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Stochastic resonance in stochastic PDEs

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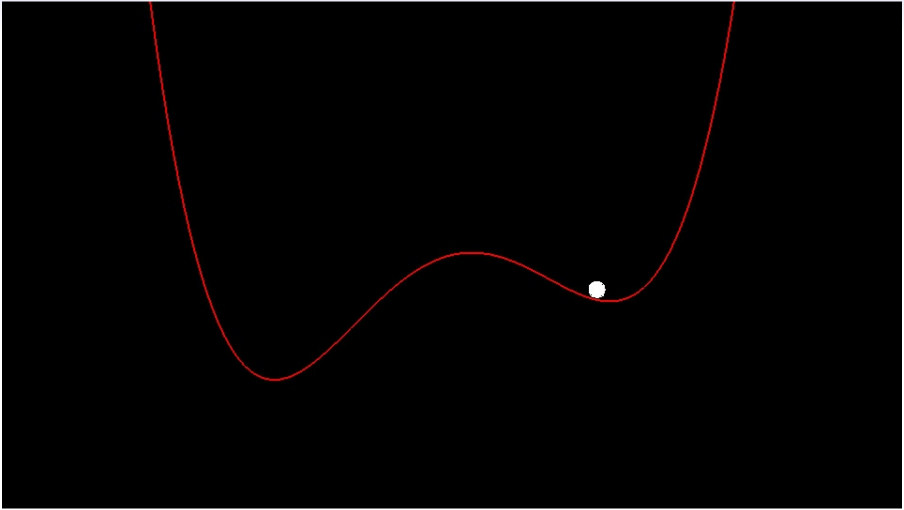
Based on joint works with Rita Nader (Orléans) and Barbara Gentz (Bielefeld)

Project
PERISTOCH

Stochastic resonance in an SDE



$$\mathrm{d}x_t = \underbrace{\left[-x_t^3 + x_t + A\cos(\varepsilon t)\right]}_{=-\frac{\partial}{\partial x}\left[\frac{1}{4}x^4-\frac{1}{2}x^2-Ax\cos(\varepsilon t)\right]\Big|_{x_t}} \mathrm{d}t + \sigma\,\mathrm{d}W_t$$

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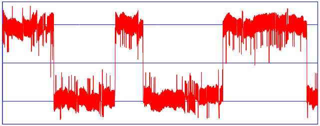
Stochastic resonance in an SDE

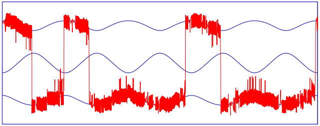
$$\mathrm{d}x_t = \underbrace{\left[-x_t^3 + x_t + A\cos(\varepsilon t)\right]}_{=-\frac{\partial}{\partial x}\left[\frac{1}{4}x^4-\frac{1}{2}x^2-Ax\cos(\varepsilon t)\right]\Big|_{x_t}} \mathrm{d}t + \sigma\,\mathrm{d}W_t$$

- ▷ Ice Ages: deterministically bistable climate [Croll, Milankovitch]
- ▷ random perturbations due to weather [Benzi-Sutera-Vulpiani, Nicolis-Nicolis]

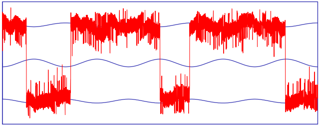
Sample paths $\{x_t\}_t$ for $\varepsilon = 0.001$:



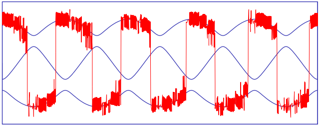
$A = 0, \ \sigma = 0.3$



$A = 0.24, \ \sigma = 0.2$



$A = 0.1, \ \sigma = 0.27$



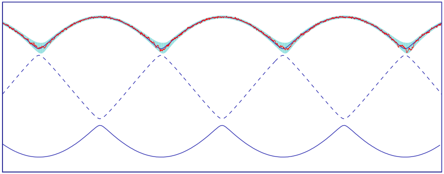
$A = 0.35, \ \sigma = 0.2$

Descriptions of stochastic resonance

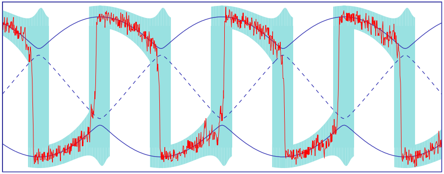
- Fokker–Planck equation: [Caroli, Caroli, Roulet & Saint-James '81]
- Two-state Markov chain: [Eckmann & Thomas '82], [Imkeller & Pavljukevich '02], [Herrmann & Imkeller '02]
- Signal-to-noise ratio: [Gammaitoni, Menichella-Saetta & . . . '89], [Fox '89], [Jung& Hänggi '89], [McNamara & Wiesenfeld '89]
- Slow forcing: [Jung & Hänggi '91], [Talkner '99], [Talkner & Łuczka '04]
- Large deviations: [Freidlin '00, Freidlin '01]
- Residence-time distributions: [Zhou, Moss & Jung '90], [Choi, Fox & Jung '98], . . .
- Overview articles: [Moss, Pierson & O'Gorman '94], [Wiesenfeld & Moss '95], [McNamara & Wiesenfeld '95], [Wiesenfeld & Jaramillo '98], [Gammaitoni, Hänggi, Jung & Marchesoni '98], [Hänggi '02], [Wellens, Shatokhin & Buchleitner '04], . . .
- Monograph: [Herrmann, Imkeller, Pavlyukevich & Peithmann '14]

The synchronisation regime

$A_c = \frac{2}{3\sqrt{3}}$, $A = A_c - \delta$, $0 < \delta \ll 1$. Critical noise intensity: $\sigma_c = \max\{\delta, \varepsilon\}^{3/4}$



$\sigma \ll \sigma_c$:
transitions unlikely



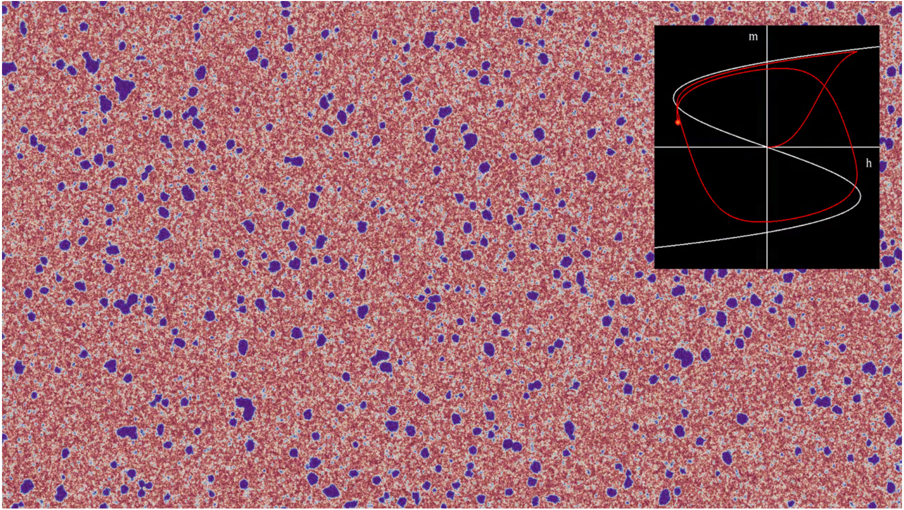
$\sigma \gg \sigma_c$:
synchronisation

Theorem [B & Gentz, Annals App. Proba 2002]

- Away from (avoided) bifurcations, sample paths concentrated in σ-neighbourhood of deterministic stable periodic solutions
- $\sigma \ll \sigma_c$: transition probability per period $\leqslant e^{-\sigma_c^2/\sigma^2}$
- $\sigma \gg \sigma_c$: transition probability per period $\geqslant 1 - e^{-c\sigma^{4/3}/(\varepsilon|\log\sigma|)}$

Stochastic resonance in stochastic PDEs

$$d\phi(t,x) = \Big[\Delta\phi(t,x) + \phi(t,x) - \phi(t,x)^3 + \underbrace{A\cos(\varepsilon t)}_{h(\varepsilon t)}\Big]dt + \sigma\, dW(t,x)$$



Simulation available at `youtu.be/eN3NWiEjBK8`

Stochastic resonance in SPDEs

$$d\phi(t,x) = \Big[\Delta\phi(t,x) + f(\varepsilon t, \phi(t,x))\Big]dt + \sigma\, dW(t,x)$$

- ▷ $\phi = \phi(t,x) \in \mathbb{R}$, $\varepsilon t \in [0, T]$ or f is T-periodic, $x \in \mathbb{T} = \mathbb{R}/L\mathbb{Z}$, $L > 0$
- ▷ $\phi \mapsto f(s,\phi)$ bistable, $\mathcal{C}^2$, confining, e.g. $f(s,\phi) = \phi - \phi^3 + A\cos(s)$
- ▷ $dW(t,x)$ space-time white noise on $\mathbb{R}_+ \times \mathbb{T}$
- ▷ $0 < \varepsilon, \sigma \ll 1$
- ▷ δ measures closeness to bifurcation (e.g. $A_{\mathrm{c}} - A$)

Theorem [B & Nader, Stoch. & PDEs: Analysis & Comput., 2022]

- ▷ Away from bifurcations, solutions are concentrated around deterministic solutions in Sobolev H^s-norm for any $s < \frac{1}{2}$
- ▷ $\sigma \ll \sigma_{\mathsf{c}} = \max\{\delta, \varepsilon\}^{3/4}$: transition probability per period $\leqslant e^{-\sigma_{\mathsf{c}}^2/\sigma^2}$
- ▷ $\sigma \gg \sigma_{\mathsf{c}}$: transition probability per period $\geqslant 1 - e^{-c\sigma^{4/3}/(\varepsilon|\log\sigma|)}$

Proof ideas, 1D SDE below threshold

On slow time scale $\varepsilon t \to t$:

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(t, x_t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

$\bar{x}(t)$ deterministic solution tracking stable equilibrium $x^*(t)$.
Write $x_t = \bar{x}(t) + \xi_t$ and Taylor-expand:

$$\mathrm{d}\xi_t = \frac{1}{\varepsilon}\Big[\bar{a}(t)\xi_t + \underbrace{b(t,\xi_t)}_{=\mathcal{O}(\xi_t^2)}\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

where $\bar{a}(t) = \partial_x f(t, \bar{x}(t)) = \partial_x f(t, x^*(t)) + \mathcal{O}(\varepsilon) < 0$

Variations of constants (Duhamel formula), if $\xi_0 = 0$:

$$\xi_t = \underbrace{\frac{\sigma}{\sqrt{\varepsilon}} \int_0^t \mathrm{e}^{\bar{\alpha}(t,s)/\varepsilon}\,\mathrm{d}W_s}_{\xi_t^0\text{: sol of linearised system}} + \underbrace{\frac{1}{\varepsilon}\int_0^t \mathrm{e}^{\bar{\alpha}(t,s)/\varepsilon}\, b(s,\xi_s)\,\mathrm{d}s}_{\text{treat as a perturbation}}$$

where $\bar{\alpha}(t,s) = \int_s^t \bar{a}(u)\,\mathrm{d}u$

Proof ideas, 1D SDE below threshold

Properties of $\xi_t^0 = \dfrac{\sigma}{\sqrt{\varepsilon}} \displaystyle\int_0^t \mathrm{e}^{\bar\alpha(t,s)/\varepsilon} \, \mathrm{d}W_s$:

- Gaussian process, $\mathbb{E}[\xi_t^0] = 0$, $\operatorname{Var}(\xi_t^0) = \frac{\sigma^2}{\varepsilon} \int_0^t \mathrm{e}^{2\bar\alpha(t,s)/\varepsilon} \, \mathrm{d}s$
- Confidence interval: $\mathbb{P}\Bigl\{|\xi_t^0| > \frac{h}{\sigma}\sqrt{\operatorname{Var}(\xi_t^0)}\Bigr\} = \mathcal{O}(\mathrm{e}^{-h^2/2\sigma^2})$
- $\sigma^{-2}\operatorname{Var}(\xi_t^0)$ satisfies ODE $\varepsilon\dot v = 2\bar a(t) v + 1$

Lemma [B & Gentz, PTRF 2002]

$\bar v(t)$ solution of ODE bounded away from 0: $\bar v(t) = \frac{1}{-2\bar a(t)} + \mathcal{O}(\varepsilon)$

$$\mathbb{P}\biggl\{\sup_{0\leqslant s\leqslant t} \frac{|\xi_s^0|}{\sqrt{\bar v(s)}} > h\biggr\} = C_0(t,\varepsilon)\, \mathrm{e}^{-h^2/2\sigma^2}$$

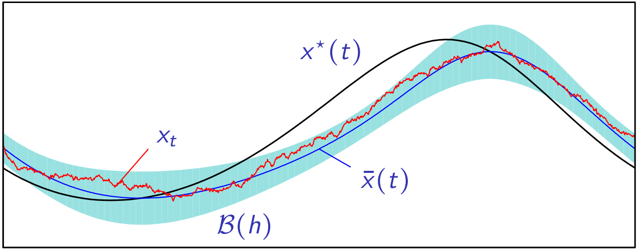
where $C_0(t,\varepsilon) = \sqrt{\frac{2}{\pi}} \frac{1}{\varepsilon} \Bigl|\int_0^t \bar a(s) \, \mathrm{d}s\Bigr| \frac{h}{\sigma} \Bigl[1 + \mathcal{O}\bigl(\varepsilon + \frac{t}{\varepsilon}\mathrm{e}^{-h^2/\sigma^2}\bigr)\Bigr]$

Proof based on Doob's submartingale inequality and partition of $[0,t]$

Proof ideas, 1D SDE below threshold

Nonlinear equation: $\mathrm{d}\xi_t = \frac{1}{\varepsilon}\big[\bar{a}(t)\xi_t + b(t,\xi_t)\big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$

Confidence strip: $\mathcal{B}(h) = \big\{|\xi| \leqslant h\sqrt{\bar{v}(t)}\ \forall t\big\} = \big\{|x - \bar{x}(t)| \leqslant h\sqrt{\bar{v}(t)}\ \forall t\big\}$



Theorem B & Gentz, PTRF 2002

$$C(t,\varepsilon)\,\mathrm{e}^{-\kappa_- h^2/2\sigma^2} \leqslant \mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \leqslant C(t,\varepsilon)\,\mathrm{e}^{-\kappa_+ h^2/2\sigma^2}$$

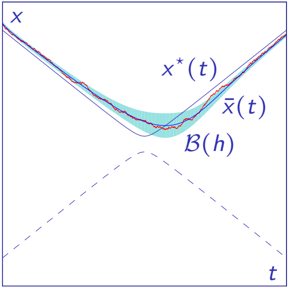
where $\kappa_\pm = 1 \mp \mathcal{O}(h)$ and $C(t,\varepsilon) = C_0(t,\varepsilon)\big[1 + \mathcal{O}(h)\big]$ (requires $h \leqslant h_0$)

Avoided transcritical bifurcation

$$\mathrm{d}x_t = \frac{1}{\varepsilon}\Big[t^2 + \delta - x_t^2 + \dots\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Equil. curve: $x^\star(t) \simeq \sqrt{t^2+\delta}$

Slow sol.: $\bar{x}(t) = x^\star(t) + \mathcal{O}(\min\{\frac{\varepsilon}{|t|}, \frac{\varepsilon}{\sqrt{\delta+\varepsilon}}\})$



$$\bar{a}(t) = \partial_x f(t, \bar{x}(t)) \asymp \begin{cases} -|t| & |t| \geqslant \sqrt{\delta+\varepsilon} \\ -\sqrt{\delta+\varepsilon} & |t| \leqslant \sqrt{\delta+\varepsilon} \end{cases}$$

Confidence strip $\mathcal{B}(h)$: width $\asymp h/\sqrt{|\bar{a}(t)|}$

Theorem [B & Gentz, AAP 2002]

$$\mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \leqslant C(t,\varepsilon)\,\mathrm{e}^{-\kappa h^2/2\sigma^2}$$

where $\kappa = 1 - \mathcal{O}(\sup_{s\leqslant t} h|\bar{a}(s)|^{-3/2}) - \mathcal{O}(\varepsilon)$ requires $h < h_0 \inf_{s\leqslant t}|\bar{a}(s)|^{3/2}$

- $\sigma < \sigma_c = \max\{\delta,\varepsilon\}^{3/4}$: result applies $\forall\, t$, $\mathbb{P}\{\text{trans}\} = \mathcal{O}(\mathrm{e}^{-\kappa\sigma_c^2/\sigma^2})$
- $\sigma > \sigma_c = \max\{\delta,\varepsilon\}^{3/4}$: result applies up to $t \asymp -\sigma^{2/3}$

Above threshold

What happens for $\sigma > \sigma_c$ and $t > -\sigma^{2/3}$?

General principle: partition $t_0 = s_0 < s_1 < s_2 < \dots < s_n = t$ of $[t_0, t]$

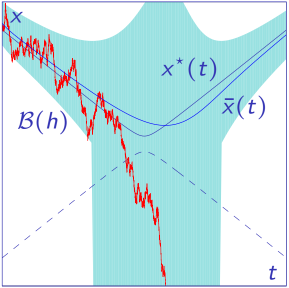
Lemma Let $P_k = \mathbb{P}\{\text{making no transition during } (s_{k-1}, s_k]\}$. Then

$$\mathbb{P}\{\text{making no transition during } [t_0, t]\} \leqslant \prod_{k=1}^{n} P_k$$

Choose partition s.t. each $P_k \leqslant q < 1 \Rightarrow \mathbb{P}\{\text{no transition}\} \leqslant e^{-n \log q}$

Define partition such that

$$\int_{s_{k-1}}^{s_k} |\bar{a}(s)| \, ds = c\varepsilon |\log \sigma| \quad \Rightarrow \quad P_k \leqslant \frac{2}{3}$$



Thm [B & Gentz, AAP 2002]

Transition probability $\geqslant 1 - e^{-\kappa\sigma^{4/3}/(\varepsilon|\log\sigma|)}$

SPDE: stable case

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}\big[\Delta\phi(t,x) + f(t,\phi(t,x))\big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x)$$

- $f(t,\phi^*(t)) = 0$ for all $t \in I = [0,T]$
- $a(t) = \partial_\phi f(t,\phi^*(t)) \leqslant -a_- < 0$ for all $t \in I$

In deterministic case $\sigma = 0$: $\exists$ particular solution $\bar\phi(t,x)$ such that

$$\big\|\bar\phi(t,\cdot) - \phi^*(t)e_0\big\|_{H^1} \leqslant C\varepsilon \qquad \forall t \in I$$

Theorem [B & Nader 2021]

Fix $s < \frac{1}{2}$, and let $\mathcal{B}(h) = \big\{(t,\phi) \colon t \in I, \|\phi - \bar\phi(t,\cdot)\|_{H^s} < h\big\}$
For any $\nu > 0$

$$\mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \leqslant C(t,\varepsilon,s)\exp\Big\{-\kappa\frac{h^2}{\sigma^2}\Big\}\Big[1 - \mathcal{O}\Big(\frac{h}{\varepsilon^\nu}\Big)\Big]$$

holds for some $\kappa > 0$, $h = \mathcal{O}(\varepsilon^\nu)$ and $C(t,\varepsilon,s) = \mathcal{O}(t/\varepsilon)$.

Ideas of proof

- $\phi(x) = \sum_{k\in\mathbb{Z}} \phi_k e_k(x) \quad \Rightarrow \quad \|\phi\|_{H^s}^2 = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s}\phi_k^2, \qquad \langle k\rangle = \sqrt{1+k^2}$
- Deterministic case: $\psi = \phi - \phi^* e_0$, $\|\psi\|_{H^1}^2$ is a Lyapunov function
- Linear stoch case:

  $$\mathrm{d}\psi_k = \frac{1}{\varepsilon} a_k(t)\psi_k \,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_k(t), \qquad a_k(t) = \bar{a}(t) - \frac{k^2\pi^2}{L^2} < 0$$

  For any decomposition $h = \sum_k h_k$,

  $$\mathbb{P}\{\tau < T\} \leqslant \sum_k \mathbb{P}\Big\{\sup_t \psi_k(t)^2 \geqslant h_k^2 \langle k\rangle^{-2s}\Big\} \leqslant \sum_k C_k(T,\varepsilon)\,\mathrm{e}^{-\kappa h_k^2 \langle k\rangle^{2-2s}/\sigma^2}$$

  Choose $h_k^2 \sim h^2 \langle k\rangle^{-2+2s+\eta}$, $\eta > 0$
- Schauder estimate: $\beta \in H^r$, $0 < r < \frac{1}{2} \quad \Rightarrow$

  $$\|\mathrm{e}^{t\Delta}\beta\|_{H^q} \leqslant M(q,r)t^{-(q-r)/2}\|\beta\|_{H^r} \qquad \forall q < r+2$$

  Consequence: $\psi = \psi^0 + \psi^1$ where nonlinear term satisfies

  $$\|\psi^1\|_{H^q} \leqslant M'\varepsilon^{(q-r)/2-1}\sup_t \|b(t,\psi(y,\cdot))\|_{H^r}$$

SPDE near a bifurcation point

$$\mathrm{d}\phi = \frac{1}{\varepsilon}\Bigl[\Delta\phi + g(t) - \phi^2 - b(t,\phi)\Bigr]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W(t,x)$$

with $g(t) = \delta + t^2 + \mathcal{O}(t^3)$ and $b = \mathcal{O}(\phi^3 + t\phi^2 + t^2\phi)$

- ▷ Decompose $\phi(t,x) = \phi_0(t)e_0(x) + \phi_\perp(t,x)$ where e_0 constant fct
- ▷ $\phi_\perp$ satisfies similar concentration result as ϕ in stable case
- ▷ ϕ_0 satisfies similar equation as in 1D, with error term of order $\|\phi_\perp\|^2_{H^s}$

Thm 1: Transverse component

$$\mathbb{P}\Bigl\{\tau_{\mathcal{B}_\perp}(h_\perp) < t \wedge \tau_{\mathcal{B}_0(h)}\Bigr\} \leqslant C(t,\varepsilon,s)\exp\Bigl\{-\kappa\frac{h^2}{\sigma^2}\Bigr\}\Bigl[1 - \mathcal{O}\Bigl(\frac{h}{\varepsilon^\nu}\Bigr)\Bigr]$$

Thm 2: Mean

$$\mathbb{P}\Bigl\{\tau_{\mathcal{B}_0(h)} < t \wedge \tau_{\mathcal{B}_\perp(h_\perp)}\Bigr\} \leqslant C(t,\varepsilon)\,\mathrm{e}^{-\kappa h^2/2\sigma^2} \qquad \kappa = 1 - \mathcal{O}\Bigl(\sup_s h|\bar{a}(s)|^{3/2}\Bigr)$$

Thm 3: Escape

$$\mathbb{P}\Bigl\{\phi_0(t_1) > -d\ \forall t \in \bigl[-\sigma^{2/3}, t \wedge \tau_{\mathcal{B}_\perp(h_\perp)}\bigr]\Bigr\} \leqslant \frac{3}{2}\,\mathrm{e}^{-\hat{\alpha}(t,-\sigma^{2/3})/[\varepsilon\log(\sigma^{-1})]}$$

Work in progress: SPDE on the 2d torus

$$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}\big[\Delta\phi(t,x) + \phi(t,x) - \phi(t,x)^3\big]\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\mathrm{d}W(t,x) \qquad x \in \mathbb{T}^2$$

- ▷ SPDE is not well-posed, needs to be renormalised
  For $N \in \mathbb{N}$, project on $\operatorname{span}\{e_k\}_{|k|<N}$:

  $$\mathrm{d}\phi(t,x) = \frac{1}{\varepsilon}\big[\Delta\phi(t,x) + \phi(t,x) - P_N{:}\phi(t,x)^3{:}\big]\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\mathrm{d}W_N(t,x)$$

  where ${:}\phi^3{:} = \phi^3 - 3C_N\phi$ Wick power, $C_N \sim \log N$ variance of GFF

  [Da Prato & Debussche '03]: $\phi - \phi^0$ cv to well-defined function

- ▷ Need to control stochastic convolution ϕ_0,
  $\mathrm{d}\phi^0 = \frac{1}{\varepsilon}\Delta\phi^0\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\mathrm{d}W_N$
  Use Besov–Hölder spaces $\mathcal{C}^\alpha$, $\alpha < 0$, instead of Sobolev spaces H^s

  Already obtained: $\mathbb{P}\Big\{\sup_{t\in[0,T]}\|\phi^0\|_{\mathcal{C}^\alpha} > h\Big\} \leqslant C(T,\varepsilon)\,\mathrm{e}^{-\kappa h^2/\sigma^2}$

  Need similar estimates for Wick powers of ϕ^0

Open questions

- Case $x \in \mathbb{T}^3$? Regularity structures or similar needed …

References

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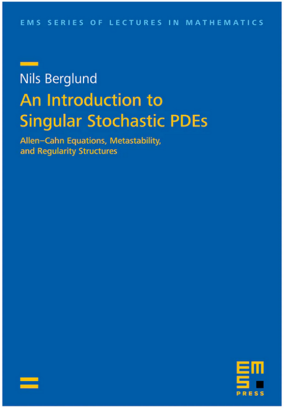

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