Multi-scale models, slow-fast differential equations, averaging in ecology

Noise-induced transitions in slow–fast dynamical systems

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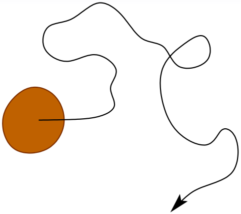
With Barbara Gentz (Bielefeld), Christian Kuehn (Vienna) and Damien Landon (Dijon)

What is noise?

Paradigm: Brownian motion

[R. Brown, 1827]



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[J. Perrin, 1909]:

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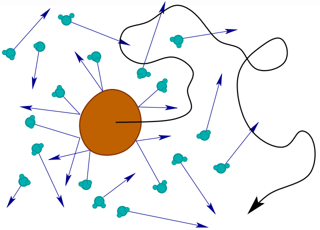

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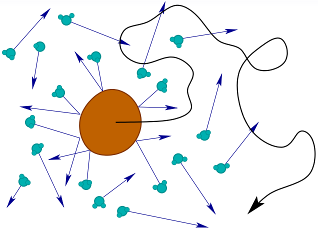

Wiener process $\{W_t\}_{t\geqslant 0}$: scaling limit of random walk $\lim_{n\to\infty} \dfrac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}$

Stochastic differential equation:

$$\mathrm{d}x_t = \underbrace{f(x_t)\,\mathrm{d}t}_{\text{exterior force}} + \underbrace{g(x_t)\,\mathrm{d}W_t}_{\text{random force}}$$

Physicist's notation: $\dot{x} = f(x) + g(x)\xi, \quad \langle \xi(s)\xi(t) \rangle = \delta(s-t)$

What is noise?

Stochastic differential equation (SDE):

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + g(x_t)\,\mathrm{d}W_t$$

Itô calculus:

define solution via $\quad x_t = x_0 + \int_0^t f(x_s)\,\mathrm{d}s + \int_0^t g(x_s)\,\mathrm{d}W_s$

Euler scheme: $\quad x_{t+\Delta t} \simeq x_t + f(x_t)\Delta t + g(x_t)\sqrt{\Delta t}\,\mathcal{N}(0,1)$

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Rigorous derivations of effective SDEs from more fundamental models:

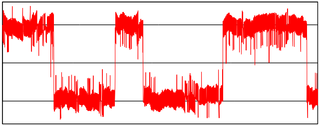
- System coupled to infinitely many harmonic oscillators
  [Ford, Kac, Mazur '65], [Lebowitz, Spohn '77],
  [Eckmann, Pillet, Rey-Bellet '99], [Rey-Bellet, Thomas '00, '02]
- Stochastic averaging for slow–fast systems
  [Khasminski '66], [Hasselmann '76], [Kifer '03]

Example 1: Stochastic resonance

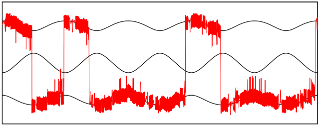
$$\mathrm{d}x_s = \underbrace{\left[-x^3 + x + A\cos\varepsilon s\right]}_{=-\frac{\partial}{\partial x}\left[\frac{1}{4}x^4-\frac{1}{2}x^2-Ax\cos\varepsilon s\right]}\,\mathrm{d}s + \sigma\,\mathrm{d}W_s$$

- deterministically bistable climate [Croll, Milankovitch]
- random perturbations due to weather [Benzi/Sutera/Vulpiani, Nicolis/Nicolis]

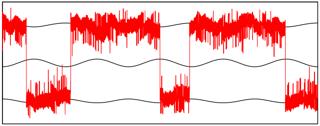
Sample paths $\{x_s\}_s$ for $\varepsilon = 0.001$:



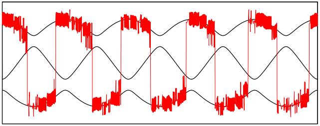
$A = 0$, $\sigma = 0.3$



$A = 0.24$, $\sigma = 0.2$



$A = 0.1$, $\sigma = 0.27$



$A = 0.35$, $\sigma = 0.2$

Example 1: Stochastic resonance

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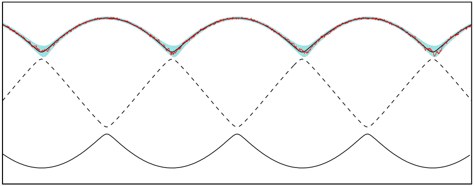
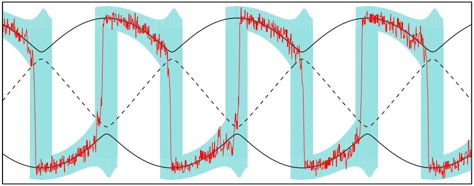
Critical noise intensity: $\sigma_{\mathrm{c}} = \max\{\delta, \varepsilon\}^{3/4}$, $\delta = A_{\mathrm{c}} - A$, $A_{\mathrm{c}} = \frac{2}{3\sqrt{3}}$

$\sigma \ll \sigma_{\mathrm{c}}$:
transitions unlikely



$\sigma \gg \sigma_{\mathrm{c}}$:
synchronisation



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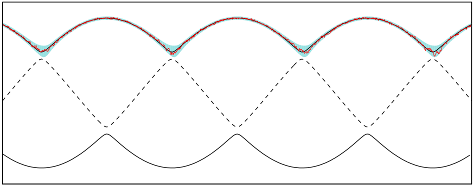

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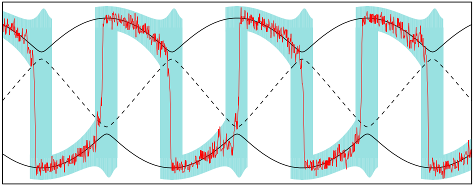

Theorem [B & Gentz, Annals App. Proba 2002]

- $\sigma < \sigma_c$: transition probability per period $\leqslant e^{-\sigma_c^2/\sigma^2}$
- $\sigma > \sigma_c$: transition probability per period $\geqslant 1 - e^{-c\sigma^{4/3}/(\varepsilon|\log\sigma|)}$

Example 2: FitzHugh–Nagumo model

$$\begin{aligned} \varepsilon \dot{x} &= x - x^3 + y \\ \dot{y} &= a - x - by \end{aligned}$$

- $x \propto$ membrane potential of neuron
- $y \propto$ proportion of open ion channels (recovery variable)
- $\varepsilon \ll 1 \Rightarrow$ fast–slow system
- $b = 0$ in the following for simplicity (but results more general)

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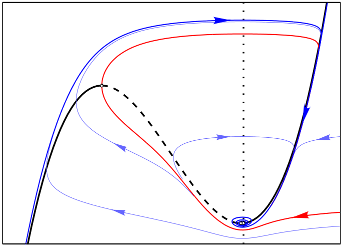
Stationary point $P = (a, a^3 - a)$
Linearisation has eigenvalues $\frac{-\delta \pm \sqrt{\delta^2 - \varepsilon}}{\varepsilon}$ where $\delta = \frac{3a^2 - 1}{2}$

- $\delta > 0$: stable node ($\delta > \sqrt{\varepsilon}$) or focus ($0 < \delta < \sqrt{\varepsilon}$)
- $\delta = 0$: singular Hopf bifurcation [Erneux & Mandel '86]
- $\delta < 0$: unstable focus ($-\sqrt{\varepsilon} < \delta < 0$) or node ($\delta < -\sqrt{\varepsilon}$)

Example 2: FitzHugh–Nagumo model

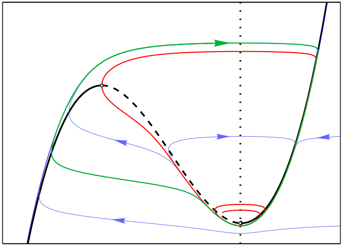
$\delta > 0$:

- P is asymptotically stable
- the system is excitable
- one can define a separatrix



$\delta < 0$:

P is unstable
$\exists$ asympt. stable periodic orbit
sensitive dependence on δ:
canard (duck) phenomenon
[Callot, Diener, Diener '78,
Benoît '81, ...]



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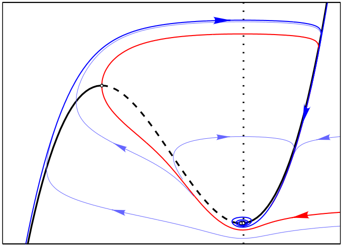

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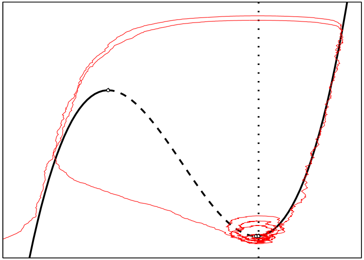
Stochastic FHN equation

$$\mathrm{d}x_t = \frac{1}{\varepsilon}[x_t - x_t^3 + y_t]\,\mathrm{d}t + \frac{\sigma_1}{\sqrt{\varepsilon}}\,\mathrm{d}W_t^{(1)}$$

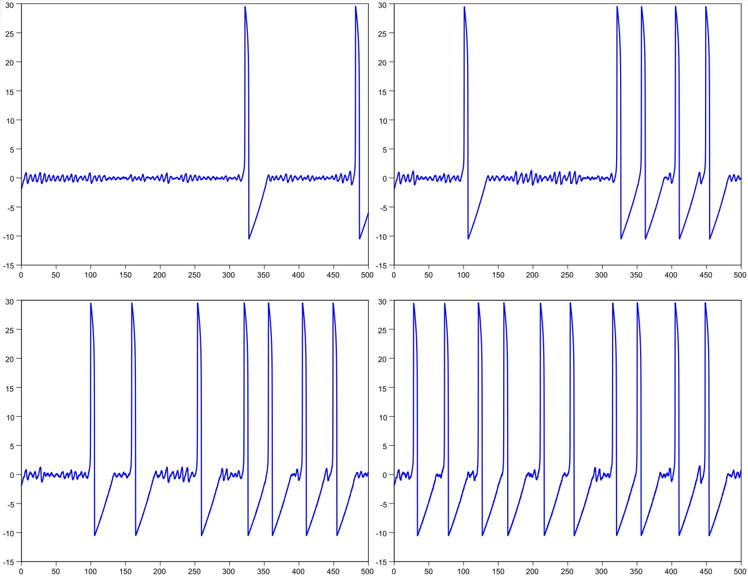
$$\mathrm{d}y_t = [a - x_t - by_t]\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_t^{(2)}$$

- Again $b = 0$ for simplicity in this talk
- $W_t^{(1)}, W_t^{(2)}$: independent Wiener processes (white noise)
- $0 < \sigma_1, \sigma_2 \ll 1$, $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$



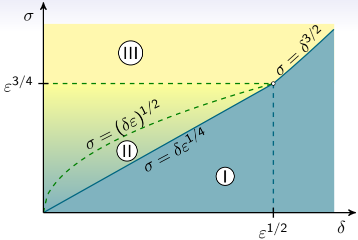
$\varepsilon = 0.1$
$\delta = 0.02$
$\sigma_1 = \sigma_2 = 0.03$

Noise-induced mixed-mode oscillations (MMOs)



Time series $t \mapsto -x_t$ for $\varepsilon = 0.01,\ \delta = 3 \cdot 10^{-3},\ \sigma = 1.46 \cdot 10^{-4}, \dots, 3.65 \cdot 10^{-4}$

Results: Parameter regimes



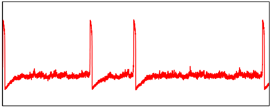
see also

[Muratov & Vanden Eijnden '08]

Regime I: rare isolated spikes

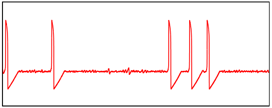
Thm [B & Landon, Nonlinearity 2012]

If $\delta \ll \varepsilon^{1/2}$: $\mathbb{E}[\#\text{ small oscil}] \simeq e^{\kappa(\varepsilon^{1/4}\delta)^2/\sigma^2}$



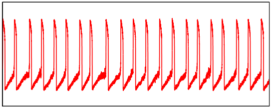
Regime II: clusters of spikes

small oscillations: asympt geometric



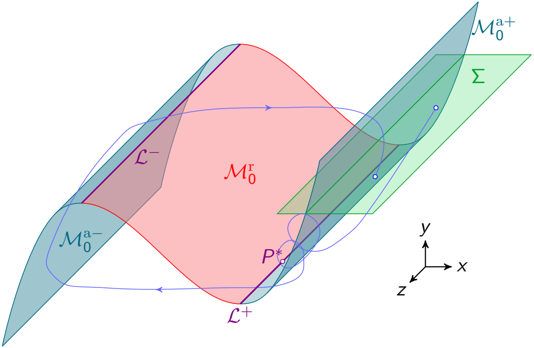
Regime III: repeated spikes

Interspike interval $\simeq$ constant



Example 3: The Koper model

$$
\begin{aligned}
\varepsilon\,\mathrm{d}x_t &= [y_t - x_t^3 + 3x_t]\,\mathrm{d}t \\
\mathrm{d}y_t &= [kx_t - 2(y_t + \lambda) + z_t]\,\mathrm{d}t \\
\mathrm{d}z_t &= [\rho(\lambda + y_t - z_t)]\,\mathrm{d}t
\end{aligned}
$$



Folded-node singularity at P^* induces mixed-mode oscillations
[Benoît, Lobry '82, Szmolyan, Wechselberger '01, Brøns, Krupa, W '06 . . .]

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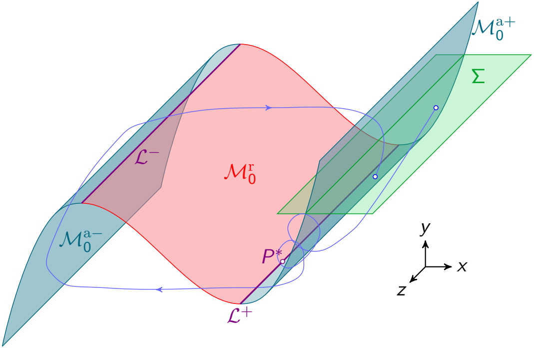
$$\begin{aligned}
\varepsilon\, \mathrm{d}x_t &= [y_t - x_t^3 + 3x_t]\,\mathrm{d}t &&+ \sqrt{\varepsilon}\sigma F(x_t, y_t, z_t)\,\mathrm{d}W_t \\
\mathrm{d}y_t &= [kx_t - 2(y_t + \lambda) + z_t]\,\mathrm{d}t &&+ \sigma' G_1(x_t, y_t, z_t)\,\mathrm{d}W_t \\
\mathrm{d}z_t &= [\rho(\lambda + y_t - z_t)]\,\mathrm{d}t &&+ \sigma' G_2(x_t, y_t, z_t)\,\mathrm{d}W_t
\end{aligned}$$



Folded-node singularity at P^* induces mixed-mode oscillations
[Benoît, Lobry '82, Szmolyan, Wechselberger '01, Brøns, Krupa, W '06 ...]
What happens if we add noise to the system?

Threshold phenomena: How to prove them

σ_c: Critical noise intensity (to be determined)

1. For $\sigma \ll \sigma_c$, the stochastic solution remains close to the deterministic one with high probability
   - ◇ slightly easier to show
   - ◇ general method available
   - ◇ bounds are (almost) sharp in 1D, less sharp in higher D

2. For $\sigma \gg \sigma_c$, the stochastic system makes noise-induced transitions with high probability
   - ◇ harder to show
   - ◇ case-by-case approach
   - ◇ less sharp results

Below threshold: 1D time-dependent case

On the slow time scale $t = \varepsilon s$:

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t)$$

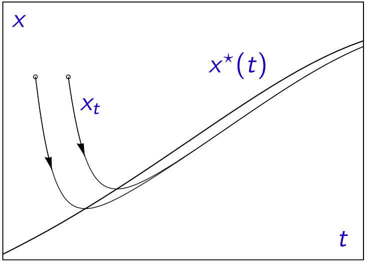
- Equilibrium branch: $\{x = x^\star(t)\}$ where $f(x^\star(t),t) = 0$ for all t
- Stable if $a^\star(t) = \partial_x f(x^\star(t),t) \leqslant -a_0 < 0$ for all t

Then [Tikhonov '52, Fenichel '79]:

- There exists particular solution

$$\bar{x}(t) = x^\star(t) + \mathcal{O}(\varepsilon)$$

- $\bar{x}$ attracts nearby orbits exp. fast
- $\bar{x}$ admits asymptotic series in ε



Below threshold: 1D time-dependent case

Stochastic perturbation:

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Write $x_t = \bar{x}(t) + \xi_t$ and Taylor-expand:

$$\mathrm{d}\xi_t = \frac{1}{\varepsilon}\Big[\bar{a}(t)\xi_t + \underbrace{b(\xi_t, t)}_{=\mathcal{O}(\xi_t^2)}\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

where $\bar{a}(t) = \partial_x f(\bar{x}(t), t) = a^\star(t) + \mathcal{O}(\varepsilon)$

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where $\bar{a}(t) = \partial_x f(\bar{x}(t), t) = a^\star(t) + \mathcal{O}(\varepsilon)$

Variations of constants (Duhamel formula), if $\xi_0 = 0$:

$$\xi_t = \underbrace{\frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{\bar{\alpha}(t,s)/\varepsilon}\, dW_s}_{\xi_t^0\text{: sol of linearised system}} + \underbrace{\frac{1}{\varepsilon} \int_0^t e^{\bar{\alpha}(t,s)/\varepsilon}\, b(\xi_s, s)\, ds}_{\text{treat as a perturbation}}$$

where $\bar{\alpha}(t,s) = \int_s^t \bar{a}(u)\, du$

Below threshold: 1D time-dependent case

Properties of $\quad \xi_t^0 = \dfrac{\sigma}{\sqrt{\varepsilon}} \displaystyle\int_0^t \mathrm{e}^{\bar\alpha(t,s)/\varepsilon} \, \mathrm{d}W_s$:

- Gaussian process, $\quad \mathbb{E}[\xi_t^0] = 0, \quad \mathrm{Var}(\xi_t^0) = \frac{\sigma^2}{\varepsilon} \int_0^t \mathrm{e}^{2\bar\alpha(t,s)/\varepsilon} \, \mathrm{d}s$
- Confidence interval: $\quad \mathbb{P}\bigl\{|\xi_t^0| > \frac{h}{\sigma}\sqrt{\mathrm{Var}(\xi_t^0)}\bigr\} = \mathcal{O}(\mathrm{e}^{-h^2/2\sigma^2})$
- $\sigma^{-2}\,\mathrm{Var}(\xi_t^0)$ satisfies ODE $\quad \varepsilon \dot v = 2\bar a(t) v + 1$

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Lemma [B & Gentz, Proba. Theory Relat. Fields 2002]

$\bar v(t)$ solution of ODE bounded away from 0: $\quad \bar v(t) = \frac{1}{-2\bar a(t)} + \mathcal{O}(\varepsilon)$

$$\mathbb{P}\biggl\{\sup_{0\leqslant s\leqslant t} \frac{|\xi_s^0|}{\sqrt{\bar v(s)}} > h\biggr\} = C_0(t,\varepsilon)\,\mathrm{e}^{-h^2/2\sigma^2}$$

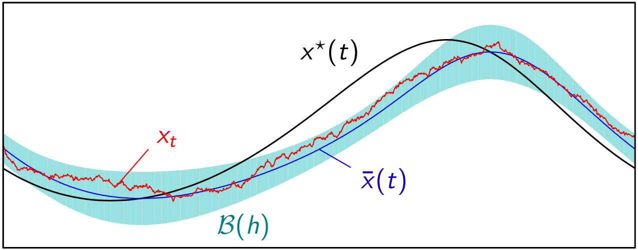
where $C_0(t,\varepsilon) = \sqrt{\frac{2}{\pi}\frac{1}{\varepsilon}}\Bigl|\int_0^t \bar a(s)\,\mathrm{d}s\Bigr|\frac{h}{\sigma}\Bigl[1 + \mathcal{O}\bigl(\varepsilon + \frac{t}{\varepsilon}\,\mathrm{e}^{-h^2/\sigma^2}\bigr)\Bigr]$

Proof based on Doob's submartingale inequality and partition of $[0,t]$

Below threshold: 1D time-dependent case

Nonlinear equation: $\mathrm{d}\xi_t = \dfrac{1}{\varepsilon}\big[\bar{a}(t)\xi_t + b(\xi_t, t)\big]\,\mathrm{d}t + \dfrac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$

Confidence strip: $\mathcal{B}(h) = \Big\{|\xi| \leqslant h\sqrt{\bar{v}(t)}\ \forall t\Big\} = \Big\{|x - \bar{x}(t)| \leqslant h\sqrt{\bar{v}(t)}\ \forall t\Big\}$



Theorem [B & Gentz, Proba. Theory Relat. Fields 2002]

$$C(t,\varepsilon)\,\mathrm{e}^{-\kappa_- h^2/2\sigma^2} \leqslant \mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \leqslant C(t,\varepsilon)\,\mathrm{e}^{-\kappa_+ h^2/2\sigma^2}$$

where $\kappa_\pm = 1 \mp \mathcal{O}(h)$ and $C(t,\varepsilon) = C_0(t,\varepsilon)\big[1 + \mathcal{O}(h)\big]$ (requires $h \leqslant h_0$)

Generalisation to the multidimensional case

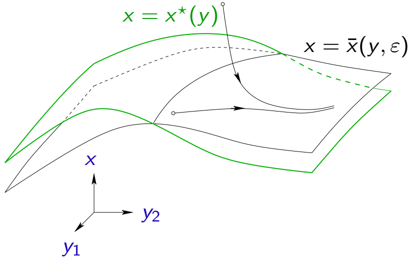
$$\varepsilon \dot{x} = f(x,y) \qquad x \in \mathbb{R}^n \text{, fast variables}$$
$$\dot{y} = g(x,y) \qquad y \in \mathbb{R}^m \text{, slow variables}$$

- Critical manifold: $f(x^\star(y), y) = 0$ (for all y in some domain)
- Stability: Eigenvalues of $A(y) = \partial_x f(x^\star(y), y)$ have negative real parts

Theorem [Tihonov '52, Fenichel '79]

$\exists$ slow manifold $x = \bar{x}(y,\varepsilon)$ s.t.

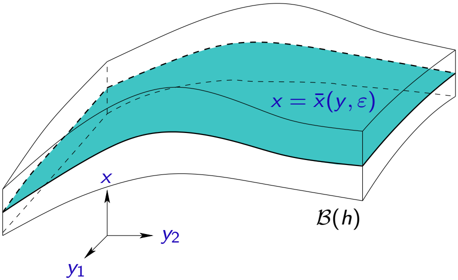
- $\bar{x}(y,\varepsilon)$ is invariant
- $\bar{x}(y,\varepsilon)$ attracts nearby solutions
- $\bar{x}(y,\varepsilon) = x^\star(y) + \mathcal{O}(\varepsilon)$



Generalisation to the multidimensional case

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, y_t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\,\mathrm{d}W_t \qquad \text{(fast variables } \in \mathbb{R}^n)$$

$$\mathrm{d}y_t = g(x_t, y_t)\,\mathrm{d}t + \sigma' G(x_t, y_t)\,\mathrm{d}W_t \qquad \text{(slow variables } \in \mathbb{R}^m)$$



$$\mathcal{B}(h) := \Big\{(x,y) \colon \big\langle [x - \bar{x}(y,\varepsilon)], \bar{X}(y)^{-1} [x - \bar{x}(y,\varepsilon)] \big\rangle < h^2 \Big\}$$

where $\bar{X}$: covariance matrix of linearisation, solution of slow–fast ODE

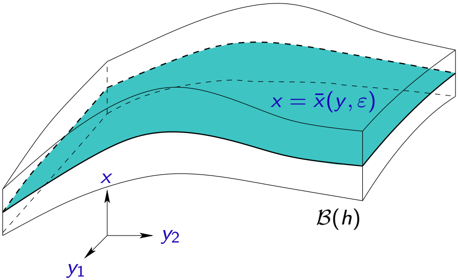
$$\varepsilon \dot{\bar{X}} = \bar{A}(y)\bar{X} + \bar{X}\bar{A}(y)^{\mathrm{T}} + F(\bar{x}(y,\varepsilon), y) F(\bar{x}(y,\varepsilon), y)^{\mathrm{T}}$$

$$\dot{y} = g(\bar{x}(y,\varepsilon), y)$$

Generalisation to the multidimensional case

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, y_t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\,\mathrm{d}W_t \qquad \text{(fast variables } \in \mathbb{R}^n\text{)}$$

$$\mathrm{d}y_t = g(x_t, y_t)\,\mathrm{d}t + \sigma' G(x_t, y_t)\,\mathrm{d}W_t \qquad \text{(slow variables } \in \mathbb{R}^m\text{)}$$



Theorem [B & Gentz, J. Diff. Equ. 2004]

$$\mathbb{P}\{\text{leaving } \mathcal{B}(h) \text{ before time } t\} \simeq C(t,\varepsilon)\,\mathrm{e}^{-\kappa h^2/2\sigma^2}$$

$\kappa = 1 - \mathcal{O}(h) - \mathcal{O}(\varepsilon)$

Back to Example 1: Avoided transcritical bif.

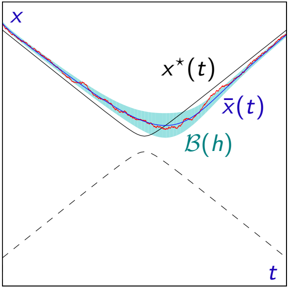
$$\mathrm{d}x_t = \frac{1}{\varepsilon}\left[t^2 + \delta - x_t^2 + \dots\right]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Equil. curve: $x^\star(t) \simeq \sqrt{t^2+\delta}$

Slow sol.: $\bar{x}(t) = x^\star(t) + \mathcal{O}(\min\{\frac{\varepsilon}{|t|}, \frac{\varepsilon}{\sqrt{\delta+\varepsilon}}\})$

$$\bar{a}(t) = \partial_x f(\bar{x}(t), \varepsilon) \asymp \begin{cases} -|t| & |t| \geqslant \sqrt{\delta+\varepsilon} \\ -\sqrt{\delta+\varepsilon} & |t| \leqslant \sqrt{\delta+\varepsilon} \end{cases}$$

Confidence strip $\mathcal{B}(h)$: width $\asymp h/\sqrt{|\bar{a}(t)|}$



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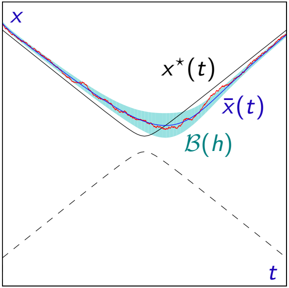

Theorem [B & Gentz, Annals App. Proba 2002]

$$\mathbb{P}\big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\big\} \leqslant C(t,\varepsilon)\,\mathrm{e}^{-\kappa h^2/2\sigma^2}$$

where $\kappa = 1 - \mathcal{O}(\sup_{s\leqslant t} h|\bar{a}(s)|^{-3/2}) - \mathcal{O}(\varepsilon)$ $\Rightarrow$ requires $h < h_0 \inf_{s\leqslant t}|\bar{a}(s)|^{3/2}$

- $\sigma < \sigma_c = \max\{\delta,\varepsilon\}^{3/4}$: result applies $\forall\, t$, $\mathbb{P}\{\text{trans}\} = \mathcal{O}(\mathrm{e}^{-\kappa\sigma_c^2/\sigma^2})$
- $\sigma > \sigma_c = \max\{\delta,\varepsilon\}^{3/4}$: result applies up to $t \asymp -\sigma^{2/3}$

Above threshold

What happens for $\sigma > \sigma_c$ and $t > -\sigma^{2/3}$?

General principle: partition $t_0 = s_0 < s_1 < s_2 < \dots < s_n = t$ of $[t_0, t]$

Lemma Let $P_k = \mathbb{P}\{\text{making no transition during } (s_{k-1}, s_k]\}$. Then

$$\mathbb{P}\{\text{making no transition during } [t_0, t]\} \leqslant \prod_{k=1}^{n} P_k$$

Choose partition s.t. each $P_k \leqslant q < 1 \Rightarrow \mathbb{P}\{\text{no transition}\} \leqslant e^{-n \log q}$

Above threshold

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Choose partition s.t. each $P_k \leqslant q < 1 \Rightarrow \mathbb{P}\{\text{no transition}\} \leqslant e^{-n \log q}$

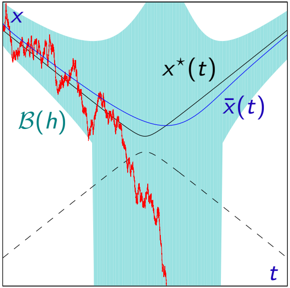
Example 1: Define partition such that

$$\int_{s_{k-1}}^{s_k} |\bar{a}(s)| \, ds = c\varepsilon |\log \sigma| \quad \Rightarrow \quad P_k \leqslant \frac{2}{3}$$

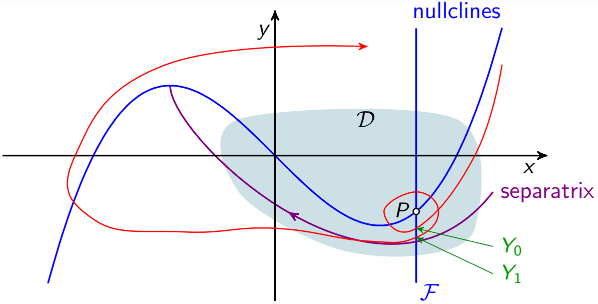
Proof uses comparison with linearised equations

Thm [B & Gentz, Ann App Proba 2002]

Transition probability $\geqslant 1 - e^{-\kappa\sigma^{4/3}/(\varepsilon|\log\sigma|)}$



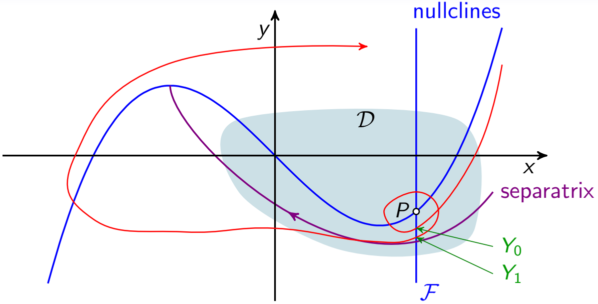
Back to Example 2: FitzHugh–Nagumo



$Y_0, Y_1, \dots$ substochastic Markov chain describing process killed on $\partial\mathcal{D}$

Number of small oscillations N = survival time of Markov chain

Back to Example 2: FitzHugh–Nagumo



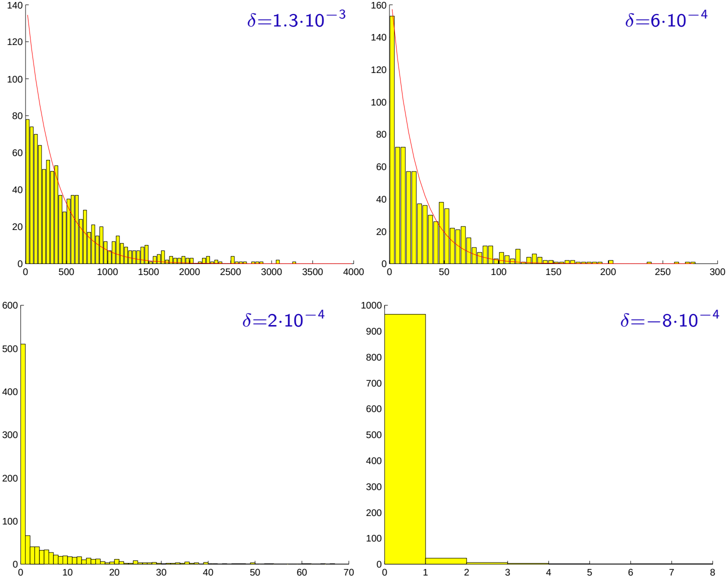
$Y_0, Y_1, \dots$ substochastic Markov chain describing process killed on $\partial\mathcal{D}$
Number of small oscillations N = survival time of Markov chain

Theorem [B & Landon, Nonlinearity 2012]

N is asymptotically geometric: $\lim_{n\to\infty} \mathbb{P}\{N = n+1 | N > n\} = 1 - \lambda_0$
where $\lambda_0 \in \mathbb{R}_+$: principal eigenvalue of the chain, $\lambda_0 < 1$ if $\sigma > 0$

Histograms of distribution of N (1000 spikes)

$\sigma = \varepsilon = 10^{-4}$



Example 2: Below threshold

Theorem [B & Landon , Nonlinearity 2012]

Assume ε and $\delta/\sqrt{\varepsilon}$ sufficiently small
There exists $\kappa > 0$ s.t. for $\sigma^2 \leqslant (\varepsilon^{1/4}\delta)^2/\log(\sqrt{\varepsilon}/\delta)$

- Principal eigenvalue:

$$1 - \lambda_0 \leqslant \exp\left\{-\kappa\frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$$

- Expected number of small oscillations:

$$\mathbb{E}^{\mu_0}[N] \geqslant C(\mu_0)\exp\left\{\kappa\frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$$

  where $C(\mu_0)$ = probability of starting on $\mathcal{F}$ above separatrix

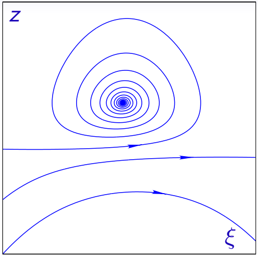
Proof:

- Construct $A \subset \mathcal{F}$ such that $K(x, A)$ exponentially close to 1 for all $x \in A$
- Use two different sets of coordinates to approximate K:
  Near separatrix, and during small oscillation

Example 2: Dynamics near the separatrix

Change of variables:

- Translate to Hopf bif. point
- Scale space and time
- Straighten nullcline $\dot{x} = 0$



$\Rightarrow$ variables (ξ, z) where nullcline: $\{z = \frac{1}{2}\}$

$$\mathrm{d}\xi_t = \left(\frac{1}{2} - z_t - \frac{\sqrt{\varepsilon}}{3}\xi_t^3\right)\mathrm{d}t$$

$$\mathrm{d}z_t = \left(\tilde{\mu} + 2\xi_t z_t + \frac{2\sqrt{\varepsilon}}{3}\xi_t^4\right)\mathrm{d}t$$

where

$$\tilde{\mu} = \frac{\delta}{\sqrt{\varepsilon}}$$

Example 2: Dynamics near the separatrix

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- ▹ Translate to Hopf bif. point
- ▹ Scale space and time
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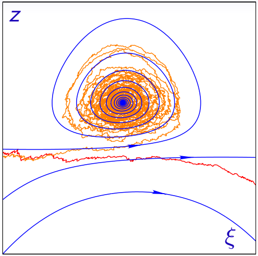

$$\mathrm{d}\xi_t = \left(\frac{1}{2} - z_t - \frac{\sqrt{\varepsilon}}{3}\xi_t^3\right)\mathrm{d}t + \tilde{\sigma}_1\,\mathrm{d}W_t^{(1)}$$

$$\mathrm{d}z_t = \left(\tilde{\mu} + 2\xi_t z_t + \frac{2\sqrt{\varepsilon}}{3}\xi_t^4\right)\mathrm{d}t - 2\tilde{\sigma}_1\xi_t\,\mathrm{d}W_t^{(1)} + \tilde{\sigma}_2\,\mathrm{d}W_t^{(2)}$$

where

$$\tilde{\mu} = \frac{\delta}{\sqrt{\varepsilon}} - \tilde{\sigma}_1^2 \qquad \tilde{\sigma}_1 = -\sqrt{3}\frac{\sigma_1}{\varepsilon^{3/4}} \qquad \tilde{\sigma}_2 = \sqrt{3}\frac{\sigma_2}{\varepsilon^{3/4}}$$

Upward drift dominates if $\tilde{\mu}^2 \gg \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 \Rightarrow (\varepsilon^{1/4}\delta)^2 \gg \sigma_1^2 + \sigma_2^2$

Rotation around P: use that $2z\,\mathrm{e}^{-2z-2\xi^2+1}$ is constant for $\tilde{\mu} = \varepsilon = 0$

Example 2: From below to above threshold

Linear approximation:

$$\mathrm{d}z_t^0 = \left(\tilde{\mu} + t z_t^0\right) \mathrm{d}t - \tilde{\sigma}_1 t \, \mathrm{d}W_t^{(1)} + \tilde{\sigma}_2 \, \mathrm{d}W_t^{(2)}$$

$$\Rightarrow \quad \mathbb{P}\{\text{no small osc}\} \simeq \Phi\left(-\pi^{1/4} \frac{\tilde{\mu}}{\sqrt{\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2}}\right) \qquad \Phi(x) = \int_{-\infty}^{x} \frac{\mathrm{e}^{-y^2/2}}{\sqrt{2\pi}} \, \mathrm{d}y$$

Example 2: From below to above threshold

Linear approximation:

$$\mathrm{d}z^0_t = \left(\tilde{\mu} + t z^0_t\right)\mathrm{d}t - \tilde{\sigma}_1 t \,\mathrm{d}W^{(1)}_t + \tilde{\sigma}_2 \,\mathrm{d}W^{(2)}_t$$

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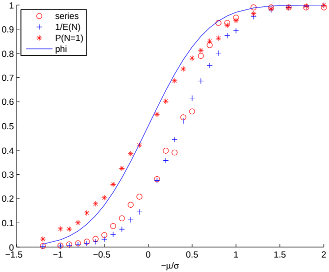
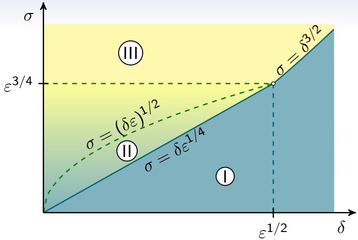


$*$: $\mathbb{P}\{\text{no small osc}\}$
$+$: $1/\mathbb{E}[N]$
$\circ$: $1 - \lambda_0$
curve: $x \mapsto \Phi(\pi^{1/4}x)$

$$x = -\frac{\tilde{\mu}}{\sqrt{\tilde{\sigma}_1^2+\tilde{\sigma}_2^2}} = -\frac{\varepsilon^{1/4}(\delta-\sigma_1^2/\varepsilon)}{\sqrt{\sigma_1^2+\sigma_2^2}}$$

Example 2: Summary of results



$\sigma_1 = \sigma_2$:

$\mathbb{P}\{N = 1\} \simeq \Phi\Big(-\frac{(\pi\varepsilon)^{1/4}(\delta-\sigma^2/\varepsilon)}{\sigma}\Big)$

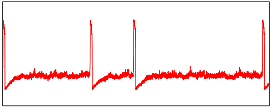
see also

[Muratov & Vanden Eijnden '08]

Regime I: rare isolated spikes

Theorem: If $\delta \ll \varepsilon^{1/2}$

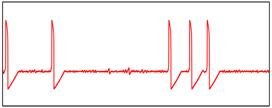
$\mathbb{P}\{\text{escape}\}^{-1} \simeq \mathbb{E}[\#\text{ small oscil}] \simeq e^{\kappa(\varepsilon^{1/4}\delta)^2/\sigma^2}$



Regime II: clusters of spikes

small oscillations: asympt geometric

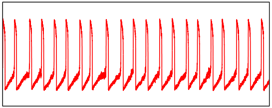
$\sigma = (\delta\varepsilon)^{1/2}$: Geom(1/2)



Regime III: repeated spikes

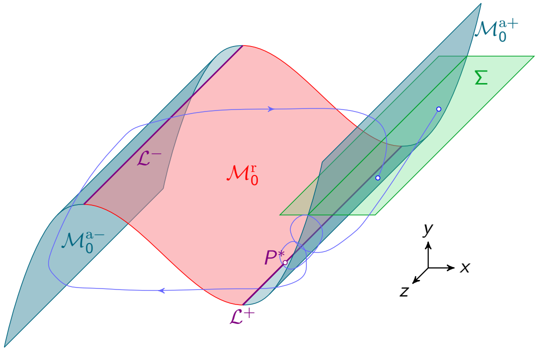
$\mathbb{P}\{N = 1\} \simeq 1$

Interspike interval $\simeq$ constant



Back to Example 3: The Koper model

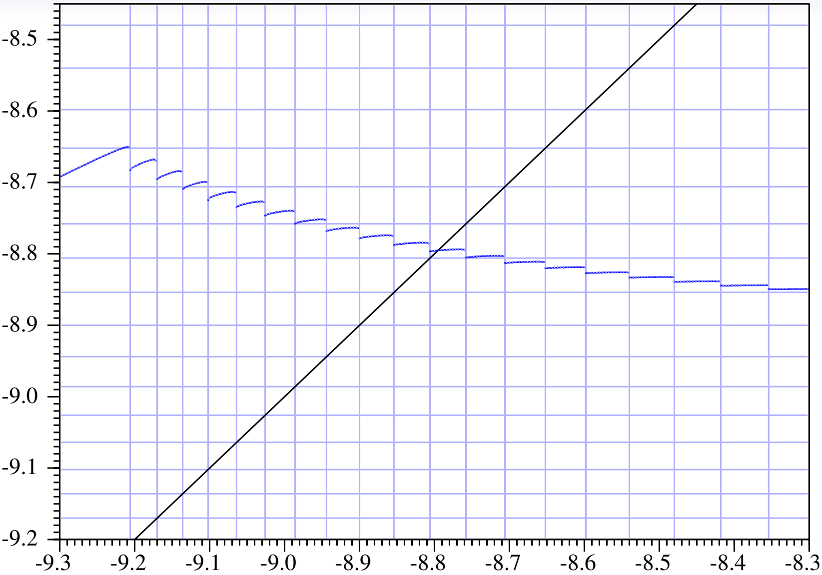
$$
\begin{aligned}
\varepsilon\,\mathrm{d}x_t &= [y_t - x_t^3 + 3x_t]\,\mathrm{d}t &&+ \sqrt{\varepsilon}\sigma F(x_t,y_t,z_t)\,\mathrm{d}W_t\\
\mathrm{d}y_t &= [kx_t - 2(y_t+\lambda) + z_t]\,\mathrm{d}t &&+ \sigma' G_1(x_t,y_t,z_t)\,\mathrm{d}W_t\\
\mathrm{d}z_t &= [\rho(\lambda + y_t - z_t)]\,\mathrm{d}t &&+ \sigma' G_2(x_t,y_t,z_t)\,\mathrm{d}W_t
\end{aligned}
$$



Folded-node singularity at P^* induces mixed-mode oscillations
[Benoît, Lobry '82, Szmolyan, Wechselberger '01, ...]

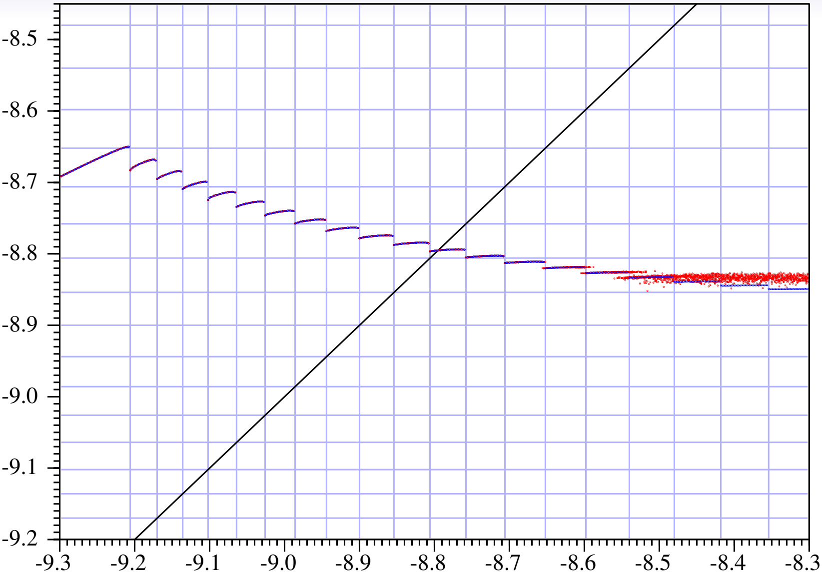
Poincaré map $\Pi : \Sigma \to \Sigma$ is almost $1d$ due to contraction in x-direction

Poincaré map $z_n \mapsto z_{n+1}$



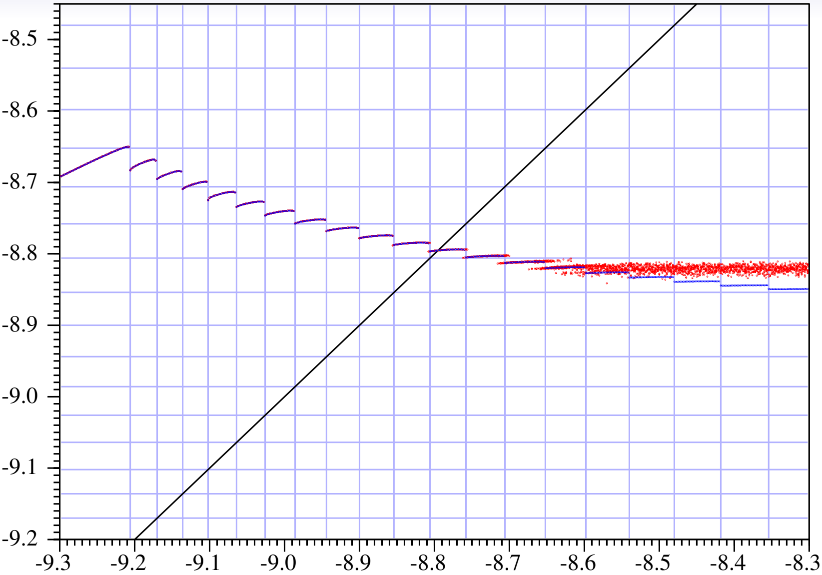
$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01, \sigma = \sigma' = 0$ – c.f. [Guckenheimer, Chaos, 2008]

Poincaré map $z_n \mapsto z_{n+1}$



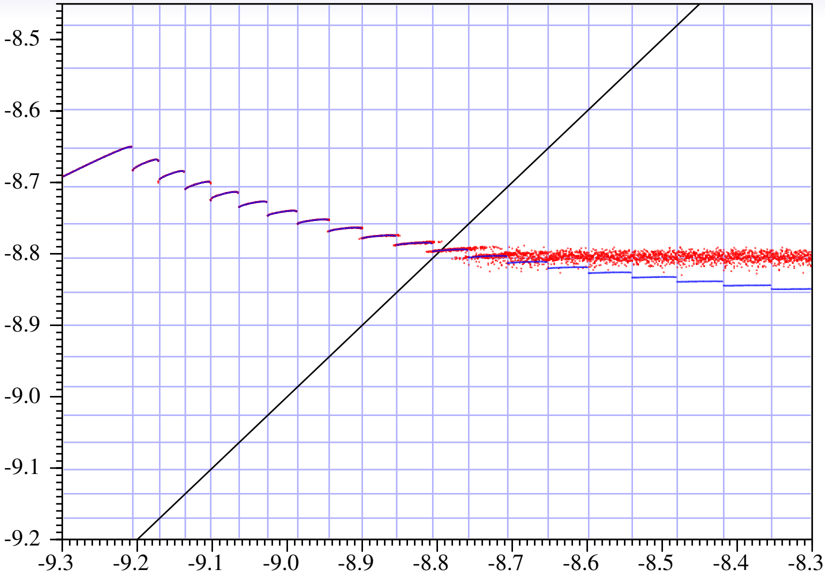
$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01,$ $\sigma = \sigma' = 2 \cdot 10^{-7}$

Poincaré map $z_n \mapsto z_{n+1}$



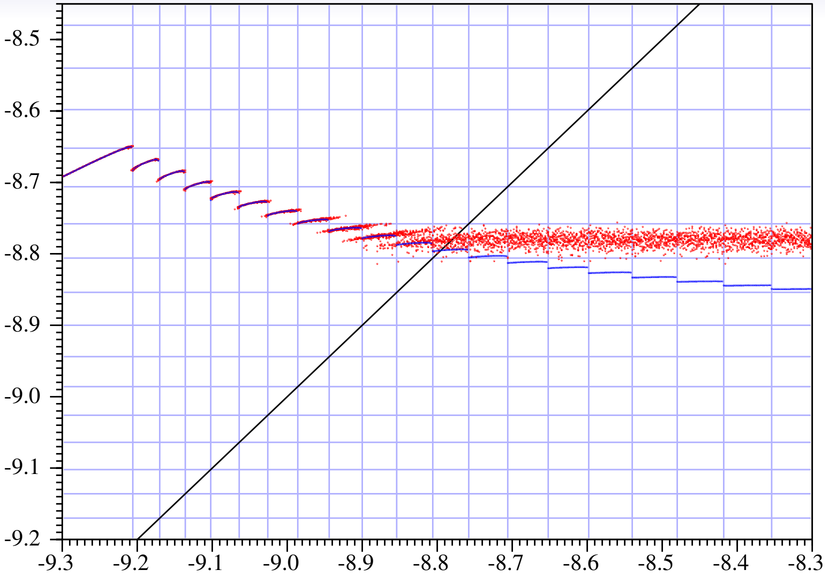
$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01,$ $\sigma = \sigma' = 2 \cdot 10^{-6}$

Poincaré map $z_n \mapsto z_{n+1}$



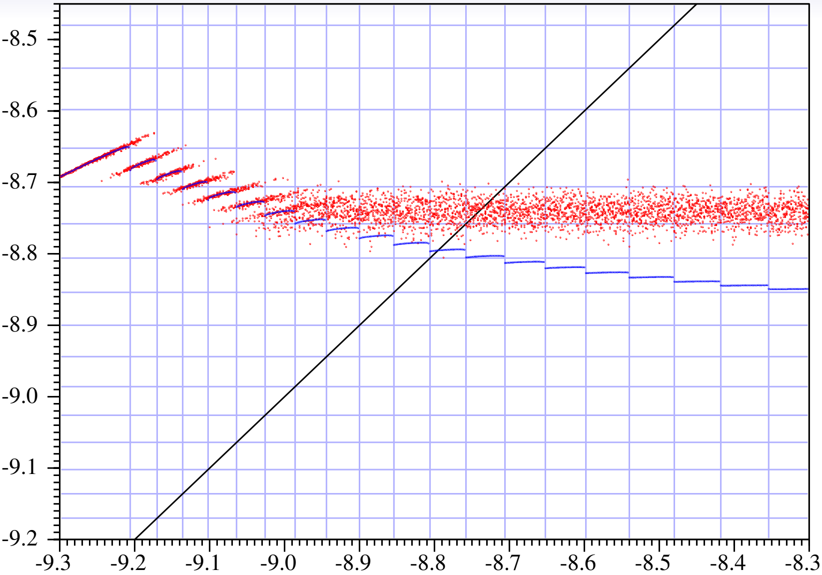
$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01,$ $\sigma = \sigma' = 2 \cdot 10^{-5}$

Poincaré map $z_n \mapsto z_{n+1}$



$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01,$ $\sigma = \sigma' = 2 \cdot 10^{-4}$

Poincaré map $z_n \mapsto z_{n+1}$



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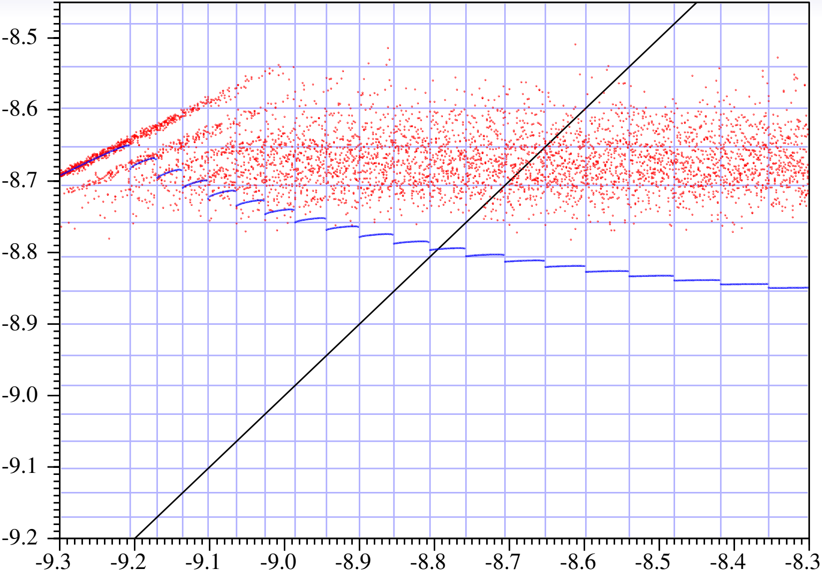
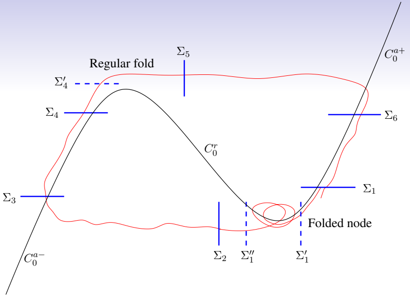
Poincaré map $z_n \mapsto z_{n+1}$



$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01, \sigma = \sigma' = 10^{-2}$

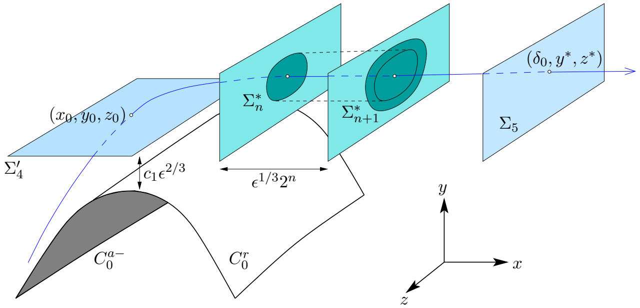
Size of fluctuations

$\mu \ll 1$: eigenvalue ratio at folded node



Transition	Δx	Δy	Δz
$\Sigma_2 \to \Sigma_3$	$\sigma + \sigma'$		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_3 \to \Sigma_4$	$\sigma + \sigma'$		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_4 \to \Sigma_4'$	$\dfrac{\sigma}{\varepsilon^{1/6}} + \dfrac{\sigma'}{\varepsilon^{1/3}}$		$\sigma\sqrt{\varepsilon\lvert\log\varepsilon\rvert} + \sigma'$
$\Sigma_4' \to \Sigma_5$		$\sigma\sqrt{\varepsilon} + \sigma'\varepsilon^{1/6}$	$\sigma\sqrt{\varepsilon} + \sigma'\varepsilon^{1/6}$
$\Sigma_5 \to \Sigma_6$	$\sigma + \sigma'$		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_6 \to \Sigma_1$	$\sigma + \sigma'$		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_1 \to \Sigma_1'$		$(\sigma + \sigma')\varepsilon^{1/4}$	σ'
$\Sigma_1' \to \Sigma_1''$ if $z = \mathcal{O}(\sqrt{\mu})$		$(\sigma + \sigma')(\varepsilon/\mu)^{1/4}$	$\sigma'(\varepsilon/\mu)^{1/4}$
$\Sigma_1'' \to \Sigma_2$		$(\sigma + \sigma')\varepsilon^{1/4}$	$\sigma'\varepsilon^{1/4}$

Example: Analysis near the regular fold



Proposition: For $h_1 = \mathcal{O}(\varepsilon^{2/3})$,

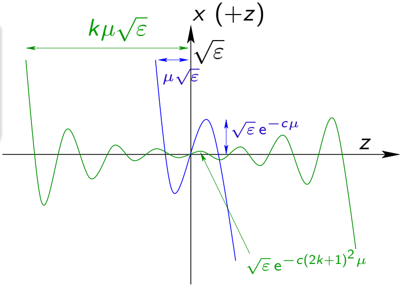
$$\mathbb{P}\left\{ \|(y_{\tau_{\Sigma_5}}, z_{\tau_{\Sigma_5}}) - (y^*, z^*)\| > h_1 \right\} \leqslant C|\log\varepsilon| \left(\exp\left\{ -\frac{\kappa h_1^2}{\sigma^2\varepsilon + (\sigma')^2\varepsilon^{1/3}} \right\} + \exp\left\{ -\frac{\kappa\varepsilon}{\sigma^2 + (\sigma')^2\varepsilon} \right\} \right)$$

Main results

[B, Gentz, Kuehn, JDE 2012 & preprint arXiv:1312.6353]

Theorem 1: canard spacing

At $z = 0$, k^{th} canard lies at distance $\sqrt{\varepsilon}\,\mathrm{e}^{-c(2k+1)^2\mu}$ from primary canard



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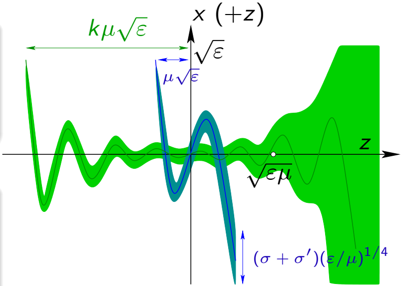
Theorem 1: canard spacing

At $z = 0$, k^{th} canard lies at distance $\sqrt{\varepsilon}\, \mathrm{e}^{-c(2k+1)^2\mu}$ from primary canard

Theorem 2: size of fluctuations

$(\sigma + \sigma')(\varepsilon/\mu)^{1/4}$ up to $z = \sqrt{\varepsilon\mu}$

$(\sigma + \sigma')(\varepsilon/\mu)^{1/4}\, \mathrm{e}^{z^2/(\varepsilon\mu)}$ for $z \geqslant \sqrt{\varepsilon\mu}$



Theorem 3: early escape

$P_0 \in \Sigma_1$ in sector with $k > 1/\sqrt{\mu} \Rightarrow$ first hitting of Σ_2 at P_2 s.t.

$\mathbb{P}^{P_0}\{z_2 \geqslant z\} \leqslant C|\log(\sigma + \sigma')|^{\gamma}\, \mathrm{e}^{-\kappa z^2/(\varepsilon\mu|\log(\sigma+\sigma')|)}$

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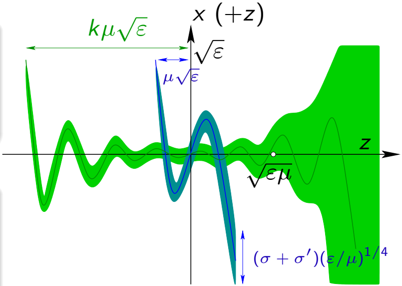

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$(\sigma+\sigma')(\varepsilon/\mu)^{1/4}$ up to $z=\sqrt{\varepsilon\mu}$

$(\sigma+\sigma')(\varepsilon/\mu)^{1/4}\,\mathrm{e}^{z^2/(\varepsilon\mu)}$ for $z\geqslant\sqrt{\varepsilon\mu}$

Theorem 3: early escape

$P_0\in\Sigma_1$ in sector with $k>1/\sqrt{\mu}$ $\Rightarrow$ first hitting of Σ_2 at P_2 s.t.

$\mathbb{P}^{P_0}\{z_2\geqslant z\}\leqslant C|\log(\sigma+\sigma')|^\gamma\,\mathrm{e}^{-\kappa z^2/(\varepsilon\mu|\log(\sigma+\sigma')|)}$

- Saturation effect occurs at $k_c\simeq\sqrt{|\log(\sigma+\sigma')|/\mu}$
- For $k>k_c$, behaviour indep. of k and $\Delta z\leqslant\mathcal{O}(\sqrt{\varepsilon\mu|\log(\sigma+\sigma')|}\,)$

Summary/Outlook

Noise can cause threshold phenomena

- Below threshold small perturbation of deterministic dynamics
- Above threshold large transitions can occur

Well understood:

- Normally hyperbolic case
- Codimension-1 bifurcations (fold, (avoided) transcritical, pitchfork, Hopf)
- Higher codimension: case studies (folded node, cf. Kuehn)

In progress: theory of random Poincaré maps

Essentially still open:

- Other types of noise (except Ornstein–Uhlenbeck)
- Equations with delay
- Infinite dimensions, in particular with continuous spectrum

Further reading

N. B. and Barbara Gentz, *Pathwise description of dynamic pitchfork bifurcations with additive noise*, Probab. Theory Related Fields **122**, 341–388 (2002)

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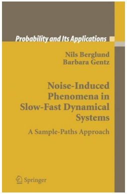
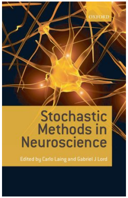
———, *Geometric singular perturbation theory for stochastic differential equations*, J. Differential Equations **191**, 1–54 (2003)

———, *Noise-induced phenomena in slow-fast dynamical systems, A sample-paths approach*, Springer, Probability and its Applications (2006)

———, *Stochastic dynamic bifurcations and excitability*, in C. Laing and G. Lord, (Eds.), *Stochastic methods in Neuroscience*, p. 65-93, Oxford University Press (2009)





N. B. and Damien Landon, *Mixed-mode oscillations and interspike interval statistics in the stochastic FitzHugh-Nagumo model*, Nonlinearity **25**, 2303–2335 (2012)

N. B., Barbara Gentz and Christian Kuehn, *Hunting French Ducks in a Noisy Environment*, J. Differential Equations **252**, 4786–4841 (2012)

———, *From random Poincaré maps to stochastic mixed-mode-oscillation patterns*, preprint `arXiv:1312.6353` (2013)