Séminaire du LMAH, Université du Havre

Phénomènes induits par le bruit dans les systèmes dynamiques lents–rapides

Nils Berglund

MAPMO, Université d'Orléans

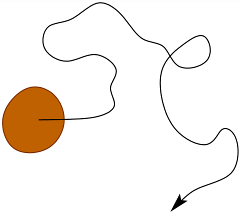
26 mars 2015

Avec Barbara Gentz (Bielefeld), Christian Kuehn (Vienne) et Damien Landon (Dijon)

What is noise?

Paradigm: Brownian motion

[R. Brown, 1827]



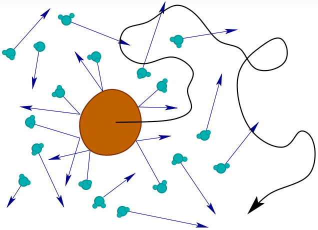
What is noise?

Paradigm: Brownian motion

[R. Brown, 1827]

[A. Einstein, 1905]: $\dfrac{\langle x^2 \rangle}{t} = \dfrac{k_{\mathrm{B}} T}{6\pi\eta r}$

[J. Perrin, 1909]:
"weighing the hydrogen atom"



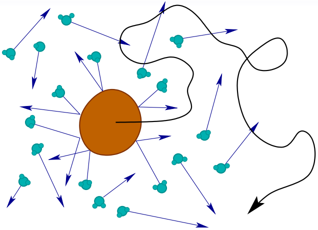
What is noise?

Paradigm: Brownian motion

[R. Brown, 1827]

[A. Einstein, 1905]: $\dfrac{\langle x^2 \rangle}{t} = \dfrac{k_B T}{6\pi\eta r}$

[J. Perrin, 1909]:
"weighing the hydrogen atom"



Wiener process $\{W_t\}_{t\geqslant 0}$: scaling limit of random walk $\lim_{n\to\infty} \dfrac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}$

Stochastic differential equation:

$$\mathrm{d}x_t = \underbrace{f(x_t)\,\mathrm{d}t}_{\text{exterior force}} + \underbrace{g(x_t)\,\mathrm{d}W_t}_{\text{random force}}$$

Physicist's notation: $\dot{x} = f(x) + g(x)\xi, \quad \langle \xi(s)\xi(t) \rangle = \delta(s-t)$

What is noise?

Stochastic differential equation (SDE):

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + g(x_t)\,\mathrm{d}W_t$$

Itô calculus:

define solution via $\displaystyle x_t = x_0 + \int_0^t f(x_s)\,\mathrm{d}s + \int_0^t g(x_s)\,\mathrm{d}W_s$

Euler scheme: $x_{t+\Delta t} \simeq x_t + f(x_t)\Delta t + g(x_t)\sqrt{\Delta t}\,\mathcal{N}(0,1)$

What is noise?

Stochastic differential equation (SDE):

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + g(x_t)\,\mathrm{d}W_t$$

Itô calculus:

define solution via $\quad x_t = x_0 + \int_0^t f(x_s)\,\mathrm{d}s + \int_0^t g(x_s)\,\mathrm{d}W_s$

Euler scheme: $\quad x_{t+\Delta t} \simeq x_t + f(x_t)\Delta t + g(x_t)\sqrt{\Delta t}\,\mathcal{N}(0,1)$

Rigorous derivations of effective SDEs from more fundamental models:

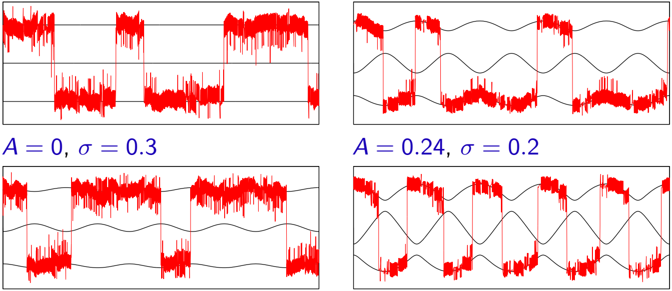
- System coupled to infinitely many harmonic oscillators
  [Ford, Kac, Mazur '65], [Lebowitz, Spohn '77],
  [Eckmann, Pillet, Rey-Bellet '99], [Rey-Bellet, Thomas '00, '02]
- Stochastic averaging for slow–fast systems
  [Khasminski '66], [Hasselmann '76], [Kifer '03]

Example 1: Stochastic resonance

$$\mathrm{d}x_s = \underbrace{\left[-x^3 + x + A\cos\varepsilon s\right]}_{=-\frac{\partial}{\partial x}\left[\frac{1}{4}x^4 - \frac{1}{2}x^2 - Ax\cos\varepsilon s\right]} \mathrm{d}s + \sigma\,\mathrm{d}W_s$$

- deterministically bistable climate [Croll, Milankovitch]
- random perturbations due to weather [Benzi/Sutera/Vulpiani, Nicolis/Nicolis]

Sample paths $\{x_s\}_s$ for $\varepsilon = 0.001$:



$A = 0$, $\sigma = 0.3$ — $A = 0.24$, $\sigma = 0.2$

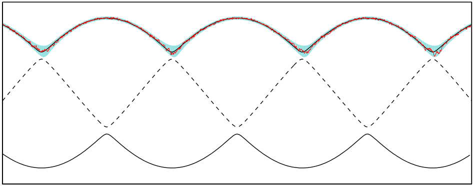
$A = 0.1$, $\sigma = 0.27$ — $A = 0.35$, $\sigma = 0.2$

Example 1: Stochastic resonance

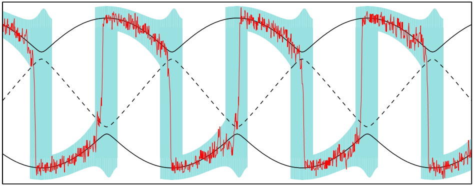
Example 1: Stochastic resonance

Critical noise intensity: $\sigma_{\rm c} = \max\{\delta, \varepsilon\}^{3/4}$, $\delta = A_{\rm c} - A$, $A_{\rm c} = \frac{2}{3\sqrt{3}}$

$\sigma \ll \sigma_{\rm c}$:
transitions unlikely



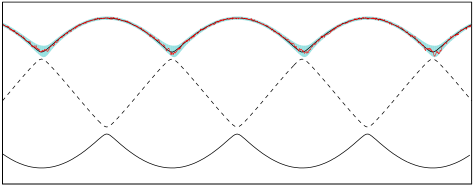
$\sigma \gg \sigma_{\rm c}$:
synchronisation



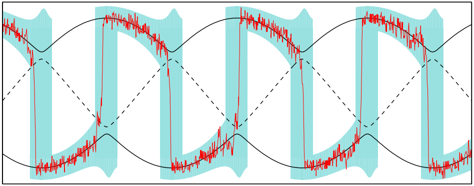
Example 1: Stochastic resonance

Critical noise intensity: $\sigma_c = \max\{\delta, \varepsilon\}^{3/4}$, $\delta = A_c - A$, $A_c = \frac{2}{3\sqrt{3}}$

$\sigma \ll \sigma_c$:
transitions unlikely



$\sigma \gg \sigma_c$:
synchronisation



Theorem [B & Gentz, Annals App. Proba 2002]

- $\sigma < \sigma_c$: transition probability per period $\leqslant e^{-\sigma_c^2/\sigma^2}$
- $\sigma > \sigma_c$: transition probability per period $\geqslant 1 - e^{-c\sigma^{4/3}/(\varepsilon|\log \sigma|)}$

Example 2: FitzHugh–Nagumo model

$$\varepsilon\dot{x} = x - x^3 + y$$
$$\dot{y} = a - x - by$$

- $x \propto$ membrane potential of neuron
- $y \propto$ proportion of open ion channels (recovery variable)
- $\varepsilon \ll 1 \Rightarrow$ fast–slow system
- $b = 0$ in the following for simplicity (but results more general)

Example 2: FitzHugh–Nagumo model

$$\varepsilon\dot{x} = x - x^3 + y$$
$$\dot{y} = a - x - by$$

- $x \propto$ membrane potential of neuron
- $y \propto$ proportion of open ion channels (recovery variable)
- $\varepsilon \ll 1 \Rightarrow$ fast–slow system
- $b = 0$ in the following for simplicity (but results more general)

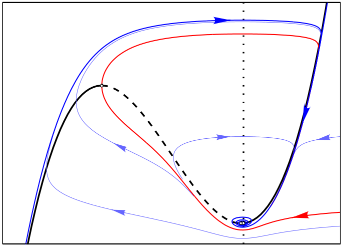
Stationary point $P = (a, a^3 - a)$
Linearisation has eigenvalues $\frac{-\delta \pm \sqrt{\delta^2 - \varepsilon}}{\varepsilon}$ where $\delta = \frac{3a^2 - 1}{2}$

- $\delta > 0$: stable node ($\delta > \sqrt{\varepsilon}$) or focus ($0 < \delta < \sqrt{\varepsilon}$)
- $\delta = 0$: singular Hopf bifurcation [Erneux & Mandel '86]
- $\delta < 0$: unstable focus ($-\sqrt{\varepsilon} < \delta < 0$) or node ($\delta < -\sqrt{\varepsilon}$)

Example 2: FitzHugh–Nagumo model

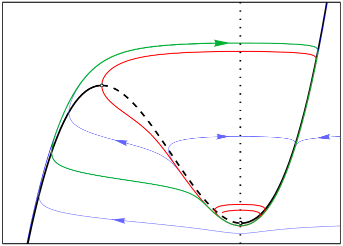
$\delta > 0$:

- P is asymptotically stable
- the system is excitable
- one can define a separatrix



$\delta < 0$:

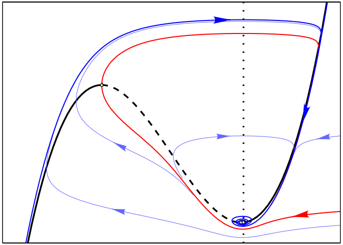
P is unstable
$\exists$ asympt. stable periodic orbit
sensitive dependence on δ:
canard (duck) phenomenon
[Callot, Diener, Diener '78, Benoît '81, ...]



Example 2: FitzHugh–Nagumo model

$\delta > 0$:

- P is asymptotically stable
- the system is excitable
- one can define a separatrix



$\delta < 0$:

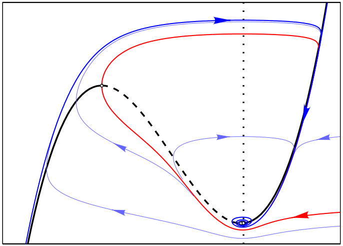
P is unstable
$\exists$ asympt. stable periodic orbit
sensitive dependence on δ:
canard (duck) phenomenon
[Callot, Diener, Diener '78,
Benoît '81, . . .]

(Link to simulation)

Example 2: FitzHugh–Nagumo model

$\delta > 0$:

- P is asymptotically stable
- the system is excitable
- one can define a separatrix



$\delta < 0$:

P is unstable
$\exists$ asympt. stable periodic orbit
sensitive dependence on δ:
canard (duck) phenomenon
[Callot, Diener, Diener '78, Benoît '81, ...]

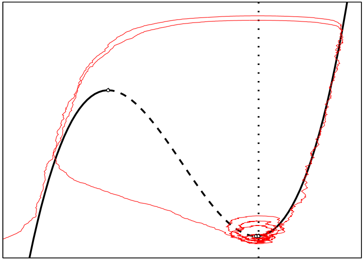
Stochastic FHN equation

$$\mathrm{d}x_t = \frac{1}{\varepsilon}[x_t - x_t^3 + y_t]\,\mathrm{d}t + \frac{\sigma_1}{\sqrt{\varepsilon}}\,\mathrm{d}W_t^{(1)}$$

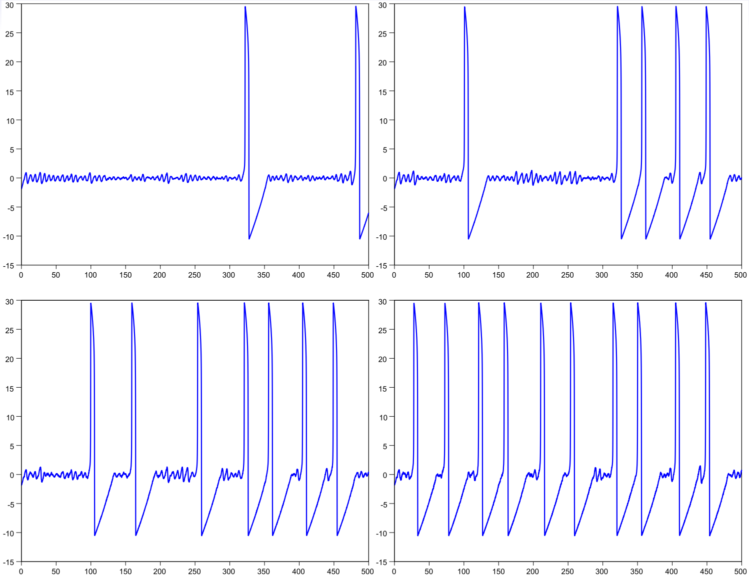
$$\mathrm{d}y_t = [a - x_t - by_t]\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_t^{(2)}$$

- Again $b = 0$ for simplicity in this talk
- $W_t^{(1)}, W_t^{(2)}$: independent Wiener processes (white noise)
- $0 < \sigma_1, \sigma_2 \ll 1$, $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$

$\varepsilon = 0.1$
$\delta = 0.02$
$\sigma_1 = \sigma_2 = 0.03$

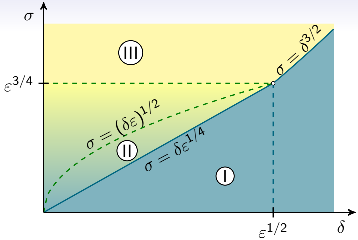


Noise-induced mixed-mode oscillations (MMOs)



Time series $t \mapsto -x_t$ for $\varepsilon = 0.01$, $\delta = 3 \cdot 10^{-3}$, $\sigma = 1.46 \cdot 10^{-4}, \dots, 3.65 \cdot 10^{-4}$

Results: Parameter regimes



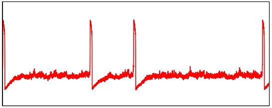
see also

[Muratov & Vanden Eijnden '08]

Regime I: rare isolated spikes

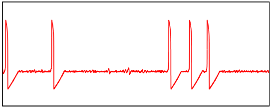
Thm [B & Landon, Nonlinearity 2012]

If $\delta \ll \varepsilon^{1/2}$: $\mathbb{E}[\text{\# small oscil}] \simeq \mathrm{e}^{\kappa(\varepsilon^{1/4}\delta)^2/\sigma^2}$



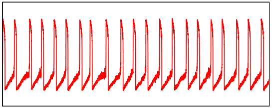
Regime II: clusters of spikes

small oscillations: asympt geometric



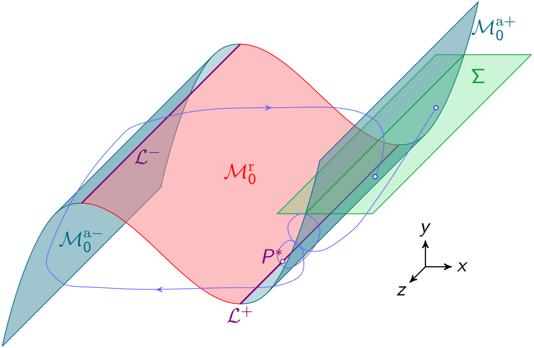
Regime III: repeated spikes

Interspike interval $\simeq$ constant



Example 3: The Koper model

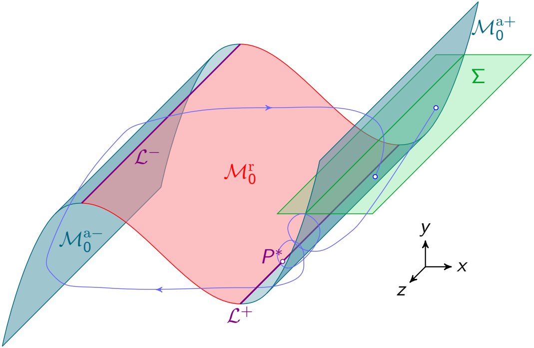
$$\varepsilon\,\mathrm{d}x_t = [y_t - x_t^3 + 3x_t]\,\mathrm{d}t$$
$$\mathrm{d}y_t = [kx_t - 2(y_t + \lambda) + z_t]\,\mathrm{d}t$$
$$\mathrm{d}z_t = [\rho(\lambda + y_t - z_t)]\,\mathrm{d}t$$



Folded-node singularity at P^* induces mixed-mode oscillations
[Benoît, Lobry '82, Szmolyan, Wechselberger '01, Brøns, Krupa, W '06 ...]

Example 3: The Koper model

$$\begin{aligned} \varepsilon\,\mathrm{d}x_t &= [y_t - x_t^3 + 3x_t]\,\mathrm{d}t &&+ \sqrt{\varepsilon}\sigma F(x_t, y_t, z_t)\,\mathrm{d}W_t \\ \mathrm{d}y_t &= [kx_t - 2(y_t + \lambda) + z_t]\,\mathrm{d}t &&+ \sigma' G_1(x_t, y_t, z_t)\,\mathrm{d}W_t \\ \mathrm{d}z_t &= [\rho(\lambda + y_t - z_t)]\,\mathrm{d}t &&+ \sigma' G_2(x_t, y_t, z_t)\,\mathrm{d}W_t \end{aligned}$$



Folded-node singularity at P^* induces mixed-mode oscillations
[Benoît, Lobry '82, Szmolyan, Wechselberger '01, Brøns, Krupa, W '06 ...]
What happens if we add noise to the system?

Threshold phenomena: How to prove them

σ_c: Critical noise intensity (to be determined)

1. For $\sigma \ll \sigma_c$, the stochastic solution remains close to the deterministic one with high probability
   - ◇ slightly easier to show
   - ◇ general method available
   - ◇ bounds are (almost) sharp in 1D, less sharp in higher D

2. For $\sigma \gg \sigma_c$, the stochastic system makes noise-induced transitions with high probability
   - ◇ harder to show
   - ◇ case-by-case approach
   - ◇ less sharp results

Below threshold: 1D time-dependent case

On the slow time scale $t = \varepsilon s$:

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t)$$

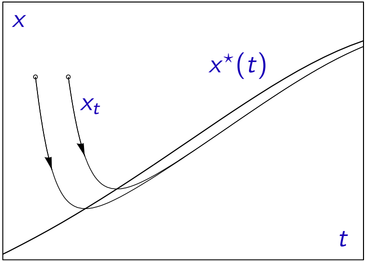
- Equilibrium branch: $\{x = x^\star(t)\}$ where $f(x^\star(t),t) = 0$ for all t
- Stable if $a^\star(t) = \partial_x f(x^\star(t),t) \leqslant -a_0 < 0$ for all t

Then [Tikhonov '52, Fenichel '79]:

- There exists particular solution

$$\bar{x}(t) = x^\star(t) + \mathcal{O}(\varepsilon)$$

- $\bar{x}$ attracts nearby orbits exp. fast
- $\bar{x}$ admits asymptotic series in ε



Below threshold: 1D time-dependent case

Stochastic perturbation:

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Write $x_t = \bar{x}(t) + \xi_t$ and Taylor-expand:

$$\mathrm{d}\xi_t = \frac{1}{\varepsilon}\Big[\bar{a}(t)\xi_t + \underbrace{b(\xi_t, t)}_{=\mathcal{O}(\xi_t^2)}\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

where $\bar{a}(t) = \partial_x f(\bar{x}(t), t) = a^\star(t) + \mathcal{O}(\varepsilon)$

Below threshold: 1D time-dependent case

Stochastic perturbation:

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Write $x_t = \bar{x}(t) + \xi_t$ and Taylor-expand:

$$\mathrm{d}\xi_t = \frac{1}{\varepsilon}\Big[\bar{a}(t)\xi_t + \underbrace{b(\xi_t, t)}_{=\mathcal{O}(\xi_t^2)}\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

where $\bar{a}(t) = \partial_x f(\bar{x}(t), t) = a^\star(t) + \mathcal{O}(\varepsilon)$

Variations of constants (Duhamel formula), if $\xi_0 = 0$:

$$\xi_t = \underbrace{\frac{\sigma}{\sqrt{\varepsilon}} \int_0^t \mathrm{e}^{\bar{\alpha}(t,s)/\varepsilon}\,\mathrm{d}W_s}_{\xi_t^0\text{: sol of linearised system}} + \underbrace{\frac{1}{\varepsilon}\int_0^t \mathrm{e}^{\bar{\alpha}(t,s)/\varepsilon}\, b(\xi_s, s)\,\mathrm{d}s}_{\text{treat as a perturbation}}$$

where $\bar{\alpha}(t,s) = \int_s^t \bar{a}(u)\,\mathrm{d}u$

Below threshold: 1D time-dependent case

Properties of $\quad \xi_t^0 = \dfrac{\sigma}{\sqrt{\varepsilon}} \displaystyle\int_0^t \mathrm{e}^{\bar{\alpha}(t,s)/\varepsilon} \, \mathrm{d}W_s$:

- Gaussian process, $\quad \mathbb{E}[\xi_t^0] = 0, \quad \operatorname{Var}(\xi_t^0) = \frac{\sigma^2}{\varepsilon} \int_0^t \mathrm{e}^{2\bar{\alpha}(t,s)/\varepsilon} \, \mathrm{d}s$
- Confidence interval: $\quad \mathbb{P}\bigl\{|\xi_t^0| > \frac{h}{\sigma}\sqrt{\operatorname{Var}(\xi_t^0)}\bigr\} = \mathcal{O}(\mathrm{e}^{-h^2/2\sigma^2})$
- $\sigma^{-2} \operatorname{Var}(\xi_t^0)$ satisfies ODE $\quad \varepsilon \dot{v} = 2\bar{a}(t) v + 1$

Below threshold: 1D time-dependent case

Properties of $\quad \xi_t^0 = \dfrac{\sigma}{\sqrt{\varepsilon}} \displaystyle\int_0^t \mathrm{e}^{\bar\alpha(t,s)/\varepsilon} \, \mathrm{d}W_s$:

- Gaussian process, $\quad \mathbb{E}[\xi_t^0] = 0, \quad \operatorname{Var}(\xi_t^0) = \frac{\sigma^2}{\varepsilon} \int_0^t \mathrm{e}^{2\bar\alpha(t,s)/\varepsilon} \, \mathrm{d}s$
- Confidence interval: $\quad \mathbb{P}\Bigl\{|\xi_t^0| > \frac{h}{\sigma}\sqrt{\operatorname{Var}(\xi_t^0)}\Bigr\} = \mathcal{O}(\mathrm{e}^{-h^2/2\sigma^2})$
- $\sigma^{-2}\operatorname{Var}(\xi_t^0)$ satisfies ODE $\quad \varepsilon\dot v = 2\bar a(t) v + 1$

Lemma [B & Gentz, Proba. Theory Relat. Fields 2002]

$\bar v(t)$ solution of ODE bounded away from 0: $\quad \bar v(t) = \frac{1}{-2\bar a(t)} + \mathcal{O}(\varepsilon)$

$$\mathbb{P}\biggl\{\sup_{0\leqslant s\leqslant t} \frac{|\xi_s^0|}{\sqrt{\bar v(s)}} > h\biggr\} = C_0(t,\varepsilon)\, \mathrm{e}^{-h^2/2\sigma^2}$$

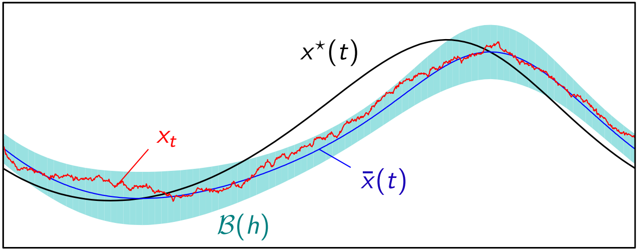
where $C_0(t,\varepsilon) = \sqrt{\frac{2}{\pi}\frac{1}{\varepsilon}} \Bigl|\int_0^t \bar a(s)\,\mathrm{d}s\Bigr| \frac{h}{\sigma}\Bigl[1 + \mathcal{O}(\varepsilon + \frac{t}{\varepsilon}\,\mathrm{e}^{-h^2/\sigma^2})\Bigr]$

Proof based on Doob's submartingale inequality and partition of $[0,t]$

Below threshold: 1D time-dependent case

Nonlinear equation: $d\xi_t = \dfrac{1}{\varepsilon}\bigl[\bar a(t)\xi_t + b(\xi_t,t)\bigr]\,dt + \dfrac{\sigma}{\sqrt{\varepsilon}}\,dW_t$

Confidence strip: $\mathcal{B}(h) = \bigl\{|\xi| \leqslant h\sqrt{\bar v(t)}\ \forall t\bigr\} = \bigl\{|x - \bar x(t)| \leqslant h\sqrt{\bar v(t)}\ \forall t\bigr\}$



Theorem [B & Gentz, Proba. Theory Relat. Fields 2002]

$$C(t,\varepsilon)\,e^{-\kappa_- h^2/2\sigma^2} \leqslant \mathbb{P}\bigl\{\text{leaving } \mathcal{B}(h) \text{ before time } t\bigr\} \leqslant C(t,\varepsilon)\,e^{-\kappa_+ h^2/2\sigma^2}$$

where $\kappa_\pm = 1 \mp \mathcal{O}(h)$ and $C(t,\varepsilon) = C_0(t,\varepsilon)\bigl[1+\mathcal{O}(h)\bigr]$ (requires $h \leqslant h_0$)

Generalisation to the multidimensional case

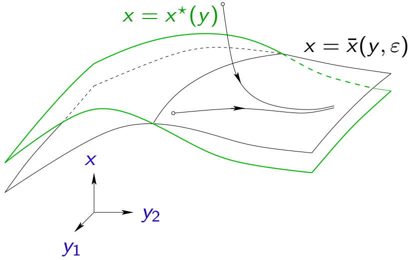
$$\varepsilon\dot{x} = f(x,y) \qquad x \in \mathbb{R}^n \text{, fast variables}$$
$$\dot{y} = g(x,y) \qquad y \in \mathbb{R}^m \text{, slow variables}$$

- Critical manifold: $f(x^\star(y),y) = 0$ (for all y in some domain)
- Stability: Eigenvalues of $A(y) = \partial_x f(x^\star(y),y)$ have negative real parts

Theorem [Tihonov '52, Fenichel '79]

$\exists$ slow manifold $x = \bar{x}(y,\varepsilon)$ s.t.

- $\bar{x}(y,\varepsilon)$ is invariant
- $\bar{x}(y,\varepsilon)$ attracts nearby solutions
- $\bar{x}(y,\varepsilon) = x^\star(y) + \mathcal{O}(\varepsilon)$



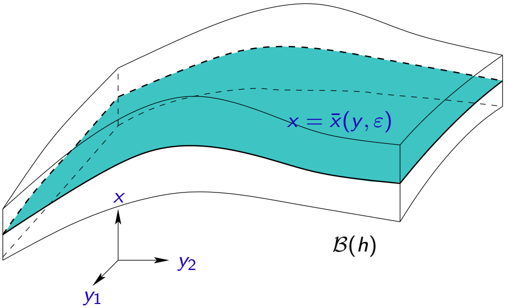
Generalisation to the multidimensional case

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, y_t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\,\mathrm{d}W_t \qquad \text{(fast variables } \in \mathbb{R}^n\text{)}$$

$$\mathrm{d}y_t = g(x_t, y_t)\,\mathrm{d}t + \sigma' G(x_t, y_t)\,\mathrm{d}W_t \qquad \text{(slow variables } \in \mathbb{R}^m\text{)}$$

$\mathcal{B}(h)$: confidence set

defined by covariance
of linearised equation
for $x - \bar{x}(y, \varepsilon)$



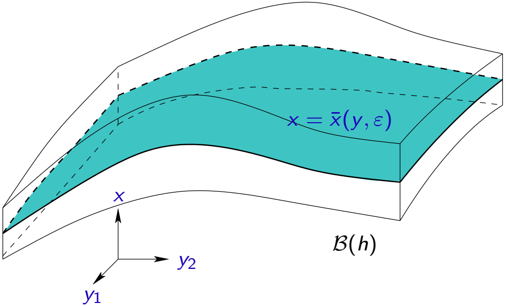
Generalisation to the multidimensional case

$$dx_t = \frac{1}{\varepsilon} f(x_t, y_t)\, dt + \frac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\, dW_t \qquad \text{(fast variables } \in \mathbb{R}^n)$$

$$dy_t = g(x_t, y_t)\, dt + \sigma' G(x_t, y_t)\, dW_t \qquad \text{(slow variables } \in \mathbb{R}^m)$$

$\mathcal{B}(h)$: confidence set

defined by covariance of linearised equation for $x - \bar{x}(y, \varepsilon)$



$\bar{A}(y) := \partial_x f(\bar{x}(y,\varepsilon), y)$

$\bar{X}$: covariance matrix of linearisation, solution of deterministic slow–fast ODE

$$\varepsilon \dot{\bar{X}} = \bar{A}(y)\bar{X} + \bar{X}\bar{A}(y)^{\mathrm{T}} + F(\bar{x}(y,\varepsilon), y)F(\bar{x}(y,\varepsilon), y)^{\mathrm{T}}$$

$$\dot{y} = g(\bar{x}(y,\varepsilon), y)$$

$$\mathcal{B}(h) := \left\{(x,y) \colon \left\langle \left[x - \bar{x}(y,\varepsilon)\right], \bar{X}(y)^{-1}\left[x - \bar{x}(y,\varepsilon)\right]\right\rangle < h^2\right\}$$

Generalisation to the multidimensional case

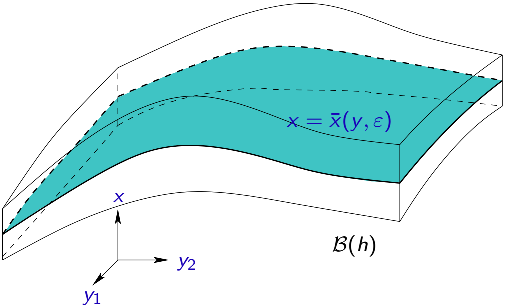
$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, y_t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\,\mathrm{d}W_t \qquad \text{(fast variables } \in \mathbb{R}^n\text{)}$$

$$\mathrm{d}y_t = g(x_t, y_t)\,\mathrm{d}t + \sigma' G(x_t, y_t)\,\mathrm{d}W_t \qquad \text{(slow variables } \in \mathbb{R}^m\text{)}$$

$\mathcal{B}(h)$: confidence set

defined by covariance of linearised equation for $x - \bar{x}(y,\varepsilon)$

$$\bar{A}(y) := \partial_x f(\bar{x}(y,\varepsilon), y)$$



Theorem [B & Gentz, J. Diff. Equ. 2004] Normally hyperbolic stable case:

$$C_-(t,\varepsilon)\,\mathrm{e}^{-\kappa h^2/2\sigma^2} \leqslant \mathbb{P}\{\text{leaving } \mathcal{B}(h) \text{ before time } t\} \leqslant C_+(t,\varepsilon)\,\mathrm{e}^{-\kappa h^2/2\sigma^2}$$

where $\kappa = 1 - \mathcal{O}(h) - \mathcal{O}(\varepsilon)$

Back to Example 1: Avoided transcritical bif.

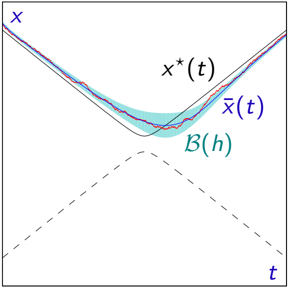
$$\mathrm{d}x_t = \frac{1}{\varepsilon}\big[t^2 + \delta - x_t^2 + \dots\big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Equil. curve: $x^\star(t) \simeq \sqrt{t^2+\delta}$

Slow sol.: $\bar{x}(t) = x^\star(t) + \mathcal{O}(\min\{\frac{\varepsilon}{|t|}, \frac{\varepsilon}{\sqrt{\delta+\varepsilon}}\})$

$$\bar{a}(t) = \partial_x f(\bar{x}(t),\varepsilon) \asymp \begin{cases} -|t| & |t| \geqslant \sqrt{\delta+\varepsilon} \\ -\sqrt{\delta+\varepsilon} & |t| \leqslant \sqrt{\delta+\varepsilon} \end{cases}$$

Confidence strip $\mathcal{B}(h)$: width $\asymp h/\sqrt{|\bar{a}(t)|}$



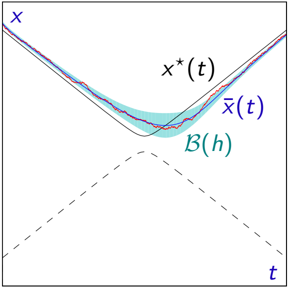
Back to Example 1: Avoided transcritical bif.

$$\mathrm{d}x_t = \frac{1}{\varepsilon}\big[t^2 + \delta - x_t^2 + \dots\big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Equil. curve: $x^\star(t) \simeq \sqrt{t^2+\delta}$
Slow sol.: $\bar{x}(t) = x^\star(t) + \mathcal{O}(\min\{\frac{\varepsilon}{|t|}, \frac{\varepsilon}{\sqrt{\delta+\varepsilon}}\})$

$$\bar{a}(t) = \partial_x f(\bar{x}(t),\varepsilon) \asymp \begin{cases} -|t| & |t| \geqslant \sqrt{\delta+\varepsilon} \\ -\sqrt{\delta+\varepsilon} & |t| \leqslant \sqrt{\delta+\varepsilon} \end{cases}$$

Confidence strip $\mathcal{B}(h)$: width $\asymp h/\sqrt{|\bar{a}(t)|}$



Theorem [B & Gentz, Annals App. Proba 2002]

$$\mathbb{P}\big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\big\} \leqslant C(t,\varepsilon)\,\mathrm{e}^{-\kappa h^2/2\sigma^2}$$

where $\kappa = 1 - \mathcal{O}(\sup_{s\leqslant t} h|\bar{a}(s)|^{-3/2}) - \mathcal{O}(\varepsilon)$ $\quad\Rightarrow$ requires $h < h_0 \inf_{s\leqslant t}|\bar{a}(s)|^{3/2}$

- $\sigma < \sigma_{\mathrm{c}} = \max\{\delta,\varepsilon\}^{3/4}$: result applies $\forall\, t$, $\mathbb{P}\{\text{trans}\} = \mathcal{O}(\mathrm{e}^{-\kappa\sigma_{\mathrm{c}}^2/\sigma^2})$
- $\sigma > \sigma_{\mathrm{c}} = \max\{\delta,\varepsilon\}^{3/4}$: result applies up to $t \asymp -\sigma^{2/3}$

Above threshold

What happens for $\sigma > \sigma_c$ and $t > -\sigma^{2/3}$?

General principle: partition $t_0 = s_0 < s_1 < s_2 < \dots < s_n = t$ of $[t_0, t]$

Lemma Let $P_k = \mathbb{P}\{\text{making no transition during } (s_{k-1}, s_k]\}$. Then

$$\mathbb{P}\{\text{making no transition during } [t_0, t]\} \leqslant \prod_{k=1}^{n} P_k$$

Choose partition s.t. each $P_k \leqslant q < 1 \Rightarrow \mathbb{P}\{\text{no transition}\} \leqslant \mathrm{e}^{-n \log q}$

Above threshold

What happens for $\sigma > \sigma_c$ and $t > -\sigma^{2/3}$?

General principle: partition $t_0 = s_0 < s_1 < s_2 < \dots < s_n = t$ of $[t_0, t]$

Lemma Let $P_k = \mathbb{P}\{\text{making no transition during } (s_{k-1}, s_k]\}$. Then

$$\mathbb{P}\{\text{making no transition during } [t_0, t]\} \leqslant \prod_{k=1}^{n} P_k$$

Choose partition s.t. each $P_k \leqslant q < 1 \Rightarrow \mathbb{P}\{\text{no transition}\} \leqslant e^{-n \log q}$

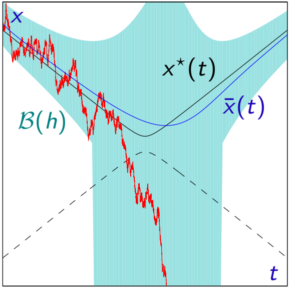
Example 1: Define partition such that

$$\int_{s_{k-1}}^{s_k} |\bar{a}(s)| \, ds = c\varepsilon |\log \sigma| \quad \Rightarrow \quad P_k \leqslant \frac{2}{3}$$

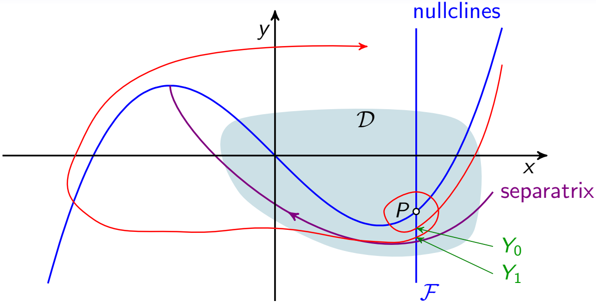
Proof uses comparison with linearised equations

Thm [B & Gentz, Ann App Proba 2002]

Transition probability $\geqslant 1 - e^{-\kappa\sigma^{4/3}/(\varepsilon|\log \sigma|)}$

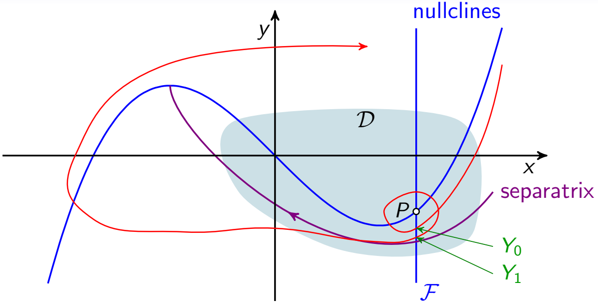


Back to Example 2: FitzHugh–Nagumo



$Y_0, Y_1, \dots$ substochastic Markov chain describing process killed on $\partial\mathcal{D}$
Number of small oscillations N = survival time of Markov chain

Back to Example 2: FitzHugh–Nagumo



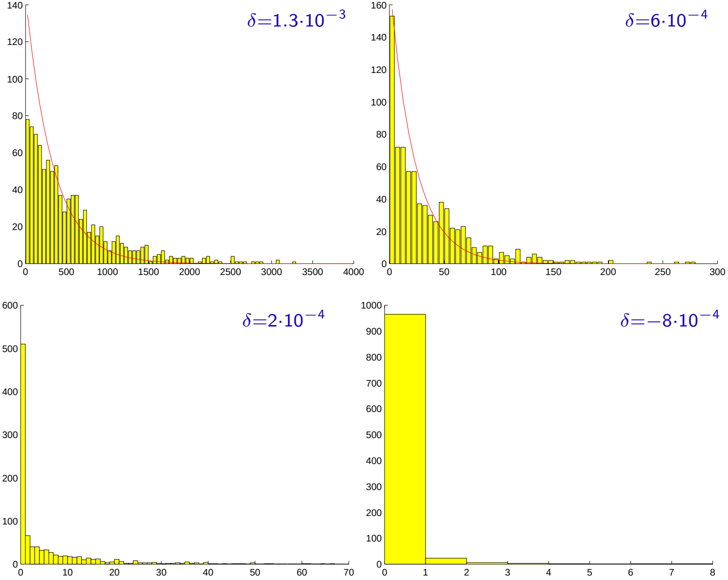
$Y_0, Y_1, \dots$ substochastic Markov chain describing process killed on $\partial\mathcal{D}$
Number of small oscillations N = survival time of Markov chain

Theorem [B & Landon, Nonlinearity 2012]

N is asymptotically geometric: $\lim\limits_{n\to\infty} \mathbb{P}\{N = n+1 | N > n\} = 1 - \lambda_0$
where $\lambda_0 \in \mathbb{R}_+$: principal eigenvalue of the chain, $\lambda_0 < 1$ if $\sigma > 0$

Histograms of distribution of N (1000 spikes)

$\sigma = \varepsilon = 10^{-4}$



Example 2: Below threshold

Theorem [B & Landon , Nonlinearity 2012]

Assume ε and $\delta/\sqrt{\varepsilon}$ sufficiently small
There exists $\kappa > 0$ s.t. for $\sigma^2 \leqslant (\varepsilon^{1/4}\delta)^2 / \log(\sqrt{\varepsilon}/\delta)$

- Principal eigenvalue:

$$1 - \lambda_0 \leqslant \exp\left\{-\kappa \frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$$

- Expected number of small oscillations:

$$\mathbb{E}^{\mu_0}[N] \geqslant C(\mu_0) \exp\left\{\kappa \frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$$

where $C(\mu_0)$ = probability of starting on $\mathcal{F}$ above separatrix

Proof:

- Construct $A \subset \mathcal{F}$ such that $K(x, A)$ exponentially close to 1 for all $x \in A$
- Use two different sets of coordinates to approximate K:
  Near separatrix, and during small oscillation

Example 2: Dynamics near the separatrix

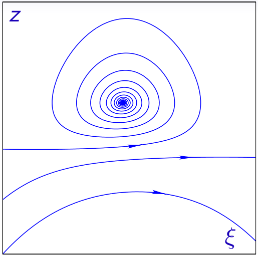
Change of variables:

- ▹ Translate to Hopf bif. point
- ▹ Scale space and time
- ▹ Straighten nullcline $\dot{x} = 0$

$\Rightarrow$ variables (ξ, z) where nullcline: $\{z = \frac{1}{2}\}$

$$d\xi_t = \left(\frac{1}{2} - z_t - \frac{\sqrt{\varepsilon}}{3}\xi_t^3\right) dt$$

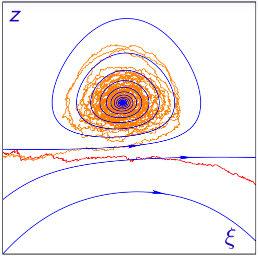
$$dz_t = \left(\tilde{\mu} + 2\xi_t z_t + \frac{2\sqrt{\varepsilon}}{3}\xi_t^4\right) dt$$



where

$$\tilde{\mu} = \frac{\delta}{\sqrt{\varepsilon}}$$

Example 2: Dynamics near the separatrix



Change of variables:

- ▹ Translate to Hopf bif. point
- ▹ Scale space and time
- ▹ Straighten nullcline $\dot{x} = 0$

$\Rightarrow$ variables (ξ, z) where nullcline: $\{z = \frac{1}{2}\}$

$$d\xi_t = \left(\frac{1}{2} - z_t - \frac{\sqrt{\varepsilon}}{3}\xi_t^3\right) dt + \tilde{\sigma}_1 \, dW_t^{(1)}$$

$$dz_t = \left(\tilde{\mu} + 2\xi_t z_t + \frac{2\sqrt{\varepsilon}}{3}\xi_t^4\right) dt - 2\tilde{\sigma}_1 \xi_t \, dW_t^{(1)} + \tilde{\sigma}_2 \, dW_t^{(2)}$$

where

$$\tilde{\mu} = \frac{\delta}{\sqrt{\varepsilon}} - \tilde{\sigma}_1^2 \qquad \tilde{\sigma}_1 = -\sqrt{3}\frac{\sigma_1}{\varepsilon^{3/4}} \qquad \tilde{\sigma}_2 = \sqrt{3}\frac{\sigma_2}{\varepsilon^{3/4}}$$

Upward drift dominates if $\tilde{\mu}^2 \gg \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 \Rightarrow (\varepsilon^{1/4}\delta)^2 \gg \sigma_1^2 + \sigma_2^2$

Rotation around P: use that $2z \, e^{-2z-2\xi^2+1}$ is constant for $\tilde{\mu} = \varepsilon = 0$

Example 2: From below to above threshold

Linear approximation:

$$\mathrm{d}z_t^0 = \left(\tilde{\mu} + t z_t^0\right) \mathrm{d}t - \tilde{\sigma}_1 t \, \mathrm{d}W_t^{(1)} + \tilde{\sigma}_2 \, \mathrm{d}W_t^{(2)}$$

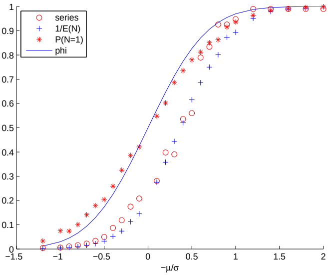
$$\Rightarrow \quad \mathbb{P}\{\text{no small osc}\} \simeq \Phi\left(-\pi^{1/4} \frac{\tilde{\mu}}{\sqrt{\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2}}\right) \qquad \Phi(x) = \int_{-\infty}^{x} \frac{\mathrm{e}^{-y^2/2}}{\sqrt{2\pi}} \, \mathrm{d}y$$

Example 2: From below to above threshold

Linear approximation:

$$\mathrm{d}z_t^0 = \left(\tilde\mu + tz_t^0\right)\mathrm{d}t - \tilde\sigma_1 t\,\mathrm{d}W_t^{(1)} + \tilde\sigma_2\,\mathrm{d}W_t^{(2)}$$

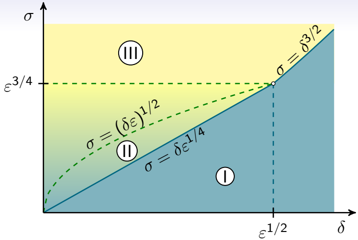
$$\Rightarrow \quad \mathbb{P}\{\text{no small osc}\} \simeq \Phi\left(-\pi^{1/4}\frac{\tilde\mu}{\sqrt{\tilde\sigma_1^2+\tilde\sigma_2^2}}\right) \qquad \Phi(x) = \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,\mathrm{d}y$$



$*$: $\mathbb{P}\{\text{no small osc}\}$
$+$: $1/\mathbb{E}[N]$
$\circ$: $1-\lambda_0$
curve: $x \mapsto \Phi(\pi^{1/4}x)$

$$x = -\frac{\tilde\mu}{\sqrt{\tilde\sigma_1^2+\tilde\sigma_2^2}} = -\frac{\varepsilon^{1/4}(\delta-\sigma_1^2/\varepsilon)}{\sqrt{\sigma_1^2+\sigma_2^2}}$$

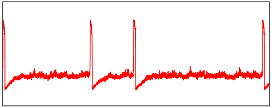
Example 2: Summary of results



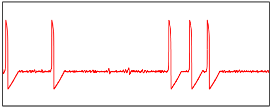
$\sigma_1 = \sigma_2$:
$\mathbb{P}\{N = 1\} \simeq \Phi\Big(-\frac{(\pi\varepsilon)^{1/4}(\delta - \sigma^2/\varepsilon)}{\sigma}\Big)$

see also
[Muratov & Vanden Eijnden '08]

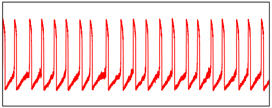
Regime I: rare isolated spikes
Theorem: If $\delta \ll \varepsilon^{1/2}$
$\mathbb{P}\{\text{escape}\}^{-1} \simeq \mathbb{E}[\#\text{ small oscil}] \simeq e^{\kappa(\varepsilon^{1/4}\delta)^2/\sigma^2}$



Regime II: clusters of spikes
small oscillations: asympt geometric
$\sigma = (\delta\varepsilon)^{1/2}$: Geom(1/2)

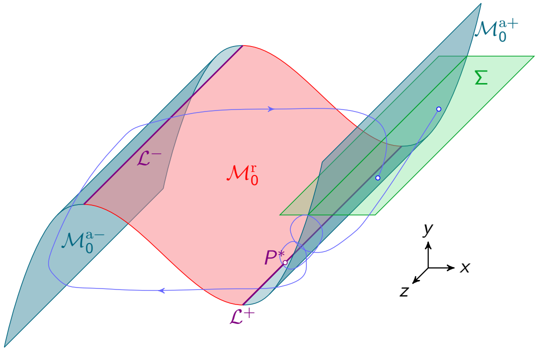


Regime III: repeated spikes
$\mathbb{P}\{N = 1\} \simeq 1$
Interspike interval $\simeq$ constant



Back to Example 3: The Koper model

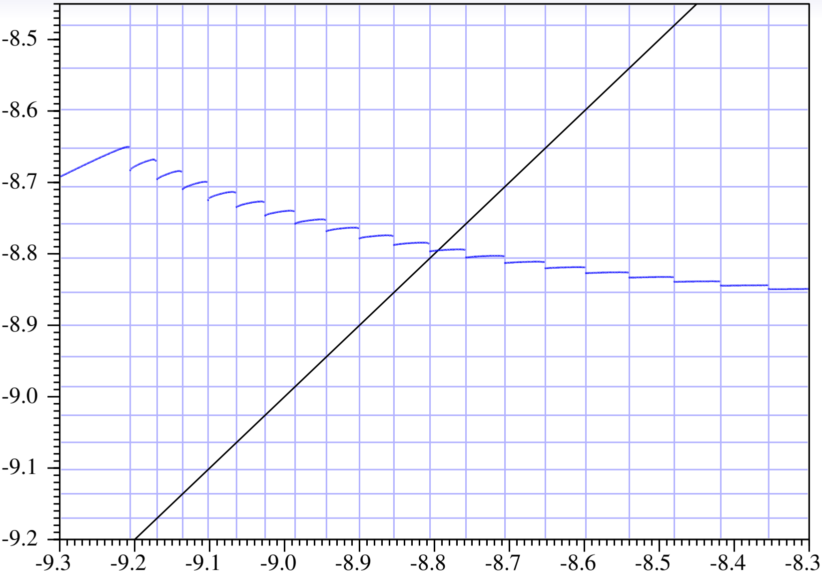
$$\begin{aligned}
\varepsilon\,\mathrm{d}x_t &= [y_t - x_t^3 + 3x_t]\,\mathrm{d}t &&+ \sqrt{\varepsilon}\sigma F(x_t,y_t,z_t)\,\mathrm{d}W_t \\
\mathrm{d}y_t &= [kx_t - 2(y_t+\lambda) + z_t]\,\mathrm{d}t &&+ \sigma' G_1(x_t,y_t,z_t)\,\mathrm{d}W_t \\
\mathrm{d}z_t &= [\rho(\lambda + y_t - z_t)]\,\mathrm{d}t &&+ \sigma' G_2(x_t,y_t,z_t)\,\mathrm{d}W_t
\end{aligned}$$



Folded-node singularity at P^* induces mixed-mode oscillations
[Benoît, Lobry '82, Szmolyan, Wechselberger '01, ...]

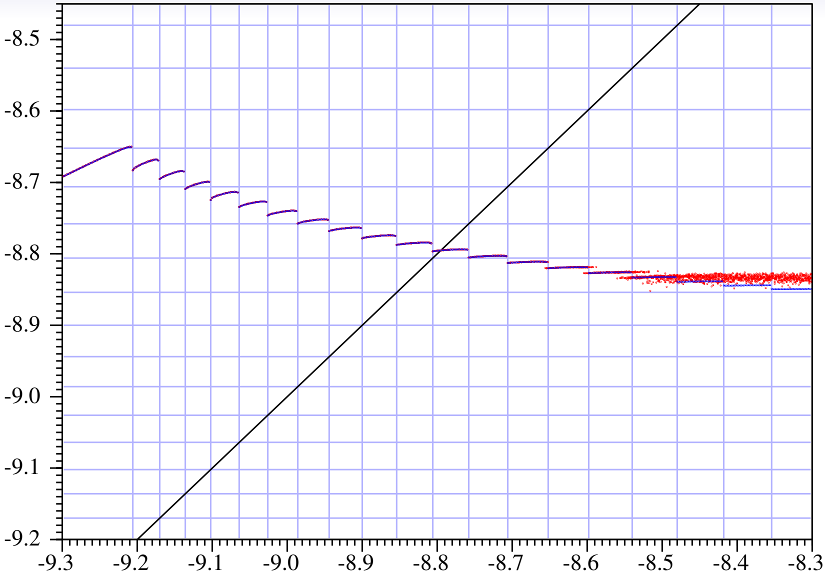
Poincaré map $\Pi : \Sigma \to \Sigma$ is almost $1d$ due to contraction in x-direction

Poincaré map $z_n \mapsto z_{n+1}$



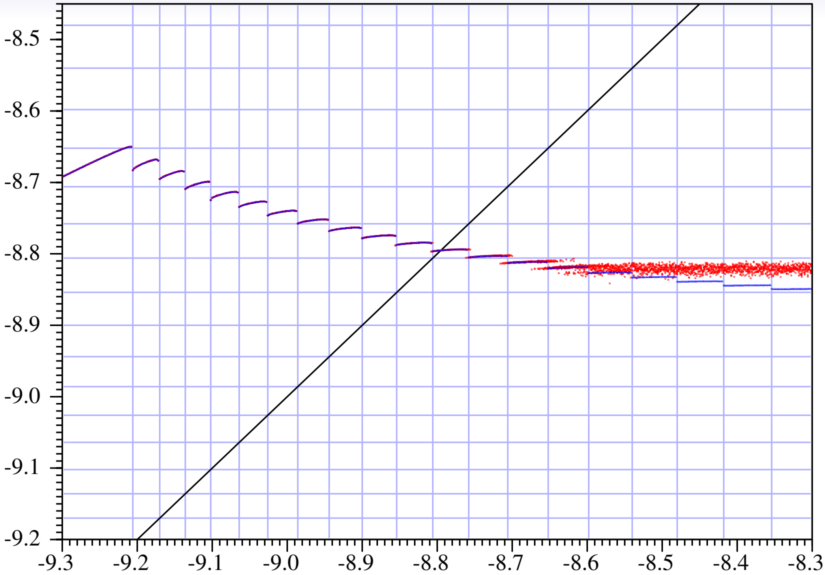
$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01, \sigma = \sigma' = 0$ – c.f. [Guckenheimer, Chaos, 2008]

Poincaré map $z_n \mapsto z_{n+1}$



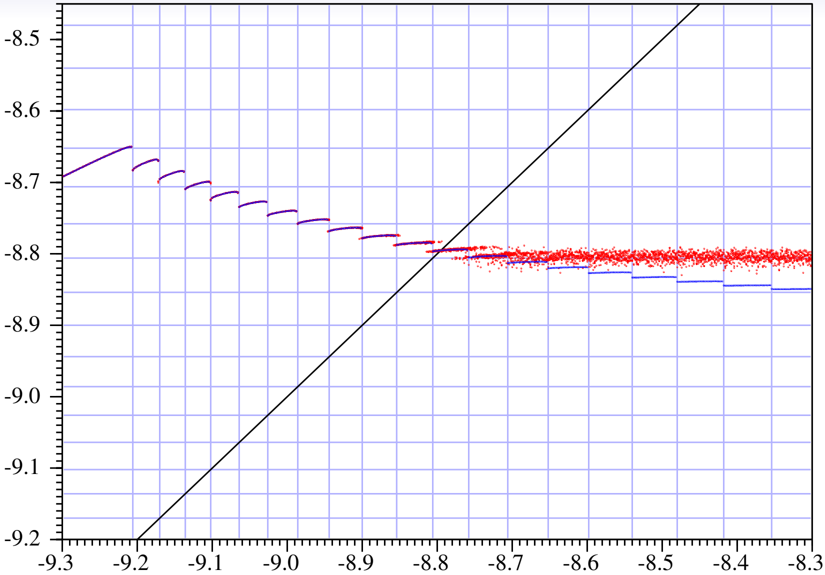
$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01,$ $\sigma = \sigma' = 2 \cdot 10^{-7}$

Poincaré map $z_n \mapsto z_{n+1}$



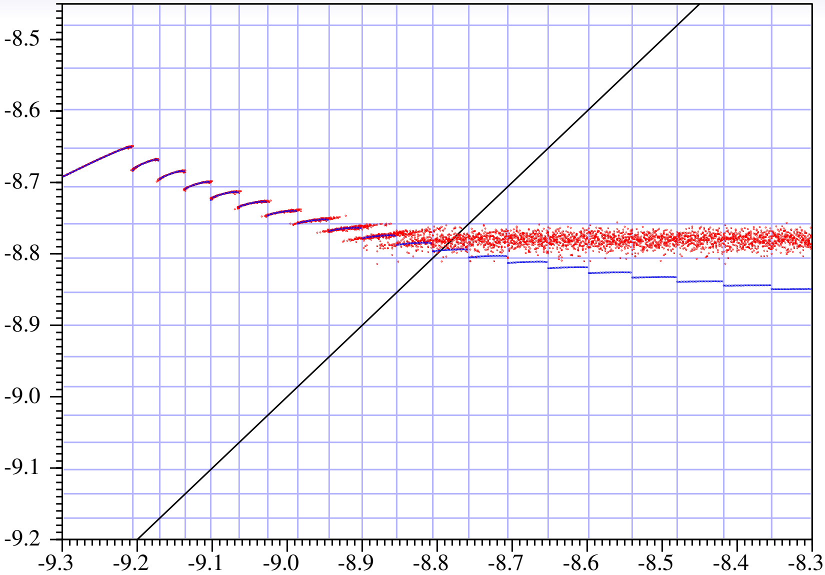
$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01,$ $\sigma = \sigma' = 2 \cdot 10^{-6}$

Poincaré map $z_n \mapsto z_{n+1}$



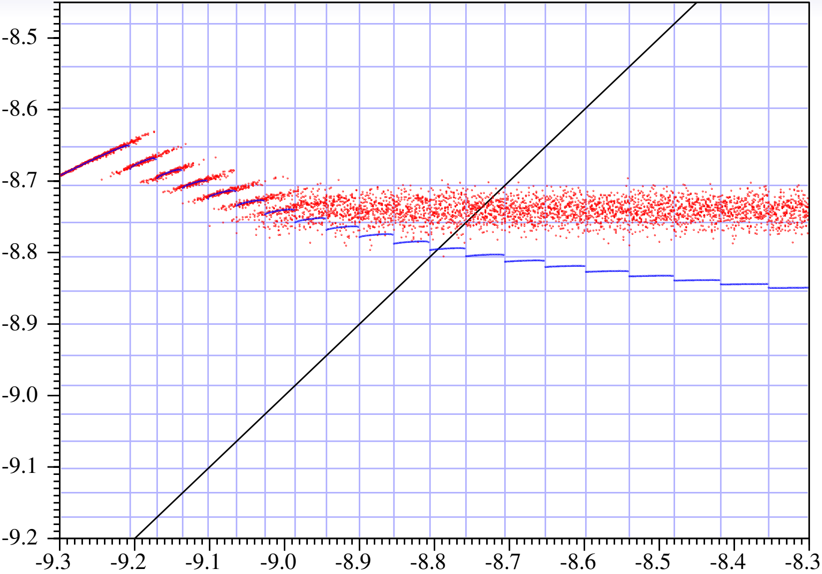
$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01,$ $\sigma = \sigma' = 2 \cdot 10^{-5}$

Poincaré map $z_n \mapsto z_{n+1}$



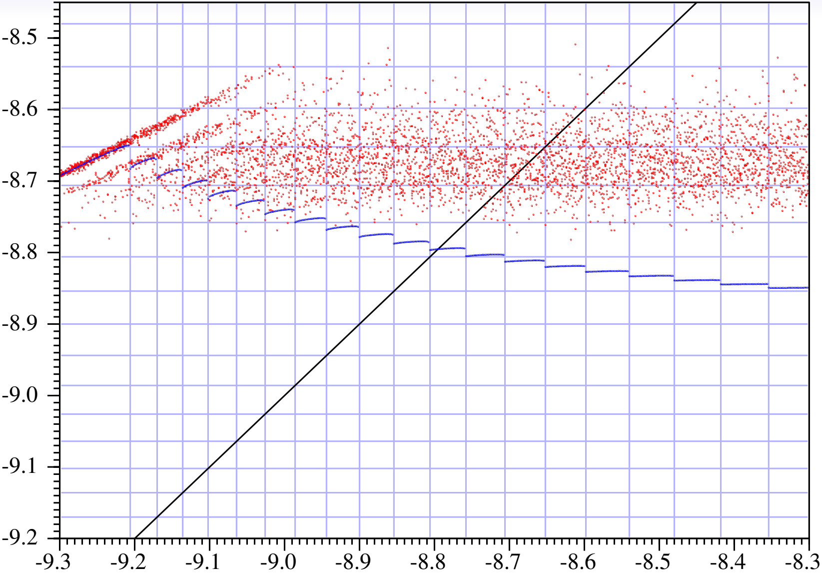
$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01,$ $\sigma = \sigma' = 2 \cdot 10^{-4}$

Poincaré map $z_n \mapsto z_{n+1}$



$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01,$ $\sigma = \sigma' = 2 \cdot 10^{-3}$

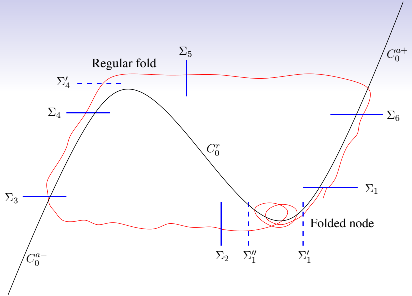
Poincaré map $z_n \mapsto z_{n+1}$



$k = -10, \lambda = -7.6, \rho = 0.7, \varepsilon = 0.01, \sigma = \sigma' = 10^{-2}$

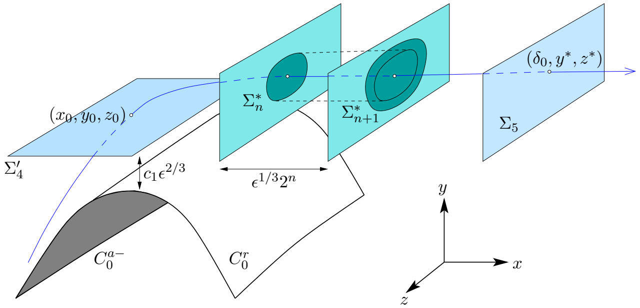
Size of fluctuations

$\mu \ll 1$: eigenvalue ratio
at folded node



Transition	Δx	Δy	Δz
$\Sigma_2 \to \Sigma_3$	$\sigma + \sigma'$		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_3 \to \Sigma_4$	$\sigma + \sigma'$		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_4 \to \Sigma_4'$	$\dfrac{\sigma}{\varepsilon^{1/6}} + \dfrac{\sigma'}{\varepsilon^{1/3}}$		$\sigma\sqrt{\varepsilon\lvert\log\varepsilon\rvert} + \sigma'$
$\Sigma_4' \to \Sigma_5$		$\sigma\sqrt{\varepsilon} + \sigma'\varepsilon^{1/6}$	$\sigma\sqrt{\varepsilon} + \sigma'\varepsilon^{1/6}$
$\Sigma_5 \to \Sigma_6$	$\sigma + \sigma'$		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_6 \to \Sigma_1$	$\sigma + \sigma'$		$\sigma\sqrt{\varepsilon} + \sigma'$
$\Sigma_1 \to \Sigma_1'$		$(\sigma + \sigma')\varepsilon^{1/4}$	σ'
$\Sigma_1' \to \Sigma_1''$ if $z = \mathcal{O}(\sqrt{\mu})$		$(\sigma + \sigma')(\varepsilon/\mu)^{1/4}$	$\sigma'(\varepsilon/\mu)^{1/4}$
$\Sigma_1'' \to \Sigma_2$		$(\sigma + \sigma')\varepsilon^{1/4}$	$\sigma'\varepsilon^{1/4}$

Example: Analysis near the regular fold



Proposition: For $h_1 = \mathcal{O}(\varepsilon^{2/3})$,

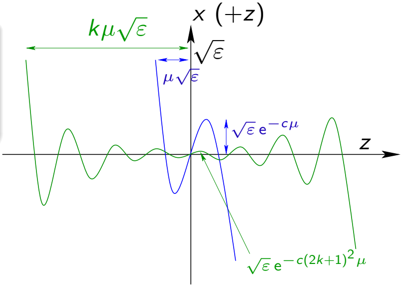
$$\mathbb{P}\Big\{ \|(y_{\tau_{\Sigma_5}}, z_{\tau_{\Sigma_5}}) - (y^*, z^*)\| > h_1 \Big\}$$
$$\leqslant C|\log\varepsilon| \bigg(\exp\Big\{ -\frac{\kappa h_1^2}{\sigma^2\varepsilon + (\sigma')^2\varepsilon^{1/3}} \Big\} + \exp\Big\{ -\frac{\kappa\varepsilon}{\sigma^2 + (\sigma')^2\varepsilon} \Big\} \bigg)$$

Main results

[B, Gentz, Kuehn, JDE 2012 & JDDE 2015]

Theorem 1: canard spacing

At $z = 0$, k^{th} canard lies at distance $\sqrt{\varepsilon}\,\mathrm{e}^{-c(2k+1)^2\mu}$ from primary canard



Main results

[B, Gentz, Kuehn, JDE 2012 & JDDE 2015]

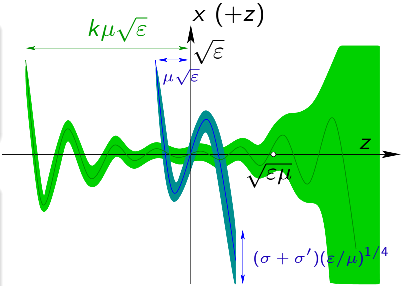
Theorem 1: canard spacing

At $z = 0$, k^{th} canard lies at distance $\sqrt{\varepsilon}\,\mathrm{e}^{-c(2k+1)^2\mu}$ from primary canard

Theorem 2: size of fluctuations

$(\sigma + \sigma')(\varepsilon/\mu)^{1/4}$ up to $z = \sqrt{\varepsilon\mu}$

$(\sigma + \sigma')(\varepsilon/\mu)^{1/4}\,\mathrm{e}^{z^2/(\varepsilon\mu)}$ for $z \geqslant \sqrt{\varepsilon\mu}$



Theorem 3: early escape

$P_0 \in \Sigma_1$ in sector with $k > 1/\sqrt{\mu}$ $\Rightarrow$ first hitting of Σ_2 at P_2 s.t.

$\mathbb{P}^{P_0}\{z_2 \geqslant z\} \leqslant C|\log(\sigma + \sigma')|^\gamma\,\mathrm{e}^{-\kappa z^2/(\varepsilon\mu|\log(\sigma+\sigma')|)}$

Main results

[B, Gentz, Kuehn, JDE 2012 & JDDE 2015]

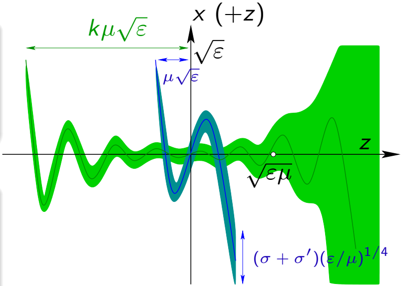
Theorem 1: canard spacing

At $z = 0$, k^{th} canard lies at distance $\sqrt{\varepsilon}\,\mathrm{e}^{-c(2k+1)^2\mu}$ from primary canard

Theorem 2: size of fluctuations

$(\sigma+\sigma')(\varepsilon/\mu)^{1/4}$ up to $z = \sqrt{\varepsilon\mu}$

$(\sigma+\sigma')(\varepsilon/\mu)^{1/4}\,\mathrm{e}^{z^2/(\varepsilon\mu)}$ for $z \geqslant \sqrt{\varepsilon\mu}$



Theorem 3: early escape

$P_0 \in \Sigma_1$ in sector with $k > 1/\sqrt{\mu} \Rightarrow$ first hitting of Σ_2 at P_2 s.t.

$\mathbb{P}^{P_0}\{z_2 \geqslant z\} \leqslant C|\log(\sigma+\sigma')|^{\gamma}\,\mathrm{e}^{-\kappa z^2/(\varepsilon\mu|\log(\sigma+\sigma')|)}$

- Saturation effect occurs at $k_c \simeq \sqrt{|\log(\sigma+\sigma')|/\mu}$
- For $k > k_c$, behaviour indep. of k and $\Delta z \leqslant \mathcal{O}(\sqrt{\varepsilon\mu|\log(\sigma+\sigma')|}\,)$

Summary/Outlook

Noise can cause threshold phenomena

- ▷ Below threshold small perturbation of deterministic dynamics
- ▷ Above threshold large transitions can occur

Well understood:

- ▷ Normally hyperbolic case
- ▷ Codimension-1 bifurcations (fold, (avoided) transcritical, pitchfork, Hopf)
- ▷ Higher codimension: case studies (folded node, cf. Kuehn)

In progress: theory of random Poincaré maps

Essentially still open:

- ▷ Other types of noise (except Ornstein–Uhlenbeck)
- ▷ Equations with delay
- ▷ Infinite dimensions, in particular with continuous spectrum

Further reading

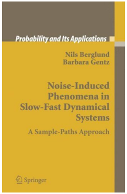
N. B. and Barbara Gentz, *Pathwise description of dynamic pitchfork bifurcations with additive noise*, Probab. Theory Related Fields **122**, 341–388 (2002)

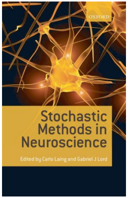
______, *A sample-paths approach to noise-induced synchronization: Stochastic resonance in a double-well potential*, Ann. Applied Probab. **12**, 1419-1470 (2002)

______, *Geometric singular perturbation theory for stochastic differential equations*, J. Differential Equations **191**, 1–54 (2003)

______, *Noise-induced phenomena in slow-fast dynamical systems, A sample-paths approach*, Springer, Probability and its Applications (2006)

______, *Stochastic dynamic bifurcations and excitability*, in C. Laing and G. Lord, (Eds.), *Stochastic methods in Neuroscience*, p. 65-93, Oxford University Press (2009)





N. B. and Damien Landon, *Mixed-mode oscillations and interspike interval statistics in the stochastic FitzHugh-Nagumo model*, Nonlinearity **25**, 2303–2335 (2012)

N. B., Barbara Gentz and Christian Kuehn, *Hunting French Ducks in a Noisy Environment*, J. Differential Equations **252**, 4786–4841 (2012)

______, *From random Poincaré maps to stochastic mixed-mode-oscillation patterns*, J. Dynam. Differential Equations **27**, 83–136 (2015)