Brandeis Mathematical Biology Seminar

Theory and applications of random Poincaré maps

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Online talk, January 27, 2021

Mainly based on joint works with Manon Baudel and Damien Landon

Outline

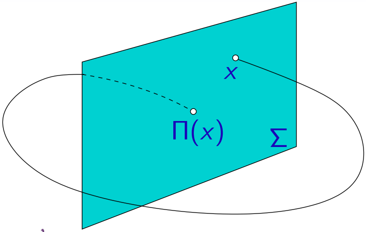
1. Definition of Random Poincaré maps
2. Application: Stochastic FitzHugh–Nagumo model
3. Spectral theory
4. Future research

1. Deterministic Poincaré maps

ODE $\dot{x} = f(x)$ $\quad x \in \mathbb{R}^n$

Flow: $x_t = \varphi_t(x_0)$

Σ transverse to f



Poincaré map: $\Pi : \Sigma \to \Sigma$

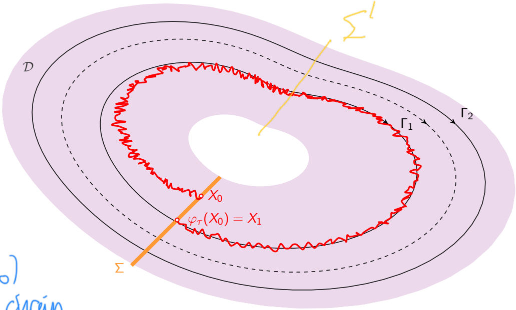
$$x \mapsto \varphi_\tau(x)$$

where $\tau = \inf\{t > 0 : \varphi_t(x) \in \Sigma\}$

Uses:
- visualization
- stability of periodic orbits
- bifurcation theory

Random Poincaré maps

SDE: $dx_t = f(x_t)\,dt + \sigma g(x_t)\,dW_t$



$\Pi(X_0) = \varphi_\tau(X_0)$

$\to$ Markov chain

[Euler scheme: $x_{t+\Delta t} \cong x_t + f(x_t)\Delta t + \sigma g(x_t)\sqrt{\Delta t}\,\underbrace{Z}_{\sim \mathcal{N}(0,1)}$]

⚠ $\inf\{t>0 : x_t \in \Sigma\} = 0$

Solution: $\tau' = \inf\{t>0 : x_t \in \Sigma'\}$

$\tau = \inf\{t>\tau' : x_t \in \Sigma\}$

Literature:

J. Weiss, E. Knobloch '90

P. Hitczenko, G. Medvedev '09

Markov chains

Discrete: $\mathcal{X} = \{1, \dots, N\}$

$\mathbb{P}\{X_{n+1} = y \mid X_n = x\} = p_{xy}$ $\qquad P = \begin{pmatrix} p_{11} & \cdots & p_{1N} \\ \vdots & & \vdots \\ p_{N1} & \cdots & p_{NN} \end{pmatrix}$ Stoch matrix

$\mathbb{P}^x\{X_1 = y\} = \mathbb{P}\{X_1 = y \mid X_0 = x\} = p_{xy}$

μ initial distribution: $\mathbb{P}^\mu\{X_1 = y\} = \sum_{x \in \mathcal{X}} \mu(x) p_{xy} = (\mu P)_y$

$f : \mathcal{X} \to \mathbb{R}$ observable: $\mathbb{E}^\mu[f(X_1)] = \sum_{x,y \in \mathcal{X}} \mu(x) p_{xy} f(y) = \mu P f$

Continuous space: Σ

kernel $k(x,y)$

$\mathbb{P}^x\{X_1 \in A\} = \int_A k(x,y)\, dy =: K(x,A)$

$\mathbb{E}^\mu[f(X_1)] = \int_\Sigma \int_\Sigma \mu(x) k(x,y) f(y)\, dx\, dy = \mu K f$

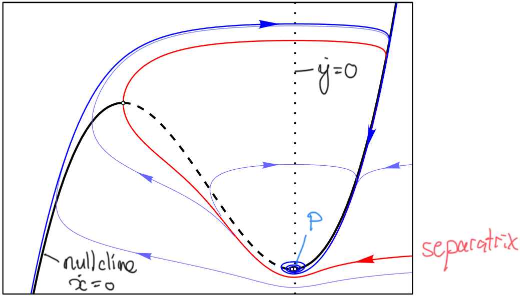
2. The FitzHugh–Nagumo equation

$$\mathrm{d}x_t = \frac{1}{\varepsilon}[x_t - x_t^3 + y_t]\,\mathrm{d}t \qquad \text{neuron membrane potential}$$

$$\mathrm{d}y_t = [a - x_t - by_t]\,\mathrm{d}t \qquad \text{open ion channels}$$

- $b = 0$ for simplicity in this talk, bifurcation parameter $\delta := \frac{3a^2-1}{2}$

Eq. pnt
$P = (a, a^3 - a)$
Focus if $|\delta| \leq \sqrt{\varepsilon}$
stable if $\delta > 0$



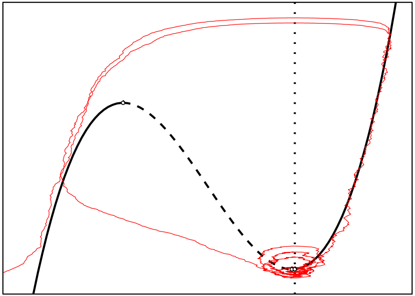
$\varepsilon = 0.1$, $\delta = 0.02$

2. The FitzHugh–Nagumo equation

$$dx_t = \frac{1}{\varepsilon}[x_t - x_t^3 + y_t]\,dt + \frac{\sigma_1}{\sqrt{\varepsilon}}\,dW_t^{(1)} \qquad \text{neuron membrane potential}$$

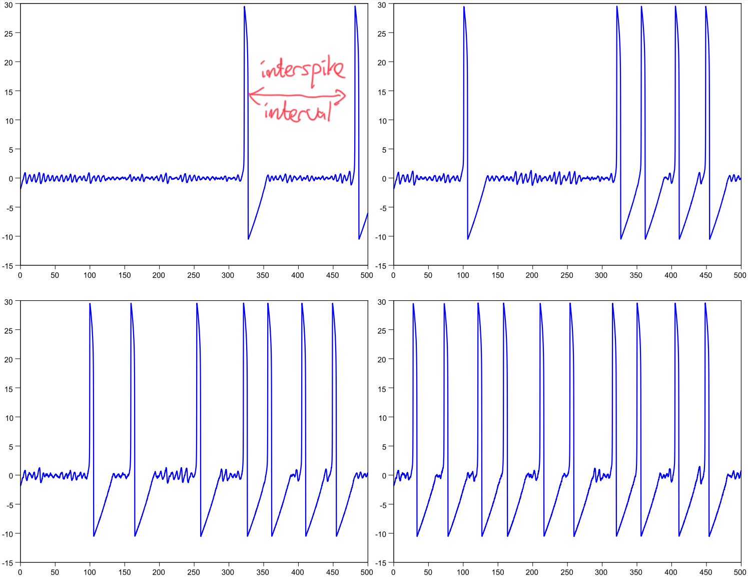
$$dy_t = [a - x_t - by_t]\,dt + \sigma_2\,dW_t^{(2)} \qquad \text{open ion channels}$$

- $b = 0$ for simplicity in this talk, bifurcation parameter $\delta := \frac{3a^2-1}{2}$



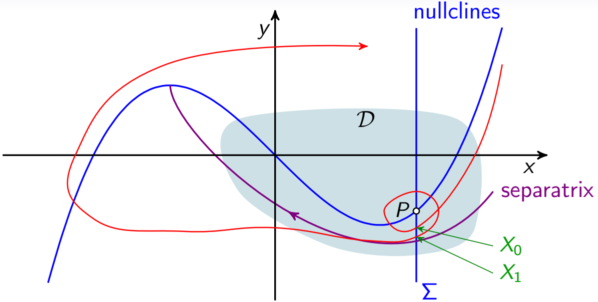
$\varepsilon = 0.1$, $\delta = 0.02$, $\sigma_1 = \sigma_2 = 0.03$

Mixed-mode oscillations (MMOs)



Time series $t \mapsto -x_t$ for $\varepsilon = 0.01$, $\delta = 3 \cdot 10^{-3}$, $\sigma = 1.46 \cdot 10^{-4}, \dots, 3.65 \cdot 10^{-4}$

Random Poincaré map



Consider process killed when leaving $\mathcal{D}$

M.C. $(X_0, X_1, \dots, X_N)$ $N = \#$ of oscillations

$$\underbrace{\int_\Sigma k(x,y)\,dy}_{\text{prob. not being killed}} < 1 \qquad \int_\Sigma k^n(x,y)\,dy = \mathbb{P}^x\{N > n\}$$

Random Poincaré map

Perron-Frobenius thm:

K has real max eigenvalue $\lambda_0 \in (0,1]$ Principal ev

Associated eigenfcts can be taken real and >0

$\pi_0 K = \lambda_0 \pi_0$ (i.e. $\int_\Sigma \pi_0(x) k(x,y)\,dx = \lambda_0 \pi_0(y)$)

$\int_\Sigma \pi_0(x)\,dx = 1$

$\Rightarrow$ if $X_0 \sim \pi_0$ then $X_n \sim \lambda_0^n \pi_0$ π_0: quasistationary distribution (QSD)

$\mathbb{P}^{\pi_0}\{N \geqslant n\} = \lambda_0^n$

$\mathbb{P}^{\pi_0}\{N = n\} = \lambda_0^n (1-\lambda_0)$ Geometric distribution

A qualitative result

Theorem 1 [B & Landon, Nonlinearity **25**:2303–2335, 2012]

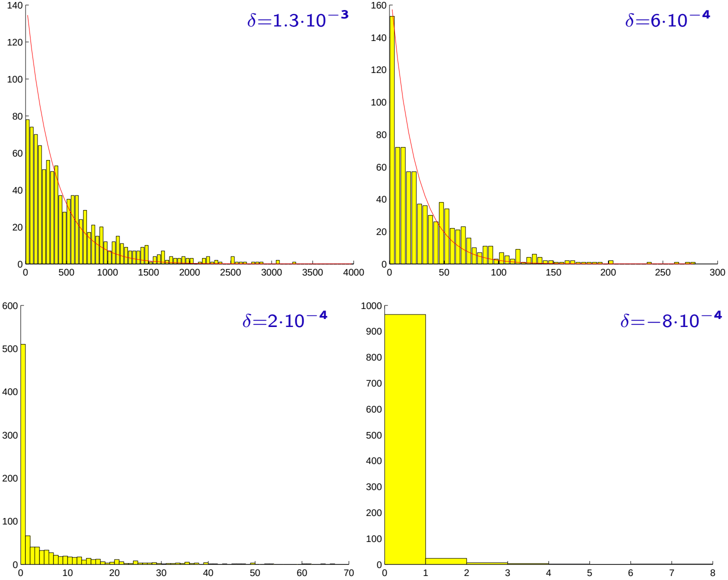
Assume $\sigma_1, \sigma_2 > 0$. Then $\lambda_0 < 1$, and for any initial distribution μ_0,

$$\lim_{n\to\infty} \mathbb{P}^{\mu_0}\{N = n+1 | N > n\} = 1 - \lambda_0$$

N is asymptotically geometric.

Histograms of distribution of N (1000 spikes)

$\sigma = \varepsilon = 10^{-4}$



Weak-noise regime

Theorem 2 [B & Landon, Nonlinearity **25**:2303–2335, 2012]

Assume $\varepsilon, \frac{\delta}{\sqrt{\varepsilon}} \ll 1$, $\sigma^2 \leqslant \frac{(\varepsilon^{1/4}\delta)^2}{\log(\sqrt{\varepsilon}\delta)}$. There exists $\kappa > 0$ s.t. for

- $1 - \lambda_0 \leqslant \exp\left\{-\kappa \frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$
- $\mathbb{E}^{\mu_0}[N] \geqslant C(\mu_0) \exp\left\{\kappa \frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$

New coords: $(x,y) \mapsto (\xi, z)$

(translation + scaling + quadratic transf)

$$\begin{cases} d\xi_t \simeq [1 - z_t]\,dt + \tilde{\sigma}_1\, dW_t^{(1)} \\ dz_t \simeq [\eta + \xi_t z_t]\,dt + \tilde{\sigma}_2\, dW_t^{(2)} \end{cases}$$

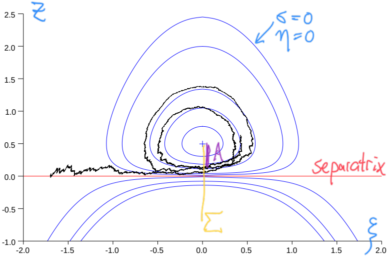
$\eta = \frac{\delta}{\sqrt{\varepsilon}} - \tilde{\sigma}_1^2$ $\qquad \tilde{\sigma}_i = \frac{\sigma_i}{\varepsilon^{3/4}}$

$A \subset \Sigma$ $\qquad \lambda_0 \pi_0(A) = (\pi_0 K)(A)$

$$\geqslant \int_A \pi_0(x) K(x,A)\,dx$$

$$\geqslant \pi_0(A) \inf_{x \in A} K(x,A)$$

$\Rightarrow \lambda_0 \geqslant \inf_{x \in A} K(x,A)$



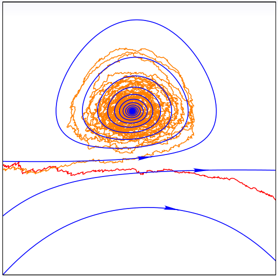
Intermediate noise regime

$$dz_t \simeq (\eta + t z_t)\,dt + \tilde{\sigma}_2\, dW_t^{(2)}$$

$$\Rightarrow \quad z_t = z_0 + \eta\, e^{2t^2} + \tilde{\sigma}_2\, e^{2t^2}\, \mathcal{N}(0,1)$$

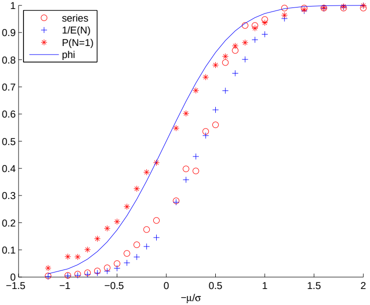
$$\Rightarrow \quad \underbrace{\mathbb{P}\{N=1\}}_{\text{spike immediately}} \simeq \lim_{t\to\infty} \mathbb{P}\{z_t < 0\} = \mathbb{P}\Big\{\mathcal{N}(0,1) < -\frac{\mu}{\tilde{\sigma}_2}\Big\} = \Phi\Big(-\frac{\mu}{\tilde{\sigma}_2}\Big)$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$

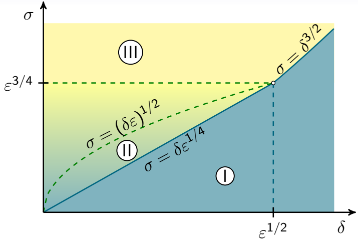


If $\mu_0 = \pi_0$ (QSD)

$$\mathbb{P}^{\pi_0}\{N=1\} = 1 - \lambda_0 = \frac{1}{\mathbb{E}^{\pi_0}[N]}$$



Summary: Parameter regimes



$\sigma_1 = \sigma_2$:

$$\mathbb{P}\{N = 1\} \simeq \Phi\Big(-\frac{(\pi\varepsilon)^{1/4}(\delta - \sigma^2/\varepsilon)}{\sigma}\Big)$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\,dy$$

see also

[Muratov & Vanden Eijnden '08]

Regime I: rare isolated spikes

Theorem 2 applies ($\delta \ll \varepsilon^{1/2}$)

Interspike interval $\simeq$ exponential

Regime II: clusters of spikes

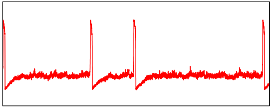
interspike osc asympt geometric

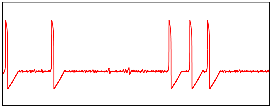
$\sigma = (\delta\varepsilon)^{1/2}$: geom(1/2)

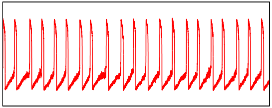
Regime III: repeated spikes

$\mathbb{P}\{N = 1\} \simeq 1$

Interspike interval $\simeq$ constant

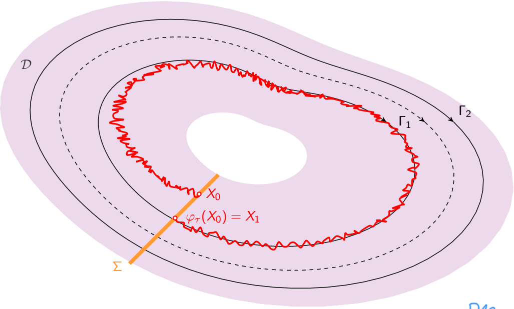




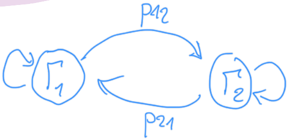


3. Spectral theory of random Poincaré maps

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sigma g(x_t)\,\mathrm{d}W_t$$



Question: Effective Markov chain?
(possibly continuous-time)



$N > 2$ orbits?

3. Spectral theory of random Poincaré maps

Spectral decomposition: $\begin{cases} K\phi_i = \lambda_i \phi_i \\ \pi_i K = \lambda_i \pi_i \end{cases}$ $\quad \langle \pi_i, \phi_j \rangle = \delta_{ij}$

$$\Rightarrow k^n(x,y) = \underbrace{\lambda_0^n}_{=1} \phi_0(x)\pi_0(y) + \lambda_1^n \phi_1(x)\pi_1(y) + \dots$$

Assumptions
1) Confining dynamics
2) Ellipticity of noise $(gg^T > 0)$
3) Finitely many det. limit sets, N stable p.o.
4) "Metastable hierarchy" of p.o.

Theorem 3: [Baudel & B, SIAM J. Math. Anal. **49**:4319–4375, 2017]

The N largest eigenvalues of K are real and positive. $\exists\, \theta_k, c > 0$ s.t.

$$\lambda_0 = 1$$

$$\lambda_k = 1 - \mathbb{P}^{\mathring{\pi}_0^{k+1}}\{\tau^+_{B_1\cup\dots\cup B_k} < \tau^+_{B_{k+1}}\}[1 + \mathcal{O}(\mathrm{e}^{-\theta_k/\sigma^2})] \qquad 1 \leqslant k \leqslant N-1$$

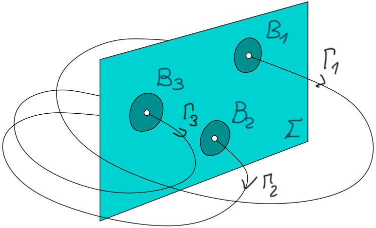
$$|\lambda_k| < 1 - \frac{c}{\log(\sigma^{-1})} \qquad k \geqslant N$$

$\mathring{\pi}_0^{k+1}$: QSD of process killed when hitting $B_1 \cup \dots \cup B_k$

$\tau^+_B = \inf\{n \geqslant 1 \colon X_n \in B\}$ (return time)

$\mathbb{P}^{\mathring{\pi}_0^{k+1}}\{\tau^+_{B_1\cup\dots\cup B_k} < \tau^+_{B_{k+1}}\} \simeq e^{-H_k/\sigma^2}$

H_k: given by variational pb (Freidlin – Wentzell)



3. Spectral theory of random Poincaré maps

Freidlin–Wentzell theory:

Rate function: $I_{[0,T]}(\gamma) = \dfrac{1}{2}\displaystyle\int_0^T (\dot{\gamma}_s - f(\gamma_s))^{\mathsf{T}} [gg^{\mathsf{T}}(\gamma_s)]^{-1} (\dot{\gamma}_s - f(\gamma_s))\,\mathrm{d}s$

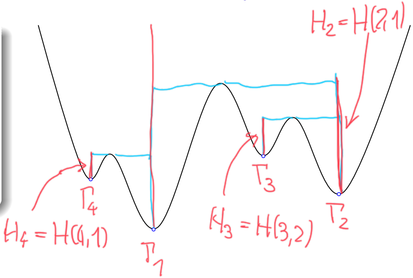
Large-deviation principle: $\mathbb{P}\bigl\{x_t \simeq \gamma_t, 0 \leqslant t \leqslant T\bigr\} \simeq \mathrm{e}^{-I_{[0,T]}(\gamma)/\sigma^2}$

Quasipotential between periodic orbits: $H(i,j) = \displaystyle\inf_{T>0} \inf_{\gamma:\Gamma_i \to \Gamma_j} I_{[0,T]}(\gamma)$

Assumption 5: Metastable hierarchy

$\exists\theta > 0$ s.t. $\forall\, 2 \leqslant k \leqslant N$

$$\min_{\ell<k} H(k,\ell) \leqslant \min_{\substack{i<k\\ j\neq i}} H(i,j) - \theta$$



Gradient case: $dx_t = -\nabla V(x_t)dt + \sigma dW_t$

$H(i,j)$ = barrier height

3. Spectral theory of random Poincaré maps

Consequence of thm:

$$k^n(x,y) \cong \pi_0(y) + \sum_{k=1}^{N-1} \lambda_k^n \phi_k(x)\pi_k(y)$$

Same expression as for N-state M.C. on $B_1, \dots, B_N$ with transition prob. $\sim e^{-H(i,j)/\sigma^2}$

Corollary:

$\forall x \in B_{k+1}$, $$\mathbb{E}^x[\tau_{B_1 \cup \dots \cup B_k}] = \frac{1 + O(e^{-\kappa/\sigma^2})}{1-\lambda_k}$$

4. Future research

- Asymptotics beyond large deviations
  (preliminary results in arXiv:2007.08443)
- Approximation of QSD / left eigenfunctions
- Phase dynamics (→ synchronisation)
- Interacting diffusions / SPDEs
- ...

References

1. N. B. & Damien Landon, *Mixed-mode oscillations and interspike interval statistics in the stochastic FitzHugh–Nagumo model*, Nonlinearity **25**, 2303-2335 (2012)

2. Manon Baudel & N. B., *Spectral theory for random Poincaré maps*, SIAM J. Math. Analysis **49**, 4319–4375 (2017)

3. N. B. & Barbara Gentz, *On the noise-induced passage through an unstable periodic orbit II: General case*, SIAM J. Math. Analysis **46**, 310–352 (2014)

4. N. B., Barbara Gentz & Christian Kuehn, *From random Poincaré maps to stochastic mixed-mode-oscillation patterns*, J. Dynam. Diff. Eq. **27**, 83–136 (2015)

5. N. B., *An Eyring-Kramers law for slowly oscillating bistable diffusions*, preprint arxiv:2007.08443 (2020)




Thanks for your attention!