New trends and applications around generalized Fokker-Planck operators

Centre Bernoulli Lausanne

An Eyring-Kramers law for slowly oscillating bistable diffusions

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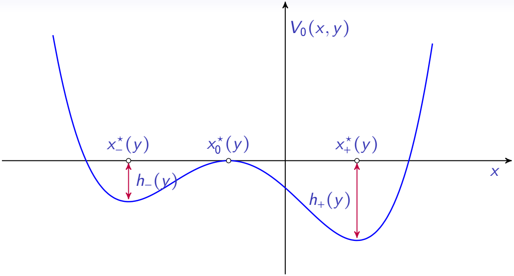
7 July 2025

partly based on joint work with Barbara Gentz (Bielefeld)

project PERISTOCH

The problem



$$\mathrm{d}x_t = -\partial_x V_0(x_t, y_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^x$$
$$\mathrm{d}y_t = \varepsilon\,\mathrm{d}t + \sigma\sqrt{\varepsilon}\varrho\,\mathrm{d}W_t^y$$

- $x \mapsto V_0(x,y)$ confining double-well potential, class $\mathcal{C}^4$
  $V_0(x,y+1) = V_0(x,y)$
- $0 \leqslant \varepsilon, \sigma \ll 1$, $\varrho > 0$
- W_t^x, W_t^y independent standard Wiener processes

Question: describe law of $\tau_+ = \inf\bigl\{t > 0 \colon x_t = x_+^*(y_t) \bigm| (x_0 = x_-^*(y_0), y_0)\bigr\}$

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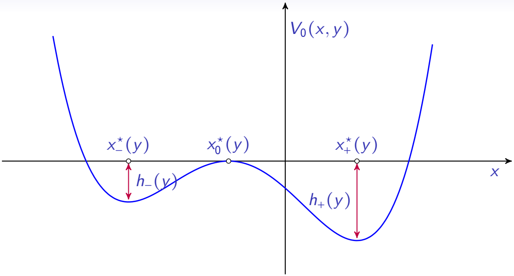

$$\begin{aligned} dx_t &= -\partial_x V_0(x_t, y_t)\,dt + \sigma\,dW_t^x \\ dy_t &= \varepsilon\,dt + \sigma\sqrt{\varepsilon}\varrho\,dW_t^y \end{aligned}$$

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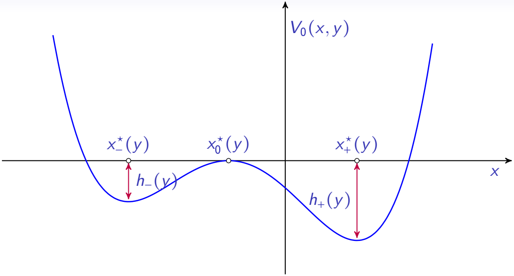

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Motivation 1: Synchronisation

See e.g. [Pikovsky, Rosenblum, Kurths 2001]

$x_i = (\theta_i, \dot\theta_i),\ i = 1, 2$

$$\begin{cases} \dot x_1 = f_1(x_1) + \varepsilon g_1(x_1, x_2) \\ \dot x_2 = f_2(x_2) + \varepsilon g_2(x_1, x_2) \end{cases}$$

ϕ_i : good parametrisation of limit cycles

$$\begin{cases} \dot\phi_1 = \omega_1 + \varepsilon Q_1(\phi_1, \phi_2) \\ \dot\phi_2 = \omega_2 + \varepsilon Q_2(\phi_1, \phi_2) \end{cases}$$

If $\omega_1 \simeq \omega_2$: $\begin{cases} \psi = \phi_1 - \phi_2 \\ \varphi = \frac{\phi_1+\phi_2}{2} \end{cases} \Rightarrow \begin{cases} \dot\psi = -\nu + \varepsilon q(\psi, \varphi) \\ \dot\varphi = \omega + \mathcal{O}(\varepsilon) \end{cases} \qquad \begin{matrix} \nu = \omega_2 - \omega_1 \\ \omega = \frac{\omega_1+\omega_2}{2} \end{matrix}$

For small detuning ν: averaging $\Rightarrow \quad \omega \dfrac{\mathrm{d}\psi}{\mathrm{d}\varphi} \simeq -\nu + \varepsilon \bar q(\psi)$

Example: Adler's equation $\bar q(\psi) = \sin(\psi)$: Fixed points for $\sin(\psi) = \nu/\varepsilon$

Remark: if $\omega_2/\omega_1 \simeq m/n$ similar behaviour for $\psi = n\phi_1 - m\phi_2$ (Arnold tongues)

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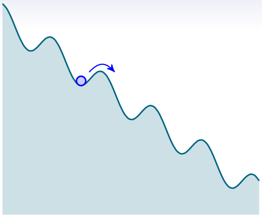
Averaged equation with noise

$$\omega \frac{\mathrm{d}\psi}{\mathrm{d}\varphi} = \underbrace{-\nu + \varepsilon \bar{q}(\psi)}_{-\frac{\partial}{\partial \psi}\left(\nu\psi - \varepsilon \int^{\psi} \bar{q}(x)\,\mathrm{d}x\right)} + \text{noise}$$



Original equations with noise

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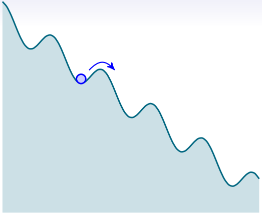

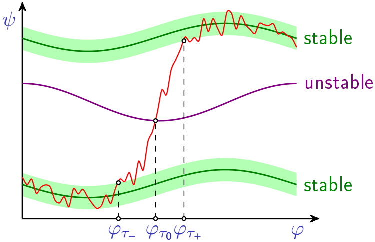

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Motivation 2: Stochastic resonance

$$\mathrm{d}x_t = \underbrace{\left[-x_t^3 + x_t + A\cos(\varepsilon t)\right]}_{=-\frac{\partial}{\partial x}\left[\frac{1}{4}x^4-\frac{1}{2}x^2-Ax\cos(\varepsilon t)\right]\Big|_{x_t}} \mathrm{d}t + \sigma\,\mathrm{d}W_t$$

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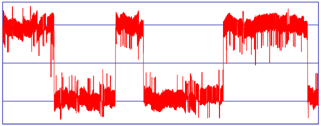
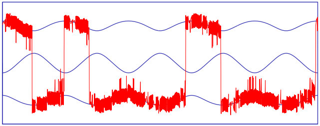
- Ice Ages: deterministically bistable climate [Croll, Milankovitch]
- random perturbations due to weather [Benzi-Sutera-Vulpiani, Nicolis-Nicolis]

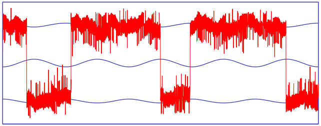
Sample paths $\{x_t\}_t$ for $\varepsilon = 0.001$:



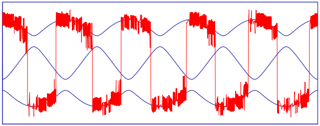
$A = 0, \ \sigma = 0.3$



$A = 0.24, \ \sigma = 0.2$



$A = 0.1, \ \sigma = 0.27$



$A = 0.35, \ \sigma = 0.2$

From SDEs to PDEs: Dynkin and Feynman–Kac

$\{z_t\}_t$ diffusion with infinitesimal generator $\mathscr{L}$ (e.g. $\mathscr{L} = \frac{\sigma^2}{2}\Delta + f \cdot \nabla$)

- Dynkin's formula: for τ stopping time, φ test function,

$$\mathbb{E}^z[\varphi(z_\tau)] = \varphi(z) + \mathbb{E}^z\Big[\int_0^\tau (\mathscr{L}\varphi)(z_s)\,\mathrm{d}s\Big]$$

- Feynman–Kac formula: For q, φ regular functions,

$$u(t,z) = \mathbb{E}^z\Big[\exp\Big(-\int_0^t q(z_s)\,\mathrm{d}s\Big)\varphi(z_t)\Big]$$

satisfies

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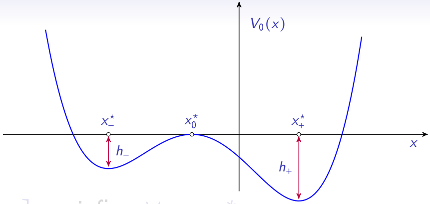
Static case

$$\mathrm{d}x_t = -V_0'(x_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

$$\omega_\pm = \sqrt{V_0''(x_\pm^*)} \qquad \omega_0 = \sqrt{-V_0''(x_0^*)}$$



▷ By Dynkin's equation, $u(x) = \mathbb{E}^x[\tau_+]$ satisfies $\forall x < x_+^*$

$$\begin{cases} \mathscr{L}u = \frac{\sigma^2}{2}u'' + V_0'u' = -1 & x < x_+^* \\ u(x_+^*) = u(-\infty) = 0 \end{cases}$$

$$\Rightarrow \qquad \mathbb{E}^x[\tau_+] = \frac{2}{\sigma^2}\int_x^{x_+^*}\int_{-\infty}^{x_2} \mathrm{e}^{2[V_0(x_2)-V_0(x_1)]/\sigma^2}\,\mathrm{d}x_1\,\mathrm{d}x_2$$

$\Rightarrow$ Eyring–Kramers law: $\mathbb{E}^{x_-^*}[\tau_+] = \dfrac{2\pi}{\omega_0\omega_-}\mathrm{e}^{2h_-/\sigma^2}\,[1+\mathcal{O}(\sigma^2)]$

▷ [Day '83]: $\forall s \geqslant 0$, $\lim_{\sigma\to 0} \mathbb{P}^{x_-^*}\left\{\tau_+ > s\,\mathbb{E}^{x_-^*}[\tau_+]\right\} = \mathrm{e}^{-s}$

(Convergence to exponential law $\mathscr{E}(1)$)

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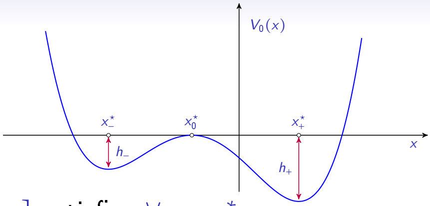

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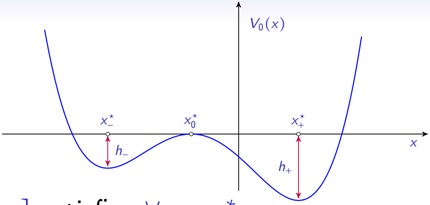

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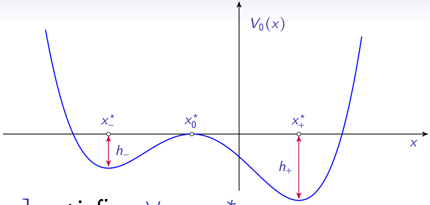

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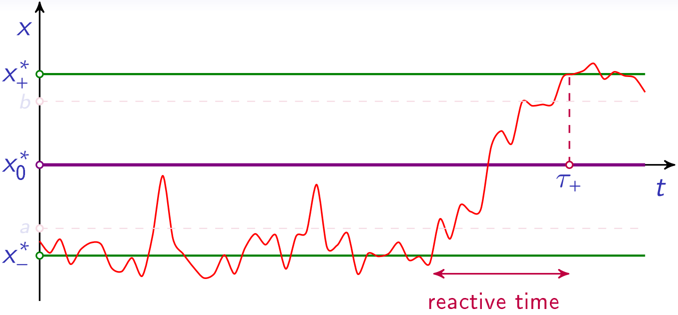
Static case: reactive time



▷ [Cérou, Guyader, Lelièvre, Malrieu '13]: $x_-^* < a < x_0 < x_0^* < b < x_+^*$

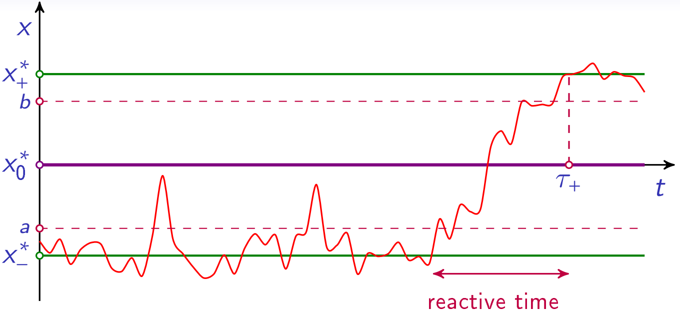
$$\lim_{\sigma\to 0} \mathrm{Law}\Big(\omega_0^2\tau_b - 2\log(\sigma^{-1}) \,\Big|\, \tau_b < \tau_a\Big) = \mathrm{Law}\Big(\underbrace{\mathcal{G}}_{\text{Gumbel}} + \underbrace{T(x_0,b)}_{\text{deterministic}}\Big)$$

Gumbel law: $\mathbb{P}\{\mathcal{G} < t\} = \mathrm{e}^{-\mathrm{e}^{-t}} \quad \forall t \in \mathbb{R}$

(max-stable distribution from extreme value theory, cf. [Bakhtin '15])

$\Rightarrow$ reactive time $\simeq \omega_0^{-2}[2\log(\sigma^{-1}) + \mathcal{G} + T(x_0,b)]$

Static case: reactive time



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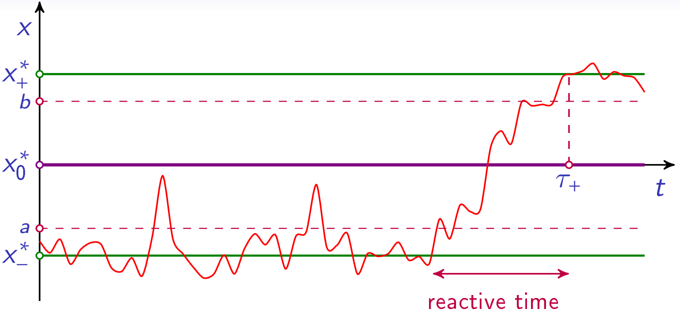

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Higher dimensions

- [Freidlin & Wentzell '79]: Large-deviation theory gives exponent of $\mathbb{E}[\tau_+]$
- Gradient case $\quad \mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$
  Invariant measure $\pi(x) = Z^{-1}\,\mathrm{e}^{-2V(x)/\sigma^2}\,\mathrm{d}x$
  Process reversible wrt π
  Eyring–Kramers law and asympt. exponential character of τ_+ known
  - [Bovier, Eckhoff, Gayrard & Klein 2004]: Potential theory
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Back to the problem

$$dx_t = -\frac{1}{\varepsilon}\partial_x V_0(x_t, y_t)\,dt + \frac{\sigma}{\sqrt{\varepsilon}}\,dW_t^x$$

$$dy_t = dt + \sigma\varrho\,dW_t^y$$

(time scaled by ε)

- Det. eq. $\varepsilon\dot{x} = -\partial_x V_0(x,t)$: $\exists!$ 3 periodic orbits $\bar{x}_i(t) = x_i^*(t) + \mathcal{O}(\varepsilon)$
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Theorem: [B & Gentz, SIAM J Math Analysis 2014]

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$p(\sigma) \simeq e^{-\mathcal{I}/\sigma^2}$, $\mathcal{I}$ Freidlin–Wentzell quasipotential, $\mathbb{E}[\tau_0] \simeq p(\sigma)^{-1}$

Back to the problem

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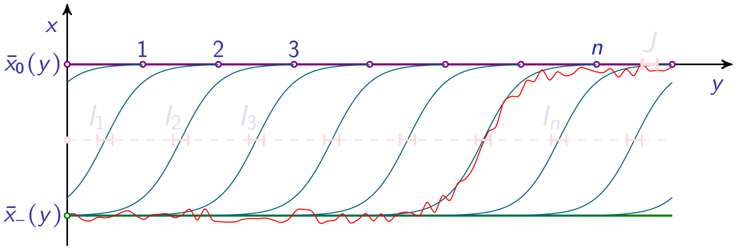

Instantons: minimize Freidlin–Wentzell large-deviation rate fct (∼ eikonal)

$$\frac{1}{2}\int_0^T \Big[\big(\dot{x}_t + \partial_x V_0(x_t,y_t)\big)^2 + \frac{1}{\varepsilon\varrho^2}(\dot{y}_t - \varepsilon)^2\Big]\,\mathrm{d}t \qquad T > 0 \text{ arbitrary}$$

$$\mathbb{P}\{y_{\tau_0}\in J\} \simeq \sum_k \underbrace{\mathbb{P}\{y_{\tau_-}\in I_k\}}_{\simeq\mathbb{P}\{Y^\sigma = k\}} \; \underbrace{\mathbb{P}^{I_k}\{y_{\tau_0}\in J\}}_{\simeq\mathbb{P}\{\frac{\mathcal{G}}{2}+const\,\in\, J-k\}}$$

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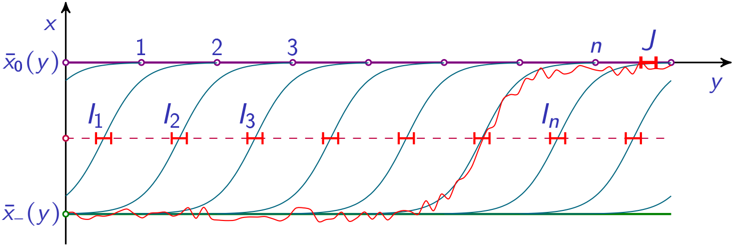

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$$\omega_\pm(y) = \sqrt{\partial_{xx} V_0(x^*_\pm(y), y)}$$

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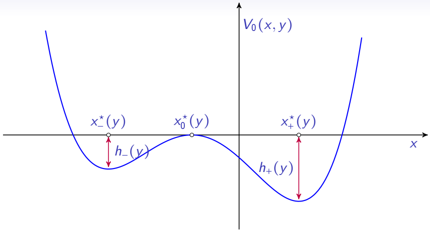
$$r_\pm(y) = \frac{\omega_\pm(y)\omega_0(y)}{2\pi}\,\mathrm{e}^{-2h_\pm(y)/\sigma^2}$$



▷ Leading eigenvalue of $-\mathscr{L}_x = -\frac{\sigma^2}{2}\partial_{xx} + \partial_x V_0 \partial_x$:

$$\lambda_1(y) = [r_+(y) + r_-(y)][1 + \mathcal{O}(\sigma^2)] \qquad \langle\lambda_1\rangle = \int_0^1 \lambda_1(y)\,\mathrm{d}y$$

Theorem: [B, Probability and Mathematical Physics 2-4:685-743 (2021)]

$$\mathbb{E}^{(x^*_-(y_0),y_0)}[\tau_+] = \frac{2\pi\varepsilon[1 + R(\varepsilon,\sigma)]}{\displaystyle\int_0^1 \omega_0(y)\omega_-(y)\,\mathrm{e}^{-2h_-(y)/\sigma^2}\,\mathrm{d}y}$$

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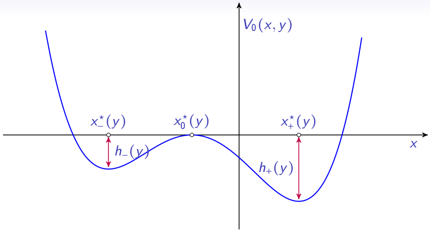

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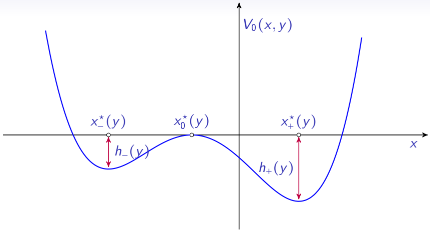

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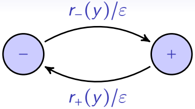
Heuristics: two-state jump process

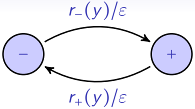


$$\frac{\mathrm{d}}{\mathrm{d}y}\mathbb{P}^{-,y_0}\{\tau_+ > y\} = -\frac{1}{\varepsilon}r_-(y)\mathbb{P}^{-,y_0}\{\tau_+ > y\}$$

$$\mathbb{P}^{-,y_0}\{\tau_+ > y\} = \mathrm{e}^{-R_-(y,y_0)/\varepsilon} \qquad \text{where } R_-(y_1,y_0) = \int_{y_0}^{y_1} r_-(y)\,\mathrm{d}y$$

$$\begin{aligned}
\mathbb{E}^{-,y_0}[\tau_+] &= \int_{y_0}^{\infty} \mathrm{e}^{-R_-(y,y_0)/\varepsilon}\,\mathrm{d}y \\
&= \frac{1}{1-\mathrm{e}^{-R_-(1,0)/\varepsilon}}\int_0^1 \mathrm{e}^{-R_-(y_0+y,y_0)/\varepsilon}\,\mathrm{d}y \qquad \text{(by periodicity of } r_-\text{)} \\
&\simeq \begin{cases}
\dfrac{\varepsilon}{R_-(1,0)} = \dfrac{2\pi\varepsilon}{\int_0^1 \omega_0(y)\omega_-(y)\,\mathrm{e}^{-2h_-(y)/\sigma^2}\,\mathrm{d}y} & \text{if } \varepsilon \gg \max\limits_{y\in[0,1]} r_-(y) \\
\dfrac{\varepsilon}{r_-(y_0)} & \text{if } \varepsilon \ll \min\limits_{y\in[0,1]} r_-(y) \\
\text{In between: Stochastic resonance}
\end{cases}
\end{aligned}$$

Heuristics: two-state jump process



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Heuristics: two-state jump process

$$\begin{tikzcd} - \arrow[r, bend left, "r_-(y)/\varepsilon"] & + \arrow[l, bend left, "r_+(y)/\varepsilon"] \end{tikzcd}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\mathbb{P}^{-,y_0}\left\{\tau_+ > y\right\} = -\frac{1}{\varepsilon} r_-(y)\mathbb{P}^{-,y_0}\left\{\tau_+ > y\right\}$$

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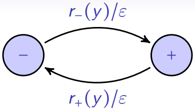
&\simeq \begin{cases} \dfrac{\varepsilon}{R_-(1,0)} = \dfrac{2\pi\varepsilon}{\int_0^1 \omega_0(y)\omega_-(y)\,\mathrm{e}^{-2h_-(y)/\sigma^2}\,\mathrm{d}y} & \text{if } \varepsilon \gg \max\limits_{y\in[0,1]} r_-(y) \\ \dfrac{\varepsilon}{r_-(y_0)} & \text{if } \varepsilon \ll \min\limits_{y\in[0,1]} r_-(y) \\ \text{In between: Stochastic resonance} & \end{cases}
\end{aligned}$$

Heuristics: two-state jump process



$$\frac{\mathrm{d}}{\mathrm{d}y}\mathbb{P}^{-,y_0}\big\{\tau_+ > y\big\} = -\frac{1}{\varepsilon}r_-(y)\mathbb{P}^{-,y_0}\big\{\tau_+ > y\big\}$$

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Potential theory for non-reversible SDEs

[Landim, Mariani & Seo 2019]:

- Generator $\mathscr{L} = \frac{1}{\varepsilon}\mathscr{L}_x + \mathscr{L}_y, \qquad \mathscr{L}_y = \frac{\varrho^2\sigma^2}{2}\partial_{yy} + \partial_y$
- Invariant measure $\mathrm{d}\pi = \mathrm{e}^{-2V(x,y)/\sigma^2}\,\mathrm{d}x\,\mathrm{d}y$, V sat. Hamilton–Jacobi eq.
- Decompose $\mathscr{L} = \mathscr{L}_{\mathrm{s}} + \mathscr{L}_{\mathrm{a}}$ where
  - $\mathscr{L}_{\mathrm{s}} = \frac{\sigma^2}{2\varepsilon}\mathrm{e}^{2V/\sigma^2}\,\nabla\cdot D\,\mathrm{e}^{-2V/\sigma^2}\,\nabla$, where $D = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon\varrho^2 \end{pmatrix}$, is self-adjoint wrt π
  - $\mathscr{L}_{\mathrm{a}} = c\cdot\nabla$ (c explicitly known) is skew-symmetric: $\mathscr{L}_{\mathrm{a}}^\dagger = -\mathscr{L}_{\mathrm{a}}$
- Adjoint system: generator $\mathscr{L}^* = \mathscr{L}_{\mathrm{s}} - \mathscr{L}_{\mathrm{a}}$

Theorem: [LMS 2019] For any $A, B \subset \mathbb{R}^2$, $A \cap B = \varnothing$

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Estimating the capacity

- For $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, define $\mathscr{D}(\varphi) = \frac{2\varepsilon}{\sigma^2} \int_{(A\cup B)^c} \varphi \cdot (D^{-1}\varphi) \frac{\mathrm{d}x\,\mathrm{d}y}{\pi(x,y)}$
- Given $f : \mathbb{R}^2 \to \mathbb{R}$, define vector fields $\Psi_f = \frac{\sigma^2}{2\varepsilon}\pi D \nabla f$, $\Phi_f = \Psi_f - \pi f c$
- $\mathscr{H}_{AB}^{\alpha,\beta}$: space of $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f|_A = \alpha, f|_B = \beta$
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Proposition: [LMS 2019] Dirichlet principle

$$\operatorname{cap}(A,B) = \inf_{f \in \mathscr{H}_{AB}^{1,0}} \inf_{\varphi \in \mathscr{F}_{AB}^{0}} \mathscr{D}(\Phi_f - \varphi)$$

Infimum reached for $f = \frac{1}{2}(h_{AB} + h_{AB}^*)$ and $\varphi = \Phi_f - \Psi_{h_{AB}}$

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Static eigenfunctions

- Eigenfunctions of $\mathscr{L}_x = \frac{\sigma^2}{2}\partial_{xx} - V_0'\partial_x = \frac{\sigma^2}{2}\,\mathrm{e}^{2V_0/\sigma^2}\,\partial_x(\mathrm{e}^{-2V_0/\sigma^2}\,\partial_x)$:
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- $\lambda_0 = 0$, $\phi_0(x|y) = 1$, $\pi_0(x|y) = \frac{1}{Z_0(y)}\,\mathrm{e}^{-2V_0(x,y)/\sigma^2}$
- Let $\tau = \tau_+ \wedge \tau_-$, $\tau_\pm = \inf\{t > 0 \colon x_t = x_\pm^*(y)\}$. By Feynman–Kac,

$$
\begin{aligned}
\phi_1(x,y) &= \mathbb{E}^x[\mathrm{e}^{\lambda_1(y)\tau}\,\phi_1(x_\tau)] \\
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- $\phi_\pm(y)$ can be determined via normalisation/orthogonality relations

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  $\mathscr{L}_x\phi_n(x|y) = -\lambda_n(y)\phi_n(x|y)$
  $\mathscr{L}_x^\dagger\pi_n(x|y) = -\lambda_n(y)\pi_n(x|y)$, where $\pi_n = \pi_0\phi_n$
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- Let $\tau = \tau_+ \wedge \tau_-$, $\tau_\pm = \inf\{t > 0 \colon x_t = x_\pm^*(y)\}$. By Feynman–Kac,

$$\begin{aligned}\phi_1(x,y) &= \mathbb{E}^x[\mathrm{e}^{\lambda_1(y)\tau}\,\phi_1(x_\tau)]\\ &= \phi_-(y)\mathbb{E}^x[\mathrm{e}^{\lambda_1(y)\tau}\,1_{\tau_-<\tau_+}] + \phi_+(y)\mathbb{E}^x[\mathrm{e}^{\lambda_1(y)\tau}\,1_{\tau_+<\tau_-}]\\ &= \phi_-(y)[h_0(x|y) + h_1(x|y)] + \phi_+(y)[1 - h_0(x|y) + \bar h_1(x|y)]\end{aligned}$$

where $\phi_\pm(y) = \phi_1(x_\pm^*(y),y)$, $h_1, \bar h_1$ are small remainders, and

$$h_0(x|y) = \mathbb{P}^x\{\tau_- < \tau_+\} = \frac{1}{N(y)}\int_x^{x_+^*(y)} \mathrm{e}^{2V_0(\bar x,y)/\sigma^2}\,\mathrm{d}\bar x$$

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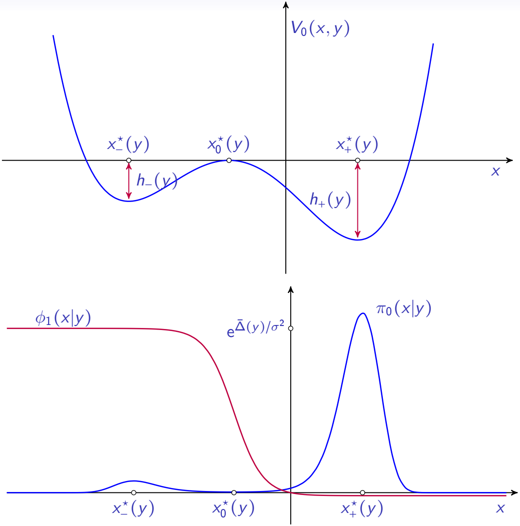

$$\bar\Delta(y) = h_+(y) - h_-(y) + \frac{\sigma^2}{2}\log\Bigl(\frac{\omega_-(y)}{\omega_+(y)}\Bigr)$$

Estimating $\pi(x,y)$

Static eigenfunctions: $\mathscr{L}_x\phi_n(x|y) = -\lambda_n(y)\phi_n(x|y)$, $\phi_0(x|y) = 1$

Proposition:

$$\pi(x,y) = \frac{e^{-2V_0(x,y)/\sigma^2}}{Z_0(y)}\Big[1 + \alpha_1(y)\phi_1(x|y) + \Phi_\perp(x,y)\Big]$$

- $\alpha_1(y)$ well-approximated in terms of jump process
- $\Phi_\perp(x,y) \perp \operatorname{span}\{\phi_0,\phi_1\}$, satisfies $\langle\pi_0,\Phi_\perp\rangle^{1/2} \lesssim \frac{\varepsilon}{\sigma^2}\cosh\Big(\frac{h_+(y)-h_-(y)}{\sigma^2}\Big)$

Case $\varrho = 0$: $\pi = \frac{e^{-2V_0/\sigma^2}}{Z_0}\big[1 + \sum_{n\geqslant 1}\alpha_n\phi_n\big]$. $\mathscr{L}^\dagger\pi = 0 \Leftrightarrow \alpha_n$ satisfy ODE

$$\varepsilon\alpha_n' = -\lambda_n(y)\alpha_n - \frac{\varepsilon}{\sigma^2}f_{n0}(y) - \frac{\varepsilon}{\sigma^2}\sum_{m\geqslant 1}f_{nm}(y)\alpha_m$$

with $f_{nm}(y) = -\sigma^2\langle\pi_0\phi_m,\partial_y\phi_n\rangle$ (Case $\varrho > 0$: 2nd order ODE)

- if $\varepsilon \gg \langle\lambda_1\rangle$, then (an affine function of) α_1 is slow variable
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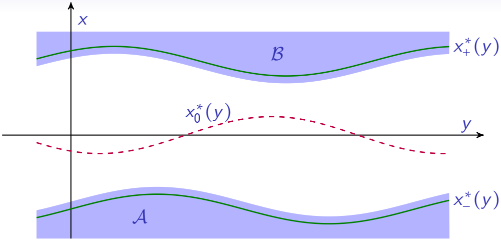
Last steps of the proof



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- Capacity: using static committors in Dirichlet and Thomson principles

$$\operatorname{cap}(\mathcal{A},\mathcal{B}) \simeq \frac{1}{4\varepsilon}\langle \lambda_1 [1 - A\delta_1]\rangle \qquad \delta_1 \simeq \frac{\langle \lambda_1 A\rangle}{\langle \lambda_1\rangle} \qquad A(y) = \frac{r_-(y) - r_+(y)}{r_-(y) + r_+(y)}$$

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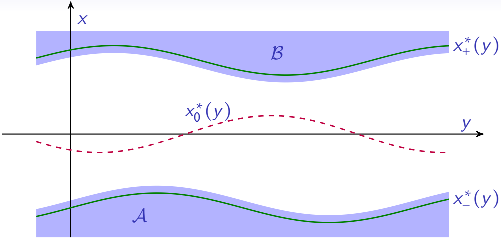

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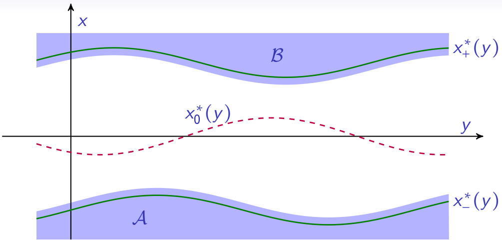

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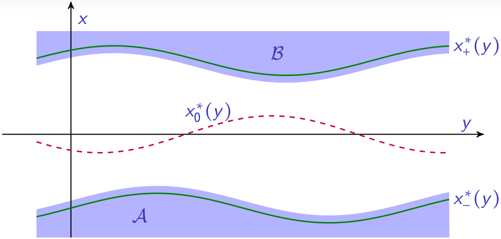

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Open questions

- ▷ Larger values of ε?
- ▷ Higher dimensions? Link with random Poincaré maps?

References

- ▷ N. B. & Barbara Gentz, *On the noise-induced passage through an unstable periodic orbit II*, SIAM J. Math. Anal., **46**(1):310–352, 2014
- ▷ N. B., *Noise-induced phase slips, log-periodic oscillations, and the Gumbel distribution*, Markov Process. Related Fields, **22**(3):467–505, 2016
- ▷ Manon Baudel & N. B., *Spectral theory for random Poincaré maps*, SIAM J. Math. Analysis **49**, 4319–4375, 2017
- ▷ C. Landim, M. Mariani, & I. Seo, *Dirichlet's and Thomson's principles for non-selfadjoint elliptic operators with application to non-reversible metastable diffusion processes*, Arch. Ration. Mech. Anal., **231**(2):887–938, 2019
- ▷ N. B., *An Eyring–Kramers law for slowly oscillating bistable diffusions*, Probability and Mathematical Physics 2-4:685-743 (2021)

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