IRTG Bielefeld–Seoul Winter School - Stochastic Dynamics

Metastable dynamics of Markov processes

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Based on joint works with Manon Baudel, Giacomo Di Gesù, Bastien Fernandez, Barbara Gentz, Damien Landon and Hendrik Weber






What is metastabilty?

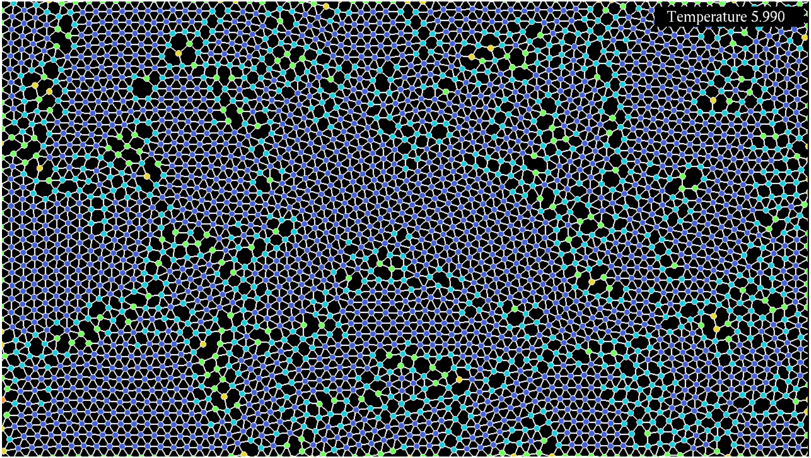
Supercooled water

(Source: https://youtu.be/fSPzMva9_CE)

What is metastabilty?

Ising model with Glauber dynamics at low temperature
(Online: https://youtu.be/_vrtfDcjfxU)

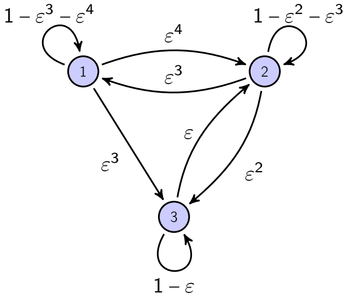
What is metastabilty?



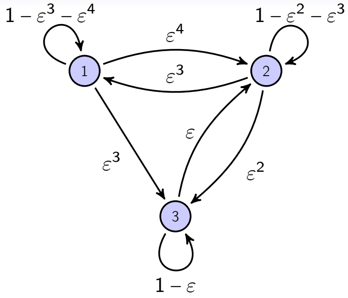
Particles interacting with a Lennard–Jones potential,
coupled to a thermostat (stochastic differential equation, or SDE)

Contents

1. Metastable Markov chains on a finite set



A simple example



$$P = \begin{pmatrix} 1-\varepsilon^3-\varepsilon^4 & \varepsilon^4 & \varepsilon^3 \\ \varepsilon^3 & 1-\varepsilon^2-\varepsilon^3 & \varepsilon^2 \\ 0 & \varepsilon & 1-\varepsilon \end{pmatrix}$$

$$0 \leqslant \varepsilon \leqslant \varepsilon_{\mathrm{max}}$$

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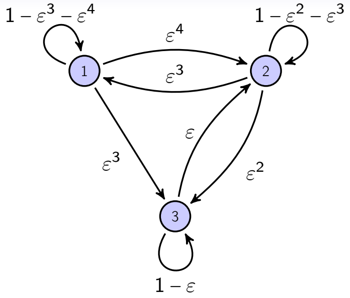

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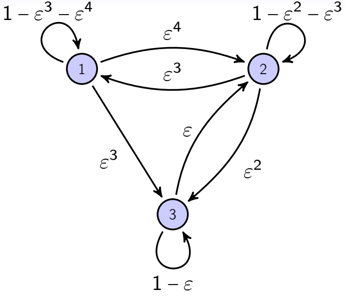

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  Stationary distribution:

  Speed of convergence to π_0?

  Eigenvalues of P:

Main question

How to easily determine leading term of spectral gap $1 - \lambda_1$?

- Linear algebra/analytic methods (singular perturbation theory), e.g. [Schweitzer 68, Hassin & Haviv 92, Avrachenkov & Lasserre 99]
- Probabilistic methods, e.g. [Wentzell 72, Freidlin & Wentzell 70s, Miclo 95, Beltràn & Landim 2010, Cameron & Vanden–Eijnden 2014, Betz & Le Roux 2016, Cameron & Gan 2016]

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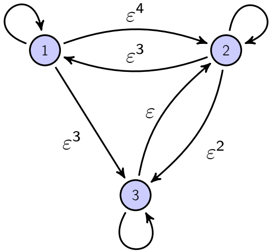
Some probabilistic tools:

- W-graphs
- Lumping of states
- Speeding up time



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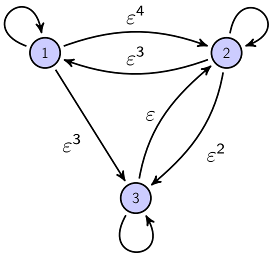
- Here: trace process



Killed process

$\mathcal{X}$ finite, $\{X_n\}_{n\in\mathbb{N}_0}$ irreducible aperiodic M.C., transition matrix P, $A \subset \mathcal{X}$

- ▷ Process killed upon leaving A: $P_A(x,y) = P(x,y)\mathbb{1}_{\{x,y\in A\}}$

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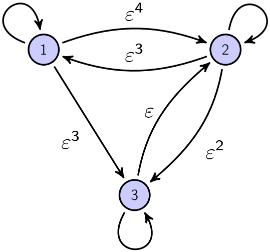

$$P = \begin{pmatrix} 1-\varepsilon^3-\varepsilon^4 & \varepsilon^4 & \varepsilon^3 \\ \varepsilon^3 & 1-\varepsilon^2-\varepsilon^3 & \varepsilon^2 \\ 0 & \varepsilon & 1-\varepsilon \end{pmatrix}$$

$$A = \{1,2\}$$

Trace process [Landim, Beltran]

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- ▷ Trace process on A: process monitored only when in A

  $_AP(x,y) = \mathbb{P}^x\{X_{\tau_A^+} = y\}, \qquad \tau_A^+ = \inf\{n \geqslant 1 : X_n \in A\}$

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Matrix representation (Schur complement)

$$P = \begin{pmatrix} P_A & P_{AA^c} \\ P_{A^cA} & P_{A^c} \end{pmatrix} \quad \Rightarrow \quad {}_AP = P_A + P_{AA^c}\left[\mathbb{1} - P_{A^c}\right]^{-1} P_{A^cA}$$

Application to the example

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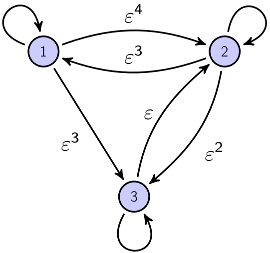

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Recall: the chain in not assumed to be reversible:
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Proposition: $\forall x, y \in A$

$$\pi_0(x)\mathbb{P}^x\{\tau_y^+ < \tau_x^+\} = \pi_0(y)\mathbb{P}^y\{\tau_x^+ < \tau_y^+\}$$

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- First proof in non-reversible case: [Betz & Le Roux 2016]
  Using $\pi_0(x) = 1/\mathbb{E}^x[\tau_x^+]$
- Alternative proof using trace process: [Baudel & B 2017]
  Remark: $\pi_0|_A$ is invariant by ${}_AP$

Good domains

Definition: For $A \subset \mathcal{X}$, let

$$p_{\text{in}}(A) = \inf_{x \in A^c} \mathbb{P}^x\{X_1 \in A\}$$

$$p_{\text{out}}(A) = \sup_{x \in A} \mathbb{P}^x\{X_1 \in A^c\}$$

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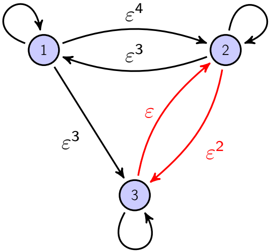
Example:



Main idea

For a good domain A,

$P = \begin{pmatrix} P_A & P_{AA^c} \\ P_{A^cA} & P_{A^c} \end{pmatrix}$ is well-approximated by $\widehat{P} = \begin{pmatrix} {}_AP & 0 \\ P_{A^cA} & P_{A^c} \end{pmatrix}$

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Norm: $\|Q\| = \sup_{\|\varphi\|_\infty = 1} \|Q\varphi\|_\infty = \sup_{\|\mu\|_1 = 1} \|\mu Q\|_1 = \sup_{x \in \mathcal{X}} \sum_{y \in \mathcal{X}} |Q(x,y)|$

Lemma: $\|P - \widehat{P}\| = 2p_{\text{out}}(A)$

Main idea

Fact from spectral theory (using complex analysis, Riesz projector):
$\hat{\lambda}$ simple eigenvalue of $\widehat{P}$ at distance $> \|P - \widehat{P}\|$ from remaining spectrum
$\Rightarrow$ P has unique eigenvalue at distance $\mathcal{O}(\|P - \widehat{P}\|)$ from $\hat{\lambda}$

Consequence: If $A^c = \{x\}$ then $p_{\text{in}}(A) = 1 - P(x,x) = 1 - \hat{\lambda}$
$\Rightarrow 1 - \lambda = 1 - \hat{\lambda} + \mathcal{O}(p_{\text{out}}(A)) = (1 - \hat{\lambda})\Big[1 + \mathcal{O}\Big(\frac{p_{\text{out}}(A)}{p_{\text{in}}(A)}\Big)\Big]$

Example: $\hat{\lambda}_2 = 1 - \varepsilon$ perturbs to $\lambda_2 = 1 - \varepsilon + \mathcal{O}(\varepsilon^2)$
The argument does not suffice to compare spectra of P_A and ${}_AP$

$$\widehat{P} = \begin{pmatrix} 1 - -\varepsilon^3 - \varepsilon^4 & \varepsilon^3 + \varepsilon^4 & 0 \\ \varepsilon^3 & 1 - \varepsilon^3 & 0 \\ 0 & \varepsilon & 1 - \varepsilon \end{pmatrix}$$

Laplace transforms

$u \in \mathbb{C} \quad \Rightarrow \quad \mathbb{E}^x[\mathrm{e}^{u\tau_A^+}]$ exists for $|\mathrm{e}^{-u}| > 1 - p_{\text{in}}(A) \quad (\star)$

Follows from $\mathbb{P}^y\{\tau_A^+ > n\} \leqslant (1 - p_{\text{in}}(A))^n \ \forall y \in A^c$

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Proposition [Feynman–Kac type relation]

Under $(\star)$,

$$\begin{cases} (P\phi)(x) = \mathrm{e}^{-u}\,\phi(x) & x \in A^c \\ \phi(x) = \bar\phi(x) & x \in A \end{cases}$$

admits unique solution $\phi(x) = \mathbb{E}^x[\mathrm{e}^{u\tau_A}\,\bar\phi(X_{\tau_A})]$, $\tau_A = \inf\{n \geqslant 0 \colon X_n \in A\}$

Proof:

Laplace transforms

Corollary [Reduction to eigenvalue problem on A]

Under $(\star)$, $P\phi = \mathrm{e}^{-u}\,\phi$ in $\mathcal{X}$ $\quad\Leftrightarrow\quad$ ${}_AP^u\phi = \mathrm{e}^{-u}\,\phi$ in A
where ${}_AP^u(x,y) = \mathbb{E}^x\Bigl[\mathrm{e}^{u(\tau_A^+-1)}\,\mathbb{1}_{\{X_{\tau_A^+}=y\}}\Bigr]$ is such that ${}_AP^0 = {}_AP$

Proof of $\Rightarrow$:

Laplace transforms

Proposition

$$\|{}_A P^u - {}_A P^0\| \leqslant \frac{|1-\mathrm{e}^{-u}| \sup_{x\in A} \mathbb{E}^x[\tau_A^+ - 1]}{1 - |1-\mathrm{e}^{-u}| \sup_{x\in A^c} \mathbb{E}^x[\tau_A^+]} \leqslant \frac{|1-\mathrm{e}^{-u}| p_{\text{out}}(A)}{p_{\text{in}}(A) - |1-\mathrm{e}^{-u}|}$$

Main result – nondegenerate case

Algorithm in nondegenerate case:

- Assume $\exists x \in \mathcal{X}$ such that $1 - P(x,x) \gg 1 - P(y,y)\ \forall y \neq x$
- Take $A = \mathcal{X} \setminus \{x\}$ (A is a good set)
- Then $\mathbb{1} - P$ has ev $1 - \lambda = P(x,x)\big[1 + \mathcal{O}(p_{\text{in}}(A)/p_{\text{out}}(A))\big] \quad \in \mathbb{R}$
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Theorem [Baudel & B, 2017]

- Non-degenerate case: $\exists A_1 \subset A_2 \subset \cdots \subset A_n = \mathcal{X}$ s.t. $\#(A_{k+1} \setminus A_k) = 1$, each A_k good set for ${}_{A_{k+1}}P$
  Renumber states s.t. $A_k = \{1, \dots, k\}$. Then
- $\lambda_0 = 1,\ \lambda_k = 1 - \mathbb{P}^{k+1}\{\tau^+_{A_k} < \tau^+_{k+1}\}\Big[1 + \mathcal{O}\Big(\frac{p_{\text{out}}(A_k|A_{k+1})}{p_{\text{in}}(A_k|A_{k+1})}\Big)\Big] \quad \in \mathbb{R}$

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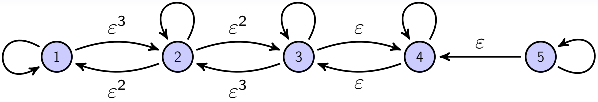
- kth right eigenvector ϕ_k close to $\mathbb{P}^x\{\tau_{k+1} < \tau_{A_k}\}$
- kth left eigenvector π_k close to quasistationary distribution (QSD) of P_{A_k} (left eigenvect of P_{A_k} for Perron–Frobenius principal eigenval)

Algorithm in degenerate case



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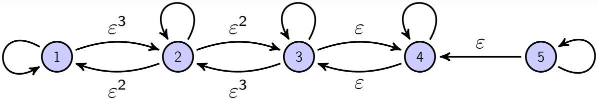
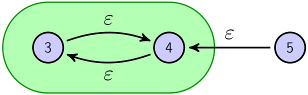
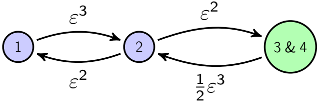


Degenerate part, leading order:



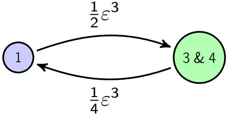
Eigenvalues: 1, $1 - \varepsilon$, $1 - 2\varepsilon$

Effective trace process:



Eigenvalues: $1 - 2\varepsilon^2$

Trace on $\{1, 3\&4\}$:



$1 - \frac{3}{4}\varepsilon^3$

1

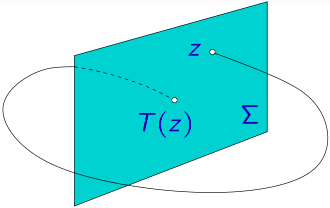
2. Continuous-space Markov chains and SDEs

$$\mathbb{P}\{X_{n+1} \in A | X_n = x\} = \int_A k_\sigma(x,y)\,\mathrm{d}y$$

Deterministic Poincaré maps

ODE $\dot{z} = f(z) \quad z \in \mathbb{R}^n$

Flow: $z_t = \varphi_t(z_0)$



$\Sigma \subset \mathbb{R}^n$: $(n-1)$-dimensional manifold

Poincaré map (or first-return map):
$T : \Sigma \to \Sigma$

$T(z) = \varphi_\tau(z)$ where $\tau = \inf\{t > 0 : \varphi_t(z) \in \Sigma\}$

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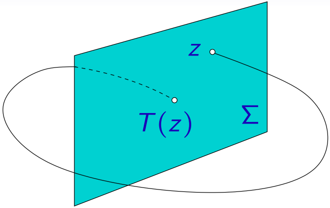


$T(z) = \varphi_\tau(z)$ where $\tau = \inf\{t > 0 : \varphi_t(z) \in \Sigma\}$

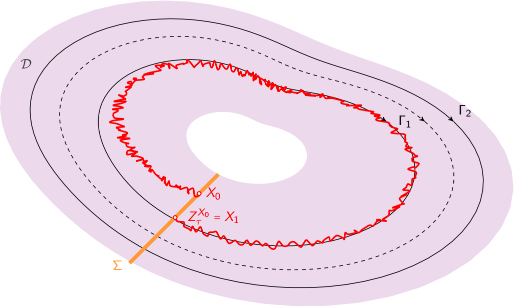
Benefits:

1. Dimension reduction: T is an $(n-1)$-dimensional map
2. Stability of periodic orbits: no neutral direction
3. Bifurcations of periodic orbits easier to study (period doubling, Hopf, ...)

Question: how about SDEs $\quad \mathrm{d}z_t = f(z_t)\,\mathrm{d}t + \sigma g(z_t)\,\mathrm{d}W_t$?

Random Poincaré maps

$\mathrm{d}z_t = f(z_t)\,\mathrm{d}t + \sigma g(z_t)\,\mathrm{d}W_t \quad \Rightarrow \quad$ Sample path $(Z_t^{z_0}(\omega))_{t\geqslant 0}$



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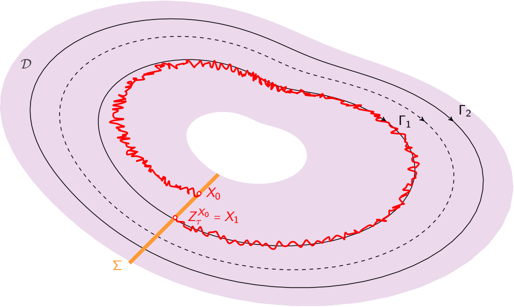

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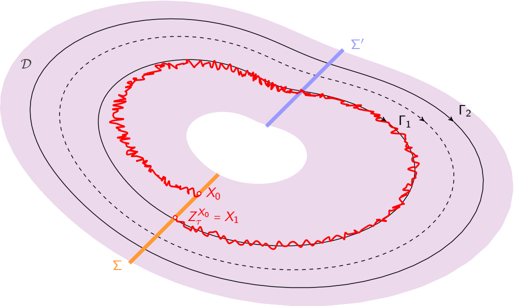

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Solution: $\tau_0 = 0,\ \tau'_{n+1} = \inf\{t > \tau_n\colon Z_t^{X_0} \in \Sigma'\}$

$\tau_{n+1} = \inf\{t > \tau'_{n+1}\colon Z_t^{X_0} \in \Sigma\}$

$X_n = Z_{\tau_n}^{X_0} \in \Sigma \quad \Rightarrow \quad (X_n)_{n\geqslant 0}$ is a Markov chain $\quad K(x,A) := \mathbb{P}^x\{X_1 \in A\}$

$(X_n,\omega) \mapsto X_{n+1}$: random Poincaré map

[J. Weiss, E. Knobloch, 1990], [P. Hitczenko, G. Medvedev, 2009]

Continuous-space Markov chains

$(X_n)_{n\in\mathbb{N}_0}$ Markov chain in $\mathcal{X} \subset \mathbb{R}^d$ with kernel K_σ:

$$\mathbb{P}\{X_{n+1} \in A | X_n = x\} = K_\sigma(x, A) = \int_A K_\sigma(x, \mathrm{d}y)$$

- ▷ $K_0(x, A) = 1\!\!1_{\{\Pi(x)\in A\}}$ defined by deterministic map $\Pi : \mathcal{X} \to \mathcal{X}$
- ▷ For $\sigma > 0$, K_σ admits continuous density k_σ

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Example 1: Randomly perturbed map

$$X_{n+1} = \Pi(X_n) + \sigma\xi_{n+1}$$

$(\xi_n)_{n\geqslant1}$ i.i.d. r.v. with density (e.g. $\sigma\xi_n$ Gaussian of variance σ^2)

Continuous-space Markov chains

$(X_n)_{n\in\mathbb{N}_0}$ Markov chain in $\mathcal{X}\subset\mathbb{R}^d$ with kernel K_σ:

$$\mathbb{P}\{X_{n+1}\in A|X_n=x\} = K_\sigma(x,A) = \int_A K_\sigma(x,\mathrm{d}y)$$

- ▷ $K_0(x,A) = 1_{\{\Pi(x)\in A\}}$ defined by deterministic map $\Pi:\mathcal{X}\to\mathcal{X}$
- ▷ For $\sigma>0$, K_σ admits continuous density k_σ

Example 1: Randomly perturbed map

$$X_{n+1} = \Pi(X_n) + \sigma\xi_{n+1}$$

$(\xi_n)_{n\geqslant1}$ i.i.d. r.v. with density (e.g. $\sigma\xi_n$ Gaussian of variance σ^2)

Example 2: Random Poincaré map

SDE

$$\mathrm{d}z_t = f(z_t)\,\mathrm{d}t + \sigma g(z_t)\,\mathrm{d}W_t$$

X_n suitably defined location of nth return to surface of section $\Sigma\subset\mathcal{X}$

Assumptions

Assumption 1: Deterministic dynamics

$\Pi : \mathcal{X} \to \mathcal{X}$ admits positively invariant compact set $\mathcal{X}_0 \subset \mathcal{X}$, finitely many limit sets in $\mathcal{X}_0$, all hyperbolic fixed points, N of which are stable

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Assumption 4: Uniform positivity (Doeblin-type condition)

$\forall x_i^\star$ stable fixed point, $\exists B_i$ nbh of $x_i^\star$ s.t. $k_i = {}_{B_1 \cup \dots \cup B_i} k_{B_i}$ satisfies

$$\sup_{x \in B_i} k_i^n(x,y) \leqslant L \inf_{x \in B_i} k_i^n(x,y) \quad \forall y \in B_i \qquad \text{for some } L \in (1,2),\ n(\sigma) \in \mathbb{N}$$

Metastable hierarchy (SDE case)

Freidlin–Wentzell theory:

Rate function: $I_{[0,T]}(\gamma) = \dfrac{1}{2}\displaystyle\int_0^T (\dot\gamma_s - f(\gamma_s))^{\mathsf{T}} [gg^{\mathsf{T}}(\gamma_s)]^{-1} (\dot\gamma_s - f(\gamma_s))\,\mathrm{d}s$

Large-deviation principle: $\mathbb{P}\big\{(z_t)_{0\leqslant t\leqslant T} \in \Lambda\big\} \simeq \mathrm{e}^{-\inf_{\gamma\in\Lambda} I_{[0,T]}(\gamma)/\sigma^2}$

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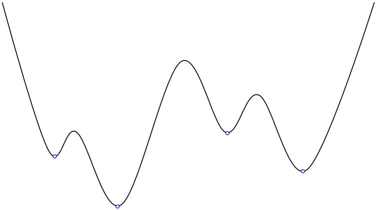
Assumption 5: Metastable hierarchy

$\exists\theta > 0$ s.t. $\forall\ 2 \leqslant k \leqslant N$

$$\min_{\ell<k} H(k,\ell) \leqslant \min_{\substack{i<k\\ j\neq i}} H(i,j) - \theta$$



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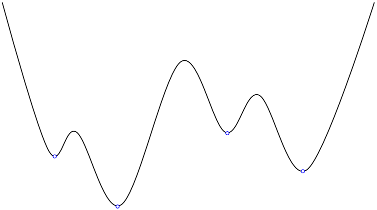

Remark: Using Doob's h-transform, one may replace Assumption 3 by

Assumption 3': Confinement

$\exists\theta' > 0$ such that $\min_i H(i,\partial\mathcal{D}) \geqslant \max_{i\neq j} H(i,j) + \theta'$

Main result

Theorem [Baudel & B, 2017]

- Non-degenerate case ($x_1^\star, \dots, x_N^\star$ in metastable order)
  - Eigenvalues of K_σ:

$$
\begin{aligned}
\lambda_0 &= 1 \\
\lambda_k &= 1 - \mathbb{P}^{\mathring{\pi}_0^{k+1}}\{\tau^+_{B_1\cup\dots\cup B_k} < \tau^+_{B_{k+1}}\}\big[1 + \mathcal{O}(\mathrm{e}^{-\theta/\sigma^2})\big] \in \mathbb{R} \qquad && 1 \leqslant k < N \\
|\lambda_k| &< \varrho = 1 - \frac{c}{\log(\sigma^{-1})} && k \geqslant N
\end{aligned}
$$

  where $\mathring{\pi}_0^{k+1}$ is a certain QSD on B_{k+1} and $c, \theta > 0$
  - kth right eigenfunction ϕ_k close to $\mathbb{P}^x\{\tau_{B_{k+1}} < \tau_{B_1\cup\dots\cup B_k}\}$
  - kth left eigenfunction π_k close to QSD of $K_{(B_1\cup\dots\cup B_k)^c}$

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    where $\mathring{\pi}_0^{k+1}$ is a certain QSD on B_{k+1} and $c, \theta > 0$
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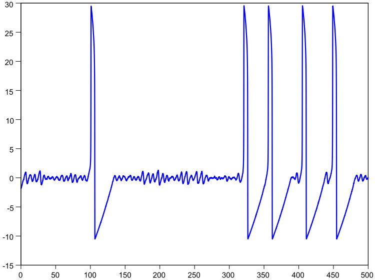
  - kth left eigenfunction π_k close to QSD of $K_{(B_1\cup\dots\cup B_k)^c}$
- Expected hitting times:
  $$\mathbb{E}^x[\tau_{B_1\cup\dots\cup B_k}] = [1 - \lambda_k]^{-1}[1 + \mathcal{O}(\mathrm{e}^{-\kappa/\sigma^2})] \qquad \forall x \in B_{k+1},\ 1 \leqslant k \leqslant N-1$$
- Degenerate case: similar to finite chain...

3. FitzHugh–Nagumo equations



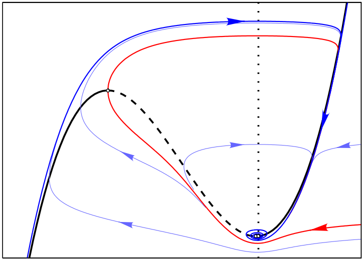
Stochastic FitzHugh–Nagumo equation

$$dx_t = \frac{1}{\varepsilon}\left[x_t - x_t^3 + y_t\right] dt$$ neuron membrane potential

$$dy_t = \left[a - x_t - by_t\right] dt$$ open ion channels

▷ $b = 0$ for simplicity in this talk, bifurcation parameter $\delta := \frac{3a^2-1}{2}$

$\varepsilon = 0.1$
$\delta = 0.02$



Stochastic FitzHugh–Nagumo equation

$$\mathrm{d}x_t = \frac{1}{\varepsilon}\left[x_t - x_t^3 + y_t\right]\mathrm{d}t + \frac{\sigma_1}{\sqrt{\varepsilon}}\,\mathrm{d}W_t^{(1)}$$ neuron membrane potential

$$\mathrm{d}y_t = \left[a - x_t - by_t\right]\mathrm{d}t + \sigma_2\,\mathrm{d}W_t^{(2)}$$ open ion channels

- ▷ $b = 0$ for simplicity in this talk, bifurcation parameter $\delta := \frac{3a^2-1}{2}$
- ▷ $W_t^{(1)}, W_t^{(2)}$: independent Wiener processes
- ▷ $0 < \sigma_1, \sigma_2 \ll 1$, $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$

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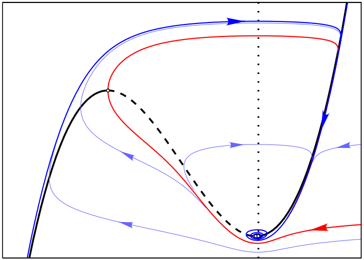

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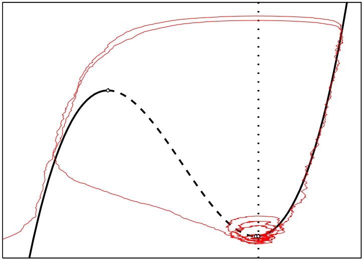
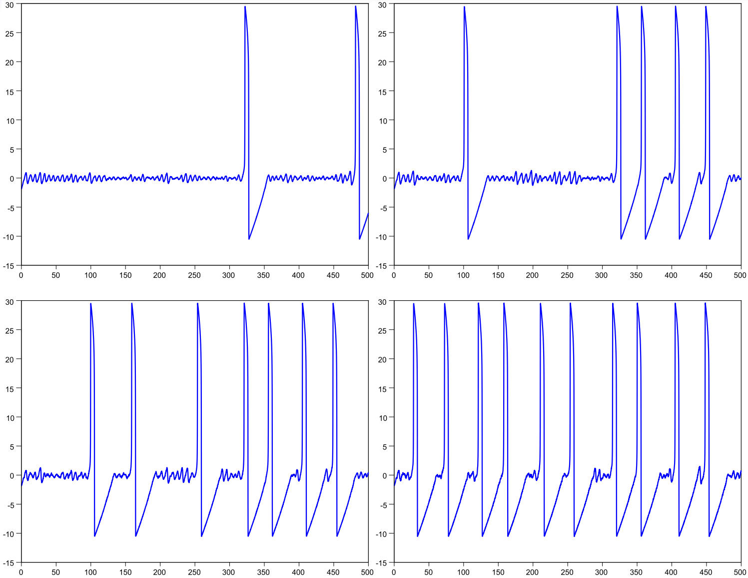
$\delta = 0.02$
$\sigma_1 = \sigma_2 = 0.03$

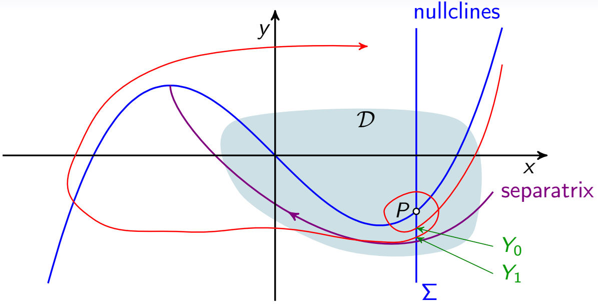


Mixed-mode oscillations (MMOs)



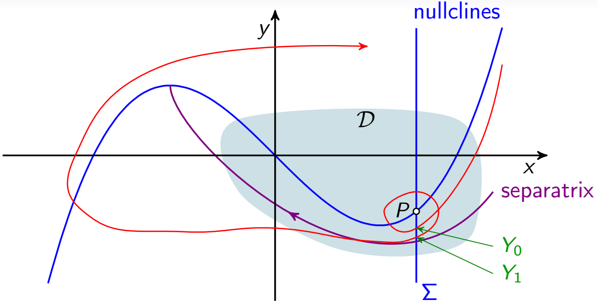
Time series $t \mapsto -x_t$ for $\varepsilon = 0.01,\ \delta = 3 \cdot 10^{-3},\ \sigma = 1.46 \cdot 10^{-4}, \dots, 3.65 \cdot 10^{-4}$

Random Poincaré map



$Y_0, Y_1, \dots$ substochastic Markov chain describing process killed on $\partial\mathcal{D}$
Number of small oscillations: $N = \inf\{n \geqslant 1 \colon Y_n \notin \Sigma\}$

Random Poincaré map



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Theorem 1 [B & Landon, 2012]

N is asymptotically geometric: $\lim_{n\to\infty} \mathbb{P}\{N = n+1 | N > n\} = 1 - \lambda_0$
where $\lambda_0 \in \mathbb{R}_+$: principal eigenvalue of the kernel K, $\lambda_0 < 1$ if $\sigma > 0$

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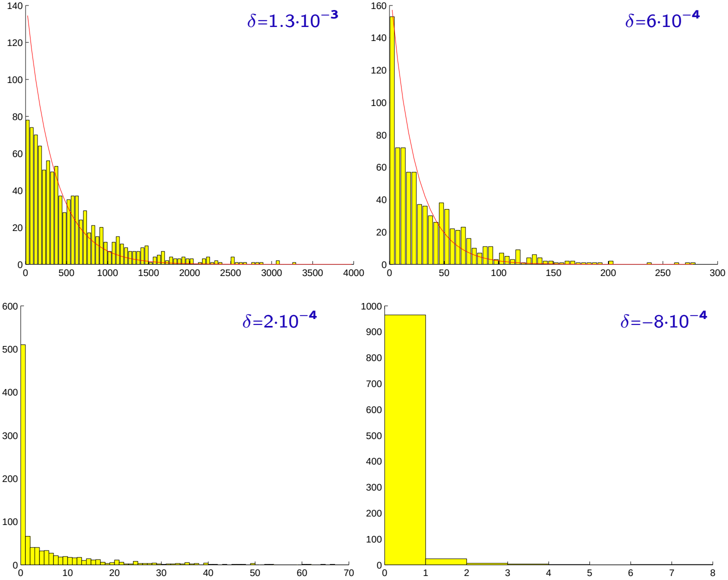
Proof:

Histograms of distribution of N (1000 spikes)

$\sigma = \varepsilon = 10^{-4}$



Weak-noise regime

Theorem 2 [B & Landon, 2012]

Assume ε and $\delta/\sqrt{\varepsilon}$ sufficiently small
There exists $\kappa > 0$ s.t. for $\sigma^2 \leqslant (\varepsilon^{1/4}\delta)^2 / \log(\sqrt{\varepsilon}/\delta)$

- Principal eigenvalue:

$$1 - \lambda_0 \leqslant \exp\left\{-\kappa \frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$$

- Expected number of small oscillations:

$$\mathbb{E}^{\mu_0}[N] \geqslant C(\mu_0) \exp\left\{\kappa \frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$$

where $C(\mu_0)$ = probability of starting on Σ above separatrix

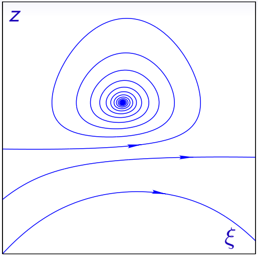
Proof:

Dynamics near the separatrix

Change of variables:

- ▷ Translate to Hopf bif. point
- ▷ Scale space and time
- ▷ Straighten nullcline $\dot{x} = 0$

$\Rightarrow$ variables (ξ, z) where nullcline: $\{z = \frac{1}{2}\}$



$$d\xi_t = \left(\frac{1}{2} - z_t - \frac{\sqrt{\varepsilon}}{3}\xi_t^3\right) dt$$

$$dz_t = \left(\mu + 2\xi_t z_t + \frac{2\sqrt{\varepsilon}}{3}\xi_t^4\right) dt$$

where

$$\mu = \frac{\delta}{\sqrt{\varepsilon}}$$

Dynamics near the separatrix

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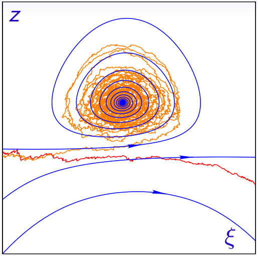

$\Rightarrow$ variables (ξ, z) where nullcline: $\{z = \frac{1}{2}\}$

$$\mathrm{d}\xi_t = \Big(\frac{1}{2} - z_t - \frac{\sqrt{\varepsilon}}{3}\xi_t^3\Big)\,\mathrm{d}t + \tilde{\sigma}_1\,\mathrm{d}W_t^{(1)}$$

$$\mathrm{d}z_t = \Big(\tilde{\mu} + 2\xi_t z_t + \frac{2\sqrt{\varepsilon}}{3}\xi_t^4\Big)\,\mathrm{d}t - 2\tilde{\sigma}_1\xi_t\,\mathrm{d}W_t^{(1)} + \tilde{\sigma}_2\,\mathrm{d}W_t^{(2)}$$

where

$$\tilde{\mu} = \frac{\delta}{\sqrt{\varepsilon}} - \tilde{\sigma}_1^2 \qquad \tilde{\sigma}_1 = -\sqrt{3}\frac{\sigma_1}{\varepsilon^{3/4}} \qquad \tilde{\sigma}_2 = \sqrt{3}\frac{\sigma_2}{\varepsilon^{3/4}}$$

Upward drift dominates if $\tilde{\mu}^2 \gg \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 \Rightarrow (\varepsilon^{1/4}\delta)^2 \gg \sigma_1^2 + \sigma_2^2$

Rotation around P: use that $2z\,\mathrm{e}^{-2z-2\xi^2+1}$ is constant for $\tilde{\mu} = \varepsilon = 0$

Take $A = \{z > \tilde{\mu}^{1-\gamma}\}$ with $0 < \gamma < \frac{1}{4}$ □

From below to above threshold

Linear approximation:

$$\mathrm{d}z_t^0 = \left(\tilde\mu + tz_t^0\right)\mathrm{d}t - \tilde\sigma_1 t\,\mathrm{d}W_t^{(1)} + \tilde\sigma_2\,\mathrm{d}W_t^{(2)}$$

$$\Rightarrow \quad \mathbb{P}\{\text{no small osc}\} \simeq \Phi\biggl(-\pi^{1/4}\frac{\tilde\mu}{\sqrt{\tilde\sigma_1^2+\tilde\sigma_2^2}}\biggr) \qquad \Phi(x) = \int_{-\infty}^{x} \frac{\mathrm{e}^{-y^2/2}}{\sqrt{2\pi}}\,\mathrm{d}y$$

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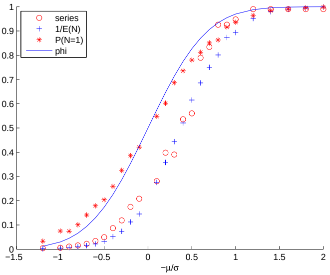


$*$: $\mathbb{P}\{\text{no small osc}\}$

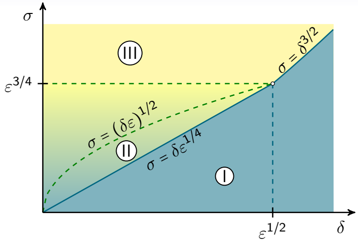
$+$: $1/\mathbb{E}[N]$

$\circ$: $1-\lambda_0$

curve: $x \mapsto \Phi(\pi^{1/4}x)$

$$x = -\frac{\tilde{\mu}}{\sqrt{\tilde{\sigma}_1^2+\tilde{\sigma}_2^2}} = -\frac{\varepsilon^{1/4}(\delta-\sigma_1^2/\varepsilon)}{\sqrt{\sigma_1^2+\sigma_2^2}}$$

Summary: Parameter regimes



$\sigma_1 = \sigma_2$:

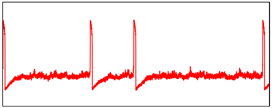
$$\mathbb{P}\{N = 1\} \simeq \Phi\Big(-\frac{(\pi\varepsilon)^{1/4}(\delta - \sigma^2/\varepsilon)}{\sigma}\Big)$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\,dy$$

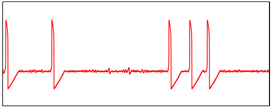
see also

[Muratov & Vanden Eijnden '08]

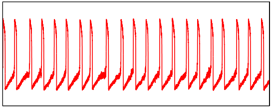
Regime I: rare isolated spikes
Theorem 2 applies ($\delta \ll \varepsilon^{1/2}$)
Interspike interval $\simeq$ exponential



Regime II: clusters of spikes
interspike osc asympt geometric
$\sigma = (\delta\varepsilon)^{1/2}$: geom(1/2)



Regime III: repeated spikes
$\mathbb{P}\{N = 1\} \simeq 1$
Interspike interval $\simeq$ constant



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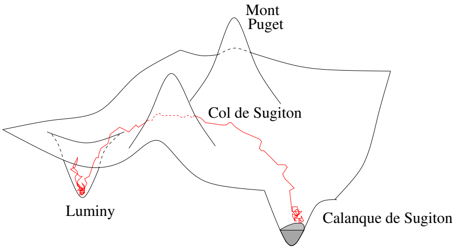
4. The case of reversible SDEs: The potential-theoretic approach

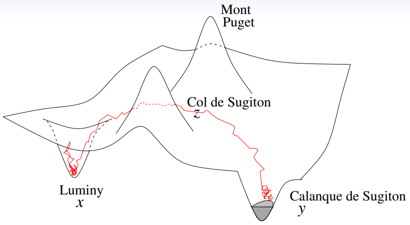


Reversible diffusion in a double-well

$$dx_t = -\nabla V(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t$$

$V : \mathbb{R}^d \to \mathbb{R}$ confining potential

$\tau_y^x = \inf\{t > 0 \colon x_t \in \mathcal{B}_\varepsilon(y)\}$
first-hitting time of small ball $\mathcal{B}_\varepsilon(y)$, when starting in x



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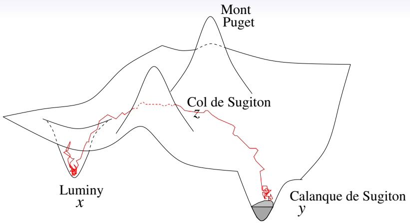

$V : \mathbb{R}^d \to \mathbb{R}$ confining potential

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first-hitting time of small ball $\mathcal{B}_\varepsilon(y)$,
when starting in x

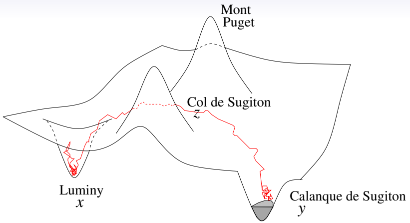
Arrhenius' law (1889): $\mathbb{E}[\tau_y^x] \simeq e^{[V(z)-V(x)]/\varepsilon}$

Eyring–Kramers law (1935, 1940):
Eigenvalues of Hessian of V at minimum x: $0 < \nu_1 \leqslant \nu_2 \leqslant \dots \leqslant \nu_d$
Eigenvalues of Hessian of V at saddle z: $\lambda_1 < 0 < \lambda_2 \leqslant \dots \leqslant \lambda_d$

$$\mathbb{E}[\tau_y^x] = 2\pi \sqrt{\frac{\lambda_2 \dots \lambda_d}{|\lambda_1| \nu_1 \dots \nu_d}}\, e^{[V(z)-V(x)]/\varepsilon}\big[1 + \mathcal{O}_\varepsilon(1)\big]$$

Reversible diffusion in a double-well

$$\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$$



$V : \mathbb{R}^d \to \mathbb{R}$ confining potential

$\tau_y^x = \inf\{t > 0 \colon x_t \in \mathcal{B}_\varepsilon(y)\}$
first-hitting time of small ball $\mathcal{B}_\varepsilon(y)$, when starting in x

Arrhenius' law (1889): $\mathbb{E}[\tau_y^x] \simeq \mathrm{e}^{[V(z)-V(x)]/\varepsilon}$

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Arrhenius' law: proved by [Freidlin, Wentzell, 1979] using large deviations

Eyring–Kramers law: [Bovier, Eckhoff, Gayrard, Klein, 2004] using potential theory, [Helffer, Klein, Nier, 2004] using Witten Laplacian, ...

Potential-theoretic proof

$$dx_t = -\nabla V(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t$$

- Generator: $\mathcal{L} = \varepsilon\Delta - \nabla V \cdot \nabla = \varepsilon\, e^{V/\varepsilon}\, \nabla \cdot e^{-V/\varepsilon}\, \nabla$
- Invariant probability: $\pi(dx) = \frac{1}{Z}\, e^{-V(x)/\varepsilon}\, dx \quad \Rightarrow \quad \mathcal{L}^\dagger \pi = 0$

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- Reversible: $\langle f, \mathcal{L}g\rangle = \langle \mathcal{L}f, g\rangle$ for $\langle f, g\rangle = \int_{\mathbb{R}^d} e^{-V(x)/\varepsilon} f(x)g(x)\,dx$
- Dirichlet form: $\mathcal{E}(f) = \langle f, -\mathcal{L}f\rangle = \varepsilon\int_{\mathbb{R}^d} e^{-V(x)/\varepsilon}|\nabla f(x)|^2\,dx$
  $\mathcal{E}(f,g) = \langle f, -\mathcal{L}g\rangle$

Expected hitting time

▷ Expected hitting time:

$$w_A(x) = \mathbb{E}^x[\tau_A] \quad \text{satisfies} \quad \begin{cases} (\mathcal{L}w_A)(x) = -1 & x \in A^c \\ w_A(x) = 0 & x \in A \end{cases}$$

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- Green function:

$$\begin{cases} (\mathcal{L}G_A)(x) = \delta(x-y) & x \in A^c \\ G_A(x,y) = 0 & x \in A \end{cases}$$

$$\Rightarrow \qquad w_A(x) = -\int_{A^c} G_A(x,y)(\mathrm{d}y)$$

Committor

- Committor:

$$h_{AB}(x) = \mathbb{P}^x\{\tau_A < \tau_B\} \quad \text{satisfies} \quad \begin{cases} (\mathcal{L}h_{AB})(x) = 0 & x \in (A \cup B)^c \\ h_{AB}(x) = 1 & x \in A \\ h_{AB}(x) = 0 & x \in B \end{cases}$$

- Equilibrium measure: $e_{AB}(\mathrm{d}x) = (-\mathcal{L}h_{AB})(\mathrm{d}x)$ measure on $x \in \partial A$

$$\Rightarrow \qquad h_{AB}(x) = -\int_A G_B(x,y) e_{AB}(\mathrm{d}y)$$

Capacity

Capacity: $\operatorname{cap}(A,B) = \displaystyle\int_{\partial A} \mathrm{e}^{-V(x)/\varepsilon}\, e_{AB}(\mathrm{d}x)$

$\Rightarrow \nu_{AB}(\mathrm{d}x) = \frac{1}{\operatorname{cap}(A,B)}\, \mathrm{e}^{-V(x)/\varepsilon}\, e_{AB}(\mathrm{d}x)$ is a probability measure on ∂A

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Theorem ("Magic" formula):

$$\mathbb{E}^{\nu_{AB}}[\tau_B] := \int_{\partial A} \mathbb{E}^x[\tau_B] \nu_{AB}(\mathrm{d}x) = \frac{1}{\text{cap}(A, B)} \int_{B^c} e^{-V(x)/\varepsilon} \, h_{AB}(x) \, \mathrm{d}x$$

Dirichlet principle

Theorem: Dirichlet principle

Let $\mathcal{H}_{AB} = \{h : \mathbb{R}^d \to [0,1] : h|_A = 1, h|_B = 0\}$. Then

$$\text{cap}(A,B) = \inf_{h \in \mathcal{H}_{AB}} \mathcal{E}(h) = \mathcal{E}(h_{AB})$$

Thomson principle

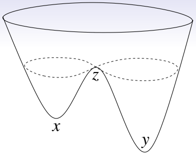
Theorem: Thomson principle [Landim, Mariani, Seo 2018]

Let $\mathcal{U}_{AB} = \{f \colon \nabla \cdot f|_{(A\cup B)^c} = 0, \int_{\partial A} f(x) \cdot n_A(x)\sigma(\mathrm{d}x) = 1\}$. Then

$$\operatorname{cap}(A,B) = \sup_{f\in\mathcal{U}_{AB}} \frac{1}{\mathcal{D}(f)} = \frac{1}{\mathcal{D}(f_{AB})} \qquad \mathcal{D}(f) = \frac{1}{\varepsilon}\int \mathrm{e}^{V(x)/\varepsilon}|f(x)|^2\,\mathrm{d}x$$

Proof of Eyring–Kramers law

$$dx_t = -\nabla V(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t$$

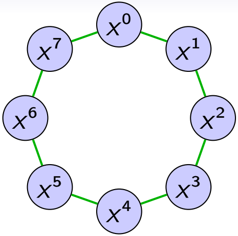


- A, B small balls around x, y

A particle system

[B, Fernandez, Gentz, Nonlinearity 2007]

- N particles on a circle $\mathbb{Z}/N\mathbb{Z}$
- Bistable local dynamics
- Ferromagnetic nearest neighbour coupling
- Independent noise on each site



$$\mathrm{d}x_t^i = \left[x_t^i - (x_t^i)^3\right]\mathrm{d}t + \frac{\gamma}{2}\left[x_t^{i+1} - 2x_t^i + x_t^{i-1}\right]\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t^i$$

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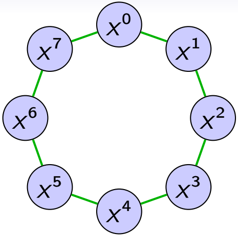

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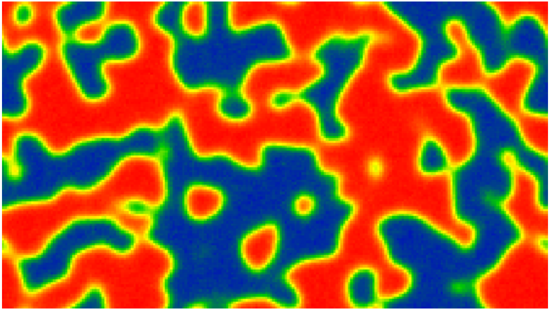
Gradient system $\quad \mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$

potential $V(x) = \sum_i U(x^i) + \frac{\gamma}{4}\sum_i (x^{i+1} - x^i)^2 \qquad U(\xi) = \frac{1}{4}\xi^4 - \frac{1}{2}\xi^2$

Coarsening dynamics with noise

(Link to simulation)

5. The stochastic Allen–Cahn PDEs



Deterministic Allen–Cahn PDE

[Chafee & Infante 74, Allen & Cahn 75]

$$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x))$$

- $x \in [0,L]$, L: bifurcation parameter
- $u(t,x) \in \mathbb{R}$
- Either periodic or zero-flux Neumann boundary conditions
- In this talk: $f(u) = u - u^3$ (results more general)

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Energy function:

$$V[u] = \int_0^L \Big[\frac{1}{2}u'(x)^2 - \frac{1}{2}u(x)^2 + \frac{1}{4}u(x)^4\Big]\,\mathrm{d}x \qquad \to \text{ min}$$

Scaling limit of particle system with $\gamma = 2\frac{N^2}{L^2}$

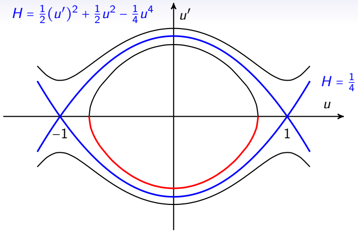
Stationary solutions: $u_0''(x) = -u_0(x) + u_0(x)^3$ critical points of V

Stability: Sturm–Liouville problem $\partial_t v_t(x) = v_t''(x) + [1 - 3u_0(x)^2]v_t(x)$

Stationary solutions

$u_0''(x) = -f(u_0(x)) = -u_0(x) + u_0(x)^3$

- $u_\pm(x) \equiv \pm 1$
- $u_0(x) \equiv 0$
- Nonconstant solutions satisfying b.c.
  (expressible in terms of Jacobi elliptic fcts)



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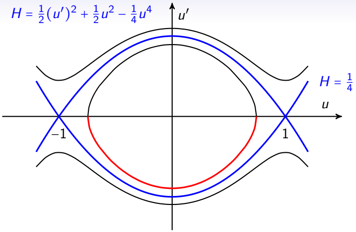
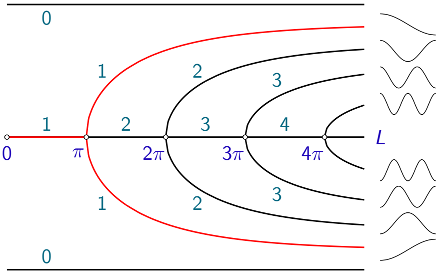


- Neumann b.c: $2k$ nonconstant solutions when $L > k\pi$

Number of positive eigenvalues
(= unstable directions)
Transition state



- Periodic b.c: k families when $L > 2k\pi$

Eyring–Kramers law for 1D SPDEs: heuristics

$$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x)) \quad + \sqrt{2\varepsilon}\,\xi(t,x) \qquad (f(u) = u - u^3)$$

Initial condition: $u_{\rm in}$ near $u_- \equiv -1$ with eigenvalues $\nu_k = (\frac{\beta k \pi}{L})^2 + 2$

Target: $u_+ \equiv 1$, $\tau_+ = \inf\{t > 0 \colon \|u_t - u_+\|_{L^\infty} < \rho\}$

Transition state: ($\beta = 1$ for Neumann b.c., $\beta = 2$ for periodic b.c.)

$$u_{\rm ts}(x) = \begin{cases} u_0(x) \equiv 0 & \text{if } L \leqslant \beta\pi \quad \text{with ev } \lambda_k = (\frac{\beta k \pi}{L})^2 - 1 \\ u_1(x) \ \beta\text{-kink stationary sol.} & \text{if } L > \beta\pi \quad \text{with ev } \lambda_k' \end{cases}$$

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[Faris & Jona-Lasinio 82]: large-deviation principle

$\Rightarrow$ Arrhenius law: $\mathbb{E}^{u_{\rm in}}[\tau_+] \simeq e^{(V[u_{\rm ts}] - V[u_-])/\varepsilon}$

[Maier & Stein 01]: formal computation; for Neumann b.c.

$\Rightarrow$ $\mathbb{E}^{u_{\rm in}}[\tau_+] \simeq 2\pi\sqrt{\frac{1}{|\lambda_0|\nu_0}\prod_{k=1}^\infty \frac{\lambda_k}{\nu_k}}\, e^{(V[u_{\rm ts}] - V[u_-])/\varepsilon}$

Eyring–Kramers law for 1D SPDEs: main result

Theorem: Neumann b.c. [B & Gentz, 2013]

- If $L < \pi - c$ with $c > 0$, then

$$\mathbb{E}^{u_{\mathrm{in}}}[\tau_+] = 2\pi \sqrt{\frac{1}{|\lambda_0|\nu_0} \prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k}}\, \mathrm{e}^{(V[u_{\mathrm{ts}}]-V[u_-])/\varepsilon} \Big[1 + \underbrace{\mathcal{O}(\varepsilon^{1/2}|\log \varepsilon|^{3/2})}_{\text{error not optimal}}\Big]$$

- If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles) and λ'_k instead of λ_k

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---

- Results also for L near π and periodic b.c.
- Prefactor involves a Fredholm determinant:
  $\Delta_\perp$ Laplacian acting on mean zero functions

$$\prod_{k=1}^{\infty}\frac{\lambda_k}{\nu_k} = \det\bigl[(-\Delta_\perp - 1)(-\Delta_\perp + 2)^{-1}\bigr] = \det\bigl[\mathbb{1} - 3(-\Delta_\perp + 2)^{-1}\bigr]$$

  converges because $\log\det = \operatorname{Tr}\log$ and $(-\Delta_\perp + 2)^{-1}$ is trace class
  (limit $= \frac{\sqrt{2}\sin(L)}{\sinh(\sqrt{2}L)}$)

Ideas of the proof ($L < \pi$)

- Spectral Galerkin approximation: $u(t,x) = \sum_{|k| \leqslant N} z_k(t) e_k(x)$ (Fourier)

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- Change of variables $u(x) = u_0 + \sqrt{\varepsilon} u_\perp(x)$ where $\int_0^L u_\perp(x)\,\mathrm{d}x = 0$

$$\Rightarrow \frac{1}{\varepsilon} V[u_0 + \sqrt{\varepsilon} u_\perp] = \frac{1}{\varepsilon} \underbrace{\left(\frac{1}{4} u_0^4 - \frac{1}{2} u_0^2\right)}_{V_0(u_0)} + Q_{u_0}[u_\perp] + \sqrt{\varepsilon} R_{u_0}[u_\perp]$$

where

$$\begin{cases} Q_{u_0}[u_\perp] = \dfrac{1}{2} \displaystyle\int_0^L \left[u_\perp'(x)^2 - (1 - 3u_0^2) u_\perp(x)^2\right] \mathrm{d}x = \dfrac{1}{2} \langle u_\perp, [-\Delta - (1 - 3u_0^2)] u_\perp \rangle \\ R_{u_0}[u_\perp] = u_0 \displaystyle\int_0^L u_\perp(x)^3 \,\mathrm{d}x + \sqrt{\varepsilon} \int_0^L u_\perp(x)^4 \,\mathrm{d}x \qquad \text{(remainder)} \end{cases}$$

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$$\Rightarrow \frac{1}{\varepsilon} V[u_0 + \sqrt{\varepsilon} u_\perp] = \frac{1}{\varepsilon}\underbrace{\left(\frac14 u_0^4 - \frac12 u_0^2\right)}_{V_0(u_0)} + Q_{u_0}[u_\perp] + \sqrt{\varepsilon} R_{u_0}[u_\perp]$$

where

$$\begin{cases} Q_{u_0}[u_\perp] = \dfrac12 \displaystyle\int_0^L \big[u_\perp'(x)^2 - (1-3u_0^2) u_\perp(x)^2\big]\,\mathrm{d}x = \dfrac12 \langle u_\perp, [-\Delta - (1-3u_0^2)] u_\perp\rangle \\ R_{u_0}[u_\perp] = u_0 \displaystyle\int_0^L u_\perp(x)^3\,\mathrm{d}x + \sqrt{\varepsilon}\int_0^L u_\perp(x)^4\,\mathrm{d}x \qquad \text{(remainder)} \end{cases}$$

- Dirichlet principle with $h = h(u_0)$ s.t. $h'(u_0) = -\frac{1}{c}\,\mathrm{e}^{V_0(u_0)/\varepsilon}$, $c \simeq \sqrt{\frac{2\pi\varepsilon}{|\lambda_0|}}$

$$\operatorname{cap}(A,B) \leqslant \mathcal{E}(h) = \frac{\varepsilon^{1+\frac{N}{2}}}{c^2}\int_{-1}^{1} \mathrm{e}^{V_0(u_0)/\varepsilon} \underbrace{\int \mathrm{e}^{-Q_{u_0}[u_\perp]}\,\mathrm{e}^{-\sqrt{\varepsilon} R_{u_0}[u_\perp]}\,\mathrm{d}u_\perp}_{=\sqrt{\frac{(2\pi)^N}{\det[-\Delta_\perp - (1-3u_0^2)]}}\,\mathbb{E}^\gamma[\mathrm{e}^{-\sqrt{\varepsilon} R_{u_0}}]}\,\mathrm{d}u_0$$

Ideas of the proof ($L < \pi$)

- Thomson principle with divergence-free unit flow $f = K^{-1}\,\mathrm{e}^{-Q_0[u_\perp]}\,e_{u_0}$

  Normalisation $K = \varepsilon^{\frac{N}{2}} \int \mathrm{e}^{-Q_0[u_\perp]}\,\mathrm{d}u_\perp = \sqrt{\frac{(2\pi\varepsilon)^N}{\det[-\Delta_\perp - 1]}}$

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- Conclusion: $\operatorname{cap}(A,B) = \varepsilon \sqrt{\frac{|\lambda_0|}{2\pi\varepsilon}} \sqrt{\frac{(2\pi\varepsilon)^N}{\det[-\Delta_\perp - 1]}} \big[1 + \mathcal{O}(\varepsilon)\big]$

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Other elements of the proof:

- A priori bounds on h_{AB}: large deviations (or symmetry argument)
- Convergence of hitting times as $N \to \infty$: a priori estimate for $\mathbb{E}[\tau_B^2]$
- Coupling argument for start in u_{in} [Martinelli, Olivieri & Scoppola]
- Bifurcation at $L = \beta\pi$

The two-dimensional case

$\partial_t u = \Delta u + u - u^3 + \sqrt{2\varepsilon}\xi$ (Link to simulation)

The two-dimensional case

- Large-deviation principle: [Hairer & Weber, 2015]
- Naive computation of prefactor fails:

$$\log \prod_{k\in(\mathbb{N}^2)^*} \frac{1-\left(\frac{L}{|k|\pi}\right)^2}{1+2\left(\frac{L}{|k|\pi}\right)^2} \simeq \sum_{k\in(\mathbb{N}^2)^*} \log\biggl(1-\frac{3L^2}{|k|^2\pi^2}\biggr)$$

$$\simeq -\sum_{k\in(\mathbb{N}^2)^*} \frac{3L^2}{|k|^2\pi^2} \simeq -\frac{3L^2}{\pi^2}\int_1^\infty \frac{r\,\mathrm{d}r}{r^2} = -\infty$$

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- In fact, the equation needs to be renormalised

Theorem: [Da Prato & Debussche 2003]
Let ξ^δ be a mollification on scale δ of white noise. Then

$$\partial_t u = \Delta u + \big[1 + 3\varepsilon C(\delta)\big]u - u^3 + \sqrt{2\varepsilon}\xi^\delta$$

with $C(\delta) \simeq \log(\delta^{-1})$ admits local solution converging as $\delta \to 0$
(Global version: [Mourrat & Weber 2015])
[Mourrat & Weber 2014]: Renormalised eq = scaling limit of Ising–Kac model

Main result in dimension 2

Theorem: [B, Di Gesù, Weber, 2017]

For $L < 2\pi$, appropriate $A \ni u_-$, $B \ni u_+$, $\exists \mu_N$ probability measures on ∂A:

$$\limsup_{N\to\infty} \mathbb{E}^{\mu_N}\big[\tau_B\big] \leqslant \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k}\, \mathrm{e}^{\frac{\nu_k-\lambda_k}{|\lambda_k|}}}\, \mathrm{e}^{(V[u_{\mathrm{ts}}]-V[u_-])/\varepsilon}\big[1 + c_+\sqrt{\varepsilon}\,\big]$$

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- Inverse of prefactor involves Carleman–Fredholm determinant:

  $$\det{}_2(\mathbb{1} + T) = \det(\mathbb{1} + T)\,\mathrm{e}^{-\operatorname{Tr} T}$$

  with $T = 3(-\Delta_\perp - 1)^{-1}$

  $\det_2$ defined whenever T is only Hilbert–Schmidt (true for $d \leqslant 3$)
- [Tsatsoulis & Weber 2018]: Same result for $\mathbb{E}^{u_0}[\tau_B]$

Renormalisation

Problem: Stoch. convolution $w_t(x) = \int_0^t \mathrm{e}^{\Delta(t-s)}\,\xi(s,x)\,\mathrm{d}s$ is distribution

- δ-mollification should be equivalent to Galerkin approx. $|k| \leqslant N = \delta^{-1}$:

$$w_N(x,t) = \sum_{|k|\leqslant N} a_k(t)\frac{1}{L}\,\mathrm{e}^{\mathrm{i}\,\Omega k\cdot x} \qquad a_k = \int_0^t \mathrm{e}^{-\mu_k(t-s)}\,\mathrm{d}W_s^{(k)}$$

$$\mu_k = (\Omega|k|)^2 \qquad \Omega = \beta\pi/L$$

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- $\lim_{t\to\infty} \int_0^t e^{(\Delta-1)(t-s)} \xi_N(s,x)\, ds = \phi_N$ is a Gaussian free field, s.t.

$$L^2 C_N := L^2 \mathbb{E}\phi_N^2 = \mathbb{E}\|\phi_N\|_{L^2}^2 = \sum_{|k|\leqslant N} \frac{1}{2(\mu_k+1)} = \frac{\mathrm{Tr}(P_N[-\Delta+\mathbb{1}]^{-1})}{2} \simeq \log(N)$$

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- Wick powers

$$:\phi_N^2: = \phi_N^2 - C_N$$
$$:\phi_N^3: = \phi_N^3 - 3C_N\phi_N$$
$$:\phi_N^4: = \phi_N^4 - 6C_N\phi_N^2 + 3C_N^2$$

have zero mean and uniformly bounded variance (when integrated)

Computation of the prefactor

- Consider for simplicity $L < \beta\pi \Rightarrow$ transition state in 0
- Galerkin-truncated renormalised potential

$$V_N = \frac{1}{2}\int_{\mathbb{T}^2}\left[\|\nabla u_N(x)\|^2 - u_N(x)^2\right]\mathrm{d}x + \frac{1}{4}\int_{\mathbb{T}^2} {:}u_N(x)^4{:}\,\mathrm{d}x$$

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- Using Nelson estimate: $\mathrm{cap}(A,B) \simeq \sqrt{\frac{|\lambda_0|\varepsilon}{2\pi}}\prod_{0<|k|\leqslant N}\sqrt{\frac{2\pi\varepsilon}{\lambda_k}}$
- Symmetry argument:

$$\int_{B^c} h_{A,B}(z)\,\mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z = \frac{1}{2}\int \mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z = \frac{1}{2}\mathcal{Z}_N(\varepsilon)$$

- $\mathcal{Z}_N(\varepsilon) \simeq 2\prod_{|k|\leqslant N}\sqrt{\frac{2\pi\varepsilon}{\nu_k}}\,\mathrm{e}^{-V_N(L,0)/\varepsilon}$ where $-V_N(L,0) = \frac{1}{4}L^2 + \frac{3}{2}L^2 C_N\varepsilon$
- Prefactor proportional to (since $\nu_k = \lambda_k + 3$)

$$\prod_{0<|k|\leqslant N}\frac{\lambda_k}{\lambda_k+3}\mathrm{e}^{3/\lambda_k} \qquad \text{converges since} \qquad \log\left[\frac{\lambda_k}{\lambda_k+3}\mathrm{e}^{3/\lambda_k}\right] = \mathcal{O}\left(\frac{1}{|k|^4}\right)$$

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Thanks for your attention!