

Inhomogeneous Random Systems

Institut Henri Poincaré, Paris

Noise-induced phase slips, log-periodic oscillations
and the Gumbel distribution

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Joint work with Barbara Gentz (Bielefeld)

Synchronization of two coupled oscillators

See e.g. [Pikovsky, Rosenblum, Kurths 2001]

$$x_i = (\theta_i, \dot{\theta}_i), i = 1, 2$$

$$\begin{cases} \dot{x}_1 = f_1(x_1) \\ \dot{x}_2 = f_2(x_2) \end{cases}$$

ϕ_i : good parametrisation of limit cycles

$$\begin{cases} \dot{\phi}_1 = \omega_1 \\ \dot{\phi}_2 = \omega_2 \end{cases}$$



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$$x_i = (\theta_i, \dot{\theta}_i), \quad i = 1, 2$$

$$\begin{cases} \dot{x}_1 = f_1(x_1) + \varepsilon g_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_2) + \varepsilon g_2(x_1, x_2) \end{cases}$$

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$$\text{If } \omega_1 \simeq \omega_2: \begin{cases} \psi = \phi_1 - \phi_2 \\ \varphi = \frac{\phi_1 + \phi_2}{2} \end{cases} \Rightarrow \begin{cases} \dot{\psi} = -\nu + \varepsilon q(\psi, \varphi) \\ \dot{\varphi} = \omega + \mathcal{O}(\varepsilon) \end{cases} \quad \begin{aligned} \nu &= \omega_2 - \omega_1 \\ \omega &= \frac{\omega_1 + \omega_2}{2} \end{aligned}$$

For small detuning ν : averaging $\Rightarrow \omega \frac{d\psi}{d\varphi} \simeq -\nu + \varepsilon \bar{q}(\psi)$

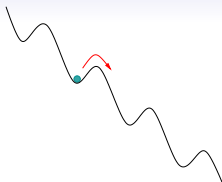
Example: Adler's equation $\bar{q}(\psi) = \sin(\psi)$: Fixed points for $\sin(\psi) = \nu/\varepsilon$

Remark: if $\omega_2/\omega_1 \simeq m/n$ similar behaviour for $\psi = n\phi_1 - m\phi_2$ (Arnold tongues)

Noise-induced phase slips

Averaged equation with noise

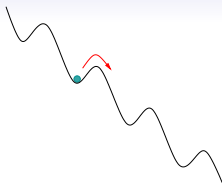
$$\omega \frac{d\psi}{d\varphi} = \underbrace{-\nu + \varepsilon \bar{q}(\psi)}_{-\frac{\partial}{\partial \psi} (\nu \psi - \varepsilon \int^{\psi} \bar{q}(x) dx)} + \text{noise}$$



Noise-induced phase slips

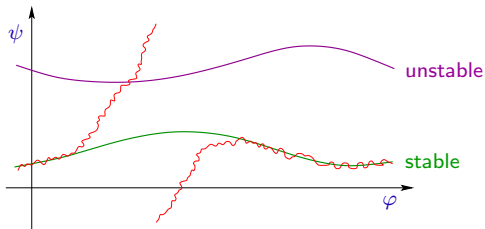
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Original equations with noise

$$\begin{cases} \dot{\psi} = -\nu + \varepsilon q(\psi, \varphi) + \text{noise} \\ \dot{\varphi} = \omega + \mathcal{O}(\varepsilon) + \text{noise} \end{cases}$$



Question: distribution of phases φ when crossing unstable orbit?
This is a **stochastic exit problem**.

Stochastic differential equations

$$dx_t = f(x_t) dt + \sigma g(x_t) dW_t \quad x \in \mathbb{R}^n$$

▷ Transition probability density: $p_t(x, y)$

▷ Markov semigroup T_t : for $\varphi \in L^\infty$,

$$(T_t \varphi)(x) = \mathbb{E}^x[\varphi(x_t)] = \int p_t(x, y) \varphi(y) dy$$

Generator: $L\varphi = \frac{d}{dt} T_t \varphi|_{t=0}$

$$(L\varphi)(x) = \sum_i f_i(x) \frac{\partial \varphi}{\partial x_i} + \frac{\sigma^2}{2} \sum_{i,j} (gg^T)_{ij}(x) \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$$

▷ Adjoint semigroup: for $\mu \in L^1$,

$$(\mu T_t)(y) = \mathbb{P}^\mu\{x_t = dy\} = \int \mu(x) p_t(x, y) dx$$

with generator L^*

▷ Kolmogorov equations: $\frac{d}{dt} p_t(x, y) = L_x p_t(x, y)$

$$\frac{d}{dt} p_t(x, y) = L_y^* p_t(x, y) \quad (\text{Fokker-Planck})$$

Stochastic exit problem

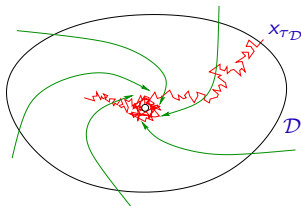
Given $\mathcal{D} \subset \mathbb{R}^n$, define first-exit time

$$\tau_{\mathcal{D}} = \inf\{t > 0: x_t \notin \mathcal{D}\}$$

First-exit location $x_{\tau_{\mathcal{D}}} \in \partial\mathcal{D}$

defines harmonic measure

$$\mu(A) = \mathbb{P}^x\{x_{\tau_{\mathcal{D}}} \in A\}$$



Stochastic exit problem

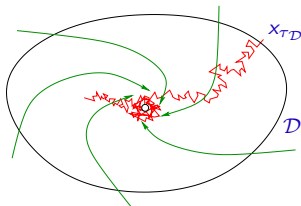
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Facts (following from Dynkin's formula):

- ▷ $u(x) = \mathbb{E}^x[\tau_{\mathcal{D}}]$ satisfies
$$\begin{cases} Lu(x) = -1 & x \in \mathcal{D} \\ u(x) = 0 & x \in \partial\mathcal{D} \end{cases}$$
- ▷ For $\varphi \in L^\infty(\partial\mathcal{D}, \mathbb{R})$, $h(x) = \mathbb{E}^x[\varphi(x_{\tau_{\mathcal{D}}})]$ satisfies

$$\begin{cases} Lh(x) = 0 & x \in \mathcal{D} \\ h(x) = \varphi(x) & x \in \partial\mathcal{D} \end{cases}$$

Wentzell–Freidlin theory

$$dx_t = f(x_t) dt + \sigma g(x_t) dW_t \quad x \in \mathbb{R}^n$$

Large-deviation principle with rate function

$$I(\gamma) = \frac{1}{2} \int_0^T (\dot{\gamma}_t - f(\gamma_t))^T D(\gamma_t)^{-1} (\dot{\gamma}_t - f(\gamma_t)) dt \quad D = gg^T$$

For a set Γ of paths $\gamma : [0, T] \rightarrow \mathbb{R}^n$: $\mathbb{P}\{(x_t)_{0 \leq t \leq T} \in \Gamma\} \simeq e^{-\inf_{\Gamma} I/\sigma^2}$

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Consider domain \mathcal{D} contained in basin of attraction of attractor \mathcal{A}

Quasipotential: $V(y) = \inf\{I(\gamma) : \gamma : \mathcal{A} \rightarrow y \in \partial\mathcal{D} \text{ in arbitrary time}\}$

▷ $\lim_{\sigma \rightarrow 0} \sigma^2 \log \mathbb{E}[\tau_{\mathcal{D}}] = \bar{V} = \inf_{y \in \partial\mathcal{D}} V(y)$ [Wentzell, Freidlin '69]

▷ If inf reached at a single point $y^* \in \mathcal{D}$ then

$$\lim_{\sigma \rightarrow 0} \mathbb{P}\{\|x_{\tau_{\mathcal{D}}} - y^*\| > \delta\} = 0 \quad \forall \delta > 0$$
 [Wentzell, Freidlin '69]

▷ Exponential distr of $\tau_{\mathcal{D}}$: $\lim_{\sigma \rightarrow 0} \mathbb{P}\{\tau_{\mathcal{D}} > s\mathbb{E}[\tau_{\mathcal{D}}]\} = e^{-s}$ [Day '83]

Application to exit through unstable periodic orbit

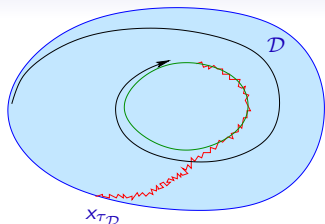
Planar SDE

$$dx_t = f(x_t) dt + \sigma g(x_t) dW_t$$

$\mathcal{D} \subset \mathbb{R}^2$: int of unstable periodic orbit

First-exit time: $\tau_{\mathcal{D}} = \inf\{t > 0: x_t \notin \mathcal{D}\}$

Law of first-exit location $x_{\tau_{\mathcal{D}}} \in \partial\mathcal{D}$?



Application to exit through unstable periodic orbit

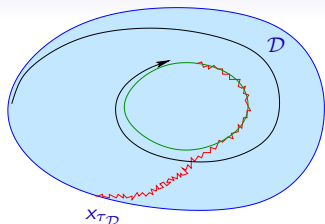
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Quasipotential:

$V(y) = \inf\{I(\gamma): \gamma: \text{stable orbit} \rightarrow y \in \partial\mathcal{D} \text{ in arbitrary time}\}$

Theorem [Wentzell, Freidlin '69]: If V reaches its min at a unique $y^* \in \partial\mathcal{D}$, then $x_{\tau_{\mathcal{D}}}$ concentrates in y^* as $\sigma \rightarrow 0$

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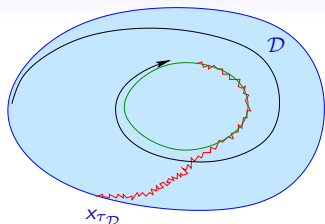
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Problem: V is constant on $\partial\mathcal{D}$!

Most probable exit paths

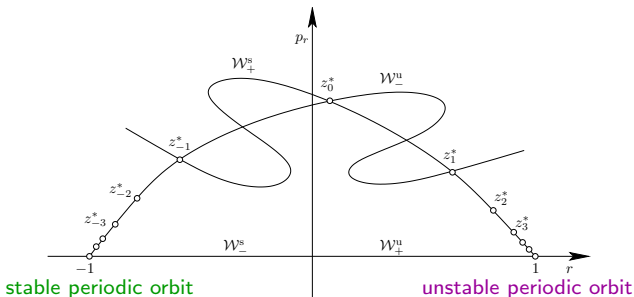
In polar-type coordinates

$$d\varphi_t = f_\varphi(\varphi_t, r_t) dt + \sigma g_\varphi(\varphi_t, r_t) dW_t \quad \varphi \in \mathbb{R} / 2\pi\mathbb{Z}$$

$$dr_t = f_r(\varphi_t, r_t) dt + \sigma g_r(\varphi_t, r_t) dW_t \quad r \in [-1, 1]$$

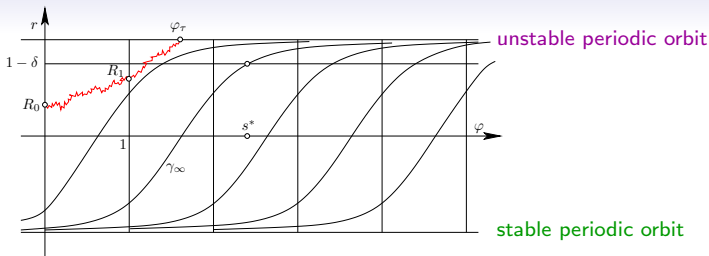
Minimisers of I obey Hamilton equations with Hamiltonian

$$H(\gamma, \psi) = \frac{1}{2} \psi^T D(\gamma) \psi + f(\gamma)^T \psi \quad \text{where } \psi = D(\gamma)^{-1}(\dot{\gamma} - f(\gamma))$$



Generically optimal path γ_∞ (for infinite time) is isolated

Random Poincaré maps



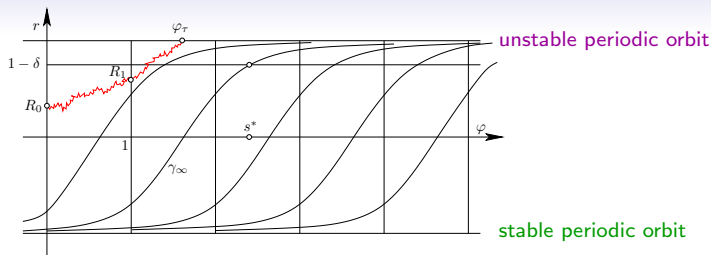
- ▷ R_0, R_1, \dots, R_N form substochastic Markov chain (killed in $r = 1$)
- ▷ Under hypoellipticity cond, transition kernel has smooth density k [Ben Arous, Kusuoka, Stroock '84]

$$\mathbb{P}^{R_0}\{R_1 \in B\} = K(R_0, B) := \int_B k(R_0, y) dy$$

- ▷ Fredholm theory: spectral decomp $k(x, y) = \sum_{k \geq 0} \lambda_k h_k(x) h_k^*(y)$
 $\lambda_0 \in [0, 1]$: principal eigenvalue [Perron, Frobenius, Jentzsch, Krein–Rutman]

$$\lim_{n \rightarrow \infty} \mathbb{P}\{R_n \in dx | N > n\} = \frac{h_0^*(x)}{\int h_0^*} = \pi_0(x) \text{ quasistationary distr (QSD)}$$

Random Poincaré maps



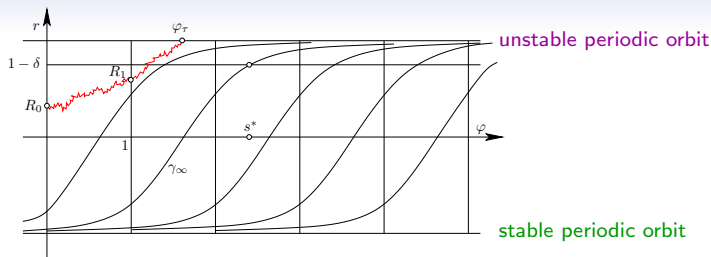
Consequences of spectral decomp $k(x, y) = \sum_{k \geq 0} \lambda_k h_k(x) h_k^*(y)$
 assuming spectral gap $|\lambda_1|/\lambda_0 < 1$:

- ▷ $\mathbb{P}^{R_0} \{R_n \in A\} = \lambda_0^n h_0(R_0) \int_A h_0^*(y) dy [1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)]$
- ▷ If $t = n + s$,

$$\mathbb{P}^{R_0} \{\varphi_t \in dt\} = \lambda_0^n h_0(R_0) \int h_0^*(y) \mathbb{P}^y \{\varphi_s \in ds\} dy [1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)]$$

Periodically modulated exponential distribution: $f(t + 1) \simeq \lambda_0 f(t)$

Computation of exit distribution



Split into two Markov chains:

- ▷ Chain killed upon r reaching $1 - \delta$ in $\varphi = \varphi_{\tau-}$

$$\mathbb{P}^0\{\varphi_{\tau-} \in [\varphi_1, \varphi_1 + \Delta]\} \simeq (\lambda_0^s)^{\varphi_1} e^{-J(\varphi_1)/\sigma^2}$$

- ▷ Chain killed at $r = 1 - 2\delta$ and on unstable orbit $r = 1$

- ◇ Principal eigenvalue: $\lambda_0^u = e^{-2\lambda_+ T_+} (1 + \mathcal{O}(\delta))$
- $\lambda_+ =$ Lyapunov exponent, $T_+ =$ period of unstable orbit
- ◇ Using LDP:

$$\mathbb{P}^{\varphi_1}\{\varphi_{\tau} \in [\varphi, \varphi + \Delta]\} \simeq (\lambda_0^u)^{\varphi - \varphi_1} e^{-[I_{\infty} + c(e^{-2\lambda_+ T_+} (\varphi - \varphi_1))]/\sigma^2}$$

Main result: log-periodic oscillations (cycling)

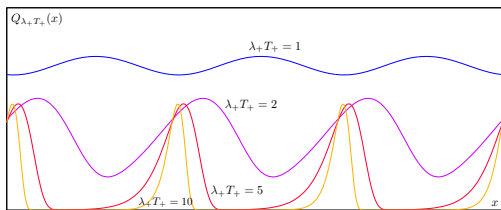
Theorem [B & Gentz, SIAM J Math Anal 2014]

$\exists \beta, c > 0: \forall \Delta, \delta > 0 \exists \sigma_0 > 0: \forall 0 < \sigma < \sigma_0$

$$\mathbb{P}^{r_0, 0} \left\{ \theta(\varphi_T) \in [t, t + \Delta] \right\} = \Delta C(\sigma) (\lambda_0)^t Q_{\lambda_+ T_+} \left(\frac{|\log \sigma|}{\lambda_+ T_+} - t + \mathcal{O}(\delta) \right) \\ \times \left[1 + \mathcal{O}(e^{-ct/|\log \sigma|}) + \mathcal{O}(\delta |\log \delta|) + \mathcal{O}(\Delta^\beta) \right]$$

▷ $Q_{\lambda T}(x) = \sum_{n=-\infty}^{\infty} G(\lambda T(n-x))$ with $G(x) = \exp\{-2x - \frac{1}{2} e^{-2x}\}$

Cycling profile, periodicised **Gumbel** distribution



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▷ $C(\sigma) = \mathcal{O}(e^{-V/\sigma^2})$

▷ **Cycling**: periodic dependence on $|\log \sigma|$

[Day'90, Maier & Stein '96, Getfert & Reimann '09]

Animations

Influence of noise intensity: cycling

$$\lambda_+ = 1, T_+ = 4, V = 1$$

$$1 \geq \sigma \geq 0.0001$$

(area under curve not normalized)

avi file

Influence of period

$$\lambda_+ = 1, \sigma = 0.4, V = 1$$

$$0.001 T_K \leq T_+ \leq T_K$$

$$T_K \simeq e^{V/\sigma^2} \text{ Kramers' time}$$

(area under curve not normalized)

avi file

See also <http://www.univ-orleans.fr/mapmo/membres/berglund/simcycling.html>

Why log-periodic oscillations?

Phase at crossing: $\mathcal{W}_\Delta(t) = \sum_{n=0}^{\infty} \mathbb{P}^{r_0,0} \{ \theta(\varphi_\tau) \in [n+t, n+t+\Delta] \}$

Corollary : $\mathcal{W}_\Delta(t) = \Delta Q_{\lambda_+ T_+} \left(\frac{|\log \sigma|}{\lambda_+ T_+} - t + \mathcal{O}(\delta) \right) [1 + \mathcal{O}(\delta |\log \delta|) + \mathcal{O}(\Delta^\beta)]$

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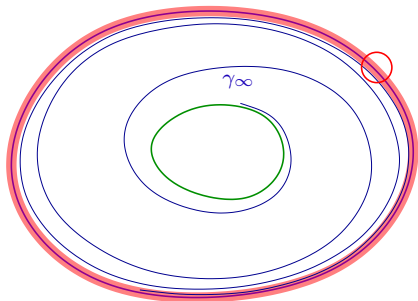
Heuristics :

$\theta(\varphi)$: parametrisation in which effective normal diffusion is constant

$\text{dist}(\gamma_\infty, \text{unst orbit}) \simeq e^{-\lambda_+ T_+ \theta(\varphi)}$

Escape when

$$e^{-\lambda_+ T_+ \theta(\varphi)} = \sigma \Rightarrow \theta(\varphi) = \frac{|\log \sigma|}{\lambda_+ T_+}$$



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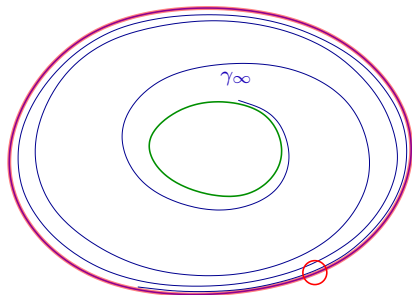
Heuristics :

$\theta(\varphi)$: parametrisation in which effective normal diffusion is constant

$\text{dist}(\gamma_\infty, \text{unst orbit}) \simeq e^{-\lambda_+ T_+ \theta(\varphi)}$

Escape when

$$e^{-\lambda_+ T_+ \theta(\varphi)} = \sigma \Rightarrow \theta(\varphi) = \frac{|\log \sigma|}{\lambda_+ T_+}$$



Why log-periodic oscillations?

Phase at crossing: $\mathcal{W}_\Delta(t) = \sum_{n=0}^{\infty} \mathbb{P}^{r_0,0} \{ \theta(\varphi_\tau) \in [n+t, n+t+\Delta] \}$

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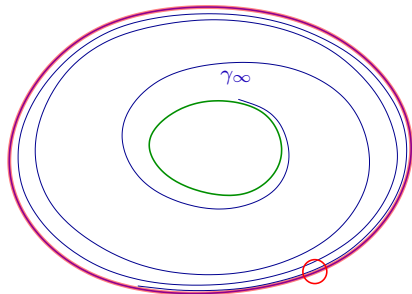
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Remark: log-periodic oscillations appear in finance, diffusion through fractals, renormalization maps ...

Why a Gumbel distribution?

Length of reactive path

[Cérou, Guyader, Lelièvre, Malrieu 2013] :

$$dx_t = -V'(x_t) dt + \sigma dW_t \quad a < x_0 < 0$$

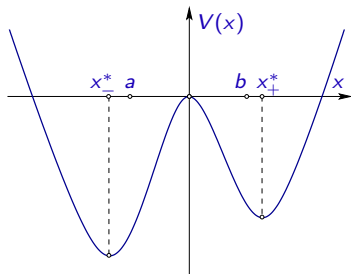
Theorem:

$$\lim_{\sigma \rightarrow 0} \mathbb{P} \left\{ \tau_b - \frac{2}{\lambda} |\log \sigma| < t \mid \tau_b < \tau_a \right\} \\ = \frac{1}{\lambda} \left(\log \frac{2|x_0|b}{\lambda} + I(x_0) + I(b) + \Lambda(t) \right)$$

where $\lambda = -V''(0)$, $I(x) = \int_x^0 \left(\frac{\lambda}{v'(y)} + \frac{1}{y} \right) dy$

and $\Lambda(t) = e^{-e^{-t}}$: distrib. function of standard Gumbel r.v.

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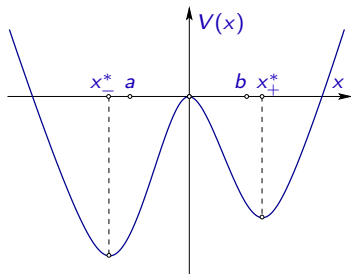
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[Bakhtin 2013] :

Link with extreme-value theory and residual lifetimes for linear case

$$dx_t = \lambda x_t dt + \sigma dW_t$$



Extreme-value theory and residual lifetime

- ▷ X_1, X_2, \dots i.i.d. real r.v. $M_n = \max\{X_1, \dots, X_n\}$
- ▷ $F(x) = \mathbb{P}\{X_1 \leq x\} = 1 - R(x) \quad \Rightarrow \quad \mathbb{P}\{M_n \leq x\} = F(x)^n$
- ▷ **Def:** $F \in D(\Phi) \Leftrightarrow \exists (a_n)_n > 0, (b_n)_n: \lim_{n \rightarrow \infty} F(a_n x + b_n)^n = \Phi(x)$

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$$\tau = \inf\{t > 0: X_t = 0\}$$

By the reflection principle:

$$\begin{aligned} \mathbb{P}\left\{\tau < t + \frac{1}{\lambda} |\log \sigma| \mid \tau < \infty\right\} &= \mathbb{P}\left\{\widetilde{X}_{t + \frac{1}{\lambda} |\log \sigma|} > 0 \mid \widetilde{X}_\infty > 0\right\} \\ &= \mathbb{P}\left\{N > \frac{x_0}{\sigma} \sqrt{\frac{2\lambda}{1-\sigma^2 e^{-2\lambda t}}} \mid N > \frac{x_0}{\sigma} \sqrt{2\lambda}\right\} \\ &\rightarrow \exp\left\{-x_0^2 \lambda e^{-2\lambda t}\right\} \quad \text{as } \sigma \rightarrow 0 \end{aligned}$$

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