Séminaire Analyse Numérique - Equations aux Dérivées Partielles

Convergence vers l'équilibre dans des EDPS paraboliques singulières

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Laboratoire Paul Painlevé Lille, 30 avril 2020
(Séminaire en visio)

Travaux en commun avec Giacomo Di Gesù (Vienne),
Barbara Gentz (Bielefeld) et Hendrik Weber (Bath)

  

Deterministic Allen–Cahn PDE

[Chafee & Infante 74, Allen & Cahn 75]

$$\partial_t u = \Delta u + u - u^3$$

- $u = u(t,x) \in \mathbb{R},\ t \geqslant 0,\ x \in \mathbb{T}_L^d = (\mathbb{R}/L\mathbb{Z})^d,\ L > 0$

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Phase separation ($d = 1$) [Carr & Pego 89, Chen 04] (Link to simulation)

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$$V[u] = \int_{\mathbb{T}_L^d} \Big[\frac{1}{2} |\nabla u(x)|^2 - \frac{1}{2} u(x)^2 + \frac{1}{4} u(x)^4 \Big] \mathrm{d}x \qquad \Rightarrow \nabla_v V[u] = -\langle \partial_t u, v \rangle$$

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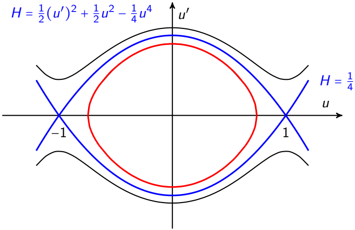
Stationary solutions ($d = 1$):

$u_0''(x) = -u_0(x) + u_0(x)^3$ critical points of V

- $u_\pm(x) \equiv \pm 1$
- $u_0(x) \equiv 0$
- Nonconstant solutions if $L > 2\pi$
  (expressible in terms of Jacobi elliptic fcts)



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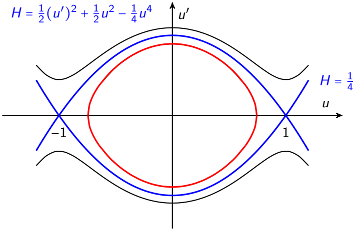

Linearisation at u_0:

$\partial_t v_t(x) = v_t''(x) + [1 - 3u_0(x)^2] v_t(x)$

Stability: Sturm–Liouville problem

Stability of stationary solutions

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Find eigenvalues of $\mathcal{L}$, $(\mathcal{L}v)(x) = v''(x) + [1 - 3u_0(x)^2]v(x)$

- $u_\pm(x) \equiv \pm 1$: stable
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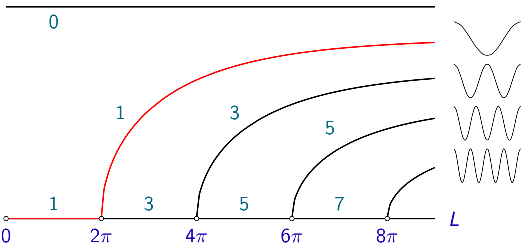
Number of positive eigenvalues
(= unstable directions)
Transition state



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$$\partial_t u = \Delta u + u - u^3 + \sqrt{2\varepsilon}\xi$$

- ξ space-time white noise: centered, Gaussian, $\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y)$

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(Link)

Reversible diffusion in a double-well

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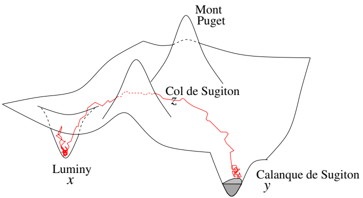
$V : \mathbb{R}^d \to \mathbb{R}$ confining 2-well potential



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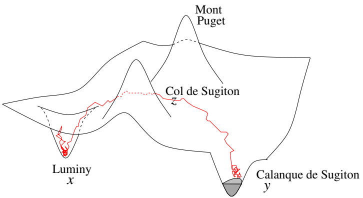

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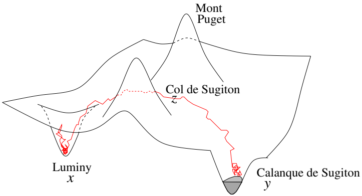

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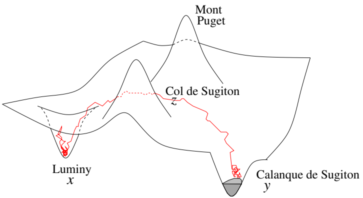

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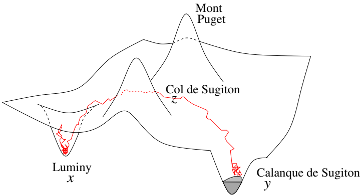

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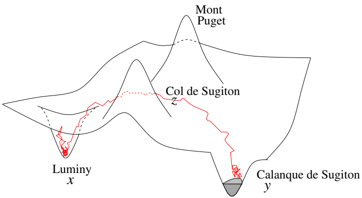

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- ▷ $h_{AB}(x) = \mathbb{P}^x\{\tau_A < \tau_B\}$ satisfies $\begin{cases} (\mathscr{L} h_{AB})(x) = 0 & x \in (A \cup B)^c \\ h_{AB}(x) = 1 & x \in A \\ h_{AB}(x) = 0 & x \in B \end{cases}$

Potential-theoretic proof of Eyring–Kramers law

Theorem: $A, B \subset \mathbb{R}^d$ disjoint. $\exists$ proba measure ν_{AB} on ∂A s.t.

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Theorem: Dirichlet principle

Let $\mathscr{H}_{AB} = \{h : \mathbb{R}^d \to [0,1] : h|_A = 1, h|_B = 0\}$. Then

$$\operatorname{cap}(A,B) = \inf_{h \in \mathcal{H}_{AB}} \mathscr{E}(h) = \mathscr{E}(h_{AB})$$

Appropriate test function yields $\operatorname{cap}(A,B) \simeq \varepsilon \sqrt{\frac{|\lambda_1|}{2\pi\varepsilon}} \sqrt{\frac{(2\pi\varepsilon)^{d-1}}{\lambda_2 \dots \lambda_d}}\,\mathrm{e}^{-V(z)/\varepsilon}$

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- Hessian of V at u_0: $-\Delta - 1$, eigenvalues $\lambda_k = (\frac{2k\pi}{L})^2 - 1$

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- Hessian of V at u_-: $-\Delta + 2$, eigenvalues $\nu_k = (\frac{2k\pi}{L})^2 + 2$

Formally, product of ratios of λ_k/ν_k converges [Maier & Stein '01]

Formal computation and Fredholm determinant

Formally (for $L < 2\pi$)

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$\Delta_\perp$ Laplacian acting on mean zero functions

$$\begin{aligned}\det\Big([-\Delta_\perp - 1][-\Delta_\perp + 2]^{-1}\Big) &= \det\Big([-\Delta_\perp + 2 - 3][-\Delta_\perp + 2]^{-1}\Big) \\ &= \underbrace{\det\Big(\mathbb{1} - 3[-\Delta_\perp + 2]^{-1}\Big)}_{\text{Fredholm determinant}}\end{aligned}$$

Formal computation and Fredholm determinant

Formally (for $L < 2\pi$)

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$$\begin{aligned}\log\det\Big(\mathbb{1} - 3[-\Delta_\perp + 2]^{-1}\Big) &= \operatorname{Tr}\log\Big(\mathbb{1} - 3[-\Delta_\perp + 2]^{-1}\Big) \\ &= -\sum_{n\geqslant 1}\frac{3^n}{n}\underbrace{\operatorname{Tr}\Big([-\Delta_\perp + 2]^{-n}\Big)}_{\lesssim\left[\left(\frac{2\pi}{L}\right)^2+2\right]^{-n}} < \infty \qquad (L < 2\pi)\end{aligned}$$

General fact: $\det(\mathbb{1} + T) < \infty$ if T is trace class

Main result for $d = 1$

Theorem: [B & Gentz, Elec. J. Proba 2013]

- If $L < 2\pi - c$ with $c > 0$, then

$$\mathbb{E}^{u_-}[\tau_+] = 2\pi\sqrt{\det(\mathbb{1} - 3[-\Delta + 2]^{-1})}\, e^{(V[u_0] - V[u_-])/\varepsilon}\Big[1 + \mathcal{O}_\varepsilon(1)\Big]$$

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Remarks:

- Proof relies on spectral Galerkin approximation
- Error more precise than $\mathcal{O}_\varepsilon(1)$
- If $u_{\rm tr} \neq u_0$, Fredholm determinant computed with techniques from path integrals [Maier & Stein]
- Similar results for Neumann b.c.
- Similar results for other nonlinearities than $-u^3$

Allen–Cahn SPDE for $d = 2$

$\partial_t u = \Delta u + u - u^3 + \sqrt{2\varepsilon}\xi \qquad x \in \mathbb{T}_L^2$ (Link to simulation)

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- In fact, the equation needs to be renormalised

Theorem: [Da Prato & Debussche 2003]
Let ξ^δ be a mollification on scale δ of white noise. Then

$$\partial_t u = \Delta u + \bigl[1 + 3\varepsilon C(\delta)\bigr]u - u^3 + \sqrt{2\varepsilon}\,\xi^\delta$$

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- $C(\delta) \sim$ variance of mollified Gaussian free field (GFF)
- Naively, one could expect $u_\pm = \pm\sqrt{1 + 3\varepsilon C(\delta)}$ but this is not the case

Main result in dimension 2

Use spectral Galerkin approximation with cut-off N instead of mollification, $C_N \sim \operatorname{Tr}(P_N[-\Delta+2]^{-1}) \sim \log(N)$

Theorem: [B, Di Gesù, Weber, Elec. J. Proba 2017]

For $L < 2\pi$, appropriate $A \ni u_-$, $B \ni u_+$, $\exists \mu_N$ probability measures on ∂A:

$$\limsup_{N\to\infty} \mathbb{E}^{\mu_N}\big[\tau_B\big] \leqslant 2\pi\sqrt{\det_2(\mathbb{1} - 3[-\Delta+2]^{-1})}\, \mathrm{e}^{(V[u_0]-V[u_-])/\varepsilon}\big[1 + c_+\sqrt{\varepsilon}\big]$$

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- [Tsatsoulis & Weber, PTRF 2018]: Same result for $\mathbb{E}^{u_-}[\tau_B]$

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Thanks for your attention!