

Berlin Probability Colloquium
Institut für Mathematik, HU Berlin

Stochastic resonance in stochastic PDEs

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Based on joint works with Rita Nader (Orléans) and Barbara Gentz (Bielefeld)



Project
PERISTOCH

Stochastic resonance in an SDE

$$\begin{aligned} dx_t &= \underbrace{\left[-x_t^3 + x_t + A \cos(\varepsilon t)\right]} dt + \sigma dW_t \\ &= -\frac{\partial}{\partial x} \left[\frac{1}{4} x^4 - \frac{1}{2} x^2 - Ax \cos(\varepsilon t) \right] \Big|_{x_t} \end{aligned}$$

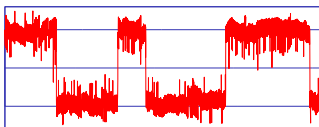
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Stochastic resonance in an SDE

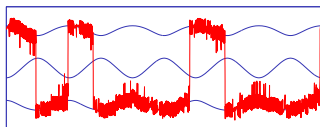
$$dx_t = \underbrace{\left[-x_t^3 + x_t + A \cos(\varepsilon t)\right]}_{= -\frac{\partial}{\partial x} \left[\frac{1}{4}x^4 - \frac{1}{2}x^2 - Ax \cos(\varepsilon t) \right] \Big|_{x_t}} dt + \sigma dW_t$$

- ▷ Ice Ages: deterministically bistable climate [Croll, Milankovitch]
- ▷ random perturbations due to weather [Benzi-Sutera-Vulpiani, Nicolis-Nicolis]

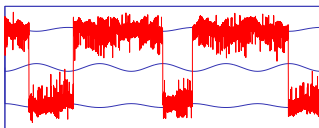
Sample paths $\{x_t\}_t$ for $\varepsilon = 0.001$:



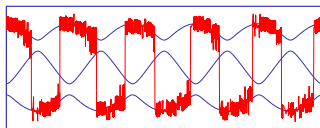
$A = 0, \sigma = 0.3$



$A = 0.24, \sigma = 0.2$



$A = 0.1, \sigma = 0.27$



$A = 0.35, \sigma = 0.2$

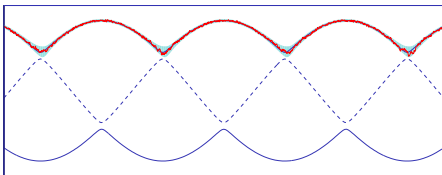
Descriptions of stochastic resonance

- ▷ Fokker–Planck equation: [Caroli, Caroli, Roulet & Saint-James '81]
- ▷ Two-state Markov chain: [Eckmann & Thomas '82], [Imkeller & Pavlyukevich '02], [Herrmann & Imkeller '02]
- ▷ Signal-to-noise ratio: [Gammaitoni, Menichella-Saetta & ... '89], [Fox '89], [Jung & Hänggi '89], [McNamara & Wiesenfeld '89]
- ▷ Slow forcing: [Jung & Hänggi '91], [Talkner '99], [Talkner & Łuczka '04]
- ▷ Large deviations: [Freidlin '00, Freidlin '01]
- ▷ Residence-time distributions: [Zhou, Moss & Jung '90], [Choi, Fox & Jung '98], ...
- ▷ Overview articles:
[Moss, Pierson & O'Gorman '94], [Wiesenfeld & Moss '95], [McNamara & Wiesenfeld '95], [Wiesenfeld & Jaramillo '98], [Gammaitoni, Hänggi, Jung & Marchesoni '98], [Hänggi '02], [Wellens, Shatokhin & Buchleitner '04], ...
- ▷ Monograph: [Herrmann, Imkeller, Pavlyukevich & Peithmann '14]

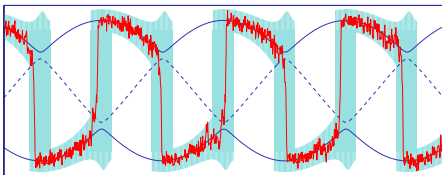
The synchronisation regime

$A_c = \frac{2}{3\sqrt{3}}$, $A = A_c - \delta$, $0 < \delta \ll 1$. Critical noise intensity: $\sigma_c = \max\{\delta, \varepsilon\}^{3/4}$

$\sigma \ll \sigma_c$:
transitions unlikely



$\sigma \gg \sigma_c$:
synchronisation



Theorem [B & Gentz, Annals App. Proba 2002]

- ▶ Away from (avoided) bifurcations, sample paths concentrated in σ -neighbourhood of deterministic stable periodic solutions
- ▶ $\sigma \ll \sigma_c$: transition probability per period $\leq e^{-\sigma_c^2/\sigma^2}$
- ▶ $\sigma \gg \sigma_c$: transition probability per period $\geq 1 - e^{-c\sigma^{4/3}/(\varepsilon|\log \sigma|)}$

Stochastic Allen–Cahn equation on \mathbb{T}^2

$$d\phi(t, x) = [\nu(\varepsilon t)\Delta\phi(t, x) + \phi(t, x) - \phi(t, x)^3] dt + \sigma dW(t, x)$$

(Online: <https://youtu.be/yX0EAxZHNCQ>)

Stochastic resonance in stochastic PDEs

$$d\phi(t, x) = \left[\Delta\phi(t, x) + \phi(t, x) - \phi(t, x)^3 + \underbrace{A \cos(\varepsilon t)}_{h(\varepsilon t)} \right] dt + \sigma dW(t, x)$$

Simulation available at youtu.be/eN3NWiEjBK8

Stochastic resonance in SPDEs

$$d\phi(t, x) = [\Delta\phi(t, x) + f(\varepsilon t, \phi(t, x))] dt + \sigma dW(t, x)$$

- ▷ $\phi = \phi(t, x) \in \mathbb{R}$, $\varepsilon t \in [0, T]$ or f is T -periodic, $x \in \mathbb{T} = \mathbb{R}/L\mathbb{Z}$, $L > 0$
- ▷ $\phi \mapsto f(s, \phi)$ bistable, \mathcal{C}^2 , confining, e.g. $f(s, \phi) = \phi - \phi^3 + A \cos(s)$
- ▷ $dW(t, x)$ space-time white noise on $\mathbb{R}_+ \times \mathbb{T}$
- ▷ $0 < \varepsilon, \sigma \ll 1$
- ▷ δ measures closeness to bifurcation (e.g. $A_c - A$)

Theorem [B & Nader, Stoch. & PDEs: Analysis & Comput., 2022]

- ▷ Away from bifurcations, solutions are concentrated around deterministic solutions in Sobolev H^s -norm for any $s < \frac{1}{2}$
- ▷ $\sigma \ll \sigma_c = \max\{\delta, \varepsilon\}^{3/4}$: transition probability per period $\leq e^{-\sigma_c^2/\sigma^2}$
- ▷ $\sigma \gg \sigma_c$: transition probability per period $\geq 1 - e^{-c\sigma^{4/3}/(\varepsilon|\log \sigma|)}$

Proof ideas, 1D SDE below threshold

On slow time scale $\varepsilon t \rightarrow t$:

$$dx_t = \frac{1}{\varepsilon} f(t, x_t) dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_t$$

$\bar{x}(t)$ deterministic solution tracking stable equilibrium $x^*(t)$.

Write $x_t = \bar{x}(t) + \xi_t$ and Taylor-expand:

$$d\xi_t = \frac{1}{\varepsilon} \left[\bar{a}(t)\xi_t + \underbrace{b(t, \xi_t)}_{=\mathcal{O}(\xi_t^2)} \right] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_t$$

where $\bar{a}(t) = \partial_x f(t, \bar{x}(t)) = \partial_x f(t, x^*(t)) + \mathcal{O}(\varepsilon) < 0$

Variations of constants (**Duhamel formula**), if $\xi_0 = 0$:

$$\xi_t = \underbrace{\frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{\bar{\alpha}(t,s)/\varepsilon} dW_s}_{\xi_t^0: \text{sol of linearised system}} + \underbrace{\frac{1}{\varepsilon} \int_0^t e^{\bar{\alpha}(t,s)/\varepsilon} b(s, \xi_s) ds}_{\text{treat as a perturbation}}$$

where $\bar{\alpha}(t, s) = \int_s^t \bar{a}(u) du$

Proof ideas, 1D SDE below threshold

Properties of $\xi_t^0 = \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{\bar{\alpha}(t,s)/\varepsilon} dW_s$:

- ▷ Gaussian process, $\mathbb{E}[\xi_t^0] = 0$, $\text{Var}(\xi_t^0) = \frac{\sigma^2}{\varepsilon} \int_0^t e^{2\bar{\alpha}(t,s)/\varepsilon} ds$
- ▷ Confidence interval: $\mathbb{P}\{|\xi_t^0| > \frac{h}{\sigma} \sqrt{\text{Var}(\xi_t^0)}\} = \mathcal{O}(e^{-h^2/2\sigma^2})$
- ▷ $\sigma^{-2} \text{Var}(\xi_t^0)$ satisfies ODE $\varepsilon \dot{v} = 2\bar{a}(t)v + 1$

Lemma [B & Gentz, PTRF 2002]

$\bar{v}(t)$ solution of ODE bounded away from 0: $\bar{v}(t) = \frac{1}{-2\bar{a}(t)} + \mathcal{O}(\varepsilon)$

$$\mathbb{P}\left\{ \sup_{0 \leq s \leq t} \frac{|\xi_s^0|}{\sqrt{\bar{v}(s)}} > h \right\} = C_0(t, \varepsilon) e^{-h^2/2\sigma^2}$$

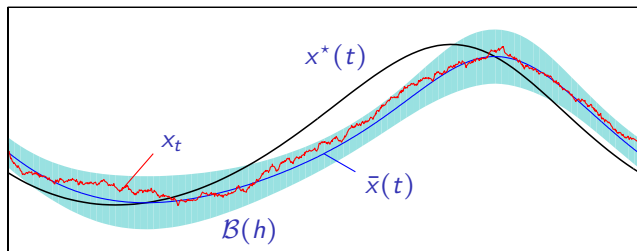
where $C_0(t, \varepsilon) = \sqrt{\frac{2}{\pi} \frac{1}{\varepsilon}} \left| \int_0^t \bar{a}(s) ds \right| \frac{h}{\sigma} \left[1 + \mathcal{O}(\varepsilon + \frac{t}{\varepsilon} e^{-h^2/\sigma^2}) \right]$

Proof based on Doob's submartingale inequality and partition of $[0, t]$

Proof ideas, 1D SDE below threshold

Nonlinear equation: $d\xi_t = \frac{1}{\varepsilon} [\bar{a}(t)\xi_t + b(t, \xi_t)] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_t$

Confidence strip: $\mathcal{B}(h) = \{|\xi| \leq h\sqrt{\bar{v}(t)} \forall t\} = \{|x - \bar{x}(t)| \leq h\sqrt{\bar{v}(t)} \forall t\}$



Theorem B & Gentz, PTRF 2002

$$C(t, \varepsilon) e^{-\kappa_- h^2 / 2\sigma^2} \leq \mathbb{P}\{\text{leaving } \mathcal{B}(h) \text{ before time } t\} \leq C(t, \varepsilon) e^{-\kappa_+ h^2 / 2\sigma^2}$$

where $\kappa_{\pm} = 1 \mp \mathcal{O}(h)$ and $C(t, \varepsilon) = C_0(t, \varepsilon)[1 + \mathcal{O}(h)]$ (requires $h \leq h_0$)

Avoided transcritical bifurcation

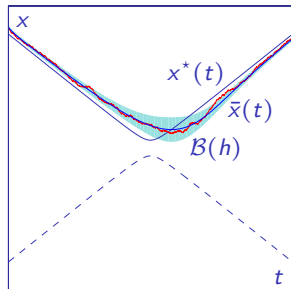
$$dx_t = \frac{1}{\varepsilon} [t^2 + \delta - x_t^2 + \dots] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_t$$

Equil. curve: $x^*(t) \simeq \sqrt{t^2 + \delta}$

Slow sol.: $\bar{x}(t) = x^*(t) + \mathcal{O}(\min\{\frac{\varepsilon}{|t|}, \frac{\varepsilon}{\sqrt{\delta+\varepsilon}}\})$

$$\bar{a}(t) = \partial_x f(t, \bar{x}(t)) \asymp \begin{cases} -|t| & |t| \geq \sqrt{\delta + \varepsilon} \\ -\sqrt{\delta + \varepsilon} & |t| \leq \sqrt{\delta + \varepsilon} \end{cases}$$

Confidence strip $\mathcal{B}(h)$: width $\asymp h/\sqrt{|\bar{a}(t)|}$



Theorem [B & Gentz, AAP 2002]

$$\mathbb{P}\{\text{leaving } \mathcal{B}(h) \text{ before time } t\} \leq C(t, \varepsilon) e^{-\kappa h^2/2\sigma^2}$$

where $\kappa = 1 - \mathcal{O}(\sup_{s \leq t} h|\bar{a}(s)|^{-3/2}) - \mathcal{O}(\varepsilon)$ requires $h < h_0 \inf_{s \leq t} |\bar{a}(s)|^{3/2}$

- ▷ $\sigma < \sigma_c = \max\{\delta, \varepsilon\}^{3/4}$: result applies $\forall t$, $\mathbb{P}\{\text{trans}\} = \mathcal{O}(e^{-\kappa\sigma_c^2/\sigma^2})$
- ▷ $\sigma > \sigma_c = \max\{\delta, \varepsilon\}^{3/4}$: result applies up to $t \asymp -\sigma^{2/3}$

Above threshold

What happens for $\sigma > \sigma_c$ and $t > -\sigma^{2/3}$?

General principle: partition $t_0 = s_0 < s_1 < s_2 < \dots < s_n = t$ of $[t_0, t]$

Lemma Let $P_k = \mathbb{P}\{\text{making no transition during } (s_{k-1}, s_k]\}$. Then

$$\mathbb{P}\{\text{making no transition during } [t_0, t]\} \leq \prod_{k=1}^n P_k$$

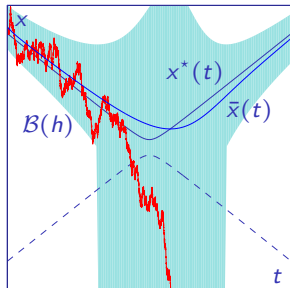
Choose partition s.t. each $P_k \leq q < 1 \Rightarrow \mathbb{P}\{\text{no transition}\} \leq e^{-n \log q}$

Define partition such that

$$\int_{s_{k-1}}^{s_k} |\bar{a}(s)| ds = c\epsilon |\log \sigma| \Rightarrow P_k \leq \frac{2}{3}$$

Thm [B & Gentz, AAP 2002]

Transition probability $\geq 1 - e^{-\kappa \sigma^{4/3} / (\epsilon |\log \sigma|)}$



SPDE: stable case

$$d\phi(t, x) = \frac{1}{\varepsilon} [\Delta\phi(t, x) + f(t, \phi(t, x))] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW(t, x)$$

- ▷ $f(t, \phi^*(t)) = 0$ for all $t \in I = [0, T]$
- ▷ $a(t) = \partial_\phi f(t, \phi^*(t)) \leq -a_- < 0$ for all $t \in I$

In deterministic case $\sigma = 0$: \exists particular solution $\bar{\phi}(t, x)$ such that

$$\|\bar{\phi}(t, \cdot) - \phi^*(t)\mathbf{e}_0\|_{H^1} \leq C\varepsilon \quad \forall t \in I$$

Theorem [B & Nader 2021]

Fix $s < \frac{1}{2}$, and let $\mathcal{B}(h) = \{(t, \phi) : t \in I, \|\phi - \bar{\phi}(t, \cdot)\|_{H^s} < h\}$

For any $\nu > 0$

$$\mathbb{P}\{\text{leaving } \mathcal{B}(h) \text{ before time } t\} \leq C(t, \varepsilon, s) \exp\left\{-\kappa \frac{h^2}{\sigma^2} \left[1 - \mathcal{O}\left(\frac{h}{\varepsilon^\nu}\right)\right]\right\}$$

holds for some $\kappa > 0$, $h = \mathcal{O}(\varepsilon^\nu)$ and $C(t, \varepsilon, s) = \mathcal{O}(t/\varepsilon)$.

Ideas of proof

▷ $\phi(x) = \sum_{k \in \mathbb{Z}} \phi_k e_k(x) \Rightarrow \|\phi\|_{H^s}^2 = \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} \phi_k^2, \quad \langle k \rangle = \sqrt{1+k^2}$

▷ Deterministic case: $\psi = \phi - \phi^* e_0$, $\|\psi\|_{H^1}^2$ is a Lyapunov function

▷ Linear stoch case:

$$d\psi_k = \frac{1}{\varepsilon} a_k(t) \psi_k dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_k(t), \quad a_k(t) = \bar{a}(t) - \frac{k^2 \pi^2}{L^2} < 0$$

For any decomposition $h = \sum_k h_k$,

$$\mathbb{P}\{\mathcal{T} < T\} \leq \sum_k \mathbb{P}\left\{ \sup_t \psi_k(t)^2 \geq h_k^2 \langle k \rangle^{-2s} \right\} \leq \sum_k C_k(T, \varepsilon) e^{-\kappa h_k^2 \langle k \rangle^{2-2s} / \sigma^2}$$

Choose $h_k^2 \sim h^2 \langle k \rangle^{-2+2s+\eta}$, $\eta > 0$

▷ Schauder estimate: $\beta \in H^r$, $0 < r < \frac{1}{2} \Rightarrow$

$$\|e^{t\Delta} \beta\|_{H^q} \leq M(q, r) t^{-(q-r)/2} \|\beta\|_{H^r} \quad \forall q < r+2$$

Consequence: $\psi = \psi^0 + \psi^1$ where nonlinear term satisfies

$$\|\psi^1\|_{H^q} \leq M' \varepsilon^{(q-r)/2-1} \sup_t \|b(t, \psi(y, \cdot))\|_{H^r}$$

SPDE near a bifurcation point

$$d\phi = \frac{1}{\varepsilon} [\Delta\phi + g(t) - \phi^2 - b(t, \phi)] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW(t, x)$$

with $g(t) = \delta + t^2 + \mathcal{O}(t^3)$ and $b = \mathcal{O}(\phi^3 + t\phi^2 + t^2\phi)$

- ▷ Decompose $\phi(t, x) = \phi_0(t)e_0(x) + \phi_\perp(t, x)$ where e_0 constant fct
- ▷ ϕ_\perp satisfies similar concentration result as ϕ in stable case
- ▷ ϕ_0 satisfies similar equation as in 1D, with error term of order $\|\phi_\perp\|_{H^s}^2$

Thm 1: Transverse component

$$\mathbb{P}\{\tau_{\mathcal{B}_\perp}(h_\perp) < t \wedge \tau_{\mathcal{B}_0}(h)\} \leq C(t, \varepsilon, s) \exp\left\{-\kappa \frac{h^2}{\sigma^2} \left[1 - \mathcal{O}\left(\frac{h}{\varepsilon^\nu}\right)\right]\right\}$$

Thm 2: Mean

$$\mathbb{P}\{\tau_{\mathcal{B}_0}(h) < t \wedge \tau_{\mathcal{B}_\perp}(h_\perp)\} \leq C(t, \varepsilon) e^{-\kappa h^2/2\sigma^2} \quad \kappa = 1 - \mathcal{O}\left(\sup_s h|\bar{a}(s)|^{3/2}\right)$$

Thm 3: Escape

$$\mathbb{P}\{\phi_0(t_1) > -d \forall t \in [-\sigma^{2/3}, t \wedge \tau_{\mathcal{B}_\perp}(h_\perp)]\} \leq \frac{3}{2} e^{-\hat{\alpha}(t, -\sigma^{2/3})/[\varepsilon \log(\sigma^{-1})]}$$

SPDE on the 2d torus

$$d\phi(t, x) = \frac{1}{\varepsilon} \left[\Delta\phi(t, x) + \sum_{j=1}^n A_j(t) \phi(t, x)^j \right] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW(t, x) \quad x \in \mathbb{T}^2$$

- ▷ SPDE is not well-posed, needs to be renormalised

For $N \in \mathbb{N}$, project on $\text{span}\{e_k\}_{|k| < N}$:

$$d\phi(t, x) = \frac{1}{\varepsilon} \left[\Delta\phi(t, x) + \sum_{j=1}^n A_j(t) P_N: \phi(t, x)^j: \right] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_N(t, x)$$

where $: \phi^n: = H_n(\phi; C_N)$ Wick power, $C_N \sim \log N$ variance of GFF

- ▷ Let ψ be stochastic convolution:

$$d\psi(t, x) = \frac{1}{\varepsilon} \Delta\psi(t, x) dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_N(t, x)$$

[Da Prato & Debussche '03]: $\phi - \psi$ cv to well-defined function

- ▷ Use Besov–Hölder spaces $\mathcal{B}_{2, \infty}^\alpha$, $\alpha < 0$, instead of Sobolev spaces H^s :

$$\|\phi\|_{\mathcal{B}_{2, \infty}^\alpha} = \sup_{q \geq 0} 2^{q\alpha} \|\delta_q \phi\|_{L^2} \quad \delta_q \phi = \sum_{2^{q-1} \leq |k| < 2^q} \phi_k e_k$$

SPDE on the 2d torus

Theorem [B & Nader 2022]

For $\alpha < 0$, $m \in \mathbb{N}$,

$$\mathbb{P} \left\{ \sup_{0 \leq t \leq T} \|\cdot \psi(t, \cdot)^m\|_{\mathcal{B}_{2,\infty}^\alpha} > h^m \right\} \leq C_m(T, \varepsilon, \alpha) e^{-\kappa_m(\alpha) h^2 / \sigma^2}$$

where

$$\kappa_m(\alpha) \geq c_0 \frac{\alpha^2}{m^7} \quad C_m(T, \varepsilon, \alpha) \leq c_1 \frac{T}{\varepsilon} \frac{m^{3/2} e^m m^m}{|\alpha|}$$

▷ Binomial formula

$$:\psi^m := H_m(\psi; C_N) = \sum_{|\mathbf{n}|=m} \frac{m!}{\mathbf{n}!} \prod_{q \geq 0} H_{\mathbf{n}_q}(\delta_q \psi; c_q) \quad c_q = \mathcal{O}(1)$$

▷ Doob submartingale inequality for $\sup_{t \in I_\ell} \|\delta_{q_0}(\prod_{q \geq 0} H_{\mathbf{n}_q}(\delta_q \hat{\psi}; c_q))\|_{L^2}^2$

where $\hat{\psi}$ martingale approximating ψ on intervals I_ℓ depending on q_0

▷ Upgrade to bound for $\sup_{t \in I_\ell} \|\delta_{q_0}(\prod_{q \geq 0} H_{\mathbf{n}_q}(\delta_q \psi; c_q))\|_{L^2}^2$

▷ Bound probability by summing over ℓ , q_0 and \mathbf{n}

Concentration estimates

Theorem [B & Nader 2022]

Let $\phi_1 = \phi - \phi^* - \psi$. Then $\forall \gamma < 2, \forall \nu < 1 - \frac{\gamma}{2}, \forall h < h_0 \varepsilon^\nu$

$$\mathbb{P}\left\{ \sup_{t \in [0, T]} \|\phi_1(t)\|_{C^{\gamma-1}} > M \varepsilon^{-\nu} h (h + \varepsilon) \right\} \leq C(T, \varepsilon) e^{-\kappa h^2 / \sigma^2}$$

- ▶ Use $\|\phi^\ell : \psi^m\|_{\mathcal{B}_{2,\infty}^{(2\ell+1)\alpha}} \leq \|\phi\|_{\mathcal{B}_{2,\infty}^\ell} \|\psi^m\|_{\mathcal{B}_{2,\infty}^\alpha}$ to bound nonlinear term in $d\phi_1$
- ▶ Use Schauder estimate and $\mathcal{B}_{2,\infty}^\gamma \hookrightarrow \mathcal{B}_{\infty,\infty}^{\gamma-1} = C^{\gamma-1}$

Example: Dynamic pitchfork bifurcation

$$d\phi(t, x) = \frac{1}{\varepsilon} [\Delta \phi(t, x) + a(t)\phi(t, x) - : \phi(t, x)^3 :] dt + \frac{\sigma}{\sqrt{\varepsilon}} dW(t, x)$$

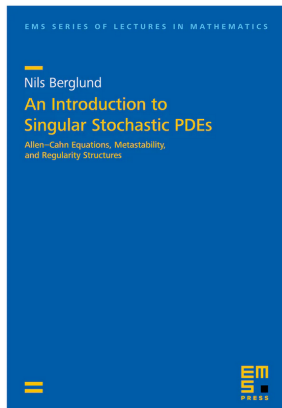
- ▶ Decompose $\phi = \psi_\perp + \phi_1$, where ψ_\perp stoch conv driven by W_\perp
- ▶ Decompose $\phi_1 = \phi_1^0 \mathbf{e}_0 + \phi_1^\perp$
- ▶ Concentration estimate for $\|\phi_1^\perp\|_{C^{\gamma-1}}$ as long as $a(t) < (2\pi)^2$
- ▶ For ϕ_1^0 , similar results as in SDE case (bif delay of order $\sqrt{\varepsilon \log(\sigma^{-1})}$)

Open questions

- ▷ Case $x \in \mathbb{T}^3$? Regularity structures or similar needed . . .

References

- ▷ N. B. & Barbara Gentz, *A sample-paths approach to noise-induced synchronization: Stochastic resonance in a double-well potential*, Ann. Appl. Probab., 12(1):1419–1470, 2002
- ▷ _____, *Noise-Induced Phenomena in Slow-Fast Dynamical Systems. A Sample-Paths Approach*, Springer, Probability and its Applications (2005)
- ▷ N. B. & Rita Nader, *Stochastic resonance in stochastic PDEs*, Stochastics & PDEs: Analysis and Computation, (2022)
- ▷ _____, *Concentration estimates for slowly time-dependent singular SPDEs on the two-dimensional torus*, arXiv/2209.15357 (2022)
- ▷ N. B., *An Introduction to Singular Stochastic PDEs*, EMS Press (2022)



Slides available at <https://www.idpoisson.fr/berglund/Berlin22.pdf>