15th AIMS Conference, Athens

Concentration results for time-dependent SPDEs with Gaussian and fractional noise

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8 July 2026

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View from Lycabettus Hill

Warm-up: Slowly time-dependent 1D SDE

$$dx_t = f(\varepsilon t, x_t)\,dt + \sigma\,dW_t$$

- On slow time scale $\varepsilon t \to t$:

$$dx_t = \frac{1}{\varepsilon} f(t, x_t)\,dt + \frac{\sigma}{\sqrt{\varepsilon}}\,dW_t$$

- Stable equilibrium branch $x^*(t)$: $f(t, x^*(t)) = 0$, $\partial_x f(t, x^*(t)) < 0$
  $\sigma = 0 \Rightarrow \exists$ slow solution $\bar{x}(t) = x^*(t) + \mathcal{O}(\varepsilon)$ [Tihonov, Fenichel]
- $\sigma > 0$: write $x_t = \bar{x}(t) + \xi_t$ and Taylor-expand:

$$d\xi_t = \frac{1}{\varepsilon}\Big[\bar{a}(t)\xi_t + \underbrace{b(t,\xi_t)}_{=\mathcal{O}(\xi_t^2)}\Big]\,dt + \frac{\sigma}{\sqrt{\varepsilon}}\,dW_t$$

  where $\bar{a}(t) = \partial_x f(t, \bar{x}(t)) = \partial_x f(t, x^*(t)) + \mathcal{O}(\varepsilon) < 0$

- Variations of constants, if $\xi_0 = 0$ and $\bar{\alpha}(t,s) = \int_s^t \bar{a}(u)\,du$:

$$\xi_t = \underbrace{\frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{\bar{\alpha}(t,s)/\varepsilon}\,dW_s}_{\xi_t^0\text{: sol of linearised system}} + \underbrace{\frac{1}{\varepsilon}\int_0^t e^{\bar{\alpha}(t,s)/\varepsilon}\, b(s,\xi_s)\,ds}_{\text{treat as a perturbation}}$$

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Concentration estimate for linearised process

Properties of OU-like process $\quad \xi_t^0 = \dfrac{\sigma}{\sqrt{\varepsilon}} \displaystyle\int_0^t \mathrm{e}^{\bar\alpha(t,s)/\varepsilon} \,\mathrm{d}W_s$:

- Gaussian process, $\quad \mathbb{E}[\xi_t^0] = 0, \quad \operatorname{Var}(\xi_t^0) = \frac{\sigma^2}{\varepsilon} \int_0^t \mathrm{e}^{2\bar\alpha(t,s)/\varepsilon} \,\mathrm{d}s$
- Confidence interval: $\quad \mathbb{P}\Bigl\{|\xi_t^0| > \frac{h}{\sigma}\sqrt{\operatorname{Var}(\xi_t^0)}\Bigr\} = \mathcal{O}(\mathrm{e}^{-h^2/2\sigma^2})$
- $\sigma^{-2}\operatorname{Var}(\xi_t^0)$ satisfies ODE $\quad \varepsilon\dot v = 2\bar a(t) v + 1$

Lemma [B & Gentz, PTRF 2002]

$\bar v(t)$ solution of ODE bounded away from 0: $\quad \bar v(t) = \frac{1}{-2\bar a(t)} + \mathcal{O}(\varepsilon)$

$$\mathbb{P}\biggl\{\sup_{0\leqslant s\leqslant t} \frac{|\xi_s^0|}{\sqrt{\bar v(s)}} > h\biggr\} = C_0(t,\varepsilon)\,\mathrm{e}^{-h^2/2\sigma^2}$$

where $C_0(t,\varepsilon) = \sqrt{\frac{2}{\pi}}\frac{1}{\varepsilon}\Bigl|\int_0^t \bar a(s)\,\mathrm{d}s\Bigr|\frac{h}{\sigma}\Bigl[1 + \mathcal{O}\bigl(\varepsilon + \frac{t}{\varepsilon}\mathrm{e}^{-h^2/\sigma^2}\bigr)\Bigr]$

Proof based on Doob's submartingale inequality and partition of $[0,t]$

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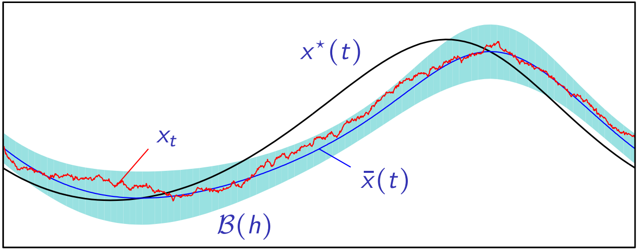
Concentration estimate for nonlinear process

Nonlinear equation: $\mathrm{d}\xi_t = \dfrac{1}{\varepsilon}\Big[\bar{a}(t)\xi_t + b(t,\xi_t)\Big]\,\mathrm{d}t + \dfrac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$

Confidence strip: $\mathcal{B}(h) = \Big\{|\xi| \leqslant h\sqrt{\bar{v}(t)}\ \forall t\Big\} = \Big\{|x - \bar{x}(t)| \leqslant h\sqrt{\bar{v}(t)}\ \forall t\Big\}$



Theorem [B & Gentz, PTRF 2002]

$$C(t,\varepsilon)\,\mathrm{e}^{-\kappa_- h^2/2\sigma^2} \leqslant \mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \leqslant C(t,\varepsilon)\,\mathrm{e}^{-\kappa_+ h^2/2\sigma^2}$$

where $\kappa_\pm = 1 \mp \mathcal{O}(h)$ and $C(t,\varepsilon) = C_0(t,\varepsilon)\big[1 + \mathcal{O}(h)\big]$ (requires $h \leqslant h_0$)

SPDE on $\mathbb{T}^1$: stable case

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}\Big[\Delta\phi(t,x) + f(t,\phi(t,x))\Big]\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\underbrace{\mathrm{d}W(t,x)}_{\text{cylindrical Wiener process}}$$

- ▷ $f(t,\phi^*(t)) = 0$ for all $t \in I = [0,T]$
- ▷ $a(t) = \partial_\phi f(t,\phi^*(t)) \leqslant -a_- < 0$ for all $t \in I$

In deterministic case $\sigma = 0$: $\exists$ particular solution $\bar\phi(t,x)$ such that

$$\left\|\bar\phi(t,\cdot) - \phi^*(t)e_0\right\|_{H^1} \leqslant C\varepsilon \qquad \forall t \in I$$

Theorem [B & Nader, SPDE 2023]

Fix $s < \frac{1}{2}$, and let $\mathcal{B}(h) = \left\{(t,\phi)\colon t \in I, \|\phi - \bar\phi(t,\cdot)\|_{H^s} < h\right\}$
For any $\nu > 0$

$$\mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \leqslant C(t,\varepsilon,s)\exp\Big\{-\kappa\frac{h^2}{\sigma^2}\Big[1 - \mathcal{O}\Big(\frac{h}{\varepsilon^\nu}\Big)\Big]\Big\}$$

holds for some $\kappa > 0$, $h = \mathcal{O}(\varepsilon^\nu)$ and $C(t,\varepsilon,s) = \mathcal{O}(t/\varepsilon)$.

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Ideas of proof

- $\phi(x) = \sum_{k\in\mathbb{Z}} \phi_k e_k(x) \quad \Rightarrow \quad \|\phi\|_{H^s}^2 = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s}\phi_k^2, \qquad \langle k\rangle = \sqrt{1+k^2}$
- Deterministic case: $\psi = \phi - \phi^* e_0$, $\|\psi\|_{H^1}^2$ is a Lyapunov function

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- Linear stoch case:

  $$\mathrm{d}\psi_k = \frac{1}{\varepsilon} a_k(t)\psi_k \,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_k(t), \qquad a_k(t) = \bar{a}(t) - \frac{k^2\pi^2}{L^2} < 0$$

  For any decomposition $h = \sum_k h_k$, τ first-exit time from $\mathcal{B}(h)$,

  $$\mathbb{P}\{\tau < T\} \leqslant \sum_k \mathbb{P}\Big\{\sup_t \psi_k(t)^2 \geqslant h_k^2 \langle k\rangle^{-2s}\Big\} \leqslant \sum_k C_k(T,\varepsilon)\,\mathrm{e}^{-\kappa h_k^2\langle k\rangle^{2-2s}/\sigma^2}$$

  Choose $h_k^2 \sim h^2\langle k\rangle^{-2+2s+\eta}$, $\eta > 0$

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  Choose $h_k^2 \sim h^2\langle k\rangle^{-2+2s+\eta}$, $\eta > 0$
- Schauder estimate: $\beta \in H^r$, $0 < r < \frac{1}{2} \quad \Rightarrow$
  $$\|\mathrm{e}^{t\Delta}\beta\|_{H^q} \leqslant M(q,r)t^{-(q-r)/2}\|\beta\|_{H^r} \qquad \forall q < r+2$$
  Consequence: $\psi = \psi^0 + \psi^1$ where nonlinear term satisfies
  $$\|\psi^1\|_{H^q} \leqslant M'\varepsilon^{(q-r)/2-1}\sup_t\|b(t,\psi(y,\cdot))\|_{H^r}$$

SPDE on the 2d torus

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}\Big[\Delta\phi(t,x) + \sum_{j=1}^{n} A_j(t)\phi(t,x)^j\Big]\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\mathrm{d}W(t,x) \qquad x \in \mathbb{T}^2$$

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▷ SPDE is not well-posed, needs to be renormalised
For $N \in \mathbb{N}$, project on $\operatorname{span}\{e_k\}_{|k|<N}$:

$$d\phi(t,x) = \frac{1}{\varepsilon}\Big[\Delta\phi(t,x) + \sum_{j=1}^{n} A_j(t)P_N{:}\phi(t,x)^j{:}\Big]\,dt + \frac{\sigma}{\sqrt{\varepsilon}}\,dW_N(t,x)$$

where $:\phi^n: = H_n(\phi; C_N)$ Wick power, $C_N \sim \log N$ variance of GFF

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- Use Besov–Hölder spaces $\mathcal{B}^{\alpha}_{2,\infty}$, $\alpha < 0$, instead of Sobolev spaces H^s:

$$\|\phi\|_{\mathcal{B}^{\alpha}_{2,\infty}} = \sup_{q\geqslant 0} 2^{q\alpha}\|\delta_q\phi\|_{L^2} \qquad \delta_q\phi = \sum_{2^{q-1}\leqslant|k|<2^q} \phi_k e_k$$

SPDE on the 2d torus

Theorem [B & Nader, EJP 2025]

For $\alpha < 0$, $m \in \mathbb{N}$,

$$\mathbb{P}\left\{\sup_{0\leqslant t\leqslant T} \|{:}\psi(t,\cdot)^m{:}\|_{\mathcal{B}^\alpha_{2,\infty}} > h^m\right\} \leqslant C_m(T,\varepsilon,\alpha)\,\mathrm{e}^{-\kappa_m(\alpha)h^2/\sigma^2}$$

where

$$\kappa_m(\alpha) \geqslant c_0\frac{\alpha^2}{m^7} \qquad C_m(T,\varepsilon,\alpha) \leqslant c_1\frac{T}{\varepsilon}\frac{m^{3/2}\,\mathrm{e}^m\,m^m}{|\alpha|}$$

- Binomial formula

$${:}\psi^m{:} = H_m(\psi; C_N) = \sum_{|\mathrm{n}|=m}\frac{m!}{\mathrm{n}!}\prod_{q\geqslant 0} H_{\mathrm{n}_q}(\delta_q\psi; c_q) \qquad c_q = \mathcal{O}(1)$$

- Doob submartingale inequality for $\sup_{t\in I_\ell}\|\delta_{q_0}(\prod_{q\geqslant 0} H_{\mathrm{n}_q}(\delta_q\hat{\psi}; c_q))\|^2_{L^2}$

  where $\hat{\psi}$ martingale approximating ψ on intervals I_l depending on q_0
- Upgrade to bound for $\sup_{t\in I_\ell}\|\delta_{q_0}(\prod_{q\geqslant 0} H_{\mathrm{n}_q}(\delta_q\psi; c_q))\|^2_{L^2}$
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Concentration estimates

Theorem [B & Nader, EJP 2025]

Let $\phi_1 = \phi - \phi^* - \psi$. Then $\forall\gamma < 2, \forall\nu < 1 - \frac{\gamma}{2}$, $\forall h < h_0\varepsilon^\nu$

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- Use Schauder estimate and $\mathcal{B}^\gamma_{2,\infty} \hookrightarrow \mathcal{B}^{\gamma-1}_{\infty,\infty} = \mathcal{C}^{\gamma-1}$

Example: Dynamic pitchfork bifurcation

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}\bigl[\Delta\phi(t,x) + a(t)\phi(t,x) - {:}\phi(t,x)^3{:}\bigr]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x)$$

- Decompose $\phi = \psi_\perp + \phi_1$, where $\psi_\perp$ stoch conv driven by $W_\perp$
- Decompose $\phi_1 = \phi_1^0 e_0 + \phi_1^\perp$
- Concentration estimate for $\|\phi_1^\perp\|_{\mathcal{C}^{\gamma-1}}$ as long as $a(t) < (2\pi)^2$
- For ϕ_1^0, similar results as in SDE case (bif delay of order $\sqrt{\varepsilon\log(\sigma^{-1})}$)

Concentration estimates

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The case of fractional noise

- Fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$:
  Centred Gaussian process with covariance

$$\mathbb{E}[W_t^H W_s^H] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

Theorem [B & Blessing, ECP 2025]

Let $\mathrm{d}x_t = \frac{1}{\varepsilon}a(t)x_t\,\mathrm{d}t + \frac{\sigma}{\varepsilon^H}\,\mathrm{d}W_t^H$, with $a(t) \leqslant a_0 < 0$ and $H \in (0,1)$. Then

$$\mathbb{P}\Big\{\sup_{t\in[0,T]} |x_t| > h\Big\} \leqslant C\Big(T,\frac{h}{\sigma},a_0\Big)\,\mathrm{e}^{-\kappa h^2/(2\sigma^2)} \qquad \kappa = \frac{1-\mathcal{O}(\varepsilon)}{2H\Gamma(2H)}$$

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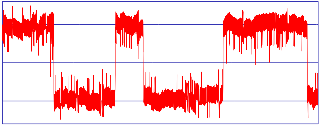
Stochastic resonance in an SDE

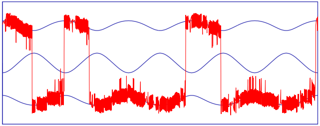
$$\mathrm{d}x_t = \underbrace{\left[-x_t^3 + x_t + A\cos(\varepsilon t)\right]}_{= -\frac{\partial}{\partial x}\left[\frac{1}{4}x^4 - \frac{1}{2}x^2 - Ax\cos(\varepsilon t)\right]\Big|_{x_t}} \mathrm{d}t + \sigma\,\mathrm{d}W_t$$

- Ice Ages: deterministically bistable climate [Croll, Milankovitch]
- random perturbations due to weather [Benzi-Sutera-Vulpiani, Nicolis-Nicolis]

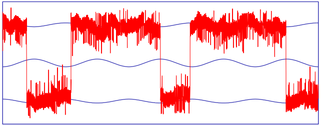
Sample paths $\{x_t\}_t$ for $\varepsilon = 0.001$:



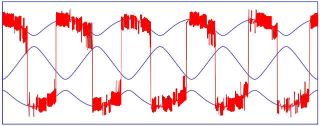
$A = 0$, $\sigma = 0.3$



$A = 0.24$, $\sigma = 0.2$



$A = 0.1$, $\sigma = 0.27$

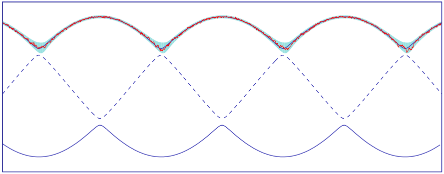


$A = 0.35$, $\sigma = 0.2$

The synchronisation regime

$A_{\mathrm{c}} = \frac{2}{3\sqrt{3}}$, $A = A_{\mathrm{c}} - \delta$, $0 < \delta \ll 1$. Critical noise intensity: $\sigma_{\mathrm{c}} = \max\{\delta, \varepsilon\}^{3/4}$

$\sigma \ll \sigma_{\mathrm{c}}$:
transitions unlikely



$\sigma \gg \sigma_{\mathrm{c}}$:
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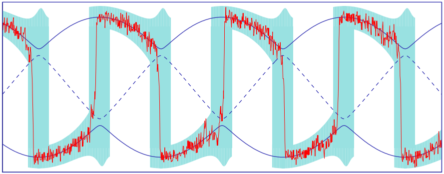

Theorem [B & Gentz, Annals App. Proba 2002]

- Away from (avoided) bifurcations, sample paths concentrated in σ-neighbourhood of deterministic stable periodic solutions
- $\sigma \ll \sigma_{\mathrm{c}}$: transition probability per period $\leqslant \mathrm{e}^{-\sigma_{\mathrm{c}}^2/\sigma^2}$
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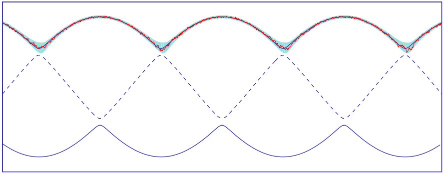

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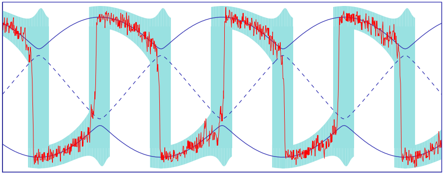

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Stochastic resonance in stochastic PDEs

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Simulation available at `youtu.be/eN3NWiEjBK8`

Stochastic resonance in bistable SPDEs on $\mathbb{T}^1$

$$\mathrm{d}\phi(t,x) = \big[\Delta\phi(t,x) + f(\varepsilon t, \phi(t,x))\big]\,\mathrm{d}t + \sigma\,\mathrm{d}W(t,x)$$

- $\phi = \phi(t,x) \in \mathbb{R}$, $\varepsilon t \in [0,T]$ or f is T-periodic, $x \in \mathbb{T} = \mathbb{R}/L\mathbb{Z}$, $L > 0$
- $\phi \mapsto f(s,\phi)$ bistable, $\mathcal{C}^2$, confining, e.g. $f(s,\phi) = \phi - \phi^3 + A\cos(s)$
- $\mathrm{d}W(t,x)$ space-time white noise on $\mathbb{R}_+ \times \mathbb{T}$
- $0 < \varepsilon, \sigma \ll 1$
- δ measures closeness to bifurcation (e.g. $A_c - A$)

Theorem [B & Nader, SPDE 2025]

- Away from bifurcations, solutions are concentrated around deterministic solutions in Sobolev H^s-norm for any $s < \frac{1}{2}$
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Stochastic resonance in bistable SPDEs on $\mathbb{T}^1$

$$d\phi(t,x) = \big[\Delta\phi(t,x) + f(\varepsilon t, \phi(t,x))\big]\,dt + \sigma\,dW(t,x)$$

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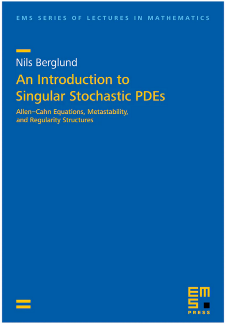
Related result

- Case $x \in \mathbb{T}^3$: Dimitri Faure, *Concentration around a stable equilibrium for the non-autonomous Φ^4_3 model*, Stoch. & PDEs: Anal. & Comp. Online first (2026)

References

- N. B. & Barbara Gentz, *A sample-paths approach to noise-induced synchronization: Stochastic resonance in a double-well potential*, Ann. Appl. Probab., **12**(1):1419–1470, 2002
- ———, *Noise-Induced Phenomena in Slow-Fast Dynamical Systems. A Sample-Paths Approach*, Springer, Probability and its Applications (2005)
- N. B. & Rita Nader, *Stochastic resonance in stochastic PDEs*, Stoch. & PDEs: Anal. & Comp. **11**:348–387 (2023)
- ———, *Concentration estimates for slowly time-dependent singular SPDEs on the two-dimensional torus*, Electronic J. Probability 29:1–35 (2024)
- N. B. & Alexandra Blessing, *Concentration estimates for SPDEs driven by fractional Brownian motion*, Electronic Comm. Probability 30:1-13 (2025)



- N. B, *An Introduction to Singular Stochastic PDEs*, EMS Press (2022)