Eighth Workshop on Random Dynamical Systems

Regularity structures and renormalisation of FitzHugh–Nagumo SPDEs in three space dimensions

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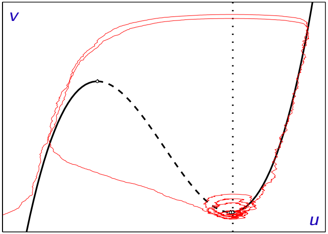
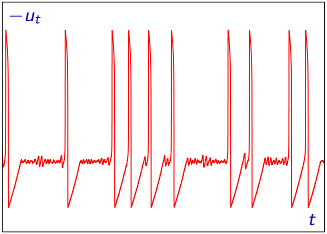
FitzHugh–Nagumo SDE

$$\begin{aligned} \mathrm{d}u_t &= [u_t - u_t^3 + v_t]\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \\ \mathrm{d}v_t &= \varepsilon[a - u_t - bv_t]\,\mathrm{d}t \end{aligned}$$

- u_t: membrane potential of neuron
- v_t: gating variable (proportion of open ion channels)

$\varepsilon = 0.1$
$b = 0$
$a = \frac{1}{\sqrt{3}} + 0.02$
$\sigma = 0.03$





FitzHugh–Nagumo SPDE

$$\partial_t u = \Delta u + u - u^3 + v + \xi$$
$$\partial_t v = a_1 u + a_2 v$$

- $u = u(t,x) \in \mathbb{R}$, $v = v(t,x) \in \mathbb{R}$ (or $\mathbb{R}^n$), $(t,x) \in D = \mathbb{R}_+ \times \mathbb{T}^d$, $d = 2,3$
- $\xi(t,x)$ Gaussian space-time white noise: $\mathbb{E}\big[\xi(t,x)\xi(s,y)\big] = \delta(t-s)\delta(x-y)$

  ξ: distribution defined by $\langle \xi, \varphi \rangle = W_\varphi$, $\{W_h\}_{h \in L^2(D)}$, $\mathbb{E}[W_h W_{h'}] = \langle h, h' \rangle$

(Link to simulation)

Main result

Mollified noise: $\xi^\varepsilon = \varrho_\varepsilon * \xi$
where $\varrho_\varepsilon(t,x) = \frac{1}{\varepsilon^{d+2}}\varrho\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big)$ with ϱ compactly supported, integral 1

Theorem [NB & C. Kuehn, preprint 2015, arXiv/1504.02953]

There exists a choice of renormalisation constant $C(\varepsilon)$, $\lim_{\varepsilon\to 0} C(\varepsilon) = \infty$, such that

$$\begin{aligned} \partial_t u^\varepsilon &= \Delta u^\varepsilon + [1 + C(\varepsilon)]u^\varepsilon - (u^\varepsilon)^3 + v^\varepsilon + \xi^\varepsilon \\ \partial_t v^\varepsilon &= a_1 u^\varepsilon + a_2 v^\varepsilon \end{aligned}$$

admits a sequence of local solutions $(u^\varepsilon, v^\varepsilon)$, converging in probability to a limit (u, v) as $\varepsilon \to 0$.

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admits a sequence of local solutions $(u^\varepsilon, v^\varepsilon)$, converging in probability to a limit (u, v) as $\varepsilon \to 0$.

- Local solution means up to a random possible explosion time
- Initial conditions should be in appropriate Hölder spaces
- $C(\varepsilon) \asymp \log(\varepsilon^{-1})$ for $d = 2$ and $C(\varepsilon) \asymp \varepsilon^{-1}$ for $d = 3$
- Similar results for more general cubic nonlinearity and $v \in \mathbb{R}^n$

Mild solutions of SPDE

$$
\begin{aligned}
\partial_t u &= \Delta u + F(u) + \xi \\
u(0,x) &= u_0(x)
\end{aligned}
$$

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Construction of mild solution via Duhamel formula:

- $\partial_t u = \Delta u \quad \Rightarrow \quad u(t,x) = \int G(t,x-y)u_0(y)\,\mathrm{d}y =: (\mathrm{e}^{\Delta t}\,u_0)(x)$

  where $G(t,x)$: heat kernel (compatible with bc)

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- $\partial_t u = \Delta u + f \quad \Rightarrow \quad u(t,x) = (\mathrm{e}^{\Delta t} u_0)(x) + \int_0^t \mathrm{e}^{\Delta(t-s)} f(s,\cdot)(x)\,\mathrm{d}s$

  Notation: $u = Gu_0 + G * f$

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- $\partial_t u = \Delta u + \xi \quad \Rightarrow \quad u = Gu_0 + G * \xi \quad$ (stochastic convolution)

- $\partial_t u = \Delta u + \xi + F(u) \quad \Rightarrow \quad u = Gu_0 + G * [\xi + F(u)]$

  Aim: use Banach's fixed-point theorem — but which function space?

Hölder spaces

Definition of $\mathcal{C}^\alpha$ for $f : I \to \mathbb{R}$, with $I \subset \mathbb{R}$ a compact interval:

- $0 < \alpha < 1$: $|f(x) - f(y)| \leqslant C|x-y|^\alpha \quad \forall x \neq y$
- $\alpha > 1$: $f \in \mathcal{C}^{\lfloor \alpha \rfloor}$ and $f' \in \mathcal{C}^{\alpha - 1}$
- $\alpha < 0$: f distribution, $|\langle f, \eta_x^\delta \rangle| \leqslant C\delta^\alpha$

  where $\eta_x^\delta(y) = \frac{1}{\delta}\eta(\frac{x-y}{\delta})$ for all test functions $\eta \in \mathcal{C}^{-\lfloor \alpha \rfloor}$

Property: $f \in \mathcal{C}^\alpha$, $0 < \alpha < 1 \quad \Rightarrow \quad f' \in \mathcal{C}^{\alpha-1}$ where $\langle f', \eta \rangle = -\langle f, \eta' \rangle$

Remark: $f \in \mathcal{C}^{1+\alpha} \not\Rightarrow |f(x) - f(y)| \leqslant C|x-y|^{1+\alpha}$. See e.g $f(x) = x + |x|^{3/2}$

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Case of the heat kernel: $(\partial_t - \Delta)u = f \quad \Rightarrow \quad u = G * f$

Parabolic scaling $\mathcal{C}_\mathfrak{s}^\alpha$: $|x-y| \longrightarrow |t-s|^{1/2} + \sum_{i=1}^d |x_i - y_i|$

$$\frac{1}{\delta}\eta\left(\frac{x-y}{\delta}\right) \longrightarrow \frac{1}{\delta^{d+2}}\eta\left(\frac{t-s}{\delta^2}, \frac{x-y}{\delta}\right)$$

Schauder estimates and fixed-point equation

Schauder estimate

$\alpha \notin \mathbb{Z},\ f \in \mathcal{C}^{\alpha}_{\mathfrak{s}} \quad \Rightarrow \quad G * f \in \mathcal{C}^{\alpha+2}_{\mathfrak{s}}$

Fact: in dimension d, space-time white noise $\xi \in \mathcal{C}^{\alpha}_{\mathfrak{s}}$ a.s. $\forall \alpha < -\frac{d+2}{2}$

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Fixed-point equation: $u = Gu_0 + G * [\xi + F(u)]$

- $d = 1$: $\xi \in \mathcal{C}^{-3/2^-}_{\mathfrak{s}} \Rightarrow G * \xi \in \mathcal{C}^{1/2^-}_{\mathfrak{s}} \Rightarrow F(u)$ defined
- $d = 3$: $\xi \in \mathcal{C}^{-5/2^-}_{\mathfrak{s}} \Rightarrow G * \xi \in \mathcal{C}^{-1/2^-}_{\mathfrak{s}} \Rightarrow F(u)$ not defined
- $d = 2$: $\xi \in \mathcal{C}^{-2^-}_{\mathfrak{s}} \Rightarrow G * \xi \in \mathcal{C}^{0^-}_{\mathfrak{s}} \Rightarrow F(u)$ not defined

  Boundary case, can be treated with Besov spaces
  [Da Prato & Debussche 2003]

Why not use mollified noise? Limit $\varepsilon \to 0$ does not exist

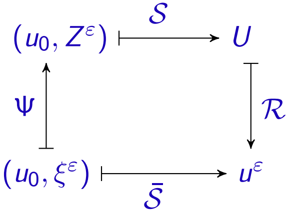
Regularity structures

Basic idea of Martin Hairer [Inventiones Math. **198**, 269–504, 2014]:

Lift mollified fixed-point equation

$$u = Gu_0 + G * [\xi^\varepsilon + F(u)]$$

to a larger space called a Regularity structure



- $u^\varepsilon = \bar{\mathcal{S}}(u_0, \xi^\varepsilon)$: classical solution of mollified equation
- $U = \mathcal{S}(u_0, Z^\varepsilon)$: solution map in regularity structure
- $\mathcal{S}$ and $\mathcal{R}$ are continuous (in suitable topology)

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$$\begin{array}{ccc}
(u_0, MZ^\varepsilon) & \xrightarrow{\;\mathcal{S}_M\;} & U_M \\
\Big\uparrow{\scriptstyle M\Psi} & & \Big\downarrow{\scriptstyle \mathcal{R}^M} \\
(u_0, \xi^\varepsilon) & \xrightarrow[\;\bar{\mathcal{S}}_M\;]{} & \hat{u}^\varepsilon
\end{array}$$

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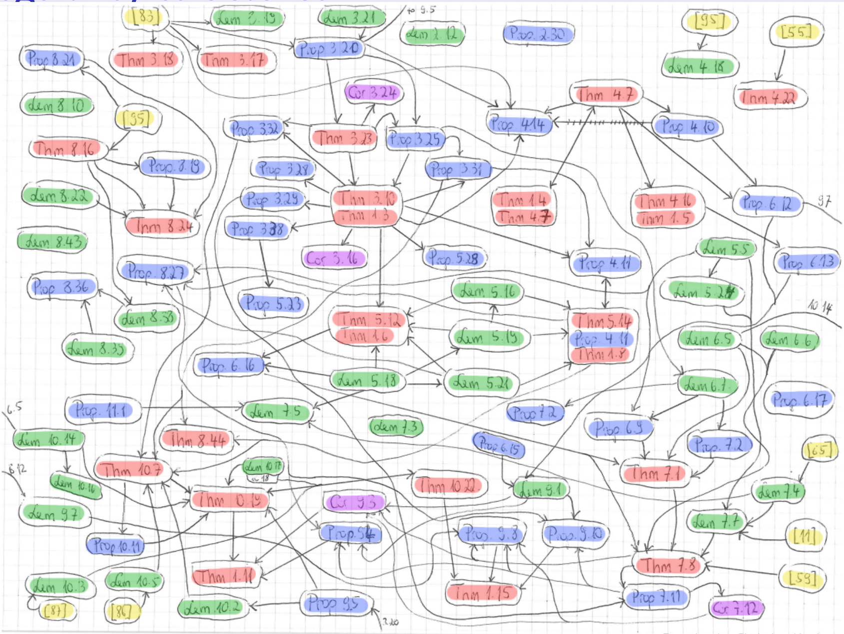
- $U = \mathcal{S}(u_0, Z^\varepsilon)$: solution map in regularity structure
- $\mathcal{S}$ and $\mathcal{R}$ are continuous (in suitable topology)
- Renormalisation: modification of the lift Ψ

Aternative approaches for $d = 3$: [Catellier & Chouk '13], [Kupiainen '15]

Regularity structures



Structure of Hairer, Invent. Math. 198:269-504 (2014)

Drawing by Christian Kuehn

Basic idea: Generalised Taylor series

$f : I \to \mathbb{R}$, $0 < \alpha < 1$

$f \in \mathcal{C}^{2+\alpha} \quad \Leftrightarrow \quad f \in \mathcal{C}^2$ and $f'' \in \mathcal{C}^\alpha$

Associate with f the triple (f, f', f'')

When does a triple (f_0, f_1, f_2) represent a function $f \in \mathcal{C}^{2+\alpha}$?

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When there is a constant C such that for all $x, y \in I$

$$\begin{aligned}
|f_0(y) - f_0(x) - (y-x)f_1(x) - \tfrac{1}{2}(y-x)^2 f_2(x)| &\leqslant C|x-y|^{2+\alpha} \\
|f_1(y) - f_1(x) - (y-x)f_2(x)| &\leqslant C|x-y|^{1+\alpha} \\
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\end{aligned}$$

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Notation: $f = f_0\mathbf{1} + f_1 X + f_2 X^2$

Regularity structure: Generalised Taylor basis whose basis elements can also be singular distributions

Definition of a regularity structure

Definition [M. Hairer, Inventiones Math 2014]

A Regularity structure is a triple $(A, T, \mathcal{G})$ where

1. Index set: $A \subset \mathbb{R}$, bdd below, locally finite, $0 \in A$
2. Model space: $T = \bigoplus_{\alpha \in A} T_\alpha$, each T_α Banach space, $T_0 = \text{span}(\mathbf{1}) \simeq \mathbb{R}$
3. Structure group: $\mathcal{G}$ group of linear maps $\Gamma : T \to T$ such that
$$\Gamma\tau - \tau \in \bigoplus_{\beta<\alpha} T_\beta \qquad \forall \tau \in T_\alpha$$
and $\Gamma\mathbf{1} = \mathbf{1}\ \forall\Gamma \in \mathcal{G}$.

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Polynomial regularity structure on $\mathbb{R}$:

- $A = \mathbb{N}_0$
- $T_k \simeq \mathbb{R}$, $T_k = \text{span}(X^k)$
- $\Gamma_h(X^k) = (X - h)^k\ \forall h \in \mathbb{R}$

Polynomial reg. structure on $\mathbb{R}^d$: $X^k = X_1^{k_1} \dots X_d^{k_d} \in T_{|k|}$, $|k| = \sum_{i=1}^d k_i$

Regularity structure for $\partial_t u = \Delta u - u^3 + \xi$

New symbols: Ξ, representing ξ, Hölder exponent $|\Xi|_\mathfrak{s} = \alpha_0 = -\frac{d+2}{2} - \kappa$
$\mathcal{I}(\tau)$, representing $G * f$, Hölder exponent $|\mathcal{I}(\tau)|_\mathfrak{s} = |\tau|_\mathfrak{s} + 2$
$\tau\sigma$, Hölder exponent $|\tau\sigma|_\mathfrak{s} = |\tau|_\mathfrak{s} + |\sigma|_\mathfrak{s}$

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τ	Symbol	$\lvert\tau\rvert_\mathfrak{s}$	$d = 3$	$d = 2$
Ξ	Ξ	α_0	$-\frac{5}{2} - \kappa$	$-2 - \kappa$
$\mathcal{I}(\Xi)^3$		$3\alpha_0 + 6$	$-\frac{3}{2} - 3\kappa$	$0 - 3\kappa$
$\mathcal{I}(\Xi)^2$		$2\alpha_0 + 4$	$-1 - 2\kappa$	$0 - 2\kappa$
$\mathcal{I}(\mathcal{I}(\Xi)^3)\mathcal{I}(\Xi)^2$		$5\alpha_0 + 12$	$-\frac{1}{2} - 5\kappa$	$2 - 5\kappa$
$\mathcal{I}(\Xi)$		$\alpha_0 + 2$	$-\frac{1}{2} - \kappa$	$0 - \kappa$
$\mathcal{I}(\mathcal{I}(\Xi)^3)\mathcal{I}(\Xi)$		$4\alpha_0 + 10$	$0 - 4\kappa$	$2 - 4\kappa$
$\mathcal{I}(\mathcal{I}(\Xi)^2)\mathcal{I}(\Xi)^2$		$4\alpha_0 + 10$	$0 - 4\kappa$	$2 - 4\kappa$
$\mathcal{I}(\Xi)^2 X_i$	 X_i	$2\alpha_0 + 5$	$0 - 2\kappa$	$1 - 2\kappa$
$\mathbf{1}$	$\mathbf{1}$	0	0	0
$\mathcal{I}(\mathcal{I}(\Xi)^3)$		$3\alpha_0 + 8$	$\frac{1}{2} - 3\kappa$	$2 - 3\kappa$
...	...	...	...	...

Fixed-point equation for $\partial_t u = \Delta u - u^3 + \xi$

$$u = G * [\xi^\varepsilon - u^3] + G u_0 \qquad \Rightarrow \qquad U = \mathcal{I}(\Xi - U^3) + \varphi \mathbf{1} + \dots$$

$U_0 = 0$

$U_1 =$  $+ \varphi \mathbf{1}$

$U_2 =$  $+ \varphi \mathbf{1} -$  $+ 3\varphi$  $+ \dots$

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To prove convergence, we need

- A model (Π, Γ): $\forall z \in \mathbb{R}^{d+1}$, $\Pi_z\tau$ is distribution describing τ near z
  $\Gamma_{z\bar{z}} \in \mathcal{G}$ describes translations: $\Pi_{\bar{z}} = \Pi_z\Gamma_{z\bar{z}}$

- Spaces of modelled distributions
  $$\mathcal{D}^\gamma = \Big\{ f : \mathbb{R}^{d+1} \to \bigoplus_{\beta<\gamma} T_\beta \colon \|f(z) - \Gamma_{z\bar{z}}f(\bar{z})\|_\beta \lesssim \|z - \bar{z}\|_{\mathfrak{s}}^{\gamma-\beta} \Big\}$$
  equipped with a seminorm

- The Reconstruction theorem: provides a unique map $\mathcal{R} : \mathcal{D}^\gamma \to \mathcal{C}_{\mathfrak{s}}^{\alpha_*}$
  ($\alpha_* = \inf A$) s.t. $|\langle \mathcal{R}f - \Pi_z f(z), \eta_{\mathfrak{s},z}^\delta \rangle| \lesssim \delta^\gamma$
  (constructed using wavelets)

Canonical model $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon)$

Defined inductively by

$$
\begin{aligned}
(\Pi_z^\varepsilon \Xi)(\bar z) &= \xi^\varepsilon(\bar z) \\
(\Pi_z^\varepsilon X^k)(\bar z) &= (\bar z - z)^k \\
(\Pi_z^\varepsilon \tau\sigma)(\bar z) &= (\Pi_z^\varepsilon \tau)(\bar z)(\Pi_z^\varepsilon \sigma)(\bar z)
\end{aligned}
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(\Pi^\varepsilon_z \mathcal{I}(\tau))(\bar z) &= \int G(\bar z - z')(\Pi^\varepsilon_z \tau)(z')\,\mathrm{d}z'
\end{aligned}$$

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Then $\exists \mathcal{K}$ s.t. $\mathcal{R}\mathcal{K}f = G * \mathcal{R}f$ and the following diagrams commute:

$$
\begin{array}{ccc}
\mathcal{D}^\gamma & \xrightarrow{\ \mathcal{K}\ } & \mathcal{D}^{\gamma+2} \\
\mathcal{R}\big\downarrow & & \big\downarrow\mathcal{R} \\
\mathcal{C}_\mathfrak{s}^{\alpha_*} & \xrightarrow[G*]{} & \mathcal{C}_\mathfrak{s}^{\alpha_*}
\end{array}
\qquad\qquad
\begin{array}{ccc}
(u_0, Z^\varepsilon) & \xmapsto{\ \mathcal{S}\ } & U \\
\mathcal{R}\big\downarrow & & \big\downarrow\mathcal{R} \\
(u_0, \xi^\varepsilon) & \xmapsto[\bar{\mathcal{S}}]{} & u^\varepsilon
\end{array}
$$

where $\alpha_* = \inf A$ and $\mathcal{K}f = \mathcal{I}f + \text{polynomial term} + \text{nonlocal term}$

Why do we need to renormalise?

Let $G_\varepsilon = G * \varrho_\varepsilon$ where ϱ_ε is the mollifier

$$(\Pi^{\varepsilon}_{\bar z} \overset{\bullet}{\mid})(z) = (G * \xi^{\varepsilon})(z) = (G_\varepsilon * \xi)(z) = \int G_\varepsilon(z - z_1)\xi(z_1)\,\mathrm{d}z_1$$

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$$(\Pi_{\bar{z}}^{\varepsilon} \overset{\bullet\,\bullet}{\vee})(z) = (G * \xi^\varepsilon)(z)^2 = \iint G_\varepsilon(z - z_1) G_\varepsilon(z - z_2)\xi(z_1)\xi(z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2$$

diverges as $\varepsilon \to 0$

Why do we need to renormalise?

Let $G_\varepsilon = G * \varrho_\varepsilon$ where ϱ_ε is the mollifier

$$(\Pi^{\varepsilon}_{z} \overset{\bullet}{|})(z) = (G * \xi^{\varepsilon})(z) = (G_\varepsilon * \xi)(z) = \int G_\varepsilon(z - z_1)\xi(z_1)\, \mathrm{d}z_1$$

belongs to first Wiener chaos, limit $\varepsilon \to 0$ well-defined

$$(\Pi^{\varepsilon}_{z} \overset{\bullet\,\bullet}{\vee})(z) = (G * \xi^{\varepsilon})(z)^2 = \iint G_\varepsilon(z - z_1) G_\varepsilon(z - z_2)\xi(z_1)\xi(z_2)\, \mathrm{d}z_1\, \mathrm{d}z_2$$

diverges as $\varepsilon \to 0$

Wick product: $\xi(z_1) \diamond \xi(z_2) = \xi(z_1)\xi(z_2) - \delta(z_1 - z_2)$

$$(\Pi^{\varepsilon}_{z} \overset{\bullet\,\bullet}{\vee})(z) = \underbrace{\iint G_\varepsilon(z - z_1) G_\varepsilon(z - z_2)\xi(z_1) \diamond \xi(z_2)\, \mathrm{d}z_1\, \mathrm{d}z_2}_{\text{in 2nd Wiener chaos, bdd}} + \underbrace{\int G_\varepsilon(z - z_1)^2\, \mathrm{d}z_1}_{C_1(\varepsilon) \to \infty}$$

Renormalised model: $(\widehat{\Pi}^{\varepsilon}_{z} \overset{\bullet\,\bullet}{\vee})(z) = (\Pi^{\varepsilon}_{z} \overset{\bullet\,\bullet}{\vee})(z) - C_1(\varepsilon)$

The case of the FitzHugh–Nagumo equations

Fixed-point equation

$$u(t,x) = G * [\xi^\varepsilon + u - u^3 + v](t,x) + Gu_0(t,x)$$
$$v(t,x) = \int_0^t u(s,x)\, \mathrm{e}^{(t-s)a_2}\, a_1 \,\mathrm{d}s + \mathrm{e}^{ta_2}\, v_0$$

Lifted version

$$U = \mathcal{I}[\Xi + U - U^3 + V] + Gu_0$$
$$V = \mathcal{E}U + Qv_0$$

where $\mathcal{E}$ is an integration map which is not regularising in space
New symbols $\mathcal{E}(\mathcal{I}(\Xi)) =$ , etc. . .

We expect U, and thus also V to be α-Hölder for $\alpha < -\frac{1}{2}$
Thus $\mathcal{I}(U - U^3 + V)$ should be well-defined

The standard theory has to be extended, because $\mathcal{E}$ does not correspond to a smooth kernel

Details

Concluding remarks

- Models with $\partial_t u$ of order $u^4 + v^4$ and $\partial_t v$ of order $u^2 + v$ should be renormalisable
  Current approach does not work when singular part (t, x)-dependent
- Global existence: recent progress by J.-C. Mourrat and H. Weber on 2D Allen–Cahn
- More quantitative results?

References

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Details on implementing $\mathcal{E}$

Problems:

- Fixed-point equation requires diagonal identity $(\Pi_{t,x}\tau)(t,x) = 0$
- Usual definition of $\mathcal{K}$ would contain Taylor series

$$\mathcal{J}(z)\tau = \sum_{|k|_{\mathfrak{s}}<\alpha} \frac{X^k}{k!} \int D^k G(z-\bar{z})(\Pi_z\tau)(\mathrm{d}\bar{z})$$

$$\mathcal{N}f(z) = \sum_{|k|_{\mathfrak{s}}<\gamma} \frac{X^k}{k!} \int D^k G(z-\bar{z})(\mathcal{R}f - \Pi_z f(z))(\mathrm{d}\bar{z})$$

Conclusion

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Solution:

- Define $\Pi\mathcal{E}\tau$ only if $-2 < |\tau|_{\mathfrak{s}} < 0$ (otherwise $\mathcal{E}\tau = 0$) $\Rightarrow$ $\mathcal{J}(z)\tau = 0$
- Define $\mathcal{K}$ only for $f = \sum_{|\tau|_{\mathfrak{s}}<0} c_\tau \tau + \sum_{|\tau|_{\mathfrak{s}}\geqslant 0} c_\tau(t,x)\tau =: f_- + f_+$
  $\Rightarrow$ can take $\mathcal{R}f = \Pi_{t,x} f(t,x)$ and thus $\mathcal{N}f = 0$ for these f
- Time-convolution with Q lifted to

$$(\mathcal{K}^Q f)(t,x) = \sum_{|\tau|_{\mathfrak{s}}<0} c_\tau \mathcal{E}\tau + \sum_{|\tau|_{\mathfrak{s}}\geqslant 0} \int Q(t-s) c_\tau(s,x)\,\mathrm{d}s\,\tau =: (\mathcal{E}f_- + \mathcal{Q}f_+)(t,x)$$

Conclusion

Fixed-point equation

Consider $\partial_t u = \Delta_u + F(u, v) + \xi$ with F a polynomial of degree 3
If (U, V) satisfies fixed-point equation

$$U = \mathcal{I}[\Xi + F(U, V)] + Gu_0 + \text{polynomial term}$$
$$V = \mathcal{E}U_- + \mathcal{Q}U_+ + Qv_0$$

then $(\mathcal{R}U, \mathcal{R}V)$ is solution, provided $\mathcal{R}F(U, V) = F(\mathcal{R}U, \mathcal{R}V)$

⏩ Conclusion

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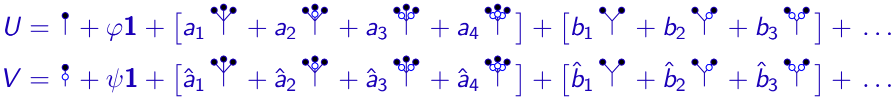
Fixed point is of the form



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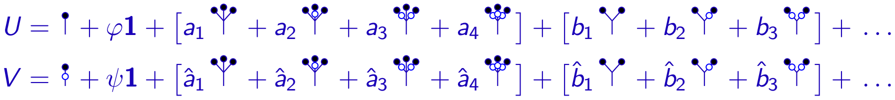

- Prove existence of fixed point in (modification of) $\mathcal{D}^\gamma$ with $\gamma = 1 + \bar\kappa$
- Extend from small interval $[0, T]$ up to first exit from large ball
- Deal with renormalisation procedure

Conclusion

Renormalisation

- Renormalisation group: group of linear maps $M : T \to T$
  Associated model: Π_z^M s.t. $\mathbf{\Pi}^M \tau = \mathbf{\Pi} M \tau$ where $\Pi_z = \mathbf{\Pi} \Gamma_{f_z}$

  Allen–Cahn eq.: $M = e^{-C_1 L_1 - C_2 L_2}$ with $L_1 :$  $\to \mathbf{1}$, $L_2 :$  $\to \mathbf{1}$
  FHN eq.: the same group suffices because Q is smoothing

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- Look for r.v. $\widehat{\Pi}_z \tau$ s.t. if $\widehat{\Pi}_z^{(\varepsilon)} = (\Pi_z^{(\varepsilon)})^{M_\varepsilon}$ then $\exists \kappa, \theta > 0$ s.t.

$$\mathbb{E} \big| \langle \widehat{\Pi}_z \tau, \eta_z^\lambda \rangle \big|^2 \lesssim \lambda^{2|\tau|_\mathfrak{s} + \kappa} \qquad \mathbb{E} \big| \langle \widehat{\Pi}_z \tau - \widehat{\Pi}_z^{(\varepsilon)} \tau, \eta_z^\lambda \rangle \big|^2 \lesssim \varepsilon^{2\theta} \lambda^{2|\tau|_\mathfrak{s} + \kappa}$$

  Then $(\widehat{\Pi}_z^{(\varepsilon)}, \widehat{\Gamma}_z^{(\varepsilon)})$ converges to limiting model, with explicit L^p bounds

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  Then $(\widehat{\Pi}_z^{(\varepsilon)}, \widehat{\Gamma}_z^{(\varepsilon)})$ converges to limiting model, with explicit L^p bounds

- Renormalised equations have nonlinearity $\widehat{F}$ s.t.
  $\widehat{F}(MU, MV) = MF(U, V) +$ terms of Hölder exponent > 0

  FHN eq. with cubic nonlinearity
  $F = \alpha_1 u + \alpha_2 v + \beta_1 u^2 + \beta_2 uv + \beta_3 v^2 + \gamma_1 u^3 + \gamma_2 u^2 v + \gamma_3 uv^2 + \gamma_4 v^3$
  $\widehat{F}(u, v) = F(u, v) - c_0(\varepsilon) - c_1(\varepsilon)u - c_2(\varepsilon)v$
  with the $c_i(\varepsilon)$ depending on C_1, C_2, provided either $d = 2$ or $\gamma_2 = 0$

Conclusion