Convergence to equilibrium in some singular parabolic SPDEs

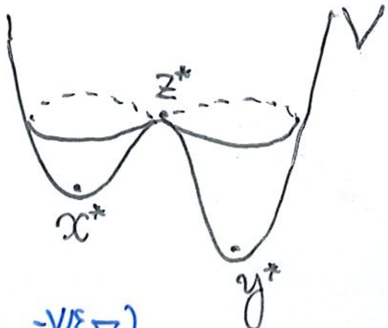
Based on: NB & B. Gentz, EJP (2013), NB, G. Di Gesù & H. Weber, EJP (2017)
See also arXiv: 1901.07420

Allen-Cahn: $\partial_t\phi - \Delta\phi = \phi - \phi^3 + \sqrt{2\varepsilon}\,\xi^\delta + C_{\delta,\varepsilon}\phi \qquad \phi(t,x),\ t\geqslant 0,\ x\in(\mathbb{R}/L\mathbb{Z})^d$

Φ^4_d model: $\partial_t\phi - \Delta\phi = -m^2\phi - \phi^3 + C_{\delta,\varepsilon}\phi + \sqrt{2\varepsilon}\,\xi^\delta \qquad m^2\geqslant 0$

	Existence/uniqueness of sol. (local/global)	Invariant measure	Large deviation principle	Eyring-Kramers law
$d=1$:	Faris & Jona-Lasinio 82	Da Prato & Zabczyk 96	Faris & Jona-Lasinio 82	Berglund & Gentz 13 Barret 15
$d=2$:	Da Prato & Debussche 03 Tsatsoulis & Weber 18	Röckner, Zhu & Zhu 17 Tsatsoulis & Weber 18 Hairer & Mattingly 18	Jona-Lasinio & Mitter 90 Hairer & Weber 15	B, Di Gesù & Weber 17 Tsatsoulis & Weber 18
$d=3$:	Hairer 14, Catellier & Chouk 18, Kupiainen 16		Hairer & Weber 15	?

1. Reversible SDE



$$dx_t = -\nabla V(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t \qquad x\in\mathbb{R}^n$$

Inv. measure: $\pi(x)\,dx = \dfrac{1}{Z}e^{-V(x)}dx$

reversible since $\langle f,\mathcal{L}g\rangle_\pi = \langle\mathcal{L}f,g\rangle_\pi \qquad (\mathcal{L} = \varepsilon e^{V/\varepsilon}\nabla\cdot e^{-V/\varepsilon}\nabla)$

Convergence to π:

(a) [Meyn & Tweedie 93] $(\mathcal{L}\tilde V)(x) \leqslant -c\tilde V(x) + d \qquad c>0 \quad d\geqslant 0$

$$\Rightarrow \sup_{|f|\leqslant V+1}\left|\mathbb{E}^x(f(x_t)) - \langle\pi,f\rangle\right| \leqslant C[V(x)+1]e^{-\beta t} \quad \forall x$$

$\beta, C>0$

(b) Expected transition time: $\mathbb{E}^{x^*}[\tau_{\mathcal{B}_r(y^*)}] = E = ?$

[Day 83] $\mathbb{P}^{x^*}(\tau_{\mathcal{B}_r(y^*)} > tE) \xrightarrow[\varepsilon\to 0]{} e^{-t}$

link with (a): $f = 1\!\!1_{\mathcal{B}_r(y^*)}$

Behaviour of E: – Arrhenius [Freidlin & Wentzell 70] via LDP $\displaystyle\lim_{\varepsilon\to 0}\varepsilon\log E = V(z^*) - V(x^*)$

– Eyring-Kramers [Bovier et al 2004] via potential theory $\displaystyle E = \frac{2\pi}{|\lambda_-(z^*)|}\sqrt{\frac{|\det \operatorname{Hess} V(z^*)|}{\det \operatorname{Hess} V(x^*)}}\, e^{[V(z^*)-V(x^*)]/\varepsilon}[1+o_\varepsilon(1)]$

(c) Spectral theory: [Bovier et al 2005] (spectral gap of $\mathcal{L}$) $= \dfrac{1}{E}(1+\mathcal{O}(e^{-c/\varepsilon}))$

2. Allen-Cahn in dim $d=1$

$$\partial_t \phi - \Delta\phi = \phi - \phi^3 + \sqrt{2\varepsilon}\,\xi$$

$$V(\phi) = \int_0^L \left[\tfrac{1}{2}\phi'(x)^2 + \tfrac{1}{4}\phi(x)^4 - \tfrac{1}{2}\phi(x)^2\right] dx$$

$$\Rightarrow \frac{\partial}{\partial\lambda} V(\phi+\lambda\psi)\Big|_{\lambda=0} = -\langle \Delta\phi + \phi - \phi^3, \psi\rangle_{L^2}$$

$\Rightarrow$ Gradient, Inv. measure "$\sim \frac{e^{-V/\varepsilon}}{Z}$" (abs. cont. w.r.t. GFF)

Stationary sol:
- $\phi = \pm 1$ (loc. min. of V, stable)
- $\phi \equiv 0$ (transition state if $L < 2\pi$)
- other, non-const. stat. states if $L > 2\pi$ ⊛ LDP

Hessian: $V(\phi) = \frac{1}{2}\langle\phi, (-\Delta-1)\phi\rangle + O(\phi^4) \Rightarrow HV(0) = -\Delta - 1$ ev $\left(\frac{2\pi}{L}k\right)^2 - 1$

$HV(\pm 1) = -\Delta + 2$ ev $\left(\frac{2\pi}{L}k\right)^2 + 2$

Δ acting on mean zero Fct

In E-K prefactor: $\det\left([-\Delta_\perp - 1][-\Delta_\perp+2]^{-1}\right) = \det\left([-\Delta_\perp + 2 - 3][-\Delta_\perp+2]^{-1}\right)$
$= \det\left(1 - 3[-\Delta_\perp+2]^{-1}\right)$ Fredholm det

$$\log\det\left([-\Delta_\perp-1][-\Delta_\perp+2]^{-1}\right) = \mathrm{Tr}\log\left(1 - 3[-\Delta_\perp+2]^{-1}\right)$$
$$= -\sum_{n\geq 1}\frac{3^n}{n}\underbrace{\mathrm{Tr}\left([-\Delta_\perp+2]^{-n}\right)}_{=\sum_{k\neq 0}\frac{1}{[(\frac{2\pi}{L}k)^2+2]^n} \leq \frac{\text{const}}{[(\frac{2\pi}{L})^2+2]^n}} < \infty$$

$$\boxed{\begin{array}{l}\text{Thm: [B, Gentz 13] EK law holds true}\\ \text{(result also exists for } L \geq 2\pi\text{, more general equ.)}\end{array}}$$

Proof uses spectral Galerkin approx.

⊛ LDP [F & JL '82]:

LDP with rate Fct $I_{[0,T]}(\gamma) = \begin{cases} \frac{1}{2}\int_0^T\int_0^L \left[\frac{\partial}{\partial t}\gamma(t,x) - \Delta\gamma(t,x) - (\phi-\phi^3)(t,x)\right]^2 dx\,dt \\ +\infty \end{cases}$

$\Rightarrow$ Arrhenius law holds $\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}^{-1}[\tau_{\mathcal{B}(+1)}] = V(0) - V(-1) = \frac{L}{4}$

$$\mathbb{P}^{-1}\{\tau_{\mathcal{B}(+1)} < T\} = \inf\{I_{[0,T]}(\gamma) : \gamma(0) = -1, \gamma(T) \in \mathcal{B}(+1)\}$$

$$I_{[0,T]}(\gamma) = \frac{1}{2}\int_0^T\int_0^L \left[\frac{\partial}{\partial t}\gamma + \Delta\gamma + (\phi-\phi^3)\right]^2 dx\,dt + 2\int_0^T\int_0^L \left[-\Delta\gamma - (\phi-\phi^3)\right]\frac{\partial}{\partial t}\gamma\, dx\,dt$$
$$\geq 2\int_0^T\int_0^T \left[\frac{\partial}{\partial t}\gamma\,\frac{\partial}{\partial t}\frac{\partial\gamma}{\partial x} - (\phi-\phi^3)\frac{\partial\gamma}{\partial t}\right] dx\,dt$$
$$= 2\left[V(\gamma_T) - V(\gamma_0)\right]$$

3. Allen-Cahn in dim $d=2$

$$\partial_t \phi - \Delta\phi = \phi - \phi^3 + 3\varepsilon C_\delta \phi + \sqrt{2\varepsilon}\,\xi^\delta \quad \Rightarrow \text{Stat. sol in } \pm\sqrt{1+3\varepsilon C_\delta}\ ??$$

$$V(\phi) = \int_{\mathbb{T}^2} \Big[\tfrac{1}{2}|\nabla\phi|^2 + \tfrac{1}{4}\phi^4 - \tfrac{1}{2}\phi^2 - \tfrac{3}{2}\varepsilon C_\delta \phi^2\Big]dx \quad \Rightarrow \text{inv. meas. "}\sim \frac{e^{-V/\varepsilon}}{Z}\text{"}$$

(can be written using $:\phi^4:$, $:\phi^3:$)

[Hairer & Weber 15] LDP with $I_{[0,T]}(\gamma) = \frac{1}{2}\int_0^T\int_{\mathbb{T}^2}\Big[\frac{\partial\gamma}{\partial t} - \Delta\gamma - (\phi-\phi^3)\Big]^2 dx\,dt$

$\Rightarrow$ does not depend on C_δ $\rightsquigarrow$ Stat sol in ± 1?

Arrhenius law $\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}^{-1}[\tau_{\mathcal{B}(+1)}] = V(0) - V(-1)\Big|_{\varepsilon=0} = \frac{L^2}{4}$

Eyring-Kramers prefactor: $\mathrm{Tr}[(-\Delta_\perp + 2)^{-n}] \cong \sum_{k\neq(0,0)} \frac{1}{\|k\|^{2n}} \sim \int_1^\infty \frac{r\,dr}{r^{2n}} \begin{cases} =+\infty & n=1 \\ <+\infty & n\geq 2\end{cases}$

$\Rightarrow (-\Delta_\perp + 2)^{-1}$ is not trace class, but Hilbert-Schmidt

Spectral Galerkin: $\partial_t \phi_N - \Delta\phi_N = \phi_N - P_N\phi_N^3 + 3\varepsilon C_N \phi_N + \sqrt{2\varepsilon}\,\xi_N$

$$\phi_N^{GFF}(x) = \sum_{|k|\leq N} \frac{Z_k}{\sqrt{\lambda_k+1}} e_k(x) \Rightarrow \mathbb{E}\big[\phi_N^{GFF}(x)\phi_N^{GFF}(y)\big] = \sum_{|k|\leq N} e_k(x)(-\Delta+1)^{-1}e_k(y)$$

$$L^2 C_N = \mathbb{E}\big[\|\phi_N^{GFF}\|_{L^2}^2\big] = \mathrm{Tr}[(-\Delta_N+1)^{-1}] \sim \log N$$

Eyring-Kramers: $V(0) - V(-1) = \frac{L^2}{4} + \frac{3}{2}L^2\varepsilon C_N$

$$\Rightarrow \sqrt{\det([-\Delta_N^\perp - 1]^{-1}[-\Delta_N^\perp + 2])}\; e^{(V(0)-V(-1))/\varepsilon}$$

$$= \Big[\det\big(1 - 3[-\Delta_N^\perp+2]^{-1}\big)\, e^{3\mathrm{Tr}[-\Delta_N^\perp+2]^{-1} + \mathcal{O}}\Big]^{1/2} e^{L^2/4\varepsilon}$$

Carleman-Fredholm det

shift in reference GFF

$$\log[\ldots] = \mathrm{Tr}\Big[\log\big(1 - 3[-\Delta_N^\perp+2]^{-1}\big) + 3(-\Delta_N^\perp+2)\Big] + \mathcal{O}$$

$$= \sum_{0\neq|k|<N}\Big(\log\Big(1 - \frac{3}{(\frac{2\pi}{L}k)^2+2}\Big) + \frac{3}{(\frac{2\pi}{L}k)^2+2}\Big) + \mathcal{O} \quad < +\infty$$

[B, Di Gesù, Weber 17]: uniform in N bounds on EK law (starting in eq. measure)

[Tsatsoulis & Weber 18]: full proof of EK law (limit $N\to\infty$ & arbitrary initial cond)