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Using Feynman diagrams in renormalisation, some new results

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Motivation: Allen–Cahn SPDE

$$\partial_t \phi(t,x) = \Delta\phi(t,x) + f(\phi(t,x)) \quad + \sqrt{2\varepsilon}\,\xi(t,x) \qquad\qquad f(\phi) = \phi - \phi^3$$

- $x \in \mathbb{T}_L^d = (\mathbb{R}/L\mathbb{Z})^d$, L bifurcation parameter
- ξ: space-time white noise
- $\varepsilon = 0$: Gradient system with potential/energy function V

$$V[\phi] = \int_0^L \Big[\tfrac{1}{2}\|\nabla\phi\|^2 - \tfrac{1}{2}\phi^2 + \tfrac{1}{4}\phi^4\Big]\,\mathrm{d}x \quad \Rightarrow \quad \partial_t\langle\phi,\psi\rangle = -\nabla_\psi V(\phi)$$

- Stationary solutions for $\varepsilon = 0$, $d = 1$:
  - $\phi_\pm(x) \equiv \pm 1$: stable
  - $\phi_0(x) \equiv 0$: unstable
  - $L > 2\pi$ ($L > \pi$ for Neumann b.c.): $\exists$ nonconstant stationary solutions
- Transition state: ϕ_{ts} lowest saddle point of V between ϕ_- and ϕ_+
  - $L \leqslant 2\pi$ ($L \leqslant \pi$ for Neumann b.c.): $\phi_{\text{ts}} = \phi_0$
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Eyring–Kramers law

$$dx_t = -\nabla V(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t$$

- $V : \mathbb{R}^n \to \mathbb{R}$ confining potential with minima x, y and saddle z
- $\tau_y^x = \inf\{t > 0 \colon x_t \in \mathcal{B}_\varepsilon(y)\}$ first-hitting time of $\mathcal{B}_\varepsilon(y)$, starting in x
- Eyring–Kramers law (1935, 1940):
  Eigenvalues of Hessian of V at minimum x: $0 < \nu_1 \leqslant \nu_2 \leqslant \dots \leqslant \nu_n$
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$$\mathbb{E}[\tau_y^x] = 2\pi \sqrt{\frac{\lambda_2 \dots \lambda_n}{|\lambda_1| \nu_1 \dots \nu_n}}\, e^{[V(z)-V(x)]/\varepsilon} \big[1 + \mathcal{O}_\varepsilon(1)\big]$$

- Potential-theoretic proof:

Theorem: $A, B \subset \mathbb{R}^n$ disjoint. $\exists$ proba measure ν_{AB} on ∂A s.t.

$$\int_{\partial A} \mathbb{E}^x[\tau_B]\,\nu_{AB}(dx) = \frac{1}{\operatorname{cap}(A,B)} \int_{B^c} e^{-V(y)/\varepsilon}\, h_{AB}(y)\,dy$$

  - $h_{AB}(y) = \mathbb{P}^y\{\tau_A < \tau_B\}$ commitor, $\operatorname{cap}(A,B)$ capacity
  - Dirichlet principle: $\operatorname{cap}(A,B) = \inf_{h : h|_A = 1, h|_B = 0} \varepsilon \int_{\mathbb{R}^d} e^{-V(x)/\varepsilon} |\nabla h(x)|^2\,dx$

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Eyring–Kramers law for 1D SPDEs: main result

- Hessian of V at $\phi_0 = 0$: $-\Delta - 1$, eigenvalues $\mu_k = \left(\frac{2k\pi}{L}\right)^2 - 1$
- Hessian of V at $\phi_- = -1$: $-\Delta + 2$, eigenvalues $\nu_k = \left(\frac{2k\pi}{L}\right)^2 + 2$

Theorem: Neumann b.c. [B & Gentz, 2013]

- If $L < \pi - c$ with $c > 0$, then

$$\mathbb{E}^{\phi_-}[\tau_+] = 2\pi \sqrt{\frac{1}{|\lambda_0|\nu_0} \prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k}}\, \mathrm{e}^{(V[\phi_{\text{ts}}]-V[\phi_-])/\varepsilon}\left[1 + \mathcal{O}(\varepsilon)\right]$$

- If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles) and λ'_k ev at nonconst transition state instead of λ_k

- Results also for L near π and periodic b.c.
- Proof uses spectral Galerkin approximation & passage to the limit
- Prefactor involves a Fredholm determinant:

$$\prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k} = \det\left[(-\Delta_\perp - 1)(-\Delta_\perp + 2)^{-1}\right] = \det\left[\mathbb{1} - 3(-\Delta_\perp + 2)^{-1}\right]$$

converges because $\log\det = \operatorname{Tr}\log$ and $(-\Delta_\perp + 2)^{-1}$ is trace class

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2. The two-dimensional case

- Large-deviation principle: [Hairer & Weber, 2015]
- Naive computation of prefactor fails:

$$\log \prod_{k\in(\mathbb{N}^2)^*} \frac{1-\left(\frac{L}{|k|\pi}\right)^2}{1+2\left(\frac{L}{|k|\pi}\right)^2} \simeq \sum_{k\in(\mathbb{N}^2)^*} \log\left(1-\frac{3L^2}{|k|^2\pi^2}\right)$$

$$\simeq -\sum_{k\in(\mathbb{N}^2)^*} \frac{3L^2}{|k|^2\pi^2} \simeq -\frac{3L^2}{\pi^2}\int_1^\infty \frac{r\,\mathrm{d}r}{r^2} = -\infty$$

- In fact, the equation needs to be renormalised ($P * \xi$ is distribution!)

Theorem: [Da Prato & Debussche 2003]

Let ξ^δ be a mollification on scale δ of white noise. Then

$$\partial_t\phi = \Delta\phi + \phi - \left[\phi^3 - 3\varepsilon C(\delta)\phi\right] + \sqrt{2\varepsilon}\xi^\delta$$

with $C(\delta) \simeq \log(\delta^{-1})$ admits local solution converging as $\delta \to 0$
Moreover, $\phi - \sqrt{2\varepsilon}(P * \xi^\delta) \in \mathscr{C}([0,T], \mathscr{B}^{\alpha}_{p,r}) \cap L^p([0,T], \mathscr{B}^{s}_{p,r})$
where $\mathscr{B}^{\bullet}_{\bullet,\bullet}$ Besov spaces with some $\alpha < 0 < s$, $p > 3$, $r \geqslant 1$

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Computation of the prefactor

- Consider for simplicity $L < 2\pi \Rightarrow$ transition state in 0
- Galerkin-truncated renormalised potential

$$V_N = \frac{1}{2}\int_{\mathbb{T}^2}\left[\|\nabla\phi_N(x)\|^2 - \phi_N(x)^2\right]\mathrm{d}x + \frac{1}{4}\int_{\mathbb{T}^2} :\phi_N(x)^4:\mathrm{d}x$$

where $:\phi_N^4: = \phi_N^4 - 6\varepsilon C_N\phi_N^2 + 3\varepsilon^2 C_N^2$, $\quad C_N = \sum_{|k|\leqslant N}\frac{1}{\lambda_k+1} \asymp \log(N)$

- Using variational principles: $\mathrm{cap}(A,B) \simeq \sqrt{\frac{|\lambda_0|\varepsilon}{2\pi}}\prod_{0<|k|\leqslant N}\sqrt{\frac{2\pi\varepsilon}{\lambda_k}}$
- Symmetry argument:

$$\int_{B^c} h_{A,B}(z)\,\mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z = \frac{1}{2}\int \mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z = \frac{1}{2}\mathcal{Z}_N(\varepsilon)$$

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Main result in dimension 2

Theorem: [B, Di Gesù, Weber, 2017]

For $L < 2\pi$, $A \ni \phi_-$, $B \ni \phi_+$ balls in $\|\cdot\|_{H^s}$, $s < 0$, $\exists \mu_N$ prob measures on ∂A:

$$\limsup_{N\to\infty} \mathbb{E}^{\mu_N}\big[\tau_B\big] \leqslant \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k}\, e^{\frac{\nu_k-\lambda_k}{|\lambda_k|}}}\; e^{(V[\phi_{ts}]-V[\phi_-])/\varepsilon}\big[1 + c_+\sqrt{\varepsilon}\,\big]$$

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  with $T = 3(-\Delta_\perp - 1)^{-1}$
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3. Allen–Cahn equation in dimension 3

- Equation needs two counterterms:

$$\partial_t \phi = \Delta\phi + \phi - \left[\phi^3 - (3\varepsilon C_1(\delta) - 9\varepsilon^2 C_2(\delta))\phi\right] + \sqrt{2\varepsilon}\xi^\delta$$

with $C_1(\delta) \sim \delta^{-1}$, $C_2(\delta) \sim \log(\delta^{-1})$

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  - Regularity structures [Hairer 2014]
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Φ^4_3 measure

- Potential (before renormalisation):

$$V_\alpha(\phi) = \int_\Lambda \left(\tfrac{1}{2} \|\nabla \phi(x)\|^2 + \tfrac{1}{2}\phi(x)^2 + \alpha \phi(x)^4 \right) \mathrm{d}x$$

  where $\Lambda = (\mathbb{R}/\mathbb{Z})^3 =: \mathbb{T}^3$, $\alpha \geqslant 0$

- Definition of Gibbs measure?

$$\nu_\alpha(\mathrm{d}\phi) \text{ “=” } \frac{1}{\mathcal{Z}_\alpha} \mathrm{e}^{-V_\alpha(\phi)} \text{ “}\mathrm{d}\phi\text{”}$$

- ν_0 is Gaussian (it's a GFF). For random variable F

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The static Φ_3^4 model

Theorem: Potential needs exactly 4 counterterms:

$$V_\alpha(\phi) = \int_\Lambda \Big(\tfrac{1}{2}\|\nabla\phi(x)\|^2 + \tfrac{1}{2}\big[1 - \alpha^2 C_N^{(2)}\big]\phi(x)^2 + \alpha :\phi(x)^4:_{C_N^{(1)}} + \alpha^2 C_N^{(3)} - \alpha^3 C_N^{(4)} \Big) \mathrm{d}x$$

where N ($\sim \delta^{-1}$) UV cut-off and

$$C_N^{(1)} = G_N(0) = \mathrm{Tr}\big((-\Delta_N + 1)^{-1}\big) = \mathcal{O}(N)$$

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and $G_N(x) = \sum_{|k|\leqslant N} \frac{1}{\lambda_k + 1} e_k(x)$ is the Green function of Δ_N

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Perturbative computation of partition function

- Wick powers: $X =$  $= \int_\Lambda :\phi(x)^4: \mathrm{d}x$, $Y =$  $= \int_\Lambda :\phi(x)^2: \mathrm{d}x$
- Potential: $V_\alpha = V_0 + \alpha X + \beta Y + \gamma$, $\beta = \frac{1}{2}\alpha^2 C_N^{(2)}$, $\gamma = \alpha^2 C_N^{(3)} - \alpha^3 C_N^{(4)}$

  Then $\dfrac{\mathcal{Z}_{N,\alpha}}{\mathcal{Z}_{N,0}} = \mathbb{E}^{\nu_0}\left[\mathrm{e}^{-\alpha X - \beta Y - \gamma}\right] = \mathrm{e}^{-\gamma}\,\mathbb{E}^{\nu_0}\left[\mathrm{e}^{-\alpha X - \beta Y}\right]$
- Let $\Gamma = (\mathscr{V}, \mathscr{E})$ be a multigraph. Its valuation is

$$\Pi_N(\Gamma) = \int_{\Lambda^{\mathscr{V}}} \prod_{e \in \mathscr{E}} G_N(x_{e_+} - x_{e_-})\,\mathrm{d}x$$

  For instance

  $C_N^{(1)} = \Pi_N$ "tadpole"

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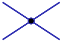
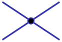
Cumulant expansion

- $\mathbb{E}^{\nu_0}\left[e^{-\alpha X - \beta Y}\right] = \sum_{n \geqslant 0} \frac{1}{n!}\mu_n$

  $\mu_n = (-1)^n \mathbb{E}^{\nu_0}\Big[\Big(\alpha$  $+ \beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\nu_0}\Big[$  ^{m}  $^{n-m}\Big]$ computed using Isserlis' thm

  Examples: $\mu_2 = \alpha^2 4! \Pi_N$ $+ \beta^2 2! \Pi_N$

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  $- 3\alpha\beta^2 4! \Pi_N$ $- 8\beta^3 \Pi_N$

- Cumulant expansion: (Leonov & Shiraev)

  $$-\log \mathbb{E}[e^{-\alpha X - \beta Y - \gamma}] = \gamma - \sum_{n=2}^{\infty} \frac{\kappa_n}{n!} \qquad \kappa_n = \mu_n - \sum_{m=2}^{n-2} \binom{n-1}{m} \kappa_m \mu_{n-m}$$

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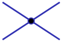
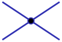
  $\mu_n = (-1)^n \mathbb{E}^{\nu_0}\Big[\Big(\alpha$  $+ \beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\nu_0}\Big[$  ^{m}  $^{n-m}\Big]$ computed using Isserlis' thm

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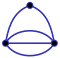
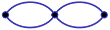
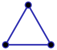
  $\mu_3 = -\alpha^3 \binom{4}{2}^3 2^3 \Pi_N$  $- 3\alpha^2\beta(4^2 \cdot 2 \cdot 3!)\Pi_N$ 

  $- 3\alpha\beta^2 4!\,\Pi_N$  $- 8\beta^3\Pi_N$ 

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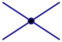
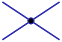
  $\mu_n = (-1)^n \mathbb{E}^{\nu_0}\Big[\Big(\alpha$  $+\beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\nu_0}\Big[$  ^{m}  $^{n-m}\Big]$ computed using Isserlis' thm

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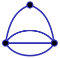
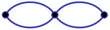
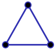
  $\mu_3 = -\alpha^3 \binom{4}{2}^3 2^3 \Pi_N$  $- 3\alpha^2\beta(4^2\cdot 2\cdot 3!)\Pi_N$ 

  $- 3\alpha\beta^2 4!\,\Pi_N$  $- 8\beta^3\Pi_N$ 

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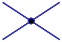
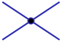
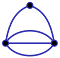
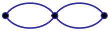
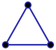
  $\mu_n = (-1)^n\mathbb{E}^{\nu_0}\Big[\Big(\alpha$  $+\beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m}\alpha^m\beta^{n-m}A_{nm}$

  where $A_{nm} = \mathbb{E}^{\nu_0}\Big[$  ^{m}  $^{n-m}\Big]$ computed using Isserlis' thm

  Examples: $\mu_2 = \alpha^2 4!\Pi_N$  $+ \beta^2 2!\Pi_N$ 

  $\mu_3 = -\alpha^3\binom{4}{2}^3 2^3\Pi_N$  $-3\alpha^2\beta(4^2\cdot 2\cdot 3!)\Pi_N$ 

  $-3\alpha\beta^2 4!\,\Pi_N$  $-8\beta^3\Pi_N$ 

- Cumulant expansion: (Leonov & Shiraev)

  $$-\log\mathbb{E}[e^{-\alpha X-\beta Y-\gamma}] = \gamma - \sum_{n=2}^{\infty}\frac{\kappa_n}{n!} \qquad \kappa_n = \mu_n - \sum_{m=2}^{n-2}\binom{n-1}{m}\kappa_m\mu_{n-m}$$

- Linked Cluster Theorem: κ_n projection of μ_n on connected graphs

  Proof: for instance Peccati & Taqqu (2011)

Divergences and subdivergences

- Degree of Γ: $\deg(\Gamma) = 3(|\mathscr{V}| - 1) - |\mathscr{E}|$. Γ divergent if $\deg(\Gamma) \leqslant 0$.
- Examples:

$\deg($ $) = 0$ $\qquad$ $\Pi_N($ $) = \mathcal{O}(\log N)$

$\deg($ $) = -1$ $\qquad$ $\Pi_N($ $) = \mathcal{O}(N)$

$\deg($ $) = 0$ $\qquad$ $\Pi_N($ $) = \mathcal{O}(\log N)$

- It looks like $\Pi_N(\Gamma) = \begin{cases} \mathcal{O}(N^{-\deg(\Gamma)}) & \text{if } \deg(\Gamma) < 0 \\ \mathcal{O}(\log N) & \text{if } \deg(\Gamma) = 0 \end{cases}$

  However, $\deg($ $) = 1$, while $\Pi_N($ $) = \mathcal{O}(\log N)$
  because it contains a subdivergence

Theorem: [Bogolyubov, Parasiuk, Hepp, Zimmermann]

$\Pi_N^{\mathrm{BPHZ}}(\Gamma) := -\Pi_N \mathscr{A}(\Gamma) \mathbb{1}_{\deg \Gamma > 0}$ is bdd uniformly in N for any Γ

where $\mathscr{A}$ defined recursively by $\mathscr{A}(\Gamma) = -\Gamma - \sum_{\mathbb{1} \neq \bar{\Gamma} \subsetneq \Gamma, \deg(\bar{\Gamma}) \leqslant 0} \mathscr{A}(\bar{\Gamma}) \cdot (\Gamma / \bar{\Gamma})$

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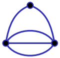
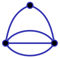
  However, deg() = 1, while Π_N() = $\mathcal{O}(\log N)$
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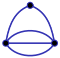
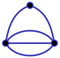
  However, deg() = 1, while $\Pi_N($  $) = \mathcal{O}(\log N)$ because it contains a subdivergence 

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Main result: Commutative diagram

$$
\begin{tikzcd}
e^{-\alpha X} \arrow[r, mapsto, "\mathscr{P}"] \arrow[d, mapsto, "\mathrm{W}"'] & \displaystyle\sum_p \frac{(-\alpha)^p}{p!}\mathscr{P}(X^p) \arrow[d, mapsto, "M^{\mathrm{BPHZ}}+\Theta"'] \arrow[dr, mapsto, "\Pi_N^{\mathrm{BPHZ}}+\Pi_N\Theta"] & \\
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- $\mathscr{P} = \Pi_{\mathrm{connected}}\left(\sum_{\mathrm{pairings}}\right)$, satisfies $\deg\mathscr{P}(X^n) = n-3$
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Theorem: [B, Klose, Tapia 25]

$$\log\frac{\mathcal{Z}_{N,\alpha}}{\mathcal{Z}_{N,0}} = \log\mathbb{E}[e^{-\alpha X-\beta Y}] - \gamma \asymp -\sum_{n\geqslant 4}\frac{(-\alpha)^n}{n!}\Pi_N\mathscr{A}(\mathscr{P}(X^n))$$

as asymptotic expansion with terms uniformly bounded in N

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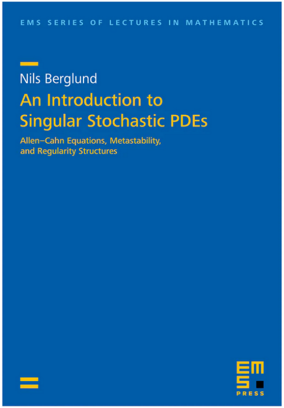

Thanks for your attention!

Slides available at https://www.idpoisson.fr/berglund/Athens26_Feynman.pdf