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Session on Complex Patterns in Biological Systems

Synchronization and noise-induced phase slips

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Joint work with Barbara Gentz (Bielefeld)

Synchronization of two coupled oscillators

See e.g. [Pikovsky, Rosenblum, Kurths 2001]

$x_i = (\theta_i, \dot{\theta}_i),\ i = 1, 2$

$$\begin{cases} \dot{x}_1 = f_1(x_1) \\ \dot{x}_2 = f_2(x_2) \end{cases}$$

ϕ_i : good parametrisation of limit cycles

$$\begin{cases} \dot{\phi}_1 = \omega_1 \\ \dot{\phi}_2 = \omega_2 \end{cases}$$

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If $\omega_1 \simeq \omega_2$: $\begin{cases} \psi = \phi_1 - \phi_2 \\ \varphi = \frac{\phi_1 + \phi_2}{2} \end{cases} \Rightarrow \begin{cases} \dot{\psi} = -\nu + \varepsilon q(\psi, \varphi) \\ \dot{\varphi} = \omega + \mathcal{O}(\varepsilon) \end{cases}$ $\quad \nu = \omega_2 - \omega_1$, $\omega = \frac{\omega_1 + \omega_2}{2}$

For small detuning ν: averaging $\Rightarrow$ $\omega \frac{d\psi}{d\varphi} \simeq -\nu + \varepsilon \bar{q}(\psi)$

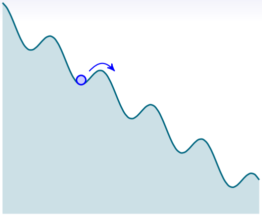
Example: Adler's equation $\bar{q}(\psi) = \sin(\psi)$: Fixed points for $\sin(\psi) = \nu/\varepsilon$

Remark: if $\omega_2/\omega_1 \simeq m/n$ similar behaviour for $\psi = n\phi_1 - m\phi_2$ (Arnold tongues)

Noise-induced phase slips

Averaged equation with noise

$$\omega \frac{\mathsf{d}\psi}{\mathsf{d}\varphi} = \underbrace{-\nu + \varepsilon \bar{\mathsf{q}}(\psi)}_{-\frac{\partial}{\partial\psi}\left(\nu\psi - \varepsilon\int^{\psi}\bar{\mathsf{q}}(x)\,\mathsf{d}x\right)} + \text{noise}$$



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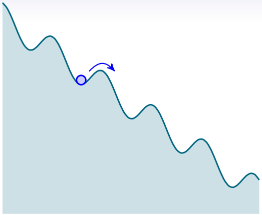

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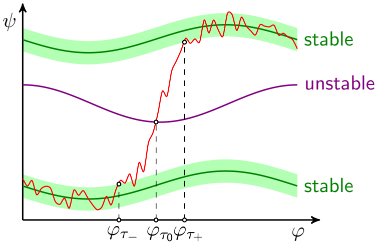


Original equations with noise

$$\begin{cases} \dot{\psi} = -\nu + \varepsilon q(\psi, \varphi) + \text{noise} \\ \dot{\varphi} = \omega + \mathcal{O}(\varepsilon) + \text{noise} \end{cases}$$

Question: distribution of phases φ_{τ_0} when crossing unstable orbit?
This is a stochastic exit problem.

Stochastic differential equations

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sigma g(x_t)\,\mathrm{d}W_t \qquad x \in \mathbb{R}^n$$

- Transition probability density: $p_t(x,y)$
- Markov semigroup T_t: for $\varphi \in L^\infty$,

$$(T_t\varphi)(x) = \mathbb{E}^x[\varphi(x_t)] = \int p_t(x,y)\varphi(y)\,\mathrm{d}y$$

  Generator: $L\varphi = \frac{\mathrm{d}}{\mathrm{d}t} T_t\varphi|_{t=0}$

$$(L\varphi)(x) = \sum_i f_i(x)\frac{\partial\varphi}{\partial x_i} + \frac{\sigma^2}{2}\sum_{i,j}(gg^T)_{ij}(x)\frac{\partial^2\varphi}{\partial x_i\partial x_j}$$

- Adjoint semigroup: for $\mu \in L^1$,

$$(\mu T_t)(y) = \mathbb{P}^\mu\{x_t = \mathrm{d}y\} = \int \mu(x)p_t(x,y)\,\mathrm{d}x$$

  with generator L^*

- Kolmogorov equations: $\frac{\mathrm{d}}{\mathrm{d}t}p_t(x,y) = L_x p_t(x,y)$

  $\frac{\mathrm{d}}{\mathrm{d}t}p_t(x,y) = L_y^* p_t(x,y)$ (Fokker–Planck)

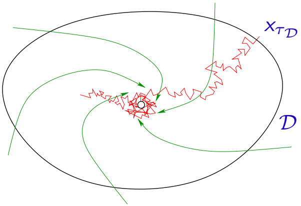
Stochastic exit problem

Given $\mathcal{D} \subset \mathbb{R}^n$, define first-exit time

$$\tau_{\mathcal{D}} = \inf\{t > 0 \colon x_t \notin \mathcal{D}\}$$

First-exit location $x_{\tau_{\mathcal{D}}} \in \partial\mathcal{D}$
defines harmonic measure

$$\mu(A) = \mathbb{P}^x\{x_{\tau_{\mathcal{D}}} \in A\}$$



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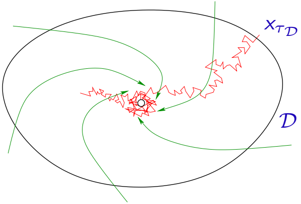

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Facts (following from Dynkin's formula):

- $u(x) = \mathbb{E}^x[\tau_{\mathcal{D}}]$ satisfies $\begin{cases} Lu(x) = -1 & x \in \mathcal{D} \\ u(x) = 0 & x \in \partial\mathcal{D} \end{cases}$

- For $\varphi \in L^\infty(\partial\mathcal{D}, \mathbb{R})$, $h(x) = \mathbb{E}^x[\varphi(x_{\tau_{\mathcal{D}}})]$ satisfies

$$\begin{cases} Lh(x) = 0 & x \in \mathcal{D} \\ h(x) = \varphi(x) & x \in \partial\mathcal{D} \end{cases}$$

Wentzell–Freidlin theory

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sigma g(x_t)\,\mathrm{d}W_t \qquad x \in \mathbb{R}^n$$

Large-deviation principle with rate function

$$I(\gamma) = \frac{1}{2}\int_0^T (\dot\gamma_t - f(\gamma_t))^T D(\gamma_t)^{-1}(\dot\gamma_t - f(\gamma_t))\,\mathrm{d}t \qquad D = gg^T$$

For a set Γ of paths $\gamma : [0, T] \to \mathbb{R}^n$: $\qquad \mathbb{P}\{(x_t)_{0\leqslant t\leqslant T} \in \Gamma\} \simeq \mathrm{e}^{-\inf_\Gamma I/\sigma^2}$

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Consider domain $\mathcal{D}$ contained in basin of attraction of attractor $\mathcal{A}$

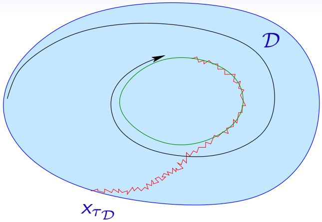
Quasipotential: $V(y) = \inf\{I(\gamma) \colon \gamma : \mathcal{A} \to y \in \partial\mathcal{D}$ in arbitrary time$\}$

- $\lim_{\sigma\to 0} \sigma^2 \log \mathbb{E}[\tau_{\mathcal{D}}] = \overline{V} = \inf_{y\in\partial\mathcal{D}} V(y)$ [Wentzell, Freidlin '69]
- If inf reached at a single point $y^* \in \mathcal{D}$ then $\lim_{\sigma\to 0} \mathbb{P}\{\|x_{\tau_{\mathcal{D}}} - y^*\| > \delta\} = 0 \quad \forall\delta > 0$ [Wentzell, Freidlin '69]
- Exponential distr of $\tau_{\mathcal{D}}$: $\lim_{\sigma\to 0} \mathbb{P}\{\tau_{\mathcal{D}} > s\mathbb{E}[\tau_{\mathcal{D}}]\} = \mathrm{e}^{-s}$ [Day '83]

Application to exit through unstable periodic orbit

Planar SDE
$dx_t = f(x_t)\,dt + \sigma g(x_t)\,dW_t$

$\mathcal{D} \subset \mathbb{R}^2$: int of unstable periodic orbit
First-exit time: $\tau_{\mathcal{D}} = \inf\{t > 0 \colon x_t \notin \mathcal{D}\}$
Law of first-exit location $x_{\tau_{\mathcal{D}}} \in \partial\mathcal{D}$?



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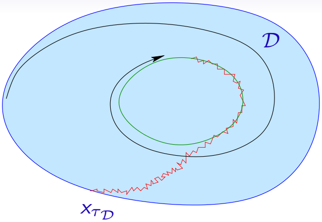

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Quasipotential:

$V(y) = \inf\{I(\gamma) \colon \gamma$: stable orbit $\to y \in \partial\mathcal{D}$ in arbitrary time$\}$

Theorem [Wentzell, Freidlin '69]: If V reaches its min at a unique $y^* \in \partial\mathcal{D}$, then $x_{\tau_{\mathcal{D}}}$ concentrates in y^* as $\sigma \to 0$

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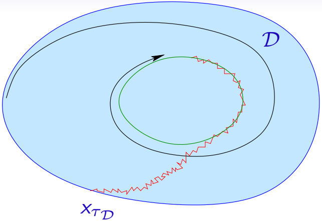

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Theorem [Wentzell, Freidlin '69]: If V reaches its min at a unique $y^* \in \partial\mathcal{D}$, then $x_{\tau_{\mathcal{D}}}$ concentrates in y^* as $\sigma \to 0$

Problem: V is constant on $\partial\mathcal{D}$!

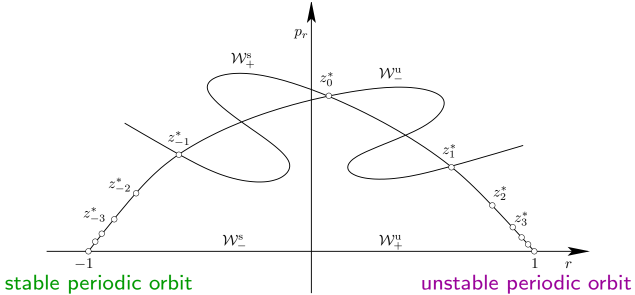
Most probable exit paths

In polar-type coordinates

$$\mathrm{d}\varphi_t = f_\varphi(\varphi_t, r_t)\,\mathrm{d}t + \sigma g_\varphi(\varphi_t, r_t)\,\mathrm{d}W_t \qquad \varphi \in \mathbb{R}/2\pi\mathbb{Z}$$
$$\mathrm{d}r_t = f_r(\varphi_t, r_t)\,\mathrm{d}t + \sigma g_r(\varphi_t, r_t)\,\mathrm{d}W_t \qquad r \in [-1, 1]$$

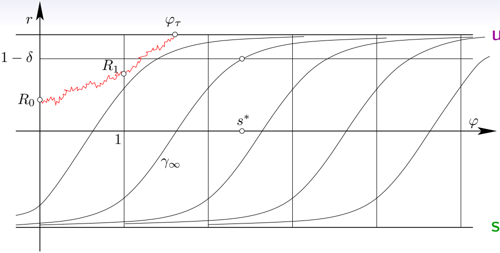
Minimisers of I obey Hamilton equations with Hamiltonian

$H(\gamma, \psi) = \frac{1}{2}\psi^T D(\gamma)\psi + f(\gamma)^T\psi$ where $\psi = D(\gamma)^{-1}(\dot{\gamma} - f(\gamma))$



Generically optimal path γ_∞ (for infinite time) is isolated

Random Poincaré maps



- $R_0, R_1, \dots R_N$ form substochastic Markov chain (killed in $r = 1$)
- Under hypoellipticity cond, transition kernel has smooth density k
  [Ben Arous, Kusuoka, Stroock '84]

$$\mathbb{P}^{R_0}\{R_1 \in B\} = K(R_0, B) := \int_B k(R_0, y)\,\mathrm{d}y$$

- Fredholm theory: spectral decomp $k(x,y) = \sum_{k \geqslant 0} \lambda_k h_k(x) h_k^*(y)$
  $\lambda_0 \in [0,1]$: principal eigenvalue [Perron, Frobenius, Jentzsch, Krein–Rutman]
  $\lim_{n\to\infty} \mathbb{P}\{R_n \in \mathrm{d}x | N > n\} = \dfrac{h_0^*(x)}{\int h_0^*} = \pi_0(x)$ quasistationary distr (QSD)

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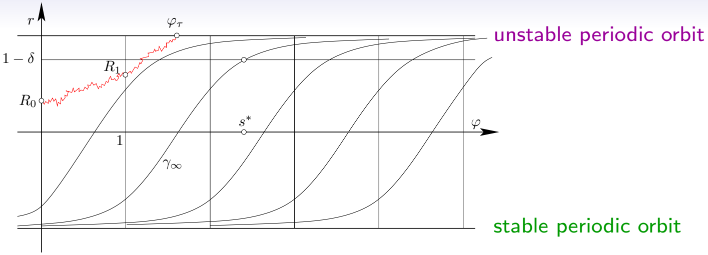


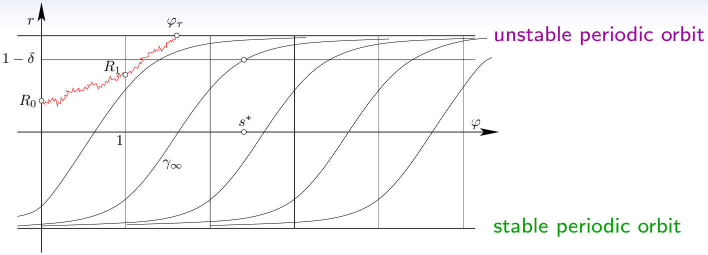
Consequences of spectral decomp $k(x,y) = \sum_{k\geqslant 0} \lambda_k h_k(x) h_k^*(y)$ assuming spectral gap $|\lambda_1|/\lambda_0 < 1$:

- $\mathbb{P}^{R_0}\{R_n \in A\} = \lambda_0^n h_0(R_0) \int_A h_0^*(y)\, \mathrm{d}y \left[1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)\right]$
- If $t = n + s$,

$$\mathbb{P}^{R_0}\{\varphi_\tau \in \mathrm{d}t\} = \lambda_0^n h_0(R_0) \int h_0^*(y) \mathbb{P}^y\{\varphi_\tau \in \mathrm{d}s\}\, \mathrm{d}y \left[1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)\right]$$

Periodically modulated exponential distribution: $f(t+1) \simeq \lambda_0 f(t)$

Computation of exit distribution



Split into two Markov chains:

- ▹ Chain killed upon r reaching $1-\delta$ in $\varphi = \varphi_{\tau_-}$

$$\mathbb{P}^0\{\varphi_{\tau_-} \in [\varphi_1, \varphi_1 + \Delta]\} \simeq (\lambda_0^{\mathsf{s}})^{\varphi_1} \, \mathrm{e}^{-J(\varphi_1)/\sigma^2}$$

- ▹ Chain killed at $r = 1 - 2\delta$ and on unstable orbit $r = 1$
  - ◇ Principal eigenvalue: $\lambda_0^{\mathsf{u}} = \mathrm{e}^{-2\lambda_+ T_+}(1 + \mathcal{O}(\delta))$
    $\lambda_+ =$ Lyapunov exponent, $T_+ =$ period of unstable orbit
  - ◇ Using LDP:

$$\mathbb{P}^{\varphi_1}\{\varphi_\tau \in [\varphi, \varphi + \Delta]\} \simeq (\lambda_0^{\mathsf{u}})^{\varphi - \varphi_1} \, \mathrm{e}^{-[I_\infty + c(\mathrm{e}^{-2\lambda_+ T_+(\varphi - \varphi_1)})]/\sigma^2}$$

Main result: log-periodic oscillations (cycling)

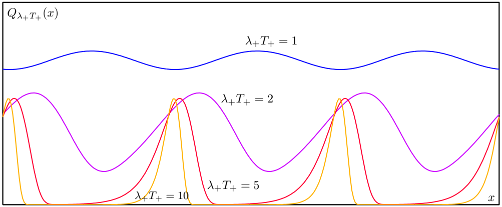
Theorem [B & Gentz, SIAM J Math Anal 2014]

$\exists \beta, c > 0 \colon \forall \Delta, \delta > 0 \; \exists \sigma_0 > 0 \colon \forall 0 < \sigma < \sigma_0$

$$\mathbb{P}^{r_0,0}\Big\{\theta(\varphi_\tau) \in [t, t+\Delta]\Big\} = \Delta C(\sigma)(\lambda_0)^t Q_{\lambda_+ T_+}\bigg(\frac{|\log\sigma|}{\lambda_+ T_+} - t + \mathcal{O}(\delta)\bigg)$$
$$\times \Big[1 + \mathcal{O}(\mathrm{e}^{-ct/|\log\sigma|}) + \mathcal{O}(\delta|\log\delta|) + \mathcal{O}(\Delta^\beta)\Big]$$

▷ $Q_{\lambda T}(x) = \displaystyle\sum_{n=-\infty}^{\infty} G(\lambda T(n-x))$ with $G(x) = \exp\{-2x - \frac{1}{2}\mathrm{e}^{-2x}\}$

Cycling profile, periodicised Gumbel distribution



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- $C(\sigma) = \mathcal{O}(\mathrm{e}^{-V/\sigma^2})$
- Cycling: periodic dependence on $|\log\sigma|$
  [Day'90, Maier & Stein '96, Getfert & Reimann '09]

Animations

Influence of noise intensity: cycling

$\lambda_+ = 1,\ T_+ = 4,\ V = 1$

$1 \geqslant \sigma \geqslant 0.0001$

(area under curve not normalized)

`avi file`

Influence of period

$\lambda_+ = 1,\ \sigma = 0.4,\ V = 1$

$0.001\, T_{\mathrm{K}} \leqslant T_+ \leqslant T_{\mathrm{K}}$

$T_{\mathrm{K}} \simeq \mathrm{e}^{V/\sigma^2}$ Kramers' time

(area under curve not normalized)

`avi file`

See also `http://www.univ-orleans.fr/mapmo/membres/berglund/simcycling.html`

Why log-periodic oscillations?

Phase at crossing: $\mathcal{W}_\Delta(t) = \sum_{n=0}^{\infty} \mathbb{P}^{r_0,0}\bigl\{\theta(\varphi_\tau) \in [n+t, n+t+\Delta]\bigr\}$

Corollary : $\mathcal{W}_\Delta(t) = \Delta Q_{\lambda_+ T_+}\Bigl(\frac{|\log\sigma|}{\lambda_+ T_+} - t + \mathcal{O}(\delta)\Bigr)\bigl[1+\mathcal{O}(\delta|\log\delta|)+\mathcal{O}(\Delta^\beta)\bigr]$

$$\lim_{\delta,\Delta\to 0}\lim_{\sigma\to 0}\frac{1}{\Delta}\mathcal{W}_\Delta\Bigl(t+\frac{|\log\sigma|}{\lambda_+ T_+}\Bigr) = Q_{\lambda_+ T_+}(-t)$$

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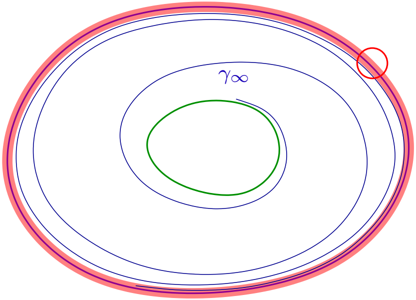
Heuristics :

$\theta(\varphi)$: parametrisation in which effective normal diffusion is constant

$\mathrm{dist}(\gamma_\infty, \text{unst orbit}) \simeq \mathrm{e}^{-\lambda_+ T_+ \theta(\varphi)}$

Escape when

$\mathrm{e}^{-\lambda_+ T_+ \theta(\varphi)} = \sigma \;\Rightarrow\; \theta(\varphi) = \dfrac{|\log \sigma|}{\lambda_+ T_+}$



Why log-periodic oscillations?

Phase at crossing: $\mathcal{W}_\Delta(t) = \sum_{n=0}^{\infty} \mathbb{P}^{r_0,0}\{\theta(\varphi_\tau) \in [n+t, n+t+\Delta]\}$

Corollary : $\mathcal{W}_\Delta(t) = \Delta Q_{\lambda_+ T_+}\left(\frac{|\log\sigma|}{\lambda_+ T_+} - t + \mathcal{O}(\delta)\right)\left[1+\mathcal{O}(\delta|\log\delta|)+\mathcal{O}(\Delta^\beta)\right]$

$$\lim_{\delta,\Delta\to 0}\lim_{\sigma\to 0}\frac{1}{\Delta}\mathcal{W}_\Delta\left(t+\frac{|\log\sigma|}{\lambda_+ T_+}\right) = Q_{\lambda_+ T_+}(-t)$$

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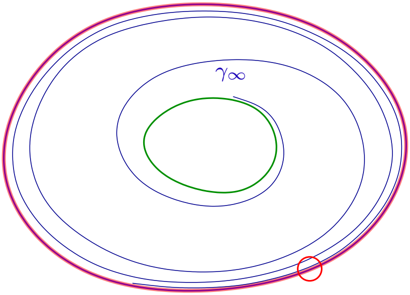

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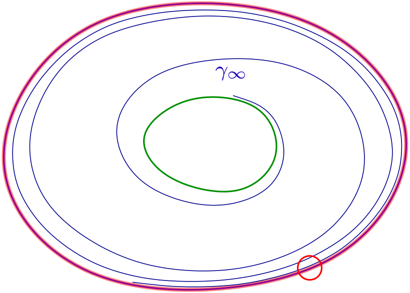

Remark: log-periodic oscillations appear in finance, diffusion through fractals, renormalization maps . . .

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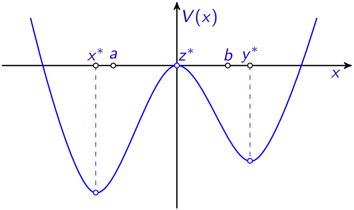
Why a Gumbel distribution?

Length of reactive path

[Cérou, Guyader, Lelièvre, Malrieu 2013] :

$\mathrm{d}x_t = -V'(x_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t \qquad a < x_0 < 0$



Theorem:

$$\lim_{\sigma\to 0} \mathbb{P}\Big\{\tau_b - \frac{2}{\lambda}|\log\sigma| < t \;\Big|\; \tau_b < \tau_a\Big\} = \frac{1}{\lambda}\Big(\log\frac{2|x_0|b}{\lambda} + I(x_0) + I(b) + \Lambda(t)\Big)$$

where $\lambda = -V''(0)$, $I(x) = \int_x^0 \big(\frac{\lambda}{V'(y)} + \frac{1}{y}\big)\,\mathrm{d}y$

and $\Lambda(t) = \mathrm{e}^{-\mathrm{e}^{-t}}$: distrib. function of standard Gumbel r.v.

Proof uses Doob's h-transform

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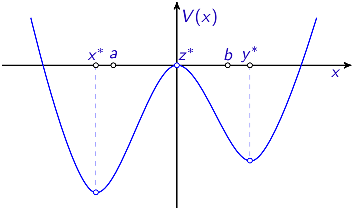

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[Bakhtin 2013] :

Link with extreme-value theory and residual lifetimes for linear case

$\mathrm{d}x_t = \lambda x_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$

Extreme-value theory and residual lifetime

- $X_1, X_2, \dots$ i.i.d. real r.v. $\qquad M_n = \max\{X_1, \dots, X_n\}$
- $F(x) = \mathbb{P}\{X_1 \leqslant x\} = 1 - R(x) \quad \Rightarrow \quad \mathbb{P}\{M_n \leqslant x\} = F(x)^n$
- **Def**: $F \in D(\Phi) \Leftrightarrow \exists (a_n)_n > 0, (b_n)_n : \lim_{n\to\infty} F(a_n x + b_n)^n = \Phi(x)$

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$$dx_t = \lambda x_t \, dt + \sigma \, dW_t \quad \Rightarrow \quad X_t = e^{\lambda t}\Big(x_0 + \widetilde{W}_{\sigma^2 \frac{1-e^{-2\lambda t}}{2\lambda}}\Big) = e^{\lambda t}\widetilde{X}_t$$

$$\tau = \inf\{t > 0\colon X_t = 0\}$$

By the reflection principle:

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$$\Lambda(\mathrm{e}^{-x}) = \mathrm{e}^{-\Lambda(x)} \qquad \to \exp\Big\{-x_0^2\lambda\,\mathrm{e}^{-2\lambda t}\Big\} \quad \text{as } \sigma \to 0$$