Random Dynamical Systems

# Theory and applications of random Poincaré maps

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Joint works with Manon Baudel (Orléans), Barbara Gentz (Bielefeld), Christian Kuehn (Munich) and Damien Landon



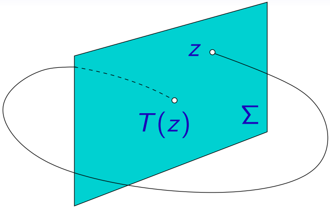

# Deterministic Poincaré maps

ODE $\dot{z} = f(z) \quad z \in \mathbb{R}^n$

Flow: $z_t = \varphi_t(z_0)$

$\Sigma \subset \mathbb{R}^n$: $(n-1)$-dimensional manifold

Poincaré map (or first-return map):
$T : \Sigma \to \Sigma$

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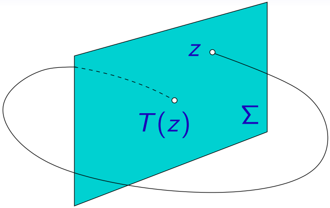


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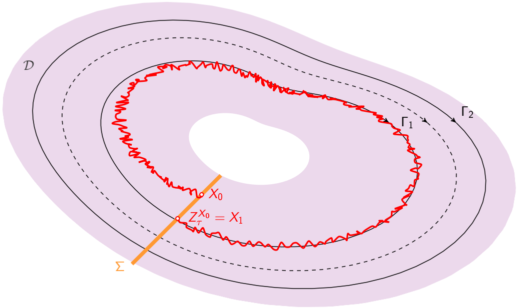
Benefits:

1. Dimension reduction: $T$ is an $(n-1)$-dimensional map
2. Stability of periodic orbits: no neutral direction
3. Bifurcations of periodic orbits easier to study (period doubling, Hopf, ...)

Question: how about SDEs $\quad \mathrm{d}z_t = f(z_t)\,\mathrm{d}t + \sigma g(z_t)\,\mathrm{d}W_t$ ?

# Random Poincaré maps

$\mathrm{d}z_t = f(z_t)\,\mathrm{d}t + \sigma g(z_t)\,\mathrm{d}W_t \quad \Rightarrow \quad$ Sample path $(Z_t^{z_0}(\omega))_{t\geqslant 0}$



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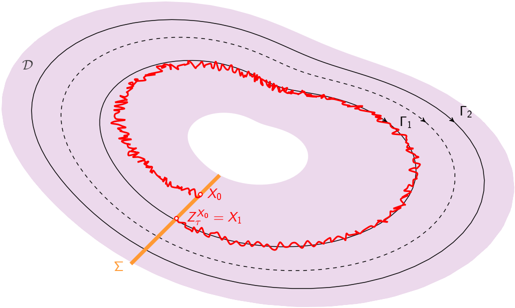


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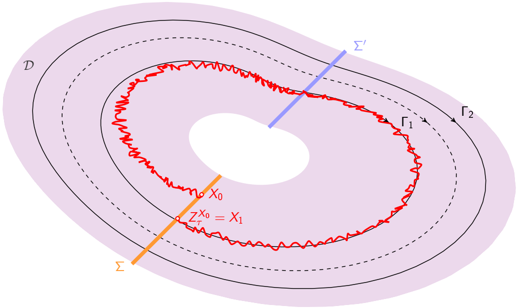


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Solution: $\tau_0 = 0$, $\tau'_{n+1} = \inf\{t > \tau_n \colon Z_t^{X_0} \in \Sigma'\}$

$\tau_{n+1} = \inf\{t > \tau'_{n+1} \colon Z_t^{X_0} \in \Sigma\}$

$X_n = Z_{\tau_n}^{X_0} \in \Sigma \quad \Rightarrow \quad (X_n)_{n\geqslant 0}$ is a Markov chain $\quad K(x, A) := \mathbb{P}^x\{X_1 \in A\}$

$(X_n, \omega) \mapsto X_{n+1}$: random Poincaré map

[J. Weiss, E. Knobloch, 1990], [P. Hitczenko, G. Medvedev, 2009]

# Application: Stochastic FitzHugh–Nagumo eq.

$$\mathrm{d}x_t = \frac{1}{\varepsilon}[x_t - x_t^3 + y_t]\,\mathrm{d}t$$ neuron membrane potential

$$\mathrm{d}y_t = [a - x_t - by_t]\,\mathrm{d}t$$ open ion channels

- $b = 0$ for simplicity in this talk, bifurcation parameter $\delta := \frac{3a^2-1}{2}$

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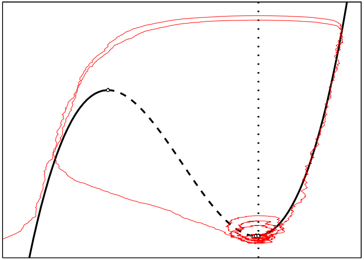
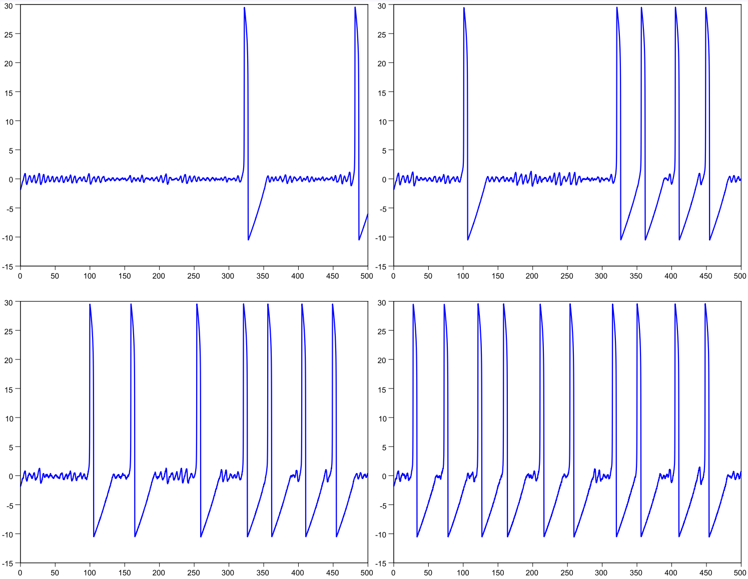
$\varepsilon = 0.1$
$\delta = 0.02$
$\sigma_1 = \sigma_2 = 0.03$

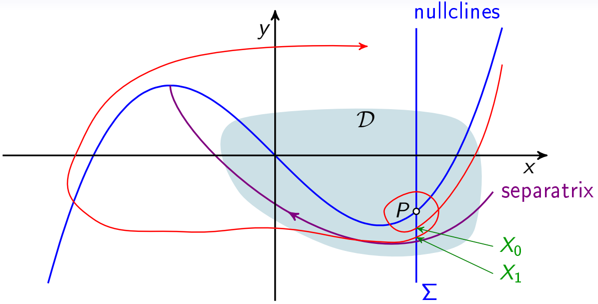


# Mixed-mode oscillations (MMOs)




Time series $t \mapsto -x_t$ for $\varepsilon = 0.01$, $\delta = 3 \cdot 10^{-3}$, $\sigma = 1.46 \cdot 10^{-4}, \dots, 3.65 \cdot 10^{-4}$

# Random Poincaré map




$X_0, X_1, \dots$ substochastic Markov chain describing process killed on $\partial\mathcal{D}$
Number of small oscillations: $N = \inf\{n \geqslant 1 \colon X_n \notin \Sigma\}$

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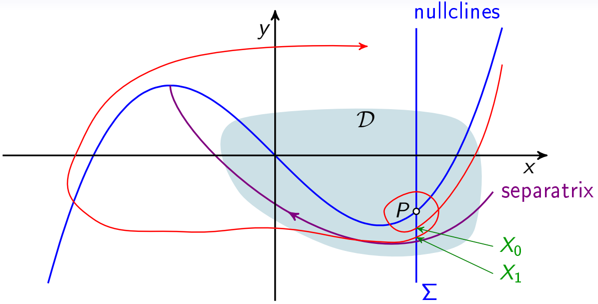


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**Theorem 1** [B & Landon, Nonlinearity **25**:2303–2335, 2012]

$N$ is asymptotically geometric: $\lim_{n\to\infty} \mathbb{P}\{N = n+1 | N > n\} = 1 - \lambda_0$

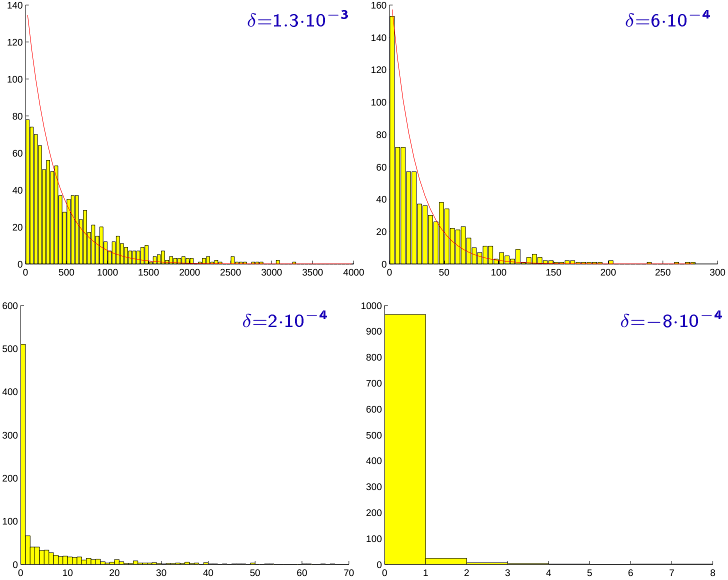
where $\lambda_0 \in \mathbb{R}_+$: principal eigenvalue of the kernel $K$, $\lambda_0 < 1$ if $\sigma > 0$

**Proof:** follows from existence of spectral gap

Details

# Histograms of distribution of $N$ (1000 spikes)

$\sigma = \varepsilon = 10^{-4}$



# Weak-noise regime

**Theorem 2** [B & Landon, Nonlinearity **25**:2303–2335, 2012]

Assume $\varepsilon$ and $\delta/\sqrt{\varepsilon}$ sufficiently small

There exists $\kappa > 0$ s.t. for $\sigma^2 \leqslant (\varepsilon^{1/4}\delta)^2 / \log(\sqrt{\varepsilon}/\delta)$

- Principal eigenvalue:

$$1 - \lambda_0 \leqslant \exp\left\{-\kappa\frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$$

- Expected number of small oscillations:

$$\mathbb{E}^{\mu_0}[N] \geqslant C(\mu_0)\exp\left\{\kappa\frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$$

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$\lambda_0 \pi_0(A) = \int_\Sigma \pi_0(\mathrm{d}x) K(x, A) \geqslant \int_A \pi_0(\mathrm{d}x) K(x, A) \Rightarrow \lambda_0 \geqslant \inf_{x \in A} K(x, A)$ □

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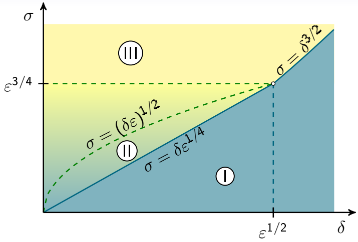
- Linear approximation around separatrix provides good description for dynamics with stronger noise

# Summary: Parameter regimes




$\sigma_1 = \sigma_2$:

$\mathbb{P}\{N = 1\} \simeq \Phi\Big(-\frac{(\pi\varepsilon)^{1/4}(\delta - \sigma^2/\varepsilon)}{\sigma}\Big)$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$

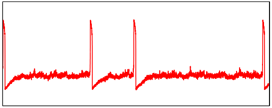
see also

[Muratov & Vanden Eijnden '08]

**Regime I:** rare isolated spikes

Theorem 2 applies ($\delta \ll \varepsilon^{1/2}$)

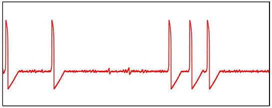
Interspike interval $\simeq$ exponential



**Regime II:** clusters of spikes

# interspike osc asympt geometric

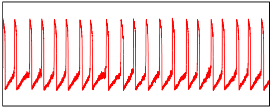
$\sigma = (\delta\varepsilon)^{1/2}$: geom(1/2)



**Regime III:** repeated spikes

$\mathbb{P}\{N = 1\} \simeq 1$

Interspike interval $\simeq$ constant



# Spectral theory of random Poincaré maps

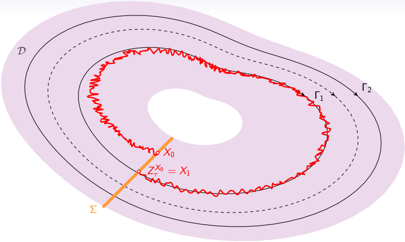
$X_n = Z^{X_0}_{\tau_n} \in \Sigma$

$(X_n)_{n\geqslant 0}$ Markov chain, of kernel

$K(x, A) = \mathbb{P}\{X_{n+1} \in A | X_n = x\}$

Under appropriate conditions

$K(x, A) = \int_A k(x, y) \, \mathrm{d}y$



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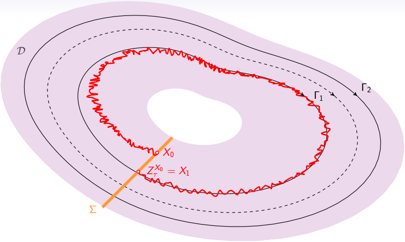


Markov semigroups:

$$(K\varphi)(x) = \int_\Sigma k(x,y)\varphi(y)\,\mathrm{d}y = \mathbb{E}^x[\varphi(X_1)] \qquad \varphi \in L^\infty$$

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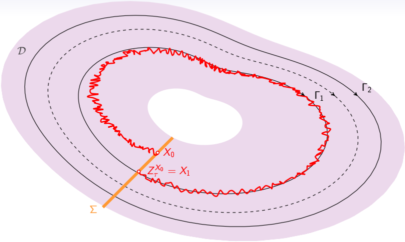


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Spectral decomposition: $\quad K\phi_i = \lambda_i\phi_i \qquad \pi_i K = \lambda_i \pi_i \qquad \langle \pi_i, \phi_j\rangle = \delta_{ij}$

$$k(x, y) = \lambda_0\phi_0(x)\pi_0(y) + \lambda_1\phi_1(x)\pi_1(y) + \dots$$

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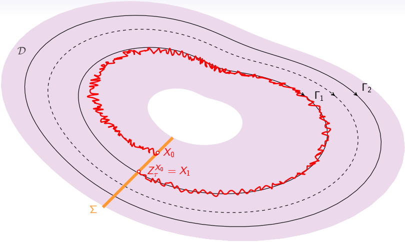


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$$k^n(x, y) = \lambda_0^n \phi_0(x) \pi_0(y) + \lambda_1^n \phi_1(x) \pi_1(y) + \dots$$

# Assumptions

$\mathrm{d}z_t = f(z_t)\,\mathrm{d}t + \sigma g(z_t)\,\mathrm{d}W_t \qquad z_t \in \mathcal{D}_0 \subset \mathbb{R}^{d+1}$

Assumption 1: Domain

$f \in \mathcal{C}^2(\mathcal{D}_0, \mathbb{R}^{d+1})$, $\mathcal{D} \subset \mathcal{D}_0$ positively invariant under deterministic flow

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Other limit sets are unstable points or orbits, no heteroclinic connections

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Assumption 3: Ellipticity

$g \in \mathcal{C}^1(\mathcal{D}_0, \mathbb{R}^{(d+1)\times k})$
$0 < c_- \|\xi\|^2 \leqslant \langle \xi, g(z) g(z)^{\mathsf{T}} \xi \rangle \leqslant c_+ \|\xi\|^2 \qquad \forall z \in \mathcal{D}\ \forall \xi \in \mathbb{R}^{d+1} \setminus \{0\}$

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Assumption 4: Confinement

$\exists$ Lyapunov function $V \in \mathcal{C}^2(\mathcal{D}_0, \mathbb{R}_+)$, $\|V(z)\| \to \infty$ as $z \to \partial\mathcal{D}_0$

$(\mathcal{L}V)(z) \leqslant -cV + d\mathbb{1}_{z\in\mathcal{D}}, \quad c > 0,\ d \geqslant 0 \qquad \mathcal{L}$: generator of diffusion

# Metastable hierarchy

Freidlin–Wentzell theory:

Rate function: $I_{[0,T]}(\gamma) = \dfrac{1}{2}\displaystyle\int_0^T (\dot\gamma_s - f(\gamma_s))^{\mathsf{T}} [gg^{\mathsf{T}}(\gamma_s)]^{-1} (\dot\gamma_s - f(\gamma_s))\,\mathrm{d}s$

Large-deviation principle: $\mathbb{P}\big\{(z_t)_{0\leqslant t\leqslant T} \in \Lambda\big\} \simeq \mathrm{e}^{-\inf_{\gamma\in\Lambda} I_{[0,T]}(\gamma)/\sigma^2}$

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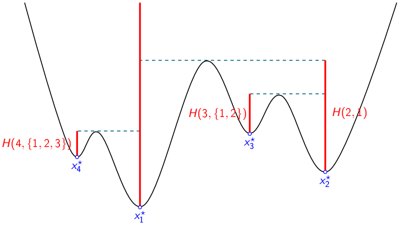
**Assumption 5: Metastable hierarchy**

$\exists\theta > 0$ s.t. $\forall\, 2 \leqslant k \leqslant N$

$$\min_{\ell<k} H(k,\ell) \leqslant \min_{\substack{i<k\\ j\neq i}} H(i,j) - \theta$$



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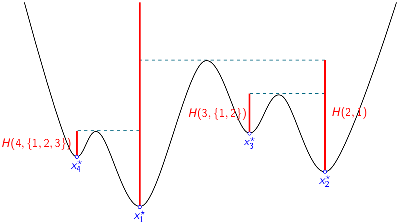


**Remark**: Using Doob's $h$-transform, one may replace Assumption 4 by

**Assumption 4': Confinement**

$\exists\theta' > 0$ such that $\min_i H(i,\partial\mathcal{D}) \geqslant \max_{i\neq j} H(i,j) + \theta'$

## Main results [Baudel & B, SIAM J. Math. Anal. **49**:4319–4375, 2017]

$B_k \subset \Sigma$: nbh of $\Gamma_k \cap \Sigma$, $\mathcal{M}_k = \bigcup_{j \leqslant k} B_k$, $\tau_A, \tau_A^+$ 1st-passage/return time to $A$

**Theorem 1**: Eigenvalues

The $N$ largest eigenvalues of $K$ are real and positive. $\exists\, \theta_k, c > 0$ s.t.

$$\lambda_0 = 1$$

$$\lambda_k = 1 - \mathbb{P}^{\mathring{\pi}_0^{k+1}}\{\tau^+_{\mathcal{M}_k} < \tau^+_{B_{k+1}}\}[1 + \mathcal{O}(e^{-\theta_k/\sigma^2})] \qquad 1 \leqslant k \leqslant N-1$$

$$|\lambda_k| < 1 - \frac{c}{\log(\sigma^{-1})} =: \rho \qquad k \geqslant N$$

$\mathring{\pi}_0^{k+1}$: QSD on $B_{k+1}$ and $\mathbb{P}^{\mathring{\pi}_0^{k+1}}\{\tau^+_{\mathcal{M}_k} < \tau^+_{B_{k+1}}\} \simeq e^{-H(k+1,\{1,\dots,k\})/\sigma^2}$

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**Theorem 2**: Eigenfunctions

$$\phi_0(x) = 1 \qquad \phi_k(x) = \mathbb{P}^x\{\tau_{B_{k+1}} < \tau_{\mathcal{M}_k}\}[1 + \mathcal{O}(\mathrm{e}^{-\bar\theta/\sigma^2})] + \mathcal{O}(\mathrm{e}^{-\bar\theta_k/\sigma^2})$$

$$\pi_k(B_{k+1}) = 1 - \mathcal{O}(\mathrm{e}^{-\kappa/\sigma^2}) \qquad \pi_k(B_j) = \mathcal{O}(\mathrm{e}^{-\bar\theta/\sigma^2}) \qquad 0 \leqslant k < j < N$$

$$k^n(x,y) = \pi_0(y) + \sum_{i=1}^{N-1} \lambda_i^n \phi_i(x)\pi_i(y) + \mathcal{O}(\rho^n) \qquad n \gg \log(\rho^{-1})$$

Proofs

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**Theorem 3**: Expected hitting times

Proofs

$$\mathbb{E}^x[\tau_{\mathcal{M}_k}] = [1 - \lambda_k]^{-1}[1 + \mathcal{O}(\mathrm{e}^{-\kappa/\sigma^2})] \qquad \forall x \in B_{k+1},\ 1 \leqslant k \leqslant N-1$$

# Outlook

- In progress: cases without metastable hierarchy (eigenvalue crossings)
- In progress: applications (Hawkes, Lotka–Volterra, Morris–Lecar, ...)
- In progress: numerical algorithms (based on QSDs, adaptive multilevel splitting)

**References**:

1. N. B. & Damien Landon, *Mixed-mode oscillations and interspike interval statistics in the stochastic FitzHugh–Nagumo model*, Nonlinearity **25**, 2303-2335 (2012)
2. Manon Baudel & N. B., *Spectral theory for random Poincaré maps*, SIAM J. Math. Analysis **49**, 4319–4375 (2017)

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- Related: exit through an unstable periodic orbit [3.]
- Related: MMOs in fast-slow systems with folded node singularity [4.]
- Open: asymptotics beyond large deviations

**References**:


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2. Manon Baudel & N. B., *Spectral theory for random Poincaré maps*, SIAM J. Math. Analysis **49**, 4319–4375 (2017)
3. N. B. & Barbara Gentz, *On the noise-induced passage through an unstable periodic orbit II: General case*, SIAM J. Math. Analysis **46**, 310–352 (2014)
4. N. B., Barbara Gentz & Christian Kuehn, *From random Poincaré maps to stochastic mixed-mode-oscillation patterns*, J. Dynam. Diff. Eq. **27**, 83–136 (2015)

# Proof of asymptotically geometric distribution

Theorem 1 [B & Landon, Nonlinearity 25:2303–2335, 2012]

$N$ is asymptotically geometric: $\lim_{n\to\infty} \mathbb{P}\{N = n+1 | N > n\} = 1 - \lambda_0$

where $\lambda_0 \in (0,1)$ if $\sigma > 0$ is principal eigenvalue of the kernel $K$

**Proof:**

Markov chain on $\Sigma$, kernel $K$ with density $k$ [Ben Arous, Kusuoka, Stroock '84]

- $\lambda_0 \leqslant \sup_{x\in\Sigma} K(x,\Sigma) < 1$ by ellipticity ($k$ bounded below)
- $\mathbb{P}^{\mu_0}\{N > n\} = \mathbb{P}^{\mu_0}\{X_n \in \Sigma\} = \int_\Sigma \mu_0(\mathrm{d}x) K^n(x,\Sigma)$
  $= \int_\Sigma \mu_0(\mathrm{d}x) \lambda_0^n \phi_0(x)[1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)]$
  $= \lambda_0^n \langle \mu_0, \phi_0 \rangle [1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)]$
- $\mathbb{P}^{\mu_0}\{N = n+1\} = \int_\Sigma \int_\Sigma \mu_0(\mathrm{d}x) K^n(x,\mathrm{d}y)[1 - K(y,\Sigma)]$
  $= \lambda_0^n (1-\lambda_0) \langle \mu_0, \phi_0 \rangle [1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)]$
- Existence of spectral gap follows from positivity condition [Birkhoff '57]

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# Proof of spectral decomposition

Feynman–Kac-type relation: For $|e^{-u}| > \sup_{x \in A^c} \mathbb{P}^x\{X_1 \in A^c\}$

$$\begin{cases} (K\psi)(x) = e^{-u}\,\psi(x) & x \in A^c \\ \psi(x) = \phi(x) & x \in A \end{cases} \qquad \Leftrightarrow \qquad \psi(x) = \mathbb{E}^x[e^{u\tau_A}\,\phi(X_{\tau_A})]$$

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**Lemma**: Though $K$ not reversible $\forall$ disjoint $A_1, A_2 \subset \Sigma$

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$$\int_{A_1} \pi_0(x) \mathbb{P}^x\{\tau_{A_2}^+ < \tau_{A_1}^+\}\, dx = \int_{A_2} \pi_0(x) \mathbb{P}^x\{\tau_{A_1}^+ < \tau_{A_2}^+\}\, dx$$