We also have

$$n_H\left(\Lambda^j(V)\right) = (e_H - 1)\binom{\ell - 1}{j - 1}.$$

Example 2 Let G be the symmetric group on n letters. When M is the irreducible representation attached to the partition $\lambda \vdash n$, the integer $n_H(M)$ has the following combinatorial interpretation : $n_H(M)$ is the number of standard λ-tableaux which have 2 on the second row (see [15, Sect. 2.7]).

Since lots of hyperplanes of reflection groups verify $e_H = 2$ (for example this is the case for Coxeter groups but not only), we focus on this specific case.

Remark 2 (Hyperplanes verifying $e_H = 2$) Let $H \in \mathscr{H}$ so that $e_H = 2$.

Then H is of a low multiplicity with respect to M if and only if H is reflection-preserving with respect to M (that is if the multiplicity of the eigenvalue -1 of s_H acting on M is not bigger than one).

If $\chi(G_H) \neq 1$ then

(i) H is multiplicity-partitionable with respect to M and χ if and only if s_H acts on M as a reflection or acts trivially on M (that is if the multiplicity of the eigenvalue -1 of s_H acting on M is not bigger than 1);
(ii) H is of a low multiplicity with respect to M and χ if and only if s_H acts trivially on M (that is if the multiplicity of the eigenvalue -1 of s_H acting on M is zero);

If $\chi(G_H) = 1$ then

(i) H is multiplicity-partitionable with respect to M and χ if and only if the multiplicity of the eigenvalue -1 of s_H acting on M is not bigger than 3;
(ii) H is of a low multiplicity with respect to M and χ if and only if s_H acts on M as a reflection or acts trivially on M (that is if the multiplicity of the eigenvalue -1 of s_H acting on M is not bigger than 1).

Following Remark 1, it is enough to show that if H is of a low multiplicity with respect to M then H is reflection-preserving with respect to M. According to hypothesis, we have $n_H(M) = n_{1,H}(M) < 2$ and thus $n_{0,H}(M) = r - n_{1,H}(M) \in \{r, r-1\}$.

Let us assume that $\chi(G_H) \neq 1$. We have $n_H(\chi) \neq 0$ and then $n_H(\chi) = 1$. The definition of being multiplicity-partitionable shows us that if H is multiplicity-partitionable with respect to M and χ then H is multiplicity-partitionable with respect to M and χ' for all linear characters χ' verifying $n_H(\chi') \leq 1$. But every linear character χ' verifies $n_H(\chi') \leq 1$, thus we have H is multiplicity-partitionable with respect to M and χ' for all linear characters χ' of G. Remark 1 shows that H is reflection-preserving with respect to M. In addition, according to the definition of being of a low multiplicity with respect to M and χ, H is of a low multiplicity with respect to M and χ if and only if $n_H(M) < 1$ that is $n_H(M) = 0$.

Let us assume that $\chi(G_H) = 1$. We have $n_H(\chi) = 0$ and then H is of a low multiplicity with respect to M and χ if and only if $n_H(M) < 2$ (that is H is of a low multiplicity with respect to M). In addition, in our case, we have $j_i \in \{0, 1\}$, the multiplicity of -1 as an eigenvalue of s_H is the number of i so that $j_i \neq 0$. If they are more than 4, we can divide them into two sets of two elements and the hyperplane is not multiplicity-partitionable with respect M and χ. If they are not more than 4, two disjoints set of $[\![1\,,\, r]\!]$ cannot both contain two integers i so that $j_i \neq 0$ and finally H is multiplicity-partitionable with respect to M and χ.

2.2 Construction of an algebra structure

Strictly following the ideas of Shepler [18], we construct an algebra structure on Ω^χ. The first step is to define a product. For this, we use the polynomial Q_χ of Definition 3 to bring back the usual product of two elements of Ω^χ into Ω^χ (by Stanley's theorem [22], Q_χ is so that $S(V^*) = Q_\chi S(V^*)^G$). Thus we are looking for divisibility conditions by Q_χ or more precisely by the non invertible part of Q_χ : this is done in Lemmas 1 and 2. The wanted divisibility is obtained under hypotheses on $\mathscr{B}$ (Hypotheses 1 and 2).

Hypothesis 1 Let us assume that $\mathscr{B}$ contains every hyperplane which is not of a low multiplicity with respect to M. Equivalently, every hyperplane contained in $\mathscr{G}$ is of a low multiplicity with respect to M.

2.2.1 Divisibility

To begin with, let us extend the following result of divisibility [5, Lemma 1] to the ring $T^{-1}S(V^*)$.

Lemma 1 (Divisibility in $T^{-1}S(V^*)$) *Let $x \in T^{-1}S(V^*)$, $H \in \mathscr{H}$ and $i \in [\![1\,,\, e_H]\!]$. Let us assume that $s_H x = {\zeta_H}^i x$. Then x is divisible by ${\alpha_H}^{e_H - i}$.*

The lemma is interesting only for $H \in \mathscr{G}$ since for $H \in \mathscr{B}$, the linear form α_H is invertible in $T^{-1}S(V^*)$.

Proof Since T is G-stable, we can write $x = P/Q$ with $Q \in T^G$. Since $S(V^*)$ is an integral domain, we deduce that $s_H P = {\zeta_H}^i P$. Lemma 1 of [5] shows that P is divisible by ${\alpha_H}^{e_H - i}$ and so is x.

We continue our study of divisibility by the α_H. Let us consider the case of Ω.

Lemma 2 (Divisibility in Ω) *Let us choose $\mu \in (\Omega^p)^\chi$ and $H \in \mathscr{H}$. We fix $(y_1, \ldots, y_r)$ a basis of M^* so that $s_H(y_i) = {\zeta_H}^{j_i} y_i$ for all $i \in [\![1\,,\, r]\!]$ (see Notation 4). We write*

$$\mu = \sum_{1 \le i_1 < \cdots < i_p \le r} \mu_{i_1,\ldots,i_p}\; y_{i_1} \wedge \cdots \wedge y_{i_p} \quad \textit{with} \quad \mu_{i_1,\ldots,i_p} \in T^{-1}S(V^*).$$

Assume that every $H \in \mathscr{G}$ is of a low multiplicity with respect to M. For $H \in \mathscr{G}$, we have

or $0 \le j_{i_1} + \cdots + j_{i_p} \le e_H - 1 - n_H(\chi)$ and ${\alpha_H}^{j_{i_1} + \cdots + j_{i_p} + n_H(\chi)} \mid \mu_{i_1,\ldots,i_p}$

or $e_H - n_H(\chi) \le j_{i_1} + \cdots + j_{i_p} \le 2e_H - 2 - n_H(\chi)$ and ${\alpha_H}^{j_{i_1} + \cdots + j_{i_p} + n_H(\chi) - e_H} \mid \mu_{i_1,\ldots,i_p}$.

Proof Since the family $(y_{i_1} \wedge \cdots \wedge y_{i_p})_{1 \le i_1 < \cdots < i_p \le r}$ is a $T^{-1}S(V^*)$-basis of Ω^p, we have

$$\mu_{i_1,\ldots,i_p}\; y_{i_1} \wedge \cdots \wedge y_{i_p} \in (\Omega^p)^\chi.$$

Thus $s_H(\mu_{i_1,\ldots,i_p}\; y_{i_1} \wedge \cdots \wedge y_{i_p}) = {\zeta_H}^{-n_H(\chi)} \mu_{i_1,\ldots,i_p}\; y_{i_1} \wedge \cdots \wedge y_{i_p}$. In addition

$$\begin{aligned} s_H(\mu_{i_1,\ldots,i_p}\; y_{i_1} \wedge \cdots \wedge y_{i_p}) &= s_H(\mu_{i_1,\ldots,i_p})\, s_H(y_{i_1}) \wedge \cdots \wedge s_H(y_{i_p}) \\ &= {\zeta_H}^{j_{i_1} + \cdots + j_{i_p}} s_H(\mu_{i_1,\ldots,i_p})\; y_{i_1} \wedge \cdots \wedge y_{i_p}. \end{aligned}$$

We then deduce $s_H(\mu_{i_1,\ldots,i_p}) = {\zeta_H}^{-j_{i_1} - \cdots - j_{i_p} - n_H(\chi)} \mu_{i_1,\ldots,i_p}$. Hypothesis 1 on $\mathscr{B}$ and $\mathscr{G}$ tells us

$$\forall\, H \in \mathscr{G}, \quad 0 \le n_H(M) = \sum_{i=1}^{r} j_i \le e_H - 1.$$

Hence $2 - 2e_H \le -j_{i_1} - \cdots - j_{i_p} - n_H(\chi) \le 0$. There are two cases to distinguish:

or $1 - e_H \le -j_{i_1} - \cdots - j_{i_p} - n_H(\chi) \le 0$ and if we set $f_H = e_H - j_{i_1} - \cdots - j_{i_p} - n_H(\chi)$, we have

$$s_H(\mu_{i_1,\ldots,i_p}) = {\zeta_H}^{f_H} \mu_{i_1,\ldots,i_p} \quad \text{with} \quad 1 \le f_H \le e_H;$$

Lemma 1 ensures that $\mu_{i_1,\ldots,i_p}$ is divisible by ${\alpha_H}^{j_{i_1} + \cdots + j_{i_p} + n_H(\chi)}$;

or $2-2e_H \leq -j_{i_1} - \cdots - j_{i_p} - n_H(\chi) \leq -e_H$ and if $f_H = 2e_H - j_{i_1} - \cdots - j_{i_p} - n_H(\chi)$, we have

$$s_H(\mu_{i_1,\ldots,i_p}) = {\zeta_H}^{f_H} \mu_{i_1,\ldots,i_p} \quad \text{with} \quad 2 \leq f_H \leq e_H;$$

Lemma 1 ensures that $\mu_{i_1,\ldots,i_p}$ is divisible by ${\alpha_H}^{j_{i_1}+\cdots+j_{i_p}+n_H(\chi)-e_H}$.

To assure the divisibility of Q_χ, we strengthen Hypothesis 1 by the following one.

Hypothesis 2 The subset $\mathscr{B}$ verifies Hypothesis 1 and $\mathscr{B}$ contains every hyperplane that is not multiplicity-partitionable with respect to M and χ. Equivalently, every hyperplane in $\mathscr{G}$ is multiplicity-partitionable with respect to M and χ and of a low multiplicity with respect to M.

We then obtain the following result of divisibility by Q_χ which is a refinement of Lemma 2 of [18].

Corollary 1 (Divisibility in Ω^χ) *Let us assume that every hyperplane in $\mathscr{G}$ is multiplicity-partitionable with respect to M and χ and of a low multiplicity with respect to M (this is Hypothesis 2). For $\mu, \omega \in \Omega^\chi$, we have $Q_\chi \mid \mu \wedge \omega$.*

Proof We fix $H \in \mathscr{G}$ and we consider the same basis $(y_1, \ldots, y_r)$ of M^* of Lemma 2. When $I = \{i_1, \ldots, i_p\}$ is a subset of $[\![1\,, \, r]\!]$ with $1 \leq i_1 < \cdots < i_p \leq r$, we set $y_I = y_{i_1} \wedge \cdots \wedge y_{i_p}$. Now, we can write

$$\mu = \sum_{I \subset [\![1\,,\,r]\!]} \mu_I \; y_I \quad \text{and} \quad \omega = \sum_{I \subset [\![1\,,\,r]\!]} \omega_I \; y_I \quad \text{with} \quad \mu_I, \omega_I \in T^{-1}S(V^*).$$

Hence

$$\mu \wedge \omega = \sum_{I \cap J = \emptyset} \varepsilon_{I,J} \mu_I \omega_J \; y_{I \cup J} \quad \text{with} \quad \varepsilon_{I,J} \in \{\pm 1\}.$$

Now, let us use Lemma 2. For this, we choose two subsets I, J of $[\![1\,, \, r]\!]$ with $I \cap J = \emptyset$.

- If $0 \leq \sum_{i \in I} j_i < e_H - n_H(\chi)$ or $0 \leq \sum_{i \in J} j_i < e_H - n_H(\chi)$ then μ_I or ω_J is divisible ${\alpha_H}^{n_H(\chi)}$ and then $\mu_I \omega_J$ is divisible by ${\alpha_H}^{n_H(\chi)}$.
- If not $\sum_{i \in I} j_i \geq e_H - n_H(\chi)$ and $\sum_{i \in J} j_i \geq e_H - n_H(\chi)$ with $I \cap J = \emptyset$ which contradicts Hypothesis 2.

Hence the product $\mu \wedge \omega$ is divisible by ${\alpha_H}^{n_H(\chi)}$ for all $H \in \mathscr{G}$. Since the family $(\alpha_H)_{H \in \mathscr{G}}$ is constituted with elements prime to each other, $\mu \wedge \omega$ is divisible by

$$\prod_{H \in \mathscr{G}} {\alpha_H}^{n_H(\chi)}.$$

In addition, for $H \in \mathscr{B}$, the element ${\alpha_H}^{n_H(\chi)}$ is invertible in $T^{-1}S(V^*)$, we finally obtain the divisibility of $\mu \wedge \omega$ by

$$\prod_{H \in \mathscr{B}} {\alpha_H}^{n_H(\chi)} \prod_{H \in \mathscr{G}} {\alpha_H}^{n_H(\chi)} = Q_\chi.$$

2.2.2 *Algebra structure*

When every hyperplane in $\mathscr{G}$ is assumed to be multiplicity-partitionable with respect to M and χ and of a low multiplicity with respect to M (Hypothesis 2), Corollary 1 applies. For $\mu, \omega \in \Omega^\chi$, we can define the twisted product $\curlywedge$ by

$$\mu \curlywedge \omega = {Q_\chi}^{-1} \mu \wedge \omega \in \Omega.$$

Actually, we have $\mu \curlywedge \omega \in \Omega^\chi$ and thus we define a law $\curlywedge$ on Ω^χ which gives to Ω^χ a structure of an associative $(T^G)^{-1}S(V^*)^G$-algebra with unit element Q_χ. Now, we have to show that $(\Omega^\chi, \curlywedge)$ is an exterior $(T^G)^{-1}S(V^*)^G$-algebra. For this, we study the structure constants of $(\Omega^\chi, \curlywedge)$ and we will show that they are those of an exterior algebra. To have more simple notation, we set $R = S(V^*)^G$.

2.3 Exterior algebra

In the previous subsection, under Hypothesis 2, we have constructed an algebra structure on Ω^χ. In this subsection, we are looking for its isomorphism class. The proof is divided into two stages : first, we give a necessary and sufficient condition for the structure constants of Ω^χ to be those of an exterior algebra (Proposition 1); subsequently, we show that this condition is verified (Proposition 3). To this perspective, we begin to generalize Stanley's theorem to the ring $T^{-1}S(V^*)$.

Corollary 2 (Stanley's theorem in $T^{-1}S(V^*)$) *We have*

$$(T^{-1}S(V^*))^\chi = (T^G)^{-1}S(V^*)^G Q_\chi \quad \textit{and} \quad (\Omega^r)^\chi = (T^G)^{-1}S(V^*)^G Q_{\chi\cdot\det_M} vol_M \quad (1)$$

where vol_M is a non-zero element of $\Lambda^r(M^*)$ once for all fixed.

Proof This is an easy consequence of the usual Stanley's theorem and of the fact that $k\mathrm{vol}_M = \Lambda^r(M^*)$ is a linear representation of G with character $\det_{M^*}$.

Notation 5 Let U be a $(T^G)^{-1}S(V^*)$-module. For $u, v \in U$, we denote by $u \doteq v$ if there exists $x \in ((T^G)^{-1}R)^\times$ so that $xu = v$. In particular, u an v generate the same $(T^G)^{-1}R$-submodule.

We now follow the method of Solomon [20] to obtain the exterior algebra structure (Propositions 1 and 3).

Proposition 1 (Necessary and Sufficient Condition) *Let us assume that every hyperplane in $\mathscr{G}$ is multiplicity-partitionable with respect to M and χ and of a low multiplicity with respect to M (Hypothesis 2). For every $\omega_1, \ldots, \omega_r \in (\Omega^1)^\chi$, the following propositions are equivalent*:

(i) *for all $p \in [\![1, r]\!]$, the family $(\omega_{i_1} \curlywedge \cdots \curlywedge \omega_{i_p})_{1 \le i_1 < \cdots < i_p \le r}$ is a $(T^G)^{-1}R$-basis of $(\Omega^p)^\chi$;*

(ii)

$$\omega_1 \curlywedge \cdots \curlywedge \omega_r \doteq Q_{\chi\cdot\det_M} vol_M. \quad (2)$$

Proof (i)$\Rightarrow$ (ii) This is an easy consequence of(1).
(ii) $\Rightarrow$ (i) We set $K = \mathrm{Frac}(T^{-1}S(V^*))$. Let us show that $\mathscr{F} = (\omega_{i_1} \curlywedge \cdots \curlywedge \omega_{i_p})_{i_1<\cdots<i_p}$ is free over K. Since $\omega_{i_1} \curlywedge \cdots \curlywedge \omega_{i_p} = {Q_\chi}^{1-p}\, \omega_{i_1} \wedge \cdots \wedge \omega_{i_p}$, it suffices to show that the family $(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p})_{1\le i_1<\cdots<i_p\le r}$ is free over K. For that, let us consider the relation

$$\sum_{1\le i_1<\cdots<i_p\le r} r_{i_1,\ldots,i_p}\, \omega_{i_1} \wedge \cdots \wedge \omega_{i_p} = 0 \quad \text{with} \quad r_{i_1,\ldots,i_p} \in K.$$

We fix $I = \{i_1, \ldots, i_p\} \subset [\![1, r]\!]$ with $1 \le i_1 < \cdots < i_p \le r$ and we set $I^c = \{i_{p+1}, \ldots, i_r\}$ the complementary of I in $[\![1, r]\!]$. Multiplying the preceding relation by $\omega_{i_{p+1}} \wedge \cdots \wedge \omega_{i_r}$, we obtain

$$r_{i_1,\ldots,i_p}\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \wedge \omega_{i_{p+1}} \wedge \cdots \wedge \omega_{i_r} = 0.$$

Hence $0 = r_{i_1,\ldots,i_p} {Q_\chi}^{r-1} \omega_1 \curlywedge \cdots \curlywedge \omega_r \doteq r_{i_1,\ldots,i_p} {Q_\chi}^{r-1} Q_{\chi\cdot\det_M}\, \mathrm{vol}_M$. We then deduce that $r_{i_1,\ldots,i_p} = 0$ and thus the K-freeness of the family $(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p})_{1\le i_1<\cdots<i_p\le r}$.

Finally, the family $\mathscr{F}$ is a basis of K-vector space $K \otimes \Lambda^p(M^*)$. Thus, if $\mu \in (\Omega^p)^\chi$, there exists $r_{i_1,\ldots,i_p} \in K$ so that

$$\mu = \sum_{1\le i_1<\cdots<i_p\le r} r_{i_1,\ldots,i_p}\, \omega_{i_1} \curlywedge \cdots \curlywedge \omega_{i_p}.$$

Let us fix again $I = \{i_1, \ldots, i_p\} \subset [\![1, r]\!]$ with $i_1 < \cdots < i_p$ and set $I^c = \{i_{p+1}, \ldots, i_r\}$ its complementary. Multiplying the defining relation of μ by $\omega_{i_{p+1}} \curlywedge \cdots \curlywedge \omega_{i_r}$, we obtain $\mu \curlywedge \omega_{i_{p+1}} \curlywedge \cdots \curlywedge \omega_{i_r} \in (\Omega^r)^\chi$. Thus, with (1), there exists $f \in (T^G)^{-1}R$ so that

$$\mu \curlywedge \omega_{i_{p+1}} \curlywedge \cdots \curlywedge \omega_{i_r} = f\, Q_{\chi\cdot\det_M}\, \mathrm{vol}_M.$$

In addition,

$$\mu \curlywedge \omega_{i_{p+1}} \curlywedge \cdots \curlywedge \omega_{i_r} = \varepsilon\, r_{i_1,\ldots,i_p}\omega_1 \curlywedge \cdots \curlywedge \omega_r \doteq \varepsilon\, r_{i_1,\ldots,i_p} Q_{\chi\cdot\det_M}\, \mathrm{vol}_M \quad \text{with} \quad \varepsilon \in \{\pm 1\}.$$

Hence $r_{i_1,\ldots,i_p} \doteq \varepsilon f \in (T^G)^{-1}R$. Therefore, the family $\mathscr{F}$ is a $(T^G)^{-1}R$-basis of $(\Omega^p)^\chi$.

Now, the aim is to show that every $(T^G)^{-1}R$-basis of $(\Omega^1)^\chi$ verifies condition 2. For this, we construct a family $(\nu_i)_{1\le i\le r}$ of G-invariants and a family $(\mu_i)_{1\le i\le r}$ of $\det_{M^*}$-invariants verifying respectively

$$\nu_1 \wedge \cdots \wedge \nu_r \doteq Q_M\, \mathrm{vol}_M \quad \text{and} \quad \mu_1 \wedge \cdots \wedge \mu_r \doteq (Q_{M^*})^{r-1}\, \mathrm{vol}_M.$$

Proposition 2 (Invariants in Ω^1) *There exists a family* $(\nu_1, \ldots, \nu_r) \in ((\Omega^1)^G)^r$ *and a family* $(\mu_1, \ldots, \mu_r) \in ((\Omega^1)^{\det_{M^*}})^r$ *verifying*

$$\nu_1 \wedge \cdots \wedge \nu_r \in k^\times Q_M\, vol_M \quad \textit{and} \quad \mu_1 \wedge \cdots \wedge \mu_r \in k^\times (Q_{M^*})^{r-1}\, vol_M.$$

Proof The proof is a rephrasing of Gutkin's theorem [5] using the notion of minimal matrix evolved by Opdam in [12, Definition 2.2 and Proposition 2.4 (ii)].

For $C = (c_{ij})_{i,j} \in \mathrm{M}_r(S(V^*))$, we denote by $g \cdot C$ the matrix $(g\, c_{ij})_{i,j}$. Let us consider C a M-minimal matrix. According to definition, $C \in \mathrm{M}_r(S(V^*))$ verifies

(i) $g \cdot C = C g_M$;
(ii) $\det C \neq 0$;
(iii) $\deg \det C$ is minimal among the matrices verifying (i) and (ii).

Let us choose $(y_i)_{1\leq i\leq r}$ a basis of M^* and define for $j \in [\![1\,,\, r]\!]$,

$$\nu_j = \sum_{i=1}^{r} c_{ji} \otimes y_i.$$

We have $\nu_1 \wedge \cdots \wedge \nu_r \in k^\times \det(C)\mathrm{vol}_M$. Although $\det C \in k^\times Q_M$ (let us see [12, Proposition 2.4 (iii)]). It remains to show that ν_j is G-invariant for all $j \in [\![1\,,\, r]\!]$. But, (i) gives

$$g\, c_{ji} = \sum_{k=1}^{r} c_{jk}\, g_{M\,ki} \quad \text{and} \quad g\, y_i = \sum_{n=1}^{r} g_{M^*\,ni}\, y_n.$$

Hence

$$g\nu_j = \sum_{i=1}^{r}\left(\sum_{k=1}^{r} c_{jk}\, g_{M\,ki} \otimes \sum_{n=1}^{r} g_{M^*\,ni}\, y_n\right) = \sum_{k=1}^{r}\sum_{n=1}^{r}\left(\sum_{i=1}^{r} g_{M\,ki} g_{M^*\,ni}\right) c_{jk} \otimes y_n.$$

Since ${}^t g_{M^*} = {g_M}^{-1}$, we have $\sum_{i=1}^{r} g_{M\,ki} g_{M^*\,ni} = \delta_{kn}$ and then

$$g\nu_j = \sum_{k=1}^{r} c_{jk} \otimes y_k = \nu_j.$$

Finally ν_j is G-invariant.

Now, let us consider D a M^*-minimal matrix. According to definition, $D \in \mathrm{M}_r(S(V^*))$ verifies

(i) $g \cdot D = D g_{M^*}$;
(ii) $\det D \neq 0$;
(iii) $\deg \det D$ is minimal among the matrices verifying (i) and (ii).

We consider $\mathrm{Com}\, D = (e_{ij})_{i,j}$ the comatrix of D. Since the action of G on $S(V^*)$ is compatible with the algebra structure, we have

$$g \cdot \mathrm{Com}\, D = \mathrm{Com}\,(g \cdot D) = \mathrm{Com}\,(D g_{M^*}) = \mathrm{Com}\, D \mathrm{Com}\, g_{M^*} = \det(g_{M^*})\mathrm{Com}(D)\, g_M.$$

We then define, for $j \in [\![1\,,\, r]\!]$,

$$\mu_j = \sum_{i=1}^{r} e_{ji} \otimes y_i.$$

We have $\mu_1 \wedge \cdots \wedge \mu_r \in k^\times \det(\mathrm{Com}\, D)\mathrm{vol}_M$. Although $\det \mathrm{Com}\, D = (\det D)^{r-1}$ and $\det D \in k^\times Q_{M^*}$ (see [12, Proposition 2.4 (iii)]). It remains to show that μ_j is $\det_{M^*}$-invariant for all $j \in [\![1\,,\, r]\!]$. But,

$$g\, e_{ji} = \det g_{M^*} \sum_{k=1}^{r} e_{jk}\, g_{M\,ki} \quad \text{and} \quad g\, y_i = \sum_{n=1}^{r} g_{M^*\,ni}\, y_n.$$

Hence

$$\begin{aligned} g\mu_j &= \det g_{M^*} \sum_{i=1}^{r}\left(\sum_{k=1}^{r} e_{jk}\, g_{M\,ki} \otimes \sum_{n=1}^{r} g_{M^*\,ni}\, y_n\right) \\ &= \det g_{M^*} \sum_{k=1}^{r}\sum_{n=1}^{r}\left(\sum_{i=1}^{r} g_{M\,ki} g_{M^*\,ni}\right) e_{jk} \otimes y_n. \end{aligned}$$

Since ${}^t g_{M^*} = g_M{}^{-1}$, we have $\sum_{i=1}^r g_{M\,ki} g_{M^*\,ni} = \delta_{kn}$ and then

$$g\mu_j = \det g_{M^*} \sum_{k=1}^r e_{jk} \otimes y_k = \det g_{M^*} \mu_j .$$

Finally μ_j is $\det_{g_{M^*}}$-invariant.

The following lemma proposes polynomial relations which generalize those of Shepler [19, Lemma 6 and Lemma 11] : the aim (with an eye to Proposition 3) is to obtain formulae to determine when $Q_\chi Q_M (Q_{\chi\cdot\det_M})^{-1}$ and $Q_{\chi\cdot\det_M} Q_{M^*} (Q_\chi)^{-1}$ are prime to each other.

Lemma 3 (Polynomial Identities) *We define*

$$\mathscr{G}_0 = \{H \in \mathscr{G},\quad n_H(M) = 0\},\quad \mathscr{G}_+ = \{H \in \mathscr{G},\quad n_H(M) \geq e_H - n_H(\chi)\},$$
$$\mathscr{G}_{\neq 0} = \mathscr{G} \backslash \mathscr{G}_0 \quad \text{ and } \quad \mathscr{G}_- = \mathscr{G} \backslash \mathscr{G}_+ .$$

We have

(i)

$$Q_{M^*} = \prod_{H\in\mathscr{B}} \alpha_H{}^{n_H(M^*)} \prod_{H\in\mathscr{G}_{\neq 0}} \alpha_H{}^{e_H(r-n_{0,H}(M))-n_H(M)} .$$

(ii)

$$Q_{\chi\cdot\det_M} = \prod_{H\in\mathscr{B}} \alpha_H{}^{n_H(\chi\cdot\det_M)} \prod_{H\in\mathscr{G}_-} \alpha_H{}^{n_H(\chi)+n_H(M)} \prod_{H\in\mathscr{G}_+} \alpha_H{}^{n_H(\chi)+n_H(M)-e_H} .$$

(iii)

$$Q_\chi Q_M (Q_{\chi\cdot\det_M})^{-1} = \prod_{H\in\mathscr{B}} \alpha_H{}^{n_H(\chi)+n_H(M)-n_H(\chi\cdot\det_M)} \prod_{H\in\mathscr{G}_+} \alpha_H{}^{e_H} .$$

(iv) $Q_{\chi\cdot\det_M} Q_{M^*} (Q_\chi)^{-1} =$

$$\prod_{H\in\mathscr{B}} \alpha_H{}^{n_H(M^*)+n_H(\chi\cdot\det_M)-n_H(\chi)} \prod_{H\in\mathscr{G}_-\backslash\mathscr{G}_0} \alpha_H{}^{e_H(r-n_{0,H}(M))} \prod_{H\in\mathscr{G}_+} \alpha_H{}^{e_H(r-1-n_{0,H}(M))} .$$

Proof We have seen in Remark 1 that $n_H(M^*) = e_H(r - n_{0,H}(M)) - n_H(M)$ for every $H \in \mathscr{H}$. Moreover, $n_H(M) = 0$ if and only if $n_H(M^*) = 0$ if and only if $n_{0,H}(M) = r$. We then obtain (i).

Let $H \in \mathscr{G}$. We have $0 \leq n_H(M) \leq e_H - 1$ and then $n_H(M) = n_H(\det_M)$. We conclude that $(\chi \cdot \det_M)(s_H) = \det(s_H)^{-n_H(\chi)-n_H(M)}$. Since $0 \leq n_H(\chi) + n_H(M) \leq 2e_H - 2$, we obtain $n_H(\chi \cdot \det{}_M) = n_H(\chi) + n_H(M)$ if $n_H(\chi) + n_H(M) \leq e_H - 1$ and

$$n_H(\chi \cdot \det{}_M) = n_H(\chi) + n_H(M) - e_H \quad \text{if} \quad n_H(\chi) + n_H(M) \geq e_H .$$

Identities (iii) and (iv) are easy consequences of (i) and (ii).

So that we can conclude on the algebra structure of Ω^χ, we need to reinforce the hypotheses.

Hypothesis 3 The subset $\mathscr{B}$ contains all the hyperplanes that are not of a low multiplicity with respect to M and χ. Equivalently, every element of $\mathscr{G}$ is of a low multiplicity with respect to M and χ, which can also be written with the notation of Lemma 3, $\mathscr{G}_+ = \emptyset$.

Hypothesis 4 The subset $\mathscr{B}$ contains all hyperplanes that are not reflection-preserving with respect to M. Equivalently, s_H acts on M as identity or as a reflection for all $H \in \mathscr{G}$.

Remark 3 (Links between Hypotheses) Remark 1 ensures that both Hypotheses 3 and 4 are stronger than Hypothesis 2. In addition, under Hypothesis 3, Lemma 3 shows that $Q_\chi Q_M (Q_{\chi\cdot\det_M})^{-1}$ are invertible in $(T^G)^{-1}R$.

Proposition 3 (Checking of the Necessary and Sufficient Condition) *Let us assume that every element of $\mathscr{G}$ is of a low multiplicity with respect to M and χ, or that s_H acts on M as identity or as a reflection for all $H \in \mathscr{G}$ (i.e. assume Hypothesis 3 or Hypothesis 4). If $\omega_1, \ldots, \omega_r$ generate $(\Omega^1)^\chi$ then*

$$\omega_1 \curlywedge \cdots \curlywedge \omega_r \doteq Q_{\chi\cdot\det_M} vol_M.$$

Proof From Remark 3, Hypothesis 2 is verified. Thus we can define the algebra structure on $(T^{-1}S(V^*) \otimes \Lambda(M^*))^\chi$. Since Ω^χ is stable by $\curlywedge$, we have $\omega_1 \curlywedge \cdots \curlywedge \omega_r \in (\Omega^r)^\chi$. The identity (1) tells us

$$\exists f \in (T^G)^{-1}R, \quad \omega_1 \curlywedge \cdots \curlywedge \omega_r = f Q_{\chi\cdot\det_M} \mathrm{vol}_M.$$

Now, we have to prove that f is invertible in $(T^{-1}S(V^*))^G$. But, actually it suffices to show that f is invertible in $T^{-1}S(V^*)$. Let us consider $(y_i)_{1\le i\le r}$ a basis of M^*. We denote by $C \in \mathrm{M}_r(T^{-1}S(V^*))$ the matrix of the family $(\omega_i)_{1\le i\le r}$ in the $T^{-1}S(V^*)$-basis $(1\otimes y_i)_{1\le i\le r}$ of Ω^1. We deduce the existence of $\lambda \in k^\times$ so that

$$\omega_1 \wedge \cdots \wedge \omega_r = \lambda \det C \,\mathrm{vol}_M \quad \text{then} \quad \lambda \det C = f Q_{\chi\cdot\det_M} (Q_\chi)^{r-1}.$$

In addition, since ν_i is G-invariant, $Q_\chi \nu_i$ is χ-invariant. We then deduce that $Q_\chi \nu_i$ is a linear combination (with coefficients in $(T^G)^{-1}R$) of the family $(\omega_i)_{1\le i\le r}$. Thus we obtain a matrix $D \in \mathrm{M}_r((T^G)^{-1}R)$ so that

$$\begin{aligned} Q_\chi \nu_1 \wedge \cdots \wedge Q_\chi \nu_r &= \det D\, \omega_1 \wedge \cdots \wedge \omega_r = \lambda \det D \det C \,\mathrm{vol}_M \\ &= \lambda \det D\, f Q_{\chi\cdot\det_M} (Q_\chi)^{r-1} \mathrm{vol}_M \end{aligned}$$

But $\nu_1 \wedge \cdots \wedge \nu_r \in k^\times Q_M \mathrm{vol}_M$, so we obtain $Q_\chi \nu_1 \wedge \cdots \wedge Q_\chi \nu_r \in k^\times (Q_\chi)^r Q_M \mathrm{vol}_M$. Hence

$$Q_\chi Q_M \in k^\times f Q_{\chi\cdot\det_M} \det D.$$

Finally $f \neq 0$ and $f \mid Q_\chi Q_M (Q_{\chi\cdot\det_M})^{-1}$.

Under Hypothesis 3, Remark 3 shows that f is invertible and (2) is verified.

Now, let us assume Hypothesis 4. For $H \in \mathscr{G}_+$, we have $n_{0,H}(M) = r - 1$. Lemma 3 shows that $\left(Q_{\chi\cdot\det_M} Q_{M^*} (Q_\chi)^{-1}\right)^{r-1}$ and $Q_\chi Q_M (Q_{\chi\cdot\det_M})^{-1}$ are prime to each other. So that, we can conclude by showing that $f \mid \left(Q_{\chi\cdot\det_M} Q_{M^*} (Q_\chi)^{-1}\right)^{r-1}$.

Let $(\mu_i)_{1\le i\le r}$ be the family of Lemma 2. Since μ_i is $\det_{M^*}$-invariant, $Q_{\chi\cdot\det_M}\mu_i \in (\Omega^1)^\chi$. Then $Q_{\chi\cdot\det_M}\mu_i$ is a linear combination (with coefficients in $(T^G)^{-1}R$) of $(\omega_i)_{1\le i\le r}$. By this way, we have constructed a matrix $D' \in \mathrm{M}_r((T^G)^{-1}R)$ so that

$$Q_{\chi\cdot\det_M}\mu_1 \wedge \cdots \wedge Q_{\chi\cdot\det_M}\mu_r = \det D'\, \omega_1 \wedge \cdots \wedge \omega_r = \det D' f Q_{\chi\cdot\det_M} (Q_\chi)^{r-1} \mathrm{vol}_M.$$

But $\mu_1 \wedge \cdots \wedge \mu_r \in k^\times (Q_{M^*})^{r-1} \mathrm{vol}_M$, so we obtain

$$Q_{\chi\cdot\det_M}\mu_1 \wedge \cdots \wedge Q_{\chi\cdot\det_M}\mu_r \in k^\times (Q_{\chi\cdot\det_M})^r (Q_{M^*})^{r-1} \mathrm{vol}_M.$$

Hence $(Q_{\chi\cdot\det_M} Q_{M^*})^{r-1} \in k^\times \det D' f(Q_\chi)^{r-1}$. Finally $f \mid \left(Q_{\chi\cdot\det_M} Q_{M^*}(Q_\chi)^{-1}\right)^{r-1}$. Thus, f divides $Q_\chi Q_{\det_M}(Q_{\chi\cdot\det_M})^{-1}$ and $\left(Q_{\chi\cdot\det_M} Q_{M^*}(Q_\chi)^{-1}\right)^{r-1}$ and identity (2) is verified under Hypothesis 4.

Theorem 1 (Exterior Algebra) *Let us assume that every element of $\mathscr{G}$ is of a low multiplicity with respect to M and χ, or that s_H acts on M as identity or as a reflection for all $H \in \mathscr{G}$ (i.e. assume Hypothesis 3 or Hypothesis 4). The $(T^G)^{-1}R$-algebra $(\Omega^\chi, \curlywedge)$ is an exterior algebra.*

Proof From Propositions 1 and 3 and Remark 3, it suffices to show that $(\Omega^1)^\chi$ can be generated by r elements. Actually, we will show that $(\Omega^1)^\chi$ is a free module of rank r over $(T^G)^{-1}R$. According to Theorem B of Chevalley [4], we have

$$(S(V^*) \otimes M^*)^\chi = (S(V^*) \otimes M^* \otimes {k_\chi}^*)^G \otimes k_\chi = (S(V^*) \otimes (M \otimes k_\chi)^*)^G \otimes k_\chi.$$

Thus we obtain

$$(S(V^*) \otimes M^*)^\chi = R \otimes (S_G \otimes (M \otimes k_\chi)^*)^G \otimes k_\chi$$

and $(S(V^*) \otimes M^*)^\chi$ is a free module of rank $\dim_k(\mathrm{Hom}_G(S_G, M \otimes k_\chi)) = \dim_k(M \otimes k_\chi) = r$. Extending the scalar to $(T^G)^{-1}R$, we obtain that $(\Omega^1)^\chi$ is free of rank r.

Remark 4 (Shepler, Orlik and Solomon) If every hyperplane of $\mathscr{H}$ is of a low multiplicity with respect to M and χ, we can choose $\mathscr{B} = \emptyset$. Similarly, if s_H acts on M as a reflection or as identity for all $H \in \mathscr{H}$, we can choose $\mathscr{B} = \emptyset$ and thus $T^{-1}S(V^*) = S(V^*)$. We obtain the results of [1] back and thus those of [13,18].

Remark 5 (When $\mathscr{B} = \mathscr{H}$) When $\mathscr{B} = \mathscr{H}$, Hypotheses 3 and 4 are trivially verified. Thus Ω^χ is an $(T^G)^{-1}R$-exterior algebra.

Example 3 Let us consider $G = \mathfrak{S}_4$ the symmetric group on four letters. We choose M to be the tensor product of the standard reflection representation by the sign representation and $\chi = 1$ the trivial character. We have $n_H(M) = 2$. So we have to invert all the hyperplanes to obtain an exterior algebra structure on Ω^χ. See the Subsect. 4.1 for more details on the case of the symmetric group.

Example 4 Let us consider $G = G(d, 1, 2)$ the group of monomial matrices with size 2 and coefficients in the dth roots of unity. We set D to be the subgroup of G of index 2 of diagonal matrices. We then choose the linear character of D defined by $\rho : \mathrm{diag}\,(\zeta, \mu) \mapsto \zeta^{-1}\mu^{-2}$ and $M = \mathrm{Ind}_D^G \rho$. For a hyperplane H associated to a diagonal reflection, we have $n_H(M) = 3$. For a hyperplane H associated to a non diagonal reflection, we have $n_H(M) = 1$.

Let us now choose χ to be the linear character of G which associates to an element g of G the inverse of the product of non-zero coefficients of g. For a hyperplane H associated to a diagonal reflection, we have $n_H(\chi) = 1$. For a hyperplane H associated to a non diagonal reflection, we have $n_H(\chi) = 0$. When $d \geq 5$, every hyperplane is of a low multiplicity with respect to M and χ and so $(S(V^*) \otimes \Lambda(M))^\chi$ is an exterior algebra but M is not a reflection representation.

Let us now choose $\chi = \det^{-1}$. For every hyperplane H, we have $n_H(\chi) = 1$. If $d \geq 5$, we have to invert only the non diagonal hyperplanes : $(T^{-1}S(V^*) \otimes \Lambda(M))^\chi$ is an exterior algebra where T is the multiplicative set associated to non diagonal hyperplanes.

See the Subsect. 4.2 for more details on the case of the rank 2 imprimitive groups.

3 Consequences of the exterior algebra structure

In this section, we take an interest in the numerical consequences of the structure of Ω^χ when $\mathscr{B}$ is empty. The first step towards numerical results is given by the Hilbert series. It allows us to transform the structure theorem into a numerical way and thus obtain a first identity between rational functions (Corollary 3) generalizing the one of Orlik and Solomon [13, equality 3.7]. The Subsect. 3.3 derives the equality of Corollary 3 to obtain polynomial identities generalizing those of [3,9,10,13] and to obtain numerical results similar to those of Shepler [19, Corollary 13]. These identities leads to characterizations of the regularity of integers.

Hypothesis 5 In this section, we assume that $\mathscr{B} = \emptyset$. Equivalently, we suppose that every hyperplane in $\mathscr{H}$ is of a low multiplicity with respect to M and χ or that s_H acts on M trivially or as a reflection for all $H \in \mathscr{H}$.

Thus $(S(V^*) \otimes \Lambda(M^*))^\chi$ is an $S(V^*)^G$-exterior algebra.

3.1 Introduction and notations

In this subsection, we introduce the objects studied next, in particular we set γ an element of the normalizer of G in $\mathrm{GL}(V)$. In addition, since the product of Ω^χ is a deformation of the usual product, we define a new degree which considers the deformation by Q_χ so that we obtain a bigraduation compatible with the algebra structure.

Notation 6 (Bigraduation) Let us consider $S_n \subset S(V^*)$ the vector space of homogeneous polynomial functions with degree n. For $p \in [\![0\,,\, r]\!]$, we set $\Omega^p = S(V^*) \otimes \Lambda^p(M^*)$ and $\Omega^p_n = S_n \otimes \Lambda^p(M^*)$. Thus, we have

$$\Omega^\chi = \bigoplus_{p=0}^{r} (\Omega^p)^\chi \quad \text{and} \quad \Omega^\chi = \bigoplus_{n\in\mathbb{N}} \bigoplus_{p=0}^{r} (\Omega^p_n)^\chi .$$

For $\omega \in (\Omega^p_n)^\chi$, we set $\deg(\omega) = (n, p)$ and $\deg'(\omega) = (n - \deg Q_\chi, p)$. If $\mu \in (\Omega^{p'}_{n'})^\chi$ then

$$\omega \curlywedge \mu \in (\Omega^{p+p'}_{n+n'-\deg Q_\chi})^\chi$$

and $\deg'(\omega) + \deg'(\mu) = \deg'(\omega \curlywedge \mu)$.

Definition 5 (Fake degree, Exponents and Degrees) Let us remind the reader what the exponents of a representation of a reflection group are. We denote by S_G the coinvariant ring of G, that is to say the quotient ring of $S(V^*)$ by the ideal generated by the polynomial invariant functions which vanish at the origin. This is a graded G-module, which affords the regular representation (see Chevalley's theorem [4]). We denote by $(S_G)_i$ the graded component of S_G of degree i and then define the fake degree of M to be the polynomial

$$F_M(T) = \sum_{i\in\mathbb{N}} \langle (S_G)_i, M\rangle_G T^i \in \mathbb{Z}[T].$$

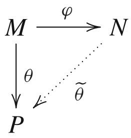
Since $\langle (S_G)_i, M\rangle_G$ are non negative integers, we can write $F_M(T) = T^{m_1(M)} + \cdots + T^{m_r(M)}$. The integers $m_1(M), \ldots, m_r(M)$ are called the M-exponents.

According to the theorem of Shephard and Todd [17], the ring $S(V^*)^G$ is generated by a family $(f_1, \ldots, f_\ell)$ of ℓ homogeneous algebraically free polynomials. We denote

$d_i = \deg f_i$. The multiset $(d_1, \ldots, d_r)$ is well determined and called the set of invariant degrees of G.

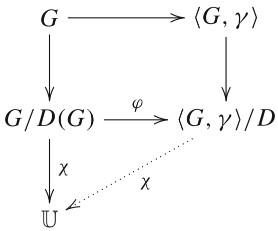
The following lemma will be useful to extend the character χ of the group $\langle G, \gamma \rangle$.

Lemma 4 (Extension) *Let M, N and P be three abelian groups and $\varphi\colon M \to N$, $\theta\colon M \to P$ be two morphisms of groups. We assume that P is a divisible group. If $\ker\varphi \subset \ker\theta$, there exists a morphism of groups $\widetilde{\theta} : N \to P$ so that the following diagram commutes*



Proof Since $\ker\varphi \subset \ker\theta$, we can define a group homomorphism $\theta_1 : M/\ker\varphi \simeq \mathrm{Im}\varphi \to P$ so that $\theta_1 \circ \varphi = \theta$. Since P is divisible, we can extend θ_1 in $\widetilde{\theta} : N \to P$. Finally we obtain $\widetilde{\theta} \circ \varphi = \theta_1 \circ \varphi = \theta$.

Let us introduce some notations and consider the normalizer $\mathscr{N}$ of G in $\mathrm{GL}(V)$. We choose a semisimple element $\gamma \in \mathscr{N}$ (see [3]). We assume that M is a $\langle G, \gamma \rangle$-module and that γ acts semisimply on M. Furthermore we assume that the derived group D of $\langle G, \gamma \rangle$ verifies $D \subset \ker\chi$. Applying Lemma 4 with $M = G/D(G)$, $N = \langle G, \gamma \rangle/D$, $P = \mathbb{U}$ the groups of complex numbers with module 1 and $\theta = \chi$, we extend χ in a linear character of $\langle G, \gamma \rangle$ (also denoted by χ).



We denote by k_χ the representation (of $\langle G, \gamma \rangle$) with character χ over k and we define $M_\chi = M \otimes k_\chi$. So M_χ is an $\langle G, \gamma \rangle$-module and, thanks to Theorem B of Chevalley [4], we obtain an isomorphism of graded G-modules and of R-modules

$$(\Omega^1)^\chi = (S(V^*) \otimes M^*)^\chi = (S(V^*) \otimes M^* \otimes {k_\chi}^*)^G \otimes k_\chi = R \otimes (S_G \otimes {M_\chi}^*)^G \otimes k_\chi.$$

Thus according to the definition of the M_χ-exponents, we can choose an R-basis $(\omega_1, \ldots, \omega_r)$ of $(\Omega^1)^\chi$ bihomogeneous with degree $\deg'(\omega_i) = (m_i(M_\chi) - \deg(Q_\chi), 1)$. Moreover, the hypothesis $D \subset \ker\chi$ ensures that γ stabilizes the vector space N^χ of χ-invariants of N, for all $\langle G, \gamma \rangle$-module N. Thus we obtain the isomorphism of graded $\langle \gamma \rangle$-modules and of R-modules

$$(\Omega^1)^\chi = (S(V^*) \otimes M^*)^\chi = (R \otimes S_G \otimes M^*)^\chi = R \otimes (S_G \otimes M^*)^\chi.$$

Finally, we can assume that the ω_i are eigenvectors for γ. We denote by $\varepsilon_{i,\gamma,\chi}(M)$ the eigenvalue of γ associated to ω_i. Both isomorphisms given above show that the multiset $(\varepsilon_{i,\gamma,\chi}(M), m_i(M_\chi))_i$ does not depend on the choice of the basis of $(\Omega^1)^\chi$.

Remark 6 (m_i, ε_i) When $\chi = 1$ is the trivial character, we set $\varepsilon_{i,\gamma}(M) := \varepsilon_{i,\gamma,1}(M)$. Similarly, when $\gamma = \mathrm{id}$, we set $\varepsilon_{i,\chi}(M) := \varepsilon_{i,\mathrm{id},\chi}(M)$. The family of $\varepsilon_{i,\gamma,\chi}(M)$ depends on the choice of the extension of χ to $\langle G, \gamma\rangle$.

The family $\varepsilon_{i,\gamma}(V)$ can also be considered as the family of eigenvalues of γ so that the associated eigenvectors $(P_1, \ldots, P_\ell)$ are a family of homogeneous and algebraically free generators of R (see [3]).

3.2 Rational functions

It is now time to transform the structure theorem into numerical considerations. So let us compute the Hilbert series : we follow the ideas of Theorem 2.1 and Equality 2.3 of [9] and of Proposition 2.3 of [10]. The following corollary will be the fundamental step for the next section.

Corollary 3 (Rational Functions) *If s_H acts on M as the identity or as a reflection for all $H \in \mathscr{H}$ or if $n_H(M) < e_H - n_H(\chi)$ for all $H \in \mathscr{H}$ then*

$$\frac{1}{|G|}\sum_{g\in G}\overline{\chi(g)}\,\frac{\det(1+(g\gamma)_M Y)}{\det(1-g\gamma X)} = X^{\deg(Q_\chi)}\,\frac{\prod_{i=1}^{r}\left(1+\varepsilon_{i,\gamma,\chi}(M)X^{m_i(M_\chi)-\deg(Q_\chi)}Y\right)}{\prod_{i=1}^{\ell}(1-\varepsilon_{i,\gamma}(V)X^{d_i})}. \tag{3}$$

Proof The hypothesis $D \subset \ker\chi$ ensures that γ stabilizes N^χ the vector space of χ-invariants of N, for all $\langle G,\gamma\rangle$-module N. In particular, γ defines a bigraded endomorphism of Ω^χ. In order to show the equality, we compute the graded trace $P_{\Omega^\chi,\gamma}(X,Y)$ of the endomorphism γ of Ω^χ in two different ways. According to definition,

$$P_{\Omega^\chi,\gamma}(X,Y) = \sum_{n\in\mathbb{N}}\sum_{p=0}^{r}\mathrm{tr}\left(\gamma_{\Omega_n^{p\,\chi}}\right)X^nY^p.$$

Since $(\Omega_n^p)^\chi$ is the χ-isotypic component of Ω_n^p,

$$\frac{1}{|G|}\sum_{g\in G}\overline{\chi(g)}\,g_{\Omega_n^p}$$

is a projector on $(\Omega_n^p)^\chi$. Hence

$$\mathrm{tr}\left(\gamma_{\Omega_n^{p\,\chi}}\right) = \frac{1}{|G|}\sum_{g\in G}\overline{\chi(g)}\,\mathrm{tr}\left((g\gamma)_{\Omega_n^p}\right).$$

Thus

$$P_{\Omega^\chi,\gamma}(X,Y) = \frac{1}{|G|}\sum_{g\in G}\overline{\chi(g)}\sum_{n\in\mathbb{N}}\sum_{p=0}^{r}\mathrm{tr}\left((g\gamma)_{\Omega_n^p}\right)X^nY^p.$$

Finally, Molien's formulae give us

$$P_{\Omega^\chi,\gamma}(X,Y) = \frac{1}{|G|}\sum_{g\in G}\overline{\chi(g)}\,\frac{\det(1+(g\gamma)_M Y)}{\det(1-g\gamma X)}. \tag{4}$$

In addition, Propositions 1 and 3 show that $\Omega^\chi = R\otimes \curlywedge((\Omega^1)^\chi)$ where $\curlywedge((\Omega^1)^\chi)$ is the k-algebra (for $\curlywedge$) generated by $(\Omega^1)^\chi$. Since the product $\curlywedge$ is compatible with $\deg'$ and since the degree of the unit element $e = Q_\chi$ for $\curlywedge$ is $\deg'(e) = (0, 0)$, we obtain

$$P_{\curlywedge((\Omega^1)^\chi),\gamma}(X, Y) = X^{\deg(Q_\chi)} \prod_{i=1}^{r} \left(1 + \varepsilon_{i,\gamma,\chi}(M) Y X^{m_i(M_\chi) - \deg(Q_\chi)}\right).$$

Moreover,

$$P_{R,\gamma}(X) = \prod_{i=1}^{\ell} \left(1 - \varepsilon_{i,\gamma}(V) X^{d_i}\right)^{-1},$$

Hence

$$P_{\Omega^\chi,\gamma}(X, Y) = X^{\deg(Q_\chi)} \frac{\prod_{i=1}^{r} \left(1 + \varepsilon_{i,\gamma,\chi}(M) Y X^{m_i(M_\chi) - \deg(Q_\chi)}\right)}{\prod_{i=1}^{\ell} (1 - \varepsilon_{i,\gamma}(V) X^{d_i})}. \tag{5}$$

The equalities 4 and 5 give the result.

Remark 7 ($m_i(M_\chi)$) If $n_H(M) < e_H - n_H(\chi)$ for all $H \in \mathscr{H}$, then the multisets

$$\left\{m_1(M) + \deg(Q_\chi), \dots, m_r(M) + \deg(Q_\chi)\right\} \quad \text{and} \quad \left\{m_1(M_\chi), \dots, m_r(M_\chi)\right\}$$

are the same. Indeed, by following the proof of Proposition 3, we notice that, under our hypothesis, the family $(Q_\chi \nu_i)_{1 \leq i \leq r}$ is a basis of $(\Omega^1)^\chi$. But the properties of minimal matrices allow us to choose ν_i bihomogeneous with degree $(m_i(M), 1)$. This result is similar to [19, Corollary 13].

3.3 Regular integers

Using the structure of exterior algebra on $(S(V^*) \otimes \Lambda(V))^\chi$ and $(S(V^*) \otimes \Lambda(V^*))^\chi$, Shepler obtains numerical consequences on exponents (see [19, Corollary 13]). In [9], Lehrer refines these results using a method of Pianzola and Weiss [14] coupled with results of Springer [21] (see also [10] for another application of the same method). This subsection is dedicated to the application of this method to the representations V^σ and $V^{*\sigma}$ where $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We thus obtain new relations between exponents.

So let us apply identity 3 to the representations V^σ and $V^{*\sigma}$ where $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let $d \in \mathbb{N}$ and ξ be a primitive dth root of unity; we then define

$$A_\gamma(d) = \{i \in [\![\, 1\,,\ \ell \,]\!],\quad \varepsilon_{i,\gamma}(V)\xi^{-d_i} = 1\} \quad \text{and} \quad a_\gamma(d) = |A_\gamma(d)|,$$

and for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$r_i(\sigma, \chi) = \deg(Q_\chi) - m_i(V^\sigma{}_\chi), \quad r_i^*(\sigma, \chi) = \deg(Q_\chi) - m_i(V^{*\sigma}{}_\chi),$$

$$B_{\sigma,\gamma}(d, \chi) = \{j \in [\![\, 1\,,\ \ell \,]\!],\quad \varepsilon_{j,\gamma,\chi}(V^\sigma)\xi^{-\sigma}\xi^{r_j(\sigma,\chi)} = 1\} \quad \text{and} \quad b_{\sigma,\gamma}(d, \chi) = |B_{\sigma,\gamma}(d, \chi)|;$$

and

$$B^*_{\sigma,\gamma}(d, \chi) = \{j \in [\![\, 1\,,\ \ell \,]\!],\quad \varepsilon_{j,\gamma,\chi}(V^{*\sigma})\xi^{\sigma}\xi^{r^*_j(\sigma,\chi)} = 1\} \quad \text{and} \quad b^*_{\sigma,\gamma}(d, \chi) = |B^*_{\sigma,\gamma}(d, \chi)|.$$

For $h \in \mathrm{End}_{\mathbb{C}}(V)$, we denote by $\det'(h)$ the product of non-zero eigenvalues of h, we denote also by $V(h, \xi) = \ker(h - \xi \mathrm{id})$ the eigenspace of h associated to the eigenvalue ξ and we set $d(h, \xi) = \dim(V(h, \xi))$.

Theorem 2 *We have $a_\gamma(d) \le b_{\sigma,\gamma}(d,\chi)$ and the following identity in $\mathbb{C}[T]$*

$$\xi^{\deg(Q_\chi)}\sum_{g\in G}\overline{\chi(g)}T^{d(g\gamma,\xi)}(\det{}'(1-\xi^{-1}g\gamma))^{\sigma-1}$$
$$=\begin{cases}\prod_{j\in B_{\sigma,\gamma}(d,\chi)}(T-r_j(\sigma,\chi))\prod_{j\notin B_{\sigma,\gamma}(d,\chi)}(1-\varepsilon_j\xi^{r_j(\sigma,\chi)-\sigma})\prod_{j\notin A_\gamma(d)}\frac{d_j}{1-\varepsilon'_j\xi^{-d_j}} & \text{if } a_\gamma(d)=b_{\sigma,\gamma}(d,\chi),\\ 0 & \text{otherwise}\end{cases}$$

where $\varepsilon_i=\varepsilon_{i,\gamma,\chi}(V^\sigma)$ and $\varepsilon'_i=\varepsilon_{i,\gamma}(V)$.

We have $a_\gamma(d)\le b^*_{\sigma,\gamma}(d,\chi)$ and the following identity in $\mathbb{C}[T]$

$$(-1)^\ell\xi^{\deg(Q_\chi)+\ell\sigma}\sum_{g\in G}\overline{\chi(g)}(-T)^{d(g\gamma,\xi)}(\det{}'(1-\xi^{-1}g\gamma))^{\sigma-1}\det(g\gamma)^{-\sigma}$$
$$=\begin{cases}\prod_{j\in B^*_{\sigma,\gamma}(d,\chi)}(T-r^*_j(\sigma,\chi))\prod_{j\notin B^*_{\sigma,\gamma}(d,\chi)}\left(1-\varepsilon_j\xi^{r^*_j(\sigma,\chi)+\sigma}\right)\prod_{j\notin A_\gamma(d)}\frac{d_j}{1-\varepsilon'_j\xi^{-d_j}} & \text{if } a_\gamma(d)=b^*_{\sigma,\gamma}(d,\chi),\\ 0 & \text{otherwise}\end{cases}$$

where $\varepsilon_i=\varepsilon_{i,\gamma,\chi}(V^{*\sigma})$ and $\varepsilon'_i=\varepsilon_{i,\gamma}(V)$.

Proof For every reflection $s\in G$, s_{V^σ} is still a reflection. Thus we can apply Corollary 3 to the G-module V^σ. For $g\in G$, we denote by $\lambda_1(g\gamma),\dots,\lambda_\ell(g\gamma)$ the eigenvalues of $g\gamma$ acting on V. We have

$$\frac{1}{|G|}\sum_{g\in G}\overline{\chi(g)}\prod_{i=1}^{\ell}\frac{(1+Y(\lambda_i(g\gamma))^\sigma)}{(1-X\lambda_i(g\gamma))}=X^{\deg(Q_\chi)}\prod_{i=1}^{\ell}\frac{(1+\varepsilon_iYX^{-r_i(\sigma,\chi)})}{(1-\varepsilon'_iX^{d_i})}.$$

We switch the indeterminate with $Y=\xi^{-\sigma}(T(1-\xi X)-1)$.

Let us begin with the left side. It becomes

$$\frac{1}{|G|}\sum_{g\in G}\overline{\chi(g)}\prod_{i=1}^{\ell}\frac{1-(\lambda_i(g\gamma)\xi^{-1})^\sigma(1-T(1-\xi X))}{1-X\lambda_i(g\gamma)}.$$

In each term of the sum, we discriminate the eigenvalues of $g\gamma$ between those equal to ξ and the others. We obtain in $\mathbb{C}(T,X)$

$$\frac{1}{|G|}\sum_{g\in G}\overline{\chi(g)}\left(\prod_{\{i\mid\lambda_i=\xi\}}T\prod_{\{i|\lambda_i\neq\xi\}}\frac{1-(\lambda_i(g\gamma)\xi^{-1})^\sigma(1-T(1-\xi X))}{1-X\lambda_i(g\gamma)}\right).$$

So ξ^{-1} is not a pole of this rational function with respect to X and evaluating at $X=\xi^{-1}$, we obtain

$$\frac{1}{|G|}\sum_{g\in G}\overline{\chi(g)}\,T^{d(g\gamma,\xi)}\left(\prod_{\{i|\lambda_i\neq\xi\}}\frac{1-(\xi^{-1}\lambda_i(g\gamma))^\sigma}{1-\xi^{-1}\lambda_i(g\gamma)}\right)=\frac{1}{|G|}\sum_{g\in G}\overline{\chi(g)}\,T^{d(g\gamma,\xi)}\,(\det{}'(1-\xi^{-1}g\gamma))^{\sigma-1}.$$

Now, let us consider the right side. After switching the indeterminate, it becomes

$$X^{\deg(Q_\chi)}\prod_{i=1}^{\ell}\frac{1-\varepsilon_i\xi^{-\sigma}(1-T(1-\xi X))X^{-r_i(\sigma,\chi)}}{1-\varepsilon'_iX^{d_i}}.$$

Let us count the multiplicity of ξ^{-1} as a root of the numerator and of the denominator of this rational function with respect to X. For the denominator, ξ^{-1} is a root of $1-\varepsilon'_iX^{d_i}$ if and only if $i \in A_\gamma(d)$. Moreover this root is simple. So ξ^{-1} is a root of order $a_\gamma(d)$ of the denominator. For the numerator,

$$1 - \varepsilon_i \xi^{-\sigma}(1 - T(1 - \xi X))X^{-r_i(\sigma,\chi)}$$

is zero for $X = \xi^{-1}$ if and only if $i \in B_\sigma(d, \chi)$. Moreover, when differentiating with respect to X, we obtain

$$-\varepsilon_i \xi^{-\sigma}\left(-r_i(\sigma, \chi)(1 - T(1 - \xi X))X^{-r_i(\sigma,\chi)-1} + T\xi X^{-r_i(\sigma,\chi)}\right)$$

which is nonzero at $X = \xi^{-1}$. So ξ^{-1} is a root of order $b_{\sigma,\gamma}(d, \chi)$ of the numerator.

Since ξ^{-1} is not a pole of the left side, we obtain $a_\gamma(d) \leq b_{\sigma,\gamma}(d, \chi)$. Moreover, we deduce that the right side is zero if $a_\gamma(d) < b_{\sigma,\gamma}(d, \chi)$.

Now, let us assume that $a_\gamma(d) = b_{\sigma,\gamma}(d, \chi)$. If $i \in B_{\sigma,\gamma}(d, \chi)$ then $\varepsilon_i \xi^{-\sigma} = \xi^{-r_i(\sigma,\chi)}$ and

$$\begin{aligned} 1 - \varepsilon_i \xi^{-\sigma}(1 - T(1 - \xi X))X^{-r_i(\sigma,\chi)} &= 1 - (1 - T(1 - \xi X))(\xi X)^{-r_i(\sigma,\chi)} \\ &= 1 - (\xi X)^{-r_i(\sigma,\chi)} + T(1 - \xi X)(\xi X)^{-r_i(\sigma,\chi)} \\ &= (1 - \xi X)\left(T(\xi X)^{-r_i(\sigma,\chi)} + \sum_{k=0}^{-r_i(\sigma,\chi)-1}(\xi X)^k\right). \end{aligned}$$

For $j \in A_\gamma(d)$, we have $\varepsilon'_j = \xi^{d_j}$ and so

$$1 - \varepsilon'_j X^{d_j} = 1 - \xi^{d_j} X^{d_j} = (1 - \xi X)\sum_{k=0}^{d_j-1}(\xi X)^k.$$

As a consequence, for $j \in A_\gamma(d)$ and $i \in B_{\sigma,\gamma}(d, \chi)$, we obtain

$$\frac{1 - \varepsilon_i \xi^{-\sigma}(1 - T(1 - \xi X))X^{-r_i(\sigma,\chi)}}{1 - X^{d_j}} = \frac{T(\xi X)^{-r_i(\sigma,\chi)} + \sum_{k=0}^{-r_i(\sigma,\chi)-1}(\xi X)^k}{\sum_{k=0}^{d_j-1}(\xi X)^k}.$$

Evaluating at $X = \xi^{-1}$, we obtain $\frac{T - r_i(\sigma,\chi)}{d_j}$. Finally, by choosing for each factor of the numerator whose index is in $B_{\sigma,\gamma}(d, \chi)$, one of the factor of the denominator whose index is in $A_\gamma(d)$ (this is possible since $a_\gamma(d) = b_{\sigma,\gamma}(d, \chi)$), we obtain, after evaluating at $X = \xi^{-1}$,

$$\xi^{-\deg(Q_\chi)} \frac{\prod_{j \in B_{\sigma,\gamma}(d,\chi)}(T - r_j(\sigma, \chi)) \prod_{j \notin B_{\sigma,\gamma}(d,\chi)}\left(1 - \varepsilon_j \xi^{r_j(\sigma,\chi)-\sigma}\right)}{\prod_{j \in A_\gamma(d)} d_j \prod_{j \notin A_\gamma(d)}(1 - \varepsilon'_j \xi^{-d_j})}.$$

The relation $|G| = \prod_{i=1}^{\ell} d_i$ give us the identity.

For the second identity, we apply Corollary 3 to $V^{*\sigma} = V^{\sigma *}$ on which s_H acts as a reflection for all $H \in \mathscr{H}$. We switch the indeterminate into $Y = \xi^\sigma(T(1 - \xi X) - 1)$ and simplify with $(1 - z^{-1})(1 - z)^{-1} = -z^{-1}$.

3.3.1 When γ is trivial

We are interested in the case where $\gamma = \mathrm{id}$. To simplify the notations, we set

$$B^*(d, \chi) := B^*_{\mathrm{id},\mathrm{id}}(d, \chi) = \left\{j \in [\![1, \ell]\!], \quad d \mid 1 + r^*_j(\mathrm{id}, \chi)\right\} \text{ and } b^*(d, \chi) = |B^*(d, \chi)|;$$
$$B(d, \chi) := B_{\mathrm{id},\mathrm{id}}(d, \chi) = \left\{j \in [\![1, \ell]\!], \quad d \mid 1 - r_j(\mathrm{id}, \chi)\right\} \text{ and } b(d, \chi) = |B(d, \chi)|;$$

and finally $A(d) := A_{\mathrm{id}}(d) = \left\{j \in [\![1, \ell]\!], \quad d \mid d_j\right\}$ and $a(d) = |A(d)|$.

Let us remind the reader that d is said to be a regular integer if one of the $V(g,\xi)$ meets the complementary of the hyperplanes of $\mathscr{H}$. The following corollary generalizes the results of [10] and the one of [9].

Corollary 4 (Consequences and Exceptional Case) *We obtain the following formulae*

(i) $\sum_{g\in G}\overline{\chi(g)}T^{d(g,1)}(\det'(1-g))^{\sigma-1}=\prod_{j=1}^{\ell}(T-r_j(\sigma,\chi))$.

(ii)

$$\xi^{\deg(Q_\chi)}\sum_{g\in G}\overline{\chi(g)}T^{d(g,\xi)} = \begin{cases} \prod\limits_{j\in B(d,\chi)}(T-r_j(\mathrm{id},\chi))\prod\limits_{j\notin B(d,\chi)}(1-\xi^{r_j(\mathrm{id},\chi)-1})\prod\limits_{j\notin A(d)}\frac{d_j}{1-\xi^{-d_j}}, & \textit{if } a(d)=b(d,\chi),\\ 0 & \textit{otherwise}.\end{cases}$$

(iii) $\sum_{g\in G}\chi(g)T^{d(g,1)}=\prod_{j=1}^{\ell}(T-r_j(\mathrm{id},\chi))$.

(iv) *We have* $a(d)\le b^*(d,\chi)$ *and*

$$(-1)^{\ell}\xi^{\ell+\deg(Q_\chi)}\sum_{g\in G}(-T)^{d(g,\xi)}(\chi\cdot\det)(g^{-1}) = \begin{cases} \prod\limits_{j\in B^*(d,\chi)}\left(T-r_j^*(\mathrm{id},\chi)\right)\prod\limits_{j\notin B^*(d,\chi)}\left(1-\xi^{r_j^*(\mathrm{id},\chi)+1}\right)\prod\limits_{j\notin A(d)}\frac{d_j}{1-\xi^{-d_j}}, & \textit{if } a(d)=b^*(d,\chi),\\ 0 & \textit{otherwise}.\end{cases}$$

(v) $\sum_{g\in G}T^{d(g,1)}(\chi\cdot\det)(g)=\prod_{j=1}^{\ell}(T+r_j^*(\mathrm{id},\chi))$.

(vi) *The multisets* $\{-r_1^*(\mathrm{id},\chi),\ldots,-r_\ell^*(\mathrm{id},\chi)\}$ *and* $\{r_1(\mathrm{id},\chi\cdot\det),\ldots,r_\ell(\mathrm{id},\chi\cdot\det)\}$ *are the same and* $b^*(d,\chi)=b(d,\chi\cdot\det)$.

(vii) *If d is regular, then* $a(d)=b_\sigma(d,\chi)$ *for every* $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *and every one dimensional character* χ.

(viii) *If for all* $H\in\mathscr{H}$*, the restriction of* $\chi\cdot\det$ *to* G_H *is non trivial, then d is a regular integer if and only if* $a(d)=b(d,\chi)$.

Proof (i) This is Theorem 2 with $d=1$ and so $\xi=1$. We have

$$A(1)=B_{\sigma,\mathrm{id}}(1,\chi)=[\![\,1\,,\,\ell\,]\!] \quad \text{and} \quad a(1)=b_{\sigma,\mathrm{id}}(1,\chi).$$

(ii) This is Theorem 2 with $\sigma=\mathrm{id}$. This Lehrer's identity 2.1 [9].

(iii) Since $\overline{\chi(g)}=\chi(g^{-1})$ and $d(g,1)=d(g^{-1},1)$, this is (i) for $\sigma=\mathrm{id}$ or (ii) with $d=1$. This is Lehrer'identity 3.2 [9].

(iv) This is the second identity of Theorem 2 for $\sigma=\mathrm{id}$.

(v) This is (iv) with $d=1=\xi$ with the remark that $d(g,1)=d(g^{-1},1)$.

(vi) This one is obtained by comparing (iii) applied to $\chi\cdot\det$ with (v).

(vii) Theorem 3.4 of Springer [21] shows us that the degree of the polynomial of the left side in Theorem 2 is at most $a(d)$. Let us compute the coefficient of $T^{a(d)}$. Since d is regular, the $g\in G$ verifying $d(g,\xi)=a(d)$ are a single conjugacy class. Thus, for every g so that $d(g,\xi)=a(d)$, the value of $\overline{\chi(g)}\det'(1-\xi^{-1}g)^{\sigma-1}$ does not depend on g. The coefficient of $T^{a(d)}$ is non zero and so $a(d)=b_\sigma(d,\chi)$.

(viii) This is (vi) coupled with Corollary 3.9 of Lehrer [9] applied to the linear character $\chi\cdot\det$.

3.3.2 When γ is not necessarily trivial

The case $\chi = 1$ is done in [3]. Let us remind the reader that d is γ-regular if one of the eigenspaces $V(g\gamma, \xi)$ meets the complementary of the hyperplanes $\mathscr{H}$. As a matter of simplification, we set $B^*_\gamma(d, \chi) := B^*_{\mathrm{id},\gamma}(d, \chi)$, $b^*_\gamma(d, \chi) := |B^*_\gamma(d, \chi)|$ and

$$B_\gamma(d, \chi) := B_{\mathrm{id},\gamma}(d, \chi) \quad \text{and} \quad b_\gamma(d, \chi) := |B_\gamma(d, \chi)|.$$

Corollary 5 (Consequences and Exceptional Cases) *We obtain the following formulae*

(i) $\xi^{\deg(Q_\chi)} \sum_{g\in G} \overline{\chi(g)} T^{d(g\gamma,\xi)}$

$$= \begin{cases} \prod_{j\in B_\gamma(d,\chi)} (T - r_j(\mathrm{id}, \chi)) \prod_{j\notin B_\gamma(d,\chi)} \left(1 - \varepsilon_j \xi^{r_j(\mathrm{id},\chi)-1}\right) \prod_{j\notin A_\gamma(d)} \frac{d_j}{1-\varepsilon'_j\xi^{-d_j}}, & \text{if } a_\gamma(d) = b_\gamma(d, \chi), \\ 0 & \text{otherwise}. \end{cases}$$

when $\varepsilon_i = \varepsilon_{i,\gamma,\chi}(V)$ *and* $\varepsilon'_i = \varepsilon_{i,\gamma}(V)$.

(ii) $(-1)^\ell \xi^{\deg(Q_\chi)+\ell} \det(\gamma^{-1}) \sum_{g\in G} (\chi \cdot \det)(g^{-1})(-T)^{d(g\gamma,\xi)}$

$$= \begin{cases} \prod_{j\in B^*_\gamma(d,\chi)} (T - r^*_j(\mathrm{id}, \chi)) \prod_{j\notin B^*_\gamma(d,\chi)} \left(1 - \varepsilon_j \xi^{r^*_j(\mathrm{id},\chi)+1}\right) \prod_{j\notin A_\gamma(d)} \frac{d_j}{1-\varepsilon'_j\xi^{-d_j}} & \text{if } a_\gamma(d) = b^*_\gamma(d, \chi), \\ 0 & \text{otherwise} \end{cases}$$

when $\varepsilon_i = \varepsilon_{i,\gamma,\chi}(V^*)$ *and* $\varepsilon'_i = \varepsilon_{i,\gamma}(V)$.

(iii) *The two multisets* $\{-r^*_i(\mathrm{id}, \chi), \quad i \in B^*_\gamma(d, \chi)\}$ *and* $\left\{r_i(\mathrm{id}, \chi \cdot \det), \quad i \in B_\gamma(d, \chi)\right\}$ *are the same and* $b^*_\gamma(d, \chi) = b_\gamma(d, \chi \cdot \det)$.

(iv) *If d is γ-regular, then* $a_\gamma(d) = b_{\sigma,\gamma}(d, \chi) = b^*_{\sigma,\gamma}(d, \chi)$ *for every linear character χ and every* $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

(v) *If for all $H \in \mathscr{H}$, the restriction of χ to G_H is non-trivial, then d is γ-regular if and only if* $a_\gamma(d) = b_\gamma(d, \chi)$.

(vi) *If for all $H \in \mathscr{H}$, the restriction of* $\chi \cdot \det$ *to G_H is non-trivial, then d is γ-regular if and only if* $a_\gamma(d) = b^*_\gamma(d, \chi)$.

Proof (i) This is Theorem 2 with $\sigma = \mathrm{id}$.

(ii) This is the second identity of Theorem 2 with $\sigma = \mathrm{id}$.

(iii) Let us compare the roots of (i) applied to $\chi \cdot \det$ and those of (ii).

(iv) The Theorem 3.4 of Springer [21] show us that the degree of polynomial of the left side in Theorem 2 is at most $a_\gamma(d)$. Let us compute the coefficient of $T^{a_\gamma(d)}$. Since d is regular, the $g \in G$ so that $d(g\gamma, \xi) = a_\gamma(d)$ are a single conjugacy class. Thus the value $\overline{\chi(g)} \det'(1 - \xi^{-1}g)^{\sigma-1}$ does not depend on g when g verifies $d(g\gamma, \xi) = a_\gamma(d)$. The coefficient of $T^{a_\gamma(d)}$ is non-zero and so $a_\gamma(d) = b_{\sigma,\gamma}(d, \chi)$.

(v) According to (iv), it suffices to show that if $a_\gamma(d) = b_\gamma(d, \chi)$ then d is γ-regular. According to (i), the coefficient of $T^{a_\gamma(d)} = T^{b_\gamma(d,\chi)}$ is non-zero. Thus, thanks to Springer's theorem, $\sum_{g\in C} \chi(g)$ is a factor of the coefficient of $T^{a_\gamma(d)}$ where

$$C = \{g \in G, \quad \forall x \in V(h\gamma, \xi), \quad gx = x\} \quad \text{with} \quad d(h\gamma, \xi) = a_\gamma(d).$$

If C is not the trivial group, then C contains one of the G_H (this is Steinberg's theorem) and since de restriction of χ to G_H is non-trivial, we have $\sum_{g\in C} \chi(g) = 0$. Finally we obtain a contradiction and $C = 1$ which means exactly that d is γ-regular.

(vi) This is (iii) and (v).

4 Types of hyperplanes

In Definition 4, we define various types of hyperplanes. In this section, we study these types of hyperplanes for some examples of reflection groups, namely the symmetric group, the wreath product $G(d, 1, n)$, the imprimitive groups of rank 2 that is $G(de, e, 2)$ and some exceptional case G_4, G_5 and G_{24} (named after the classification of Shephard and Todd [17]). The details of the computations can be found in [2].

4.1 The symmetric group

The symmetric group $\mathfrak{S}_n$ acts faithfully as a reflection group over $\mathbb{C}^n/\langle(1, \dots, 1)\rangle$ by permuting the coordinates. The reflections are the transpositions. They are of order 2 and conjugate to each other. Hence there is a unique conjugacy class of hyperplane. The linear characters of $\mathfrak{S}_n$ are the trivial one (denoted by 1) and the sign character (denoted by ε). In addition, the representations of $\mathfrak{S}_n$ are described by the partitions of n (see for example [11]).

Proposition 4 (Symmetric Group) *Let H be a hyperplane of the reflection group $\mathfrak{S}_n$ and ρ be an irreducible representation of $\mathfrak{S}_n$.*

The hyperplane H is reflection-preserving with respect to ρ, of a low multiplicity with respect to ρ, of a low multiplicity with respect to ρ and 1, *multiplicity-partitionable with respect to ρ and ε if and only if $\rho = 1$ or $\rho = \varepsilon$ or ρ is the reflection representation or $n = 4$ and ρ is associated to the partition* (2, 2).

The hyperplane H is of a low multiplicity with respect to ρ and ε if and only if $\rho = 1$.

The hyperplane H is multiplicity-partitionable with respect to ρ and 1 *if and only if $\rho = 1$ or $\rho = \varepsilon$ or ρ is the reflection representation or $n \leq 5$ or $n = 6$ and ρ is associated to one of the partition* (3, 3), (2, 2, 2), (4, 2).

4.2 The rank 2 imprimitive reflection groups

For d, e, r nonzero integers, we define the group $G(de, e, r)$ to be the group of r-dimensional monomial matrices (with one nonzero element on each row and column) whose nonzero entries are deth root of unity such that their product is a dth root of unity. These groups are called the imprimitive groups of reflection. The integer r is called the rank of $G(de, e, r)$.

The reflections of $G(de, e, 2)$ are of the form

$$\begin{bmatrix} \xi & \\ & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & \\ & \xi \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} & \zeta \\ \zeta^{-1} & \end{bmatrix}$$

where ξ is a non trivial dth root of unity and ζ a deth root of unity. If e is odd, there are two conjugacy classes of hyperplanes : one given by the hyperplanes of the diagonal reflections, the other given by the nondiagonal reflections. If e is even, there are three conjugacy classes of hyperplanes. One given by the hyperplanes of the diagonal reflections. The hyperplanes of nondiagonal reflections split into two classes : one associated to the hyperplane of

$$s = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$$

the other one associated to the hyperplane of

$$s' = \begin{bmatrix} & \zeta \\ \zeta^{-1} & \end{bmatrix}$$

where ζ is a deth primitive root of unity.

We will describe the character of $G(de, e, 2)$ following the method of small groups of Wigner and Mackey (see [16, paragraph 8.2]). So we set D the subgroup of diagonal matrix of $G(de, e, 2)$. This is an abelian normal subgroup of $G(de, e, 2)$ of index 2. To use the method of Wigner and Mackey, we have to describe the one-dimensional character of D.

Lemma 5 (Linear Character of D) *For $d, e \in \mathbb{N}^*$, the map*

$$\Delta : \begin{cases} \mathbb{Z}/de\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z} \to \widehat{D} = \mathrm{Hom}_{gr}(D, \mathbb{C}^\times) \\ (k, k') \to \big(diag(\alpha, \beta) \mapsto \alpha^{-k}(\alpha\beta)^{-k'}\big) \end{cases}$$

is a group isomorphism.

Let us now describe the irreducible representation of $G(de, e, 2)$. We distinguish cases when d and e are odd or even.

Proposition 5 (d, e odd) *Let $d, e \in \mathbb{N}^*$ be odd numbers.*

For $k' \in \mathbb{Z}/d\mathbb{Z}$, we extend $\Delta(0, k')$ to $G(de, e, 2)$ by $\Delta(0, k')(dx) = \Delta(0, k')(d)$ for every $d \in D$ and $x \in \langle 1, s\rangle$. We extend any irreducible representation ρ of $\langle 1, s\rangle$ to $G(de, e, 2)$ by $\rho(dx) = \rho(x)$ for every $d \in D$ and $x \in \langle 1, s\rangle$. We then define

$$\beta_{k',\rho}(dx) = \rho(dx)\Delta(0, k')(dx) = \rho(x)\Delta(0, k')(d)$$

for $d \in D$ and $x \in \langle 1, s\rangle$. For $(k, k') \in [\![1, de/2]\!] \times \mathbb{Z}/d\mathbb{Z}$, we set $\beta_{k,k'} = Ind_D^{G(de,e,2)} \Delta(k, k')$.

The family $((\beta_{k',1}, \beta_{k',\varepsilon})_{k' \in \mathbb{Z}/d\mathbb{Z}}, (\beta_{k,k'})_{(k,k') \in [\![1, de/2]\!] \times \mathbb{Z}/d\mathbb{Z}})$ is a complete set for the irreducible representations of $G(de, e, 2)$.

Proposition 6 (e odd, d even) *Let $d, e \in \mathbb{N}^*$ with $d = 2d'$ even and e odd.*

For $k' \in \mathbb{Z}/d\mathbb{Z}$, we extend $\Delta(0, k')$ to $G(de, e, 2)$ by $\Delta(0, k')(dx) = \Delta(0, k')(d)$ for every $d \in D$ and $x \in \langle 1, s\rangle$. We extend any irreducible representation ρ of $\langle 1, s\rangle$ to $G(de, e, 2)$ by $\rho(dx) = \rho(x)$ for every $d \in D$ and $x \in \langle 1, s\rangle$. We then define

$$\beta_{k',\rho}(dx) = \rho(dx)\Delta(0, k')(dx) = \rho(x)\Delta(0, k')(d)$$

for every $d \in D$ and $x \in \langle 1, s\rangle$.

We denote by A the set $A = \big\{[\![1, d'e - 1]\!] \times \mathbb{Z}/d\mathbb{Z}\big\} \cup \{d'e\} \times [\![0, d' - 1]\!]$. For $(k, k') \in A$, we set

$$\beta_{k,k'} = Ind_D^{G(de,e,2)} \Delta(k, k').$$

The family $((\beta_{k',1}, \beta_{k',\varepsilon})_{k' \in \mathbb{Z}/d\mathbb{Z}}, (\beta_{k,k'})_{(k,k') \in A})$ is a complete set of irreducible representations of $G(de, e, 2)$.

Proposition 7 (e even) *Let $d, e \in \mathbb{N}^*$ with $e = 2e'$ even.*

For $k' \in \mathbb{Z}/d\mathbb{Z}$ and $\delta \in \{0, de'\}$, we extend the character $\Delta(\delta, k')$ to $G(de, e, 2)$ by $\Delta(\delta, k')(dx) = \Delta(\delta, k')(d)$ for $d \in D$ and $x \in \langle 1, s\rangle$. We extend any irreducible representation ρ of $\langle 1, s\rangle$ to $G(de, e, 2)$ by $\rho(dx) = \rho(x)$ for every $d \in D$ and $x \in \langle 1, s\rangle$. We then define, for $d \in D$ and $x \in \langle 1, s\rangle$,

$$\beta_{\delta,k',\rho}(dx) = \rho(dx)\Delta(\delta, k')(dx) = \rho(x)\Delta(\delta, k')(d).$$

For $(k,k') \in [\![1\,,\, de'-1]\!] \times \mathbb{Z}/d\mathbb{Z}$*, we set*

$$\beta_{k,k'} = Ind_D^{G(de,e,2)} \Delta(k,k').$$

The family $((\beta_{\delta,k',1}, \beta_{\delta,k',\varepsilon})_{(\delta,k')\in\{0,de'\}\times\mathbb{Z}/d\mathbb{Z}}, (\beta_{k,k'})_{(k,k')\in[\![1\,,\,de'-1]\!]\times\mathbb{Z}/d\mathbb{Z}})$ *is a complete set of irreducible representations of* $G(de,e,2)$.

Corollary 6 (Non Diagonal Hyperplanes) *Let* $d, e \in \mathbb{N}^*$, ρ *be an irreducible representation of* $G(de,e,2)$, χ *a linear character of* $G(de,e,r)$ *and* H *the hyperplane of a non-diagonal reflection.*

The hyperplane H *is of a low multiplicity with respect to* ρ*, reflection-preserving with respect to* ρ *and multiplicity-partitionable with respect to* ρ *and* χ *for every* ρ *and* χ.

The following tables summarize the situation for H *to be of a low multiplicity with respect to* ρ *and* χ*. The first row gives the irreducible representations* ρ *of* $G(de,e,2)$*. The first column gives the linear characters* χ*. In the table, l.m. means low multiplicity with respect to* ρ *and* χ.

e odd	$\dim \rho = 2$	$\beta_{k',1}$	$\beta_{k',\varepsilon}$
$\beta_{k'',1}$	*l.m*	*l.m*	*l.m*
$\beta_{k'',\varepsilon}$	—	*l.m*	—

e even , H hyp. of s	$\dim \rho = 2$	$\beta_{u,k',1}$	$\beta_{u,k',\varepsilon}$
$\beta_{v,k'',1}$	*l.m*	*l.m*	*l.m*
$\beta_{v,k'',\varepsilon}$	—	*l.m*	—

$e = 2e'$, *H hyp. of* s'	$\dim \rho = 2$	$\beta_{0,k',1}$	$\beta_{0,k',\varepsilon}$	$\beta_{de',k',1}$	$\beta_{de',k',\varepsilon}$
$\beta_{0,k'',1}$	*l.m*	*l.m*	*l.m*	*l.m*	*l.m*
$\beta_{0,k'',\varepsilon}$	—	*l.m*	—	—	*l.m*
$\beta_{de',k'',1}$	—	*l.m*	—	—	*l.m*
$\beta_{de',k'',\varepsilon}$	*l.m*	*l.m*	*l.m*	*l.m*	*l.m*

Corollary 7 (Diagonal Hyperplanes) *Let* $d, e \in \mathbb{N}^*$ *and* $\beta_{k,k'}$ *be a* 2*-dimensional irreducible representation of* $G(de,e,2)$, χ *a linear character of* $G(de,e,r)$ *and* H *a hyperplane associated to a diagonal reflection.*

For $n \in \mathbb{Z}$*, we denote by* $\overline{n}$ *the unique integer such that* $0 \le \overline{n} \le d-1$ *and* $d \mid (n-\overline{n})$.

With these notations, we obtain $n_H(\beta_{k,k'}) = \overline{k+k'} + k'$*. For e odd and* $u \in \{1,\varepsilon\}$*, we have* $n_H(\beta_{k',u}) = k'$ *For* $e = 2e'$ *even,* $\delta \in \{0, de'\}$ *and* $u \in \{1,\varepsilon\}$*, we have* $n_H(\beta_{\delta,k',u}) = k'$.

- *Let us start with* $\rho = \beta_{k,k'}$ *a* 2*-dimensional representation of* $G(de,e,2)$.
  - *The case* $k' = 0$.
    - (i) *The hyperplane* H *is reflection-preserving with respect to* $\beta_{k,k'}$ *and so of a low multiplicity with respect to* $\beta_{k,k'}$ *and multiplicity-partitionable with respect to* $\beta_{k,k'}$ *and* χ *for every* χ.
    - (ii) *Moreover* $n_H(\beta_{k,k'}) = \overline{k}$*. Thus, for* $\chi = \beta_{\delta,k'',u}$ *or* $\chi = \beta_{k'',u}$ *with* $\delta \in \{0, de'\}$ *and* $u \in \{1,\varepsilon\}$*, the hyperplane* H *is of a low multiplicity with respect to* $\beta_{k,k'}$ *and* χ *if and only if* $\overline{k} + k'' < d$.
  - *The case* $k' \neq 0$*. For* $\chi = \beta_{\delta,k'',u}$ *or* $\chi = \beta_{k'',u}$ *with* $\delta \in \{0, de'\}$ *and* $u \in \{1,\varepsilon\}$*, the hyperplane* H *is*

  (i) *reflection-preserving with respect to* $\beta_{k,k'}$ *if and only if* $\overline{k+k'} = 0$;
  (ii) *of a low multiplicity with respect to* $\beta_{k,k'}$ *if and only if* $\overline{k+k'} + k' < d$;
  (iii) *of a low multiplicity with respect to* $\beta_{k,k'}$ *and* χ *if and only if* $\overline{k+k'} + k' + k'' < d$;
  (iv) *multiplicity-partitionable with respect to* $\beta_{k,k'}$ *and* χ *if and only if* $d - k'' > \overline{k+k'}$ *or* $d - k'' > k'$.

- *Let us now consider the case where ρ is a 1-dimension representation of $G(de, e, 2)$.*
  - *The case e odd. If $\rho = \beta_{k',\rho'}$ and $\chi = \beta_{k'',\rho''}$, then H is of a low multiplicity with respect to ρ and χ if and only if $k' + k'' < d$.*
  - *The case e even. If $\rho = \beta_{u,k',\rho'}$ and χ a linear character of $G(de, e, 2)$, then the hyperplane H is of a low multiplicity with respect to ρ and χ if and only if $k' + k'' < d$.*

4.3 The wreath product

Let us now study the imprimitive reflection group $G(d, 1, r)$ which is also the wreath product $\mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_r$. Let us assume $r \geq 3$. The reflections of $G(d, 1, r)$ are of the form

$$\operatorname{diag}(1, \ldots, 1, \xi, 1, \ldots, 1)$$

with ξ a non trivial dth of unity and of the form

$$\operatorname{diag}(1, \ldots, 1, \zeta, 1, \ldots, 1, \zeta^{-1}, 1, \ldots, 1)\tau_{ij}$$

where τ_{ij} is the transposition matrix swapping i and j and ζ is a dth root of unity. The associated hyperplanes split into two conjugacy class : the diagonal one and the non-diagonal one.

The irreducible character of $G(d, 1, r)$ can be described by the method of Wigner and Mackey with the normal abelian subgroup of diagonal matrices of $G(d, 1, r)$. Thus the representation of $G(d, 1, r)$ are given by the d-multipartitions of r, that is to say families of d partitions so that the sum of the length of the partitions is r. One can also describe the irreducible representations of $G(d, 1, r)$ by giving a family of d integers $\underline{r} = (n_0, \ldots, n_{d-1})$ such that $n_0 + \cdots + n_{d-1} = r$ and ρ a representation of $\mathfrak{S}_{n_0} \times \cdots \times \mathfrak{S}_{n_{d-1}}$. We denote by $\beta_{\underline{r},\rho}$ the corresponding representation of $G(d, 1, r)$.

Corollary 8 (Non Diagonal Hyperplanes) *Let $d \geq 2, r \geq 3$, H be a non diagonal hyperplane of $G(d, 1, r)$ (we denote by G_H the subgroup of G of reflections whose hyperplane is H), $\rho' = \beta_{\underline{r},\rho}$ be an irreducible representation of $G(d, 1, r)$ and χ a linear character of $G(d, 1, r)$.*

(i) *The hyperplane H is reflection-preserving with respect to ρ', of a low multiplicity with respect to ρ', of a low multiplicity with respect to ρ' and χ (for $\chi(G_H) = 1$), multiplicity-partitionable with respect to ρ' and χ (for $\chi(G_H) \neq 1$) if and only if $\rho' = \beta_{\underline{r},\rho}$ is of the form*

  (a) $\underline{r} = (0, \ldots, 0, r, 0, \ldots, 0)$ *and $\rho = 1$ or $\rho = \varepsilon$ or ρ is the standard representation or $\rho = (2, 2)$ if $r = 4$.*
  (b) $\underline{r} = (0, \ldots, 0, 1, 0 \ldots, 0, r - 1, 0, \ldots, 0)$ *and $\rho = 1$.*
  (c) $\underline{r} = (0, \ldots, 0, r - 1, 0 \ldots, 0, 1, 0, \ldots, 0)$ *and $\rho = 1$.*

(ii) *H is of a low multiplicity with respect to ρ' and χ (for $\chi(G_H) \neq 1$) if and only if $\underline{r} = (0, \ldots, 0, r, 0, \ldots, 0)$ and $\rho = 1$.*

(iii) *The hyperplane H is multiplicity-partitionable with respect to ρ' and χ (for $\chi(G_H) = 1$) if and only if ρ' is one the following representations*

  (a) $\underline{r} = (0, \ldots, 0, r, 0, \ldots, 0)$ *and $\rho = 1$ or $\rho = \varepsilon$ or ρ is the standard representation or $r \leq 5$ or $\rho \in \{(3, 3), (2, 2, 2), (4, 2)\}$ if $r = 6$.*
  (b) $\underline{r} = (0, \ldots, 0, 1, 0 \ldots, 0, r - 1, 0, \ldots, 0)$ *and $\rho = 1$ or $\rho = \varepsilon$ if $r \in \{3, 4\}$ or ρ is the standard representation if $r = 3$.*

(c) $\underline{r} = (0, \ldots, 0, r-1, 0 \ldots, 0, 1, 0, \ldots, 0)$ *and* $\rho = 1$ *or* $\rho = \varepsilon$ *if* $r \in \{3, 4\}$ *or* ρ *is the standard representation if* $r = 3$.
(d) $\underline{r} = (0, \ldots, 0, 2, 0 \ldots, 0, 3, 0, \ldots, 0)$ *and* $\rho = 1$.
(e) $\underline{r} = (0, \ldots, 0, 3, 0 \ldots, 0, 2, 0, \ldots, 0)$ *and* $\rho = 1$.
(f) $\underline{r} = (0, \ldots, 0, 2, 0 \ldots, 0, 2, 0, \ldots, 0)$ *and* $\rho = 1$ *or* $\rho = \varepsilon \otimes 1$ *or* $\rho = 1 \otimes \varepsilon$.
(g) $\underline{r} = (0, \ldots, 0, 1, 0 \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$ *and* $\rho = 1$.

Corollary 9 (Diagonal Hyperplanes) *Let* $d \geq 2, r \geq 3$, H *be a diagonal hyperplane of* $G(d, 1, r)$ *and* $\rho' = \beta_{\underline{r},\rho}$ *be an irreducible representation of* $G(d, 1, r)$ *with* $\underline{r} = (n_0, \ldots, n_{d-1})$ *and* χ *a linear character of* $G(d, 1, r)$.

(i) *The hyperplane* H *is of a low multiplicity with respect to* ρ' *if and only if*

$$\dim \rho \left(\sum_{j=0}^{d-1} \frac{j n_j}{r} \frac{r!}{n_0! \cdots n_{d-1}!} \right) < d.$$

(ii) *The hyperplane* H *is reflection-preserving with respect to* ρ' *if and only if* ρ' *is one of the following representations*

(a) $\underline{r} = (r, 0, \ldots, 0)$;
(b) $\underline{r} = (0, 0, \ldots, 0, r, 0, \ldots, 0)$ *and* $\rho = 1$ *or* $\rho = \varepsilon$;
(c) $\underline{r} = (r-1, 0, \ldots, 0, 1, 0, \ldots, 0)$ *and* $\rho = 1$ *or* $\rho = \varepsilon$.

(iii) *The hyperplane* H *is of a low multiplicity with respect to* ρ' *and* χ *if*

$$\dim \rho \left(\sum_{j=0}^{d-1} \frac{j n_j}{r} \frac{r!}{n_0! \cdots n_{d-1}!} \right) < d - n_H(\chi).$$

4.4 The group G_4

The group G_4 named after the Shephard and Todd classification [17] is a rank 2 reflection group. There is only one class of hyperplanes which is of order 3. The linear character of G_4 are given by the trivial one, the determinant and the square of the determinant. As an abstract group, G_4 is nothing else but $SL(2, \mathbb{F}_3)$. The irreducible representations of G_4 are then given by the 3 one-dimensional representation, the standard reflection representation named V, and two other 2-dimensional representations $V \det$ (whose character is real) and $V \det^2$ and one 3-dimensional representation (whose character is real).

Corollary 10 (Hyperplanes of G_4) *Let* H *be a hyperplane of* G_4, ρ *be an irreducible representation of* G_4.

Then H *is reflection-preserving with respect to* ρ, *of a low multiplicity with respect to* ρ, *multiplicity-partitionable with respect to* ρ *and* det *and of a low multiplicity with respect to* ρ *and* 1 *when* $\dim \rho = 1$ *or* $\rho = V$ *or* $\rho = V \det^2$.

The hyperplane H *is of a low multiplicity with respect to* ρ *and* det *if and only if* $\rho = 1$.

The hyperplane H *is of a low multiplicity with respect to* ρ *and* $\det^2$ *if and only if* $\rho = 1$, $\rho = \det^2$ *or* $\rho = V \det^2$.

The hyperplane H *is multiplicity-partitionable with respect to* ρ *and* 1 *and multiplicity-partitionable with respect to* ρ *and* $\det^2$ *for every* ρ.

4.5 The group G_5

The group G_5 named after the Shephard and Todd classification [17] is a rank 2 reflection group. There are two classes of hyperplanes which are both of order 3. In fact,

$$G_5 = \{j^k G_4,\ j = \exp(2i\pi/3), \quad k \in \{0, 1, 2\}\} = G_4 \times \left\{\mathrm{id}, j\mathrm{id}, j^2\mathrm{id}\right\}.$$

The irreducible representations of G_5 are then given by tensor product of representation of G_4 and of the three one-dimensional representation of $\{\mathrm{id}, j\mathrm{id}, j^2\mathrm{id}\}$ which are given by $(1, \det, \det^2)$. One class of hyperplanes is in fact the class of hyperplanes of G_4. The other one is a new class.

Corollary 11 (The Hyperplanes of G_5 which are in G_4) *Let H be a hyperplane of G_5 which is a hyperplane of G_4, ρ be an irreducible representation of G_5 and χ a linear character of G_5. Write $\rho = \rho_1 \otimes \rho_2$ (resp. $\chi = \chi_1 \otimes \chi_2$) where ρ_1 (resp. χ_1) is an (resp. one-dimensional) irreducible representation of G_4 and ρ_2 (resp. χ_2) an irreducible representation of* $\{\mathrm{id}, j\mathrm{id}, j^2\mathrm{id}\}$.

Then H is reflection-preserving with respect to ρ if and only if H is reflection-preserving with respect to ρ_1 for G_4.

The hyperplane H is of a low multiplicity with respect to ρ if and only if H is of a low multiplicity with respect to ρ_1 for G_4.

The hyperplane H is multiplicity-partitionable with respect to ρ and χ if and only if H is multiplicity-partitionable with respect to ρ_1 and χ_1 for G_4.

The hyperplane H is of a low multiplicity with respect to ρ and χ if and only if H is of a low multiplicity with respect to ρ_1 and χ_1 for G_4.

Corollary 12 (The Hyperplanes of G_5 which are not in G_4) *Let H be a hyperplane of G_5 which is not a hyperplane of G_4, ρ be an irreducible representation of G_5 and χ a linear character of G_5.*

We denote by $\widehat{G_5}$ the set of linear characters of G_5 and by $\mathscr{I}'' \subset \mathscr{I}' \subset \mathscr{I}$ the following sets of irreducible representations of G_5:

$$\mathscr{I} = \widehat{G_5} \cup \left\{V \otimes 1,\ V \otimes \det{}^2,\ V \det \otimes \det,\ V \det \otimes \det{}^2,\ V \det{}^2 \otimes 1,\ V \det{}^2 \otimes \det\right\};$$

$$\mathscr{I}' = \left\{\det{}^i \otimes \det{}^k \textit{ with } i+k = 0[3] \textit{ and } i+k = 1[3],\ V \otimes 1,\ V \det \otimes \det{}^2,\ V \det{}^2 \otimes \det\right\};$$

$$\mathscr{I}'' = \left\{\det{}^i \otimes \det{}^k \textit{ with } i+k = 0[3]\right\}.$$

The hyperplane H is reflection-preserving with respect to ρ if and only if H is of a low multiplicity with respect to ρ if and only if $\rho \in \mathscr{I}$.

The following table summarizes the situation for H to be multiplicity-partitionable and of a low multiplicity with respect to ρ and χ. The first row gives the irreducible representations ρ of G_5. The first column gives the linear characters $\chi = \det^i \otimes \det^k$. In the table, m.p. means multiplicity-partitionable with respect to ρ and χ and l.m. means low multiplicity with respect to ρ and χ.

	$\rho \in \mathscr{I} \backslash \mathscr{I}'$	$\rho \in \mathscr{I}' \backslash \mathscr{I}''$	$\rho \in \mathscr{I}''$	$\rho \notin \mathscr{I}$
$i+k = 0[3]$	*m.p. − l.m.*	*m.p. − l.m.*	*m.p. − l.m.*	*m.p.*
$i+k = 1[3]$	*m.p.*	*m.p. − l.m.*	*m.p. − l.m.*	*m.p.*
$i+k = 2[3]$	*m.p.*	*m.p.*	*m.p. − l.m.*	–

4.6 The group G_{24}

The group G_{24} named after the Shephard and Todd classification [17] is a rank 3 reflection group. There is only one class of hyperplanes which is of order 2. The linear character of G_{24} are given by the determinant and the trivial one. As an abstract group, G_{24} is nothing else but the product of the simple groups $GL(3, \mathbb{F}_2) \times \{-1, 1\}$. Let us denote by 1 and ε the irreducible representations of $\{-1, 1\}$ and $1, 3_1, 3_2, 6, 7, 8$ the irreducible representations of $GL(3, \mathbb{F}_2)$ (determined by their dimension). The irreducible representations of G_{24} are then given by the tensor products of an irreducible representation of $GL(3, \mathbb{F}_2)$ and $\{-1, 1\}$.

Corollary 13 (Hyperplanes of G_{24}) *Let H be a hyperplane of G_{24}.*

The hyperplane H is reflection-preserving with respect to ρ, of a low multiplicity with respect to ρ, of a low multiplicity with respect to ρ and 1 *and multiplicity-partitionable with respect to ρ and* det *if and only if $\rho = 1 \otimes 1$, $\rho = 1 \otimes \varepsilon$, $\rho = 3_1 \otimes \varepsilon$ and $\rho = 3_2 \otimes \varepsilon$.*

The hyperplane H is of a low multiplicity with respect to ρ and det *if and only if $\rho = 1 \otimes 1$.*

The hyperplane H is multiplicity-partitionable with respect to ρ and 1 *if and only if $\rho = 1 \otimes 1$, $\rho = 3_1 \otimes 1$, $\rho = 3_2 \otimes 1$ and $\rho = 6 \otimes 1$ and $\rho = 1 \otimes \varepsilon$, $\rho = 3_1 \otimes \varepsilon$ and $\rho = 3_2 \otimes \varepsilon$ and $\rho = 7 \otimes \varepsilon$.*

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