RPQFTR23, NTNU, Gjøvik, Norway

Perturbation theory for the Φ^4_3 model, revisited with Hopf Algebras

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Joint works with Tom Klose (Warwick) and Yvain Bruned (Nancy)

Project PERISTOCH

The Φ^4_d model

- Lattice system: $\Lambda_N = (\mathbb{Z}/N\mathbb{Z})^d$, $y \in \mathbb{R}^{\Lambda_N}$

$$V_{N,\varepsilon}(y) = \frac{1}{2}N^2 \sum_{\substack{i,j\in\Lambda \\ \|i-j\|=1}} (y_i - y_j)^2 + \sum_{i\in\Lambda} U_\varepsilon(y_i) \qquad U_\varepsilon(\xi) = \frac{1}{2}\xi^2 + \frac{\varepsilon}{4}\xi^4$$

Gibbs measure $\mu_{N,\varepsilon}(\mathrm{d}y) = \frac{1}{Z_{N,\varepsilon}} \mathrm{e}^{-V_{N,\varepsilon}(y)}\,\mathrm{d}y$

- Continuum limit: $y_i = \phi(i/N)$, $N \to \infty$,

$$V_\varepsilon(\phi) = \int_\Lambda \Big(\tfrac{1}{2}\|\nabla\phi(x)\|^2 + \tfrac{1}{2}\phi(x)^2 + \tfrac{\varepsilon}{4}\phi(x)^4\Big)\,\mathrm{d}x$$

where $\Lambda = (\mathbb{R}/\mathbb{Z})^d =: \mathbb{T}^d$

Definition of Gibbs measure?

$$\mu_\varepsilon(\mathrm{d}\phi) \text{ “=” } \frac{1}{Z_\varepsilon} \mathrm{e}^{-V_\varepsilon(\phi)} \text{ “}\mathrm{d}\phi\text{”}$$

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The case $d = 1$

- $\varepsilon = 0$: $V_0(\phi) = \int_\Lambda \Big(\frac12 \|\nabla\phi(x)\|^2 + \frac12 \phi(x)^2 \Big) \,\mathrm{d}x = \frac12 \langle \phi, (-\Delta + 1)\phi \rangle$

  μ_0 is Gaussian free field with covariance $(-\Delta + 1)^{-1}$
  (well-defined since $(-\Delta+1)^{-1}$ trace class: $\lambda_k = (2\pi k)^2$, $\sum_{k\in\mathbb{Z}} \frac{1}{\lambda_k + 1} < \infty$)

- $\varepsilon > 0$:

$$\frac{\mathrm{d}\mu_\varepsilon}{\mathrm{d}\mu_0} = \frac{Z_0}{Z_\varepsilon} \mathrm{e}^{-[V_\varepsilon - V_0]} = \frac{Z_0}{Z_\varepsilon} \mathrm{e}^{-\frac{\varepsilon}{4}\int_\Lambda \phi(x)^4 \,\mathrm{d}x}$$

  where

$$\frac{Z_\varepsilon}{Z_0} = \mathbb{E}^{\mu_0}\Big[\mathrm{e}^{-\frac{\varepsilon}{4}\int_\Lambda \phi(x)^4 \,\mathrm{d}x}\Big] \text{ “=” } \frac{1}{Z_0} \int \mathrm{e}^{-V_0(\phi) - \frac{\varepsilon}{4}\int_\Lambda \phi(x)^4 \,\mathrm{d}x} \,\mathrm{d}\phi$$

  Fourier representation:

$$\phi_{\mathrm{GFF}}(x) = \sum_{k\in\mathbb{Z}} \frac{Z_k}{\sqrt{\lambda_k + 1}} e_k(x), \qquad Z_k \sim \mathcal{N}(0,1) \text{ iid}$$

$$\Rightarrow \qquad \mathbb{E}\Big[\int_\Lambda \phi_{\mathrm{GFF}}(x)^{2n} \,\mathrm{d}x\Big] \lesssim \Big(\sum_{k\in\mathbb{Z}} \frac{1}{\lambda_k + 1}\Big)^n < C^n$$

  so that $\frac{Z_\varepsilon}{Z_0} = 1 + \mathcal{O}(\varepsilon)$

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The case $d = 2$

- $(-\Delta + 1)^{-1}$ no longer trace class, since $\lambda_k = (2\pi\|k\|)^2$, $\sum_{k\in\mathbb{Z}^2} \frac{1}{\lambda_k+1} = \infty$
- Truncated GFF:

$$\phi_{\mathrm{GFF},N}(x) = \sum_{k\in\mathbb{Z}^2, |k|\leqslant N} \frac{Z_k}{\sqrt{\lambda_k + 1}} e_k(x)$$

$$C_N = \int_\Lambda \mathbb{E}[\phi_{\mathrm{GFF},N}(x)^2]\,\mathrm{d}x = \sum_{|k|\leqslant N} \frac{1}{\lambda_k + 1} = \mathrm{Tr}\big[(-\Delta_N + 1)^{-1}\big] \sim \log N$$

- Wick calculus: $:\phi(x)^n: = H_n(\phi(x); C_N)$ where H_n Hermite polynomials

If (X, Y) centred jointly Gaussian rv, $\mathbb{E}[X^2] = C$, $\mathbb{E}[Y^2] = C'$ then $\mathbb{E}[H_n(X;C)H_m(Y;C')] = n!\delta_{nm}\mathbb{E}[XY]^n$

Consequence: $\sup_N \mathbb{E}\bigg[\bigg(\int_\Lambda :\phi_{\mathrm{GFF},N}(x)^n:\mathrm{d}x\bigg)^2\bigg] < \infty \qquad \forall n$

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$$V_\varepsilon(\phi) = \int_\Lambda \Big(\tfrac{1}{2}\|\nabla\phi(x)\|^2 + \tfrac{1}{2}\phi(x)^2 + \tfrac{\varepsilon}{4}:\phi(x)^4:\Big)\,\mathrm{d}x$$

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The case $d = 3$

Theorem: Potential needs exactly 4 counterterms:

$$V(\phi) = \int_\Lambda \Big(\tfrac{1}{2} \|\nabla\phi(x)\|^2 + \tfrac{1}{2}\big[1 - \varepsilon^2 C_N^{(2)}\big]\phi(x)^2 + \tfrac{\varepsilon}{4} {:}\phi(x)^4{:}_{C_N^{(1)}} + \varepsilon^2 C_N^{(3)} - \varepsilon^3 C_N^{(4)} \Big) \,\mathrm{d}x$$

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$$C_N^{(1)} = G_N(0) = \mathrm{Tr}\big((-\Delta_N + 1)^{-1}\big) = \mathcal{O}(N)$$

$$C_N^{(2)} = 3! \int_\Lambda G_N(x)^3 \,\mathrm{d}x = \mathcal{O}(\log N)$$

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and $G_N(x) = \sum_{|k| \leqslant N} \frac{1}{\lambda_k + 1} e_k(x)$ if the Green function of Δ_N

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Some literature

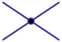
- Glimm & Jaffe (1968, 1973), Feldman (1974):
  Combinatorics of Feynman diagrams
- Benfatto, Cassandro, Gallavotti, Nicolò & Olivieri (1978, 1980):
  Renormalisation group (integrating out scales)
- Brydges, Fröhlich & Sokal (1983):
  Generating function and skeleton inequalities
- Brydges, Dimock & Hurd (1995):
  Polymer expansions
- Connes & Kreimer (2000, 2001):
  Hopf algebras
- ...
- Barashkov & Gubinelli (2020):
  Boué–Dupuis formula

Singular stochastic PDEs

$$\partial_t \phi(t,x) = \Delta\phi(t,x) - \phi(t,x) - \varepsilon\phi(t,x)^3 + \xi(t,x)$$

- Parisi & Wu (1981):
  Stochastic quantization
- Da Prato & Debussche (2003):
  2d case: Besov spaces, fixed-point argument for difference between ϕ and stochastic convolution
- Hairer (2014):
  3d case: regularity structures

Graphical notations

- Wick powers: $X =$  $= \int_\Lambda :\phi(x)^4: \mathrm{d}x$, $Y =$  $= \int_\Lambda :\phi(x)^2: \mathrm{d}x$
- Parameters: $\alpha = \frac{\varepsilon}{4}$, $\beta = \frac{1}{2}\varepsilon^2 C_N^{(2)}$, $\gamma = \varepsilon^2 C_N^{(3)} - \varepsilon^3 C_N^{(4)}$

  Then $\dfrac{Z_{N,\varepsilon}}{Z_{N,0}} = \mathbb{E}^{\mu_0}\left[\mathrm{e}^{-\alpha X - \beta Y - \gamma}\right] = \mathrm{e}^{-\gamma}\,\mathbb{E}^{\mu_0}\left[\mathrm{e}^{-\alpha X - \beta Y}\right]$
- Let $\Gamma = (\mathscr{V}, \mathscr{E})$ be a multigraph, $\mathscr{G} = \operatorname{span}\{\Gamma\}$. Its valuation is

$$\Pi_N(\Gamma) = \int_{\Lambda^{\mathscr{V}}} \prod_{e\in\mathscr{E}} G_N(x_{e_+} - x_{e_-})\,\mathrm{d}x$$

  For instance

$$C_N^{(1)} = \Pi_N$$

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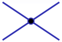
- Wick powers: $X =$  $= \int_\Lambda :\phi(x)^4:\mathrm{d}x$, $Y =$  $= \int_\Lambda :\phi(x)^2:\mathrm{d}x$
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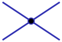
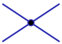
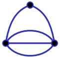
Cumulant expansion

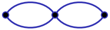
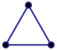
- $\mu_n = (-1)^n \mathbb{E}^{\mu_0}\Big[\Big(\alpha$  $+ \beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\mu_0}\Big[$  ^{m}  $^{n-m}\Big]$

  Examples:

  $\mu_2 = \alpha^2 4! \Pi_N$  $+ \beta^2 2! \Pi_N$ 

  $\mu_3 = -\alpha^3 \binom{4}{2}^3 2^3 \Pi_N$  $- 3\alpha^2\beta(4^2 \cdot 2 \cdot 3!)\Pi_N$ 

  $- 3\alpha\beta^2 4!\, \Pi_N$  $- 8\beta^3 \Pi_N$ 

- Cumulant expansion: (Leonov & Shiraev)

  $$-\log \mathbb{E}[e^{-\alpha X - \beta Y - \gamma}] = \gamma - \sum_{n=2}^{\infty} \frac{\kappa_n}{n!} \qquad \kappa_n = \mu_n - \sum_{m=2}^{n-2} \binom{n-1}{m} \kappa_m \mu_{n-m}$$

- Linked Cluster Theorem: κ_n projection of μ_n on connected graphs

  Proof: for instance Peccati & Taqqu (2011)

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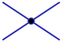
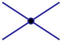
- $\mu_n = (-1)^n \mathbb{E}^{\mu_0}\Big[\Big(\alpha$  $+ \beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\mu_0}\Big[$  ^{m}  $^{n-m}\Big]$

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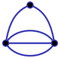
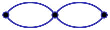
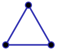
  $\mu_3 = -\alpha^3 \binom{4}{2}^3 2^3 \Pi_N$  $- 3\alpha^2\beta(4^2 \cdot 2 \cdot 3!)\Pi_N$ 

  $- 3\alpha\beta^2 4!\, \Pi_N$  $- 8\beta^3 \Pi_N$ 

- Cumulant expansion: (Leonov & Shiraev)

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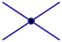
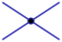
- Linked Cluster Theorem: κ_n projection of μ_n on connected graphs

  Proof: for instance Peccati & Taqqu (2011)

Cumulant expansion

- $\mu_n = (-1)^n \mathbb{E}^{\mu_0}\Big[\Big(\alpha$  $+ \beta$  $\Big)^n\Big] = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} A_{nm}$

  where $A_{nm} = \mathbb{E}^{\mu_0}\Big[$  ^{m}  $^{n-m}\Big]$

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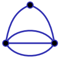
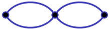
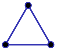
  $\mu_3 = -\alpha^3 \binom{4}{2}^3 2^3 \Pi_N$  $- 3\alpha^2\beta(4^2 \cdot 2 \cdot 3!)\Pi_N$ 

  $- 3\alpha\beta^2 4!\, \Pi_N$  $- 8\beta^3 \Pi_N$ 

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Divergences and subdivergences

- Degree of Γ: $\deg(\Gamma) = 3(|\mathscr{V}| - 1) - |\mathscr{E}|$. Γ divergent if $\deg(\Gamma) \leqslant 0$.
- Examples:

$\deg($ $) = 0$ $\Pi_N($ $) = \mathcal{O}(\log N)$

$\deg($ $) = -1$ $\Pi_N($ $) = \mathcal{O}(N)$

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However, $\deg($ $) = 1$, while $\Pi_N($ $) = \mathcal{O}(\log N)$ ‘
because it contains a subdivergence

Theorem: [Dyson]

If $\deg \bar{\Gamma} > 0$ for all subgraphs $\bar{\Gamma} \subset \Gamma$, then $\Pi_N(\Gamma)$ is bounded unif in N

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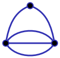
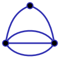
  However, deg() = 1, while $\Pi_N($  $) = \mathcal{O}(\log N)$ ‘
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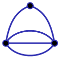
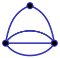
  However, $\deg($  $) = 1$, while $\Pi_N($  $) = \mathcal{O}(\log N)$ ‘
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Hopf algebras and renormalisation

- Connes–Kreimer extraction–contraction coproduct: $\Delta : \mathscr{G} \to \mathscr{G} \otimes \mathscr{G}$

  $$\Delta(\Gamma) = \Gamma \otimes \mathbf{1} + \mathbf{1} \otimes \Gamma + \sum_{\substack{\mathbf{1} \neq \bar\Gamma \subsetneq \Gamma \\ \deg(\bar\Gamma) \leqslant 0}} \bar\Gamma \otimes (\Gamma/\bar\Gamma) \qquad (\mathbf{1}\text{: empty graph})$$

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  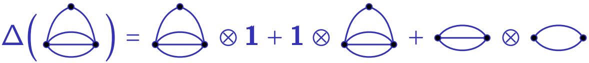

- (Twisted) antipode: $\mathscr{A} : \mathscr{G} \to \mathscr{G}$, $\quad \mathscr{A}(\Gamma) = -\Gamma - \sum_{\substack{\mathbf{1} \neq \bar\Gamma \subsetneq \Gamma \\ \deg(\bar\Gamma) \leqslant 0}} \mathscr{A}(\bar\Gamma) \cdot (\Gamma/\bar\Gamma)$

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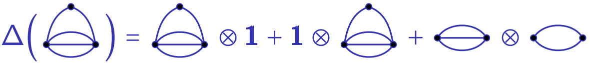

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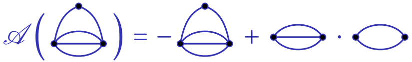

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  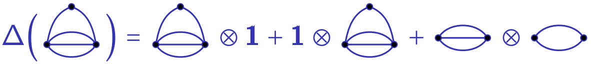

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  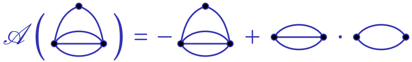

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BPHZ renormalisation

- BPHZ character: $\langle g^{\mathrm{BPHZ}}, \Gamma \rangle = \Pi_N \mathscr{A}(\Gamma) \mathbf{1}_{\deg \Gamma \leqslant 0}$
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Theorem: [Bogolyubov, Parasiuk, Hepp, Zimmermann]

If $\deg \Gamma > 0$ then $\Pi_N^{\mathrm{BPHZ}}(\Gamma)$ bdd uniformly in N

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Write $\kappa_n = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \beta^{n-m} \sum_k b_{nm}^{(k)} \Pi_N(\Gamma_{nm}^{(k)})$ Then

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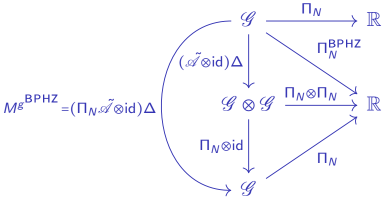

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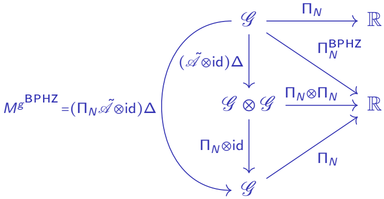

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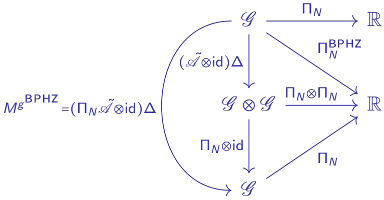

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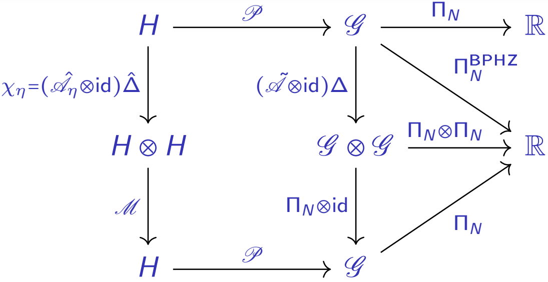
Commutative diagram



- $H = \mathrm{span}\{X^{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{N}^2\}$ $\quad X^{\boldsymbol{n}} := X^{n_1} Y^{n_2}$ (Ebrahimi-Fard et al)

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Lemma: $(\mathscr{M} \circ \chi_\eta)\, e^{-\alpha X} = e^{-\alpha X - \beta Y}$

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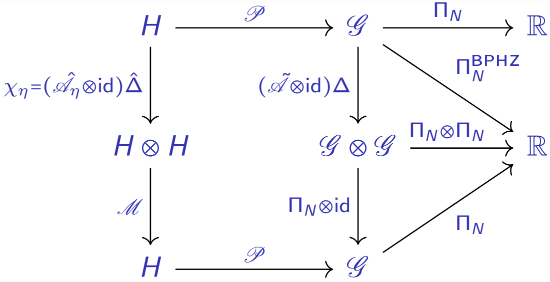

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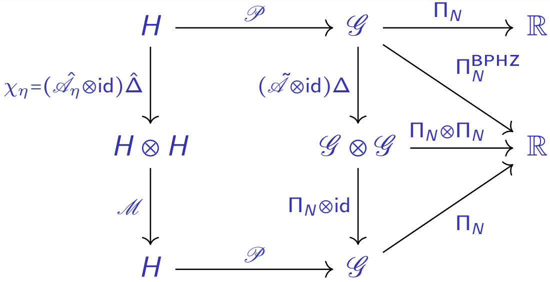

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Zimmermann's forest formula

- Zimmermann forest formula: $\mathscr{A}(\Gamma) = -\sum_{\mathscr{F}} (-1)^{|\mathscr{F}|} \mathscr{C}_{\mathscr{F}} \Gamma$

  where sum ranges over all forests $\mathscr{F}$ (set of subgraphs, pairwise vertex-disjoint or included) and $\mathscr{C}_{\mathscr{F}}$ extracts all subgraphs in $\mathscr{F}$

- Our case: $\mathscr{A}(\Gamma) = -\sum_{S \subset \{1,\dots,g\}} (-1)^{|S|}$ $^{|S|} \mathscr{C}_S \Gamma$

  where $\mathscr{C}_S$ extracts all "bubbles" labelled by element of S, g is number of bubbles

  Since $\Pi_N($ $) = \frac{\beta}{3\varepsilon^2} = \frac{\beta}{48\alpha^2}$ we have

$$\Pi_N^{\mathrm{BPHZ}}(\Gamma_{pp}^{(k)}) = -\sum_{S \subset \{1,\dots,g\}} \left(-\frac{\beta}{48\alpha^2}\right)^{|S|} \Pi_N(\mathscr{C}_S \Gamma_{pp}^{(k)})$$

Lemma: The diagram commutes, that is,

$\mathscr{P} \circ \mathscr{M} \circ \chi_\eta = (\Pi_N \tilde{\mathscr{A}} \otimes \mathrm{id})\Delta \circ \mathscr{P}$

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Borel resummation: The Φ_0^4 model

- $V(\phi) = \frac{1}{2}\phi^2 + \frac{\varepsilon}{4}\phi^4$

$$Z(\varepsilon) = \int_{-\infty}^{\infty} \mathrm{e}^{-V(\phi)}\,\mathrm{d}\phi = \int_{-\infty}^{\infty} \mathrm{e}^{-\phi^2/2}\,\mathrm{e}^{-\varepsilon\phi^4/4}\,\mathrm{d}\phi$$

$$Z(\varepsilon) \asymp \sqrt{2\pi}\sum_{n\geqslant 0}\Big(-\frac{\varepsilon}{4}\Big)^n \frac{(4n-1)!!}{n!} = \sum_{n\geqslant 0} a_n\varepsilon^n, \qquad a_n \sim n!$$

- Borel transform:

$$Z(\varepsilon) \asymp \sum_{n\geqslant 0} a_n\varepsilon^n \frac{\Gamma(n+1)}{n!} = \sum_{n\geqslant 0}\frac{a_n\varepsilon^n}{n!}\int_0^\infty t^n\,\mathrm{e}^{-t}\,\mathrm{d}t$$

$$Z_{\mathrm{Borel}}(\varepsilon) = \int_0^\infty \mathrm{e}^{-t}\sum_{n\geqslant 0}\frac{a_n\varepsilon^n t^n}{n!}\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-t}\,\mathcal{B}Z(\varepsilon t)\,\mathrm{d}t$$

where $\mathcal{B}Z(t) = \sum_{n\geqslant 0}\frac{a_n}{n!}t^n$

Theorem (Watson 1912, Sokal 1980) $D_R = \{\varepsilon\colon \mathrm{Re}\,\varepsilon^{-1} > R^{-1}\}$

If Z analytic in D_R and $Z(\varepsilon) = \sum_{k=0}^n a_k\varepsilon^k + R_n(\varepsilon)$ with $|R_n(\varepsilon)| \leqslant Cr^n n!|\varepsilon|^n$ unif in n and ε, then $\mathcal{B}Z(t)$ cv for $|t| < \frac{1}{r}$ and $Z(\varepsilon) = Z_{\mathrm{Borel}}(\varepsilon)$ in D_R

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- Borel summability proved by Magnen and Sénéor (1977)
- Need to prove
  - Analycity in D_R: hard?
  - Bound $|R_n(\varepsilon)| \leqslant Cr^n n! |\varepsilon|^n$ in D_R: doable?
- $e^{-\alpha X} = P(X) + F(X), \quad P(X) = \sum_{p=0}^{3} \frac{(-\alpha)^p}{p!} X^p$

  $\log \mathbb{E}[e^{-\alpha X - \beta Y - \gamma}] = -\gamma - (\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}) e^{-\alpha X} = -(\Pi_N^{\mathrm{BPHZ}} \circ \mathscr{P}) F(X)$
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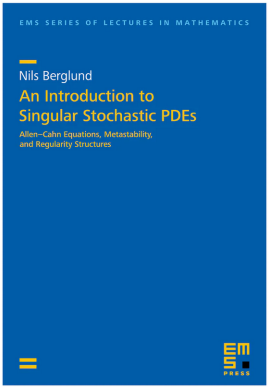

Thanks for your attention!

Slides available at

https://www.idpoisson.fr/berglund/Gjovik23.pdf

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