New trends of stochastic nonlinear systems:
well-posedeness, dynamics and numerics. CIRM, Marseille

Metastable Allen–Cahn SPDEs & perturbative renormalisation of Φ^4 measures

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Joint works with Barbara Gentz (Bielefeld), Giacomo Di Gesù (Rome), Hendrik Weber (Münster), Tom Klose (Oxford) and Nikolas Tapia (Berlin)

    

Plan

1. Metastability in Allen–Cahn equations on $\mathbb{T}^1$ and $\mathbb{T}^2$

   Joint works with Barbara Gentz ($\mathbb{T}^1$)
   Electronic J. Probability **18**, (24):1–58 (2013)

   and Giacomo Di Gesù & Hendrik Weber ($\mathbb{T}^2$)
   Electronic J. Probability **22**, (41):1–27 (2017)

2. Perturbative renormalisation of Φ^4 measures

   Joint works with Tom Klose and Nikolas Tapia
   `arXiv/2207.08555` (2022) (Proceedings, Vienna trimester)
   `arXiv/2507.03820` (2025)

Allen–Cahn equation on $\mathbb{T}^2$

$\partial_t \phi(t,x) = \nu(\varepsilon t)\Delta\phi(t,x) + \phi(t,x) - \phi(t,x)^3$

(Online: https://youtu.be/yX0EAxZHNCQ)

Deterministic Allen–Cahn PDE in $d = 1$

[Chafee & Infante 74, Allen & Cahn 75]

$$\partial_t \phi(t,x) = \Delta\phi(t,x) + f(\phi(t,x))$$

- $x \in [0, L]$, L: bifurcation parameter
- $\phi(t,x) \in \mathbb{R}$
- Either periodic or zero-flux Neumann boundary conditions
- In this talk: $f(\phi) = \phi - \phi^3$ (results more general)

Energy function:

$$V[\phi] = \int_0^L \Big[\frac{1}{2}\phi'(x)^2 - \frac{1}{2}\phi(x)^2 + \frac{1}{4}\phi(x)^4\Big]\,\mathrm{d}x \quad \Rightarrow \quad \partial_t\langle\phi,\psi\rangle = -\nabla_\psi V(\phi)$$

Stationary solutions: $\phi_0''(x) = -\phi_0(x) + \phi_0(x)^3$ critical points of V

Stability: Sturm–Liouville problem $\partial_t\psi_t(x) = \psi_t''(x) + [1 - 3\phi_0(x)^2]\psi_t(x)$

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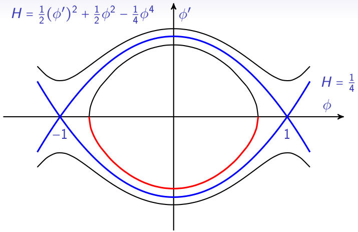
Stationary solutions



$\phi_0''(x) = -f(\phi_0(x)) = -\phi_0(x) + \phi_0(x)^3$

- $\phi_\pm(x) \equiv \pm 1$: stable
- $\phi_0(x) \equiv 0$: unstable
- Nonconstant solutions satisfying b.c. (expressible in terms of Jacobi elliptic fcts)
- Neumann b.c: $2k$ nonconstant solutions when $L > k\pi$

Number of positive eigenvalues (= unstable directions)

Transition state

- Periodic b.c: k families when $L > 2k\pi$

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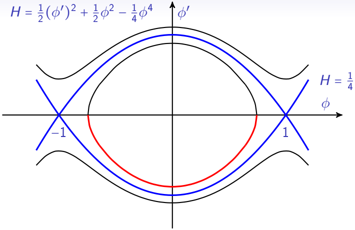

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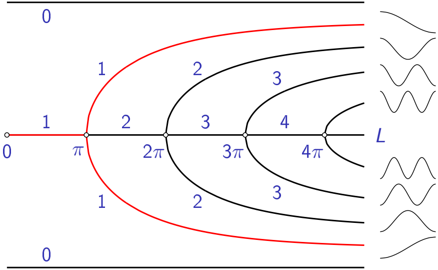

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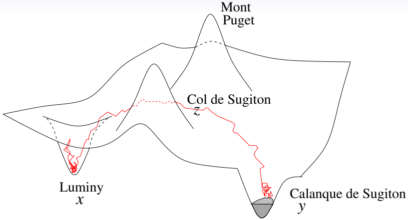
Reversible diffusion in a double-well

$$dx_t = -\nabla V(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t$$

$V : \mathbb{R}^d \to \mathbb{R}$ confining potential

$\tau_y^x = \inf\{t > 0 \colon x_t \in \mathcal{B}_\varepsilon(y)\}$
first-hitting time of small ball $\mathcal{B}_\varepsilon(y)$, when starting in x



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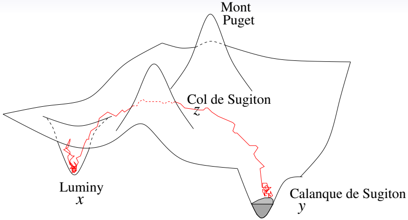

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Arrhenius' law (1889): $\mathbb{E}[\tau_y^x] \simeq e^{[V(z)-V(x)]/\varepsilon}$

Eyring–Kramers law (1935, 1940):

Eigenvalues of Hessian of V at minimum x: $0 < \nu_1 \leqslant \nu_2 \leqslant \dots \leqslant \nu_d$

Eigenvalues of Hessian of V at saddle z: $\lambda_1 < 0 < \lambda_2 \leqslant \dots \leqslant \lambda_d$

$$\mathbb{E}[\tau_y^x] = 2\pi \sqrt{\frac{\lambda_2 \dots \lambda_d}{|\lambda_1| \nu_1 \dots \nu_d}}\, e^{[V(z)-V(x)]/\varepsilon}\big[1 + \mathcal{O}_\varepsilon(1)\big]$$

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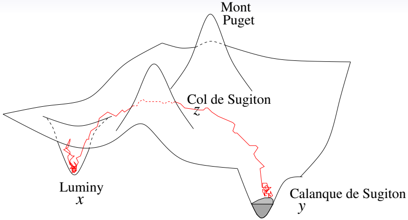

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Arrhenius' law: proved by [Freidlin, Wentzell, 1979] using large deviations

Eyring–Kramers law: [Bovier, Eckhoff, Gayrard, Klein, 2004] using potential theory, [Helffer, Klein, Nier, 2004] using Witten Laplacian, ...

Potential-theoretic proof of Eyring–Kramers law

- $w_A(x) = \mathbb{E}^x[\tau_A]$ satisfies $\begin{cases} (\mathscr{L} w_A)(x) = -1 & x \in A^c \\ w_A(x) = 0 & x \in A \end{cases}$

- $h_{AB}(x) = \mathbb{P}^x\{\tau_A < \tau_B\}$ satisfies $\begin{cases} (\mathscr{L} h_{AB})(x) = 0 & x \in (A \cup B)^c \\ h_{AB}(x) = 1 & x \in A \\ h_{AB}(x) = 0 & x \in B \end{cases}$

Theorem: $A, B \subset \mathbb{R}^d$ disjoint. $\exists$ proba measure ν_{AB} on ∂A s.t.

$$\int_{\partial A} \mathbb{E}^x[\tau_B] \nu_{AB}(\mathrm{d}x) = \frac{1}{\operatorname{cap}(A,B)} \int_{B^c} \mathrm{e}^{-V(y)/\varepsilon} h_{AB}(y) \, \mathrm{d}y$$

Apply to A, B neighbourhoods of x, y

- Laplace asymptotics: $\displaystyle\int_{B^c} h_{A,B}(y) \, \mathrm{e}^{-V(y)/\varepsilon} \, \mathrm{d}y \simeq \sqrt{\frac{(2\pi\varepsilon)^d}{\nu_1 \dots \nu_d}} \, \mathrm{e}^{-V(x)/\varepsilon}$
- Capacity: $\operatorname{cap}(A,B) = \mathscr{E}(h_{AB})$
- Dirichlet form: $\mathscr{E}(f) = \langle f, -\mathscr{L} f \rangle = \varepsilon \displaystyle\int_{\mathbb{R}^d} \mathrm{e}^{-V(x)/\varepsilon} |\nabla f(x)|^2 \, \mathrm{d}x$

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Theorem: Dirichlet principle

Let $\mathscr{H}_{AB} = \{h : \mathbb{R}^d \to [0,1] : h|_A = 1, h|_B = 0\}$. Then

$$\operatorname{cap}(A,B) = \inf_{h \in \mathcal{H}_{AB}} \mathscr{E}(h) = \mathscr{E}(h_{AB})$$

Appropriate test function yields $\operatorname{cap}(A,B) \lesssim \varepsilon \sqrt{\frac{|\lambda_1|}{2\pi\varepsilon}} \sqrt{\frac{(2\pi\varepsilon)^{d-1}}{\lambda_2 \dots \lambda_d}} \, \mathrm{e}^{-V(z)/\varepsilon}$

Theorem: Thomson principle [Landim, Mariani, Seo 2018]

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Eyring–Kramers law for 1D SPDEs: heuristics

$$\partial_t \phi(t,x) = \Delta\phi(t,x) + f(\phi(t,x)) \;\; + \sqrt{2\varepsilon}\,\xi(t,x) \qquad (f(\phi) = \phi - \phi^3)$$

Initial condition: ϕ_{in} near $\phi_- \equiv -1$ with eigenvalues $\nu_k = (\frac{\beta k \pi}{L})^2 + 2$

Target: $\phi_+ \equiv 1$, $\tau_+ = \inf\{t > 0 \colon \|\phi(t,\cdot) - \phi_+\|_{L^\infty} < \rho\}$

Transition state: ($\beta = 1$ for Neumann b.c., $\beta = 2$ for periodic b.c.)

$$\phi_{\text{ts}}(x) = \begin{cases} \phi_0(x) \equiv 0 & \text{if } L \leqslant \beta\pi \quad \text{with ev } \lambda_k = (\frac{\beta k\pi}{L})^2 - 1 \\ \phi_1(x)\ \beta\text{-kink stationary sol.} & \text{if } L > \beta\pi \quad \text{with ev } \lambda_k' \end{cases}$$

[Faris & Jona-Lasinio 82]: large-deviation principle

$\Rightarrow$ Arrhenius law: $\mathbb{E}^{\phi_{\text{in}}}[\tau_+] \simeq e^{(V[\phi_{\text{ts}}] - V[\phi_-])/\varepsilon}$

[Maier & Stein 01]: formal computation; for Neumann b.c. and $L < \pi$

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Eyring–Kramers law for 1D SPDEs: main result

Theorem: Neumann b.c. [B & Gentz, 2013]

- If $L < \pi - c$ with $c > 0$, then

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- If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles) and λ_k' instead of λ_k

- Results also for L near π and periodic b.c.
- Proof uses spectral Galerkin approximation & passage to the limit
- Prefactor involves a Fredholm determinant:
  $\Delta_\perp$ Laplacian acting on mean zero functions

$$\prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k} = \det\big[(-\Delta_\perp - 1)(-\Delta_\perp + 2)^{-1}\big] = \det\big[\mathbb{1} - 3(-\Delta_\perp + 2)^{-1}\big]$$

converges because $\log\det = \operatorname{Tr}\log$ and $(-\Delta_\perp + 2)^{-1}$ is trace class (limit $= \frac{\sqrt{2}\sin(L)}{\sinh(\sqrt{2}L)}$)

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  $\Delta_\perp$ Laplacian acting on mean zero functions

$$\prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k} = \det\big[(-\Delta_\perp - 1)(-\Delta_\perp + 2)^{-1}\big] = \det\big[\mathbb{1} - 3(-\Delta_\perp + 2)^{-1}\big]$$

  converges because $\log\det = \operatorname{Tr}\log$ and $(-\Delta_\perp + 2)^{-1}$ is trace class (limit $= \frac{\sqrt{2}\sin(L)}{\sinh(\sqrt{2}L)}$)

The two-dimensional case

- Large-deviation principle: [Hairer & Weber, 2015]
- Naive computation of prefactor fails:

$$\log \prod_{k\in(\mathbb{N}^2)^*} \frac{1-\left(\frac{L}{|k|\pi}\right)^2}{1+2\left(\frac{L}{|k|\pi}\right)^2} \simeq \sum_{k\in(\mathbb{N}^2)^*} \log\left(1-\frac{3L^2}{|k|^2\pi^2}\right)$$

$$\simeq -\sum_{k\in(\mathbb{N}^2)^*} \frac{3L^2}{|k|^2\pi^2} \simeq -\frac{3L^2}{\pi^2}\int_1^\infty \frac{r\,\mathrm{d}r}{r^2} = -\infty$$

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- In fact, the equation needs to be renormalised

Theorem: [Da Prato & Debussche 2003]

Let ξ^δ be a mollification on scale δ of white noise. Then

$$\partial_t\phi = \Delta\phi + \left[1+3\varepsilon C(\delta)\right]\phi - \phi^3 + \sqrt{2\varepsilon}\xi^\delta$$

with $C(\delta) \simeq \log(\delta^{-1})$ admits local solution converging as $\delta \to 0$
(Global version: [Mourrat & Weber 2015])
[Mourrat & Weber 2014]: Renormalised eq = scaling limit of Ising–Kac model

Computation of the prefactor

- Consider for simplicity $L < \beta\pi \Rightarrow$ transition state in 0
- Galerkin-truncated renormalised potential

$$V_N = \frac{1}{2}\int_{\mathbb{T}^2}\left[\|\nabla\phi_N(x)\|^2 - \phi_N(x)^2\right]\mathrm{d}x + \frac{1}{4}\int_{\mathbb{T}^2} :\phi_N(x)^4:\mathrm{d}x$$

where $:\phi_N^4: = \phi_N^4 - 6\varepsilon C_N \phi_N^2 + 3\varepsilon^2 C_N^2, \qquad C_N \asymp \log(N)$

- Using variational principles: $\operatorname{cap}(A,B) \simeq \sqrt{\frac{|\lambda_0|\varepsilon}{2\pi}} \prod_{0<|k|\leqslant N} \sqrt{\frac{2\pi\varepsilon}{\lambda_k}}$
- Symmetry argument:

$$\int_{B^c} h_{A,B}(z)\, \mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z = \frac{1}{2}\int \mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z = \frac{1}{2}\mathcal{Z}_N(\varepsilon)$$

- $\mathcal{Z}_N(\varepsilon) \simeq 2\prod_{|k|\leqslant N}\sqrt{\frac{2\pi\varepsilon}{\nu_k}}\,\mathrm{e}^{-V_N(L,0)/\varepsilon}$ where $-V_N(L,0) = \frac{1}{4}L^2 + \frac{3}{2}L^2 C_N\varepsilon$
- Prefactor proportional to (since $\nu_k = \lambda_k + 3$)

$$\prod_{0<|k|\leqslant N}\frac{\lambda_k}{\lambda_k+3}\mathrm{e}^{3/\lambda_k} \qquad \text{converges since} \qquad \log\left[\frac{\lambda_k}{\lambda_k+3}\mathrm{e}^{3/\lambda_k}\right] = \mathcal{O}\left(\frac{1}{|k|^4}\right)$$

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Main result in dimension 2

Theorem: [B, Di Gesù, Weber, 2017]

For $L < 2\pi$, $A \ni \phi_-$, $B \ni \phi_+$ balls in $\|\cdot\|_{H^s}$, $s < 0$, $\exists \mu_N$ prob measures on ∂A:

$$\limsup_{N\to\infty} \mathbb{E}^{\mu_N}\big[\tau_B\big] \leqslant \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k}\,\mathrm{e}^{\frac{\nu_k-\lambda_k}{|\lambda_k|}}}\;\mathrm{e}^{(V[\phi_{\mathrm{ts}}]-V[\phi_-])/\varepsilon}\big[1 + c_+\sqrt{\varepsilon}\,\big]$$

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- Inverse of prefactor involves Carleman–Fredholm determinant:

$$\det{}_2(\mathbb{1} + T) = \det(\mathbb{1} + T)\,\mathrm{e}^{-\operatorname{Tr} T}$$

with $T = 3(-\Delta_\perp - 1)^{-1}$

$\det_2$ defined whenever T is only Hilbert–Schmidt (true for $d \leqslant 3$)

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Allen–Cahn equation in dimension 3

- Equation needs two counterterms:

$$\partial_t\phi = \Delta\phi + \left[1 + 3\varepsilon C_1(\delta) - 9\varepsilon^2 C_2(\delta)\right]\phi - \phi^3 + \sqrt{2\varepsilon}\xi^\delta$$

with $C_1(\delta) \sim \delta^{-1}$, $C_2(\delta) \sim \log(\delta^{-1})$

- Solution theories:
  - Regularity structures [Hairer 2014]
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Φ^4_3 measure

- Potential (before renormalisation):

$$V_\alpha(\phi) = \int_\Lambda \Big(\tfrac{1}{2} \|\nabla\phi(x)\|^2 + \tfrac{1}{2}\phi(x)^2 + \alpha\phi(x)^4 \Big)\,\mathrm{d}x$$

where $\Lambda = (\mathbb{R}/\mathbb{Z})^3 =: \mathbb{T}^3$, $\alpha \geqslant 0$

- Definition of Gibbs measure?

$$\nu_\alpha(\mathrm{d}\phi) \text{ ``$=$'' } \frac{1}{\mathcal{Z}_\alpha} \mathrm{e}^{-V_\alpha(\phi)} \text{ ``$\mathrm{d}\phi$''}$$

- ν_0 is Gaussian. For random variable F

$$\mathbb{E}^{\nu_\alpha}[F] = \frac{\mathcal{Z}_0}{\mathcal{Z}_\alpha} \mathbb{E}^{\nu_0}\Big[F\,\mathrm{e}^{-\alpha\int_\Lambda \phi^4(x)\,\mathrm{d}x}\Big]$$

In particular, $\dfrac{\mathcal{Z}_\alpha}{\mathcal{Z}_0} = \mathbb{E}^{\nu_0}\Big[\mathrm{e}^{-\alpha\int_\Lambda \phi^4(x)\,\mathrm{d}x}\Big]$

- Link with Allen–Cahn equation: write $\phi(x) = \phi_0 + \sqrt{\varepsilon}\phi_\perp(x)$, where ϕ_0 is average of ϕ (and $\alpha = \frac{\varepsilon}{4}$)

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The static Φ^4_3 model

Theorem: Potential needs exactly 4 counterterms:

$$V_\alpha(\phi) = \int_\Lambda \Big(\tfrac{1}{2}\|\nabla\phi(x)\|^2 + \tfrac{1}{2}\big[1 - \alpha^2 C_N^{(2)}\big]\phi(x)^2 + \alpha :\phi(x)^4:_{C_N^{(1)}} + \alpha^2 C_N^{(3)} - \alpha^3 C_N^{(4)} \Big)\,\mathrm{d}x$$

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and $G_N(x) = \sum_{|k| \leqslant N} \frac{1}{\lambda_k + 1} e_k(x)$ is the Green function of Δ_N

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Some literature on the static Φ^4_3 model

- Glimm & Jaffe (1968, 1973), Feldman (1974):
  Combinatorics of Feynman diagrams
- Benfatto, Cassandro, Gallavotti, Nicolò & Olivieri (1978, 1980):
  Renormalisation group (integrating out scales)
- Brydges, Fröhlich & Sokal (1983):
  Generating function and skeleton inequalities
- Brydges, Dimock & Hurd (1995):
  Polymer expansions
- Connes & Kreimer (2000, 2001):
  Hopf algebras
- ...
- Barashkov & Gubinelli (2020):
  Boué–Dupuis formula

Perturbative computation of partition function

- Wick powers: $X =$  $= \int_\Lambda :\phi(x)^4: \mathrm{d}x$, $Y =$  $= \int_\Lambda :\phi(x)^2: \mathrm{d}x$
- Parameters: $\beta = \frac{1}{2}\alpha^2 C_N^{(2)}$, $\gamma = \alpha^2 C_N^{(3)} - \alpha^3 C_N^{(4)}$

  Then $\dfrac{\mathcal{Z}_{N,\alpha}}{\mathcal{Z}_{N,0}} = \mathbb{E}^{\nu_0}\left[\mathrm{e}^{-\alpha X - \beta Y - \gamma}\right] = \mathrm{e}^{-\gamma}\,\mathbb{E}^{\nu_0}\left[\mathrm{e}^{-\alpha X - \beta Y}\right]$

- Let $\Gamma = (\mathscr{V}, \mathscr{E})$ be a multigraph, $\mathscr{G} = \mathrm{span}\{\Gamma\}$. Its valuation is

$$\Pi_N(\Gamma) = \int_{\Lambda^{\mathscr{V}}} \prod_{e \in \mathscr{E}} G_N(x_{e_+} - x_{e_-})\,\mathrm{d}x$$

  For instance

$C_N^{(1)} = \Pi_N$

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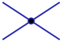
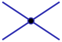
Cumulant expansion

- $\mathbb{E}^{\nu_0}\left[\mathrm{e}^{-\alpha X-\beta Y}\right] = \sum_{n\geqslant 0} \frac{1}{n!}\mu_n$

  $\mu_n = (-1)^n \mathbb{E}^{\nu_0}\Big[\Big(\alpha$  $+\beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\nu_0}\Big[$  ^{m}  $^{n-m}\Big]$ computed using Isserlis' thm

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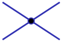
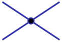
  $\mu_n = (-1)^n \mathbb{E}^{\nu_0}\Big[\Big(\alpha$  $+\beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\nu_0}\Big[$  ^{m}  $^{n-m}\Big]$ computed using Isserlis' thm

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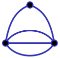
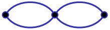
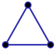
  $\mu_3 = -\alpha^3 \binom{4}{2}^3 2^3 \Pi_N$  $- 3\alpha^2\beta(4^2\cdot 2\cdot 3!)\Pi_N$ 

  $- 3\alpha\beta^2 4!\,\Pi_N$  $- 8\beta^3\Pi_N$ 

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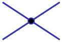
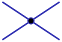
  $\mu_n = (-1)^n \mathbb{E}^{\nu_0}\Big[\Big(\alpha$  $+ \beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\nu_0}\Big[$  ^{m}  $^{n-m}\Big]$ computed using Isserlis' thm

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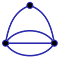
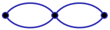
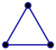
  $\mu_3 = -\alpha^3 \binom{4}{2}^3 2^3 \Pi_N$  $- 3\alpha^2\beta(4^2 \cdot 2 \cdot 3!)\Pi_N$ 

  $- 3\alpha\beta^2 4!\, \Pi_N$  $- 8\beta^3 \Pi_N$ 

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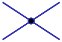
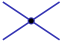
  $\mu_n = (-1)^n \mathbb{E}^{\nu_0}\Big[\Big(\alpha$  $+\beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\nu_0}\Big[$  ^{m}  $^{n-m}\Big]$ computed using Isserlis' thm

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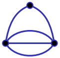
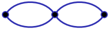
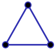
  $\mu_3 = -\alpha^3\binom{4}{2}^3 2^3\Pi_N$  $-3\alpha^2\beta(4^2\cdot 2\cdot 3!)\Pi_N$ 

  $-3\alpha\beta^2 4!\,\Pi_N$  $-8\beta^3\Pi_N$ 

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Divergences and subdivergences

- Degree of Γ: $\deg(\Gamma) = 3(|\mathscr{V}| - 1) - |\mathscr{E}|$. Γ divergent if $\deg(\Gamma) \leqslant 0$.
- Examples:

$\deg($ $) = 0$ $\qquad$ $\Pi_N($ $) = \mathcal{O}(\log N)$

$\deg($ $) = -1$ $\qquad$ $\Pi_N($ $) = \mathcal{O}(N)$

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- It looks like $\Pi_N(\Gamma) = \begin{cases} \mathcal{O}(N^{-\deg(\Gamma)}) & \text{if } \deg(\Gamma) < 0 \\ \mathcal{O}(\log N) & \text{if } \deg(\Gamma) = 0 \end{cases}$

However, $\deg($ $) = 1$, while $\Pi_N($ $) = \mathcal{O}(\log N)$ because it contains a subdivergence

Theorem: [Dyson]

If $\deg \bar{\Gamma} > 0$ for all subgraphs $\bar{\Gamma} \subset \Gamma$, then $\Pi_N(\Gamma)$ is bounded unif in N

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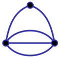
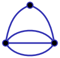
  However, deg() $= 1$, while $\Pi_N($  $) = \mathcal{O}(\log N)$ because it contains a **subdivergence** 

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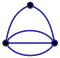
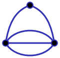
  However, deg() $= 1$, while Π_N() $= \mathcal{O}(\log N)$ because it contains a subdivergence 

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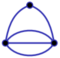
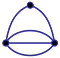
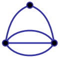
Hopf algebras and renormalisation

- Connes–Kreimer extraction–contraction coproduct: $\Delta : \mathscr{G} \to \mathscr{G} \otimes \mathscr{G}$

$$\Delta(\Gamma) = \Gamma \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + \sum_{\substack{\mathbf{1} \neq \overline{\Gamma} \subsetneq \Gamma \\ \deg(\overline{\Gamma}) \leqslant 0}} \overline{\Gamma} \otimes (\Gamma/\overline{\Gamma}) \qquad (\mathbf{1}\text{: empty graph})$$

Example: $\Delta\Big($  $\Big) =$  $\otimes \mathbf{1} + \mathbf{1} \otimes$  $+$  $\otimes$ 

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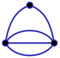
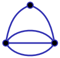
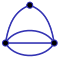
Example: $\Delta\Big($  $\Big) =$  $\otimes \mathbf{1} + \mathbf{1} \otimes$  $+$  $\otimes$ 

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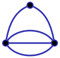
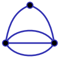
Example: $\mathscr{A}\Big($  $\Big) = -$  $+$  $\cdot$ 

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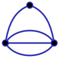
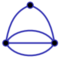
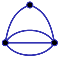
  Example: $\Delta\Big($  $\Big) =$  $\otimes \mathbf{1} + \mathbf{1} \otimes$  $+$  $\otimes$ 

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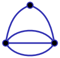
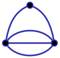
  Example: $\mathscr{A}\Big($  $\Big) = -$  $+$  $\cdot$ 

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BPHZ renormalisation

- BPHZ character: $\langle g^{\mathrm{BPHZ}}, \Gamma \rangle = \Pi_N \mathscr{A}(\Gamma) \mathbf{1}_{\deg \Gamma \leqslant 0}$
- Renormalised valuation:

$$
\begin{aligned}
\Pi_N^{\mathrm{BPHZ}}(\Gamma) &= \Pi_N M^{g^{\mathrm{BPHZ}}}(\Gamma) \\
&= (g^{\mathrm{BPHZ}} \otimes \Pi_N)\Delta\Gamma \\
&= (\Pi_N \tilde{\mathscr{A}} \otimes \Pi_N)\Delta\Gamma
\end{aligned}
$$

$$\tilde{\mathscr{A}}(\Gamma) = \mathscr{A}(\Gamma)\mathbf{1}_{\deg \Gamma \leqslant 0}$$

$$
\begin{tikzcd}
\mathscr{G} \arrow[r, "\Pi_N"] \arrow[d, "(\tilde{\mathscr{A}}\otimes\mathrm{id})\Delta"'] \arrow[dr, "\Pi_N^{\mathrm{BPHZ}}"] \arrow[dd, bend right=60, "M^{g^{\mathrm{BPHZ}}}=(\Pi_N\tilde{\mathscr{A}}\otimes\mathrm{id})\Delta"'] & \mathbb{R} \\
\mathscr{G}\otimes\mathscr{G} \arrow[r, "\Pi_N\otimes\Pi_N"] \arrow[d, "\Pi_N\otimes\mathrm{id}"'] & \mathbb{R} \\
\mathscr{G} \arrow[ur, "\Pi_N"'] &
\end{tikzcd}
$$

Theorem: [Bogolyubov, Parasiuk, Hepp, Zimmermann]

$$
\Pi_N^{\mathrm{BPHZ}}(\Gamma) = \begin{cases} 0 & \text{if } \deg \Gamma \leqslant 0 \\ -\Pi_N \mathscr{A}(\Gamma) & \text{if } \deg \Gamma > 0 \end{cases}
$$

Furthermore, $\Pi_N^{\mathrm{BPHZ}}(\Gamma)$ is bdd uniformly in N for any Γ

For a modern proof, see [Hairer, An analyst's take on the BPHZ theorem (2018)]

BPHZ renormalisation

- BPHZ character: $\langle g^{\mathrm{BPHZ}}, \Gamma \rangle = \Pi_N \mathscr{A}(\Gamma) \mathbf{1}_{\deg \Gamma \leqslant 0}$
- Renormalised valuation:

$$\begin{aligned}
\Pi_N^{\mathrm{BPHZ}}(\Gamma) &= \Pi_N M^{g^{\mathrm{BPHZ}}}(\Gamma) \\
&= (g^{\mathrm{BPHZ}} \otimes \Pi_N)\Delta\Gamma \\
&= (\Pi_N \tilde{\mathscr{A}} \otimes \Pi_N)\Delta\Gamma
\end{aligned}$$

$$\tilde{\mathscr{A}}(\Gamma) = \mathscr{A}(\Gamma)\mathbf{1}_{\deg \Gamma \leqslant 0}$$

$$\begin{tikzcd}
\mathscr{G} \arrow[r, "\Pi_N"] \arrow[d, "(\tilde{\mathscr{A}} \otimes \mathrm{id})\Delta"'] \arrow[dr, "\Pi_N^{\mathrm{BPHZ}}"] \arrow[dd, bend right=60, "M^{g^{\mathrm{BPHZ}}} = (\Pi_N \tilde{\mathscr{A}} \otimes \mathrm{id})\Delta"'] & \mathbb{R} \\
\mathscr{G} \otimes \mathscr{G} \arrow[r, "\Pi_N \otimes \Pi_N"] \arrow[d, "\Pi_N \otimes \mathrm{id}"'] & \mathbb{R} \\
\mathscr{G} \arrow[ur, "\Pi_N"'] &
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Main result: Commutative diagram

$$
\begin{tikzcd}
e^{-\alpha X} \arrow[r, mapsto, "\mathscr{P}"] \arrow[d, mapsto, "\mathrm{W}"'] & \displaystyle\sum_p \frac{(-\alpha)^p}{p!}\mathscr{P}(X^p) \arrow[d, mapsto, "M^{g^{\mathrm{BPHZ}}}+\Theta"'] \arrow[dr, mapsto, "\Pi_N^{\mathrm{BPHZ}}+\Pi_N\Theta"] & \\
e^{-\alpha X-\beta Y} \arrow[r, mapsto, "\mathscr{P}"] & \displaystyle\sum_{n,m} \frac{(-\alpha)^m(-\beta)^{n-m}}{m!(n-m)!}\mathscr{P}(X^m Y^{n-m}) \arrow[r, mapsto, "\Pi_N"] & \log \mathbb{E}[e^{-\alpha X-\beta Y}]
\end{tikzcd}
$$

- $\mathscr{P} = \Pi_{\text{connected}}\left(\sum_{\text{pairings}}\right)$, satisfies $\deg \mathscr{P}(X^n) = n-3$
- W: Wick map, $\mathrm{W}(X^n) = H_n(X;\beta Y)$ is Hermite polynomial (for more general models: Bell polynomial)
- Θ associated with energy renormalisation, $\Pi_N\Theta(e^{-\alpha X}) = \gamma$

Theorem: [B, Klose, Tapia 25]

$$\log \frac{\mathcal{Z}_{N,\alpha}}{\mathcal{Z}_{N,0}} = \log \mathbb{E}[e^{-\alpha X-\beta Y}] - \gamma \asymp -\sum_{n\geqslant 4} \frac{(-\alpha)^n}{n!}\Pi_N \mathscr{A}(\mathscr{P}(X^n))$$

as asymptotic expansion with terms uniformly bounded in N

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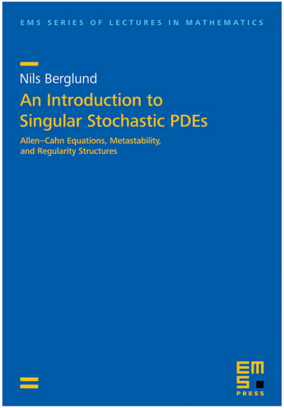
References

- N. B. & Barbara Gentz, *Sharp estimates for metastable lifetimes in parabolic SPDEs: Kramers' law and beyond*, Electronic J. Probability **18**, (24):1–58 (2013)
- N. B., Giacomo Di Gesù & Hendrik Weber, *An Eyring–Kramers law for the stoch. Allen–Cahn equation in dimension two*, Electronic J. Probability **22**, 1–27 (2017)
- N. B. & Tom Klose, *Perturbation theory for the Φ^4_3 measure, revisited with Hopf algebras*, arXiv/2207.08555 (2022)
- N. B., Tom Klose & Nikolas Tapia, *Perturbative renormalisation of the $\Phi^4_{4-\varepsilon}$ model via generalized Wick maps*, arXiv/2507.03820 (2025)
- N. B, *An Introduction to Singular Stochastic PDEs*, EMS Press (2022)



Thanks for your attention!

Slides available at https://www.idpoisson.fr/berglund/CIRM_oct_2025.pdf