Stochastics Meeting Lunteren

I. Trace process and metastability

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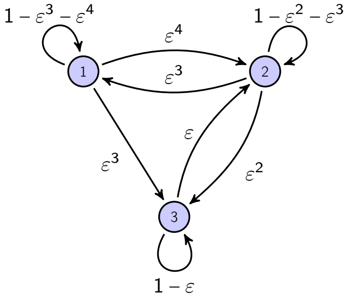
Lunteren, November 12, 2018

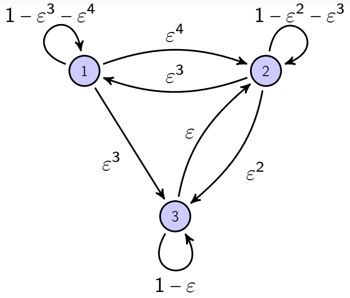
Joint works with Manon Baudel (Ecole des Ponts, Paris) and Damien Landon

Contents

1. Metastable Markov chains on a finite set



A simple example



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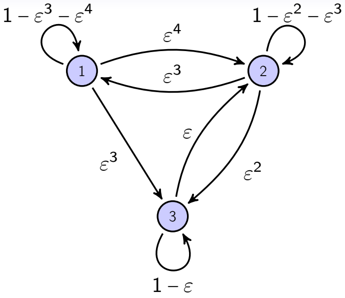

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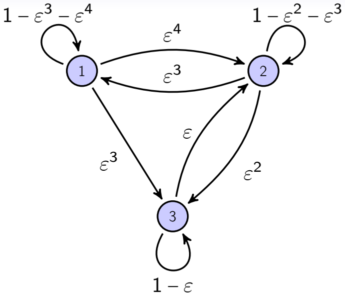

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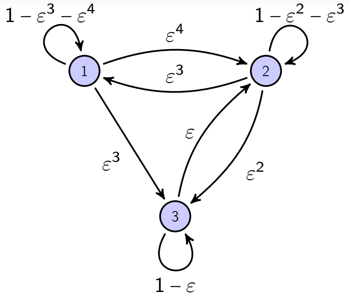

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  Eigenvalues of P:
  $$\lambda_0 = 1$$
  $$\lambda_1 = 1 - 2\varepsilon^3 + \mathcal{O}(\varepsilon^5)$$
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Main question

How to easily determine leading term of spectral gap $1 - \lambda_1$?

- ▷ Linear algebra/analytic methods (singular perturbation theory), e.g. [Schweitzer 68, Hassin & Haviv 92, Avrachenkov & Lasserre 99]
- ▷ Probabilistic methods, e.g. [Wentzell 72, Freidlin & Wentzell 70s, Miclo 95, Beltràn & Landim 2010, Cameron & Vanden–Eijnden 2014, Betz & Le Roux 2016, Cameron & Gan 2016]

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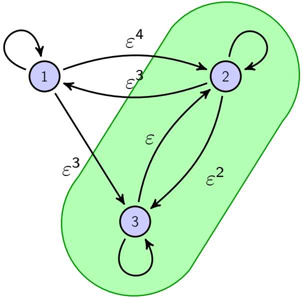
Some probabilistic tools:

- ▷ W-graphs
- ▷ Lumping of states
- ▷ Speeding up time



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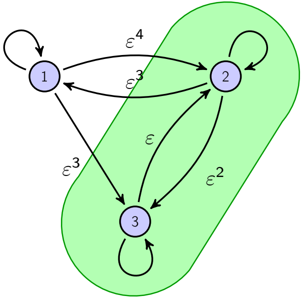
- Here: trace process



Trace process

$\mathcal{X}$ finite, $\{X_n\}_{n\in\mathbb{N}_0}$ irreducible aperiodic M.C., transition matrix P, $A \subset \mathcal{X}$

- Process killed upon leaving A: $P_A(x,y) = P(x,y)\mathbb{1}_{\{x,y\in A\}}$
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&= P(x,y) + \sum_{z\in A^c} P(x,z) \sum_{n\geqslant 1} \mathbb{P}^z\{\tau_A^+ = n, X_{\tau_A^+} = y\} \\
&= P_A(x,y) + \sum_{z,z'\in A^c} P(x,z) \underbrace{\sum_{n\geqslant 1} P_{A^c}^{n-1}(z,z')}_{[\mathbb{1}-P_{A^c}]^{-1}(z,z')} P(z',y)
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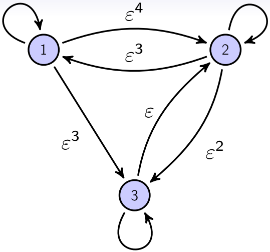
Matrix representation (Schur complement)

$$P = \begin{pmatrix} P_A & P_{AA^c} \\ P_{A^cA} & P_{A^c} \end{pmatrix} \quad \Rightarrow \quad {}_AP = P_A + P_{AA^c}[\mathbb{1} - P_{A^c}]^{-1} P_{A^cA}$$

Application to the example



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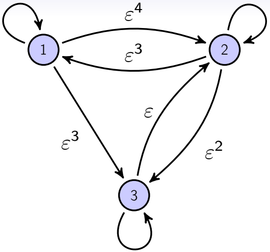

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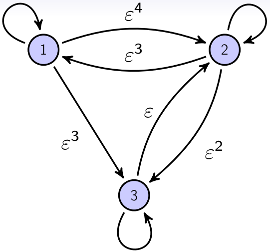

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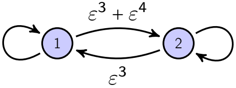

$$\begin{aligned} {}_A\lambda_0 &= 1 \\ {}_A\lambda_1 &= 1 - 2\varepsilon^3 - \varepsilon^4 \end{aligned} \qquad \begin{aligned} \lambda_0 &= 1 \\ \lambda_1 &= 1 - 2\varepsilon^3 + \mathcal{O}(\varepsilon^5) \\ \lambda_2 &= 1 - \varepsilon + \mathcal{O}(\varepsilon^2) \end{aligned}$$

A nice application of the trace process

Recall: the chain in not assumed to be reversible:
$\pi_0(x)P(x,y) \neq \pi_0(y)P(y,x)$ in general

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  Using $\pi_0(x) = 1/\mathbb{E}^x[\tau_x^+]$
- Alternative proof using trace process:
  Remark: $\pi_0|_A$ is invariant by ${}_AP$
  Take $A = \{x, y\}$. Then

$$\begin{aligned}\pi_0(x) &= (\pi_0 {}_AP)(x) \\ &= \pi_0(x)\mathbb{P}^x\{X_{\tau_A^+} = x\} + \pi_0(y)\mathbb{P}^y\{X_{\tau_A^+} = x\} \\ &= \pi_0(x)\big[1 - \mathbb{P}^x\{\tau_y^+ < \tau_x^+\}\big] + \pi_0(y)\mathbb{P}^y\{\tau_x^+ < \tau_y^+\} \quad \square\end{aligned}$$

Good domains

Definition: For $A \subset \mathcal{X}$, let

$$p_{\text{in}}(A) = \inf_{x \in A^c} \mathbb{P}^x\{X_1 \in A\}$$

$$p_{\text{out}}(A) = \sup_{x \in A} \mathbb{P}^x\{X_1 \in A^c\}$$

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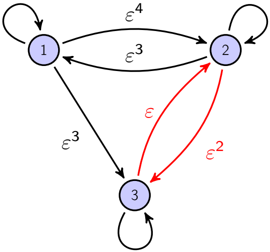
Example:



$A = \{1, 2\}$

$p_{\text{in}}(A) = \varepsilon$

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A is a good domain

Main idea

For a good domain A,

$P = \begin{pmatrix} P_A & P_{AA^c} \\ P_{A^cA} & P_{A^c} \end{pmatrix}$ is well-approximated by $\widehat{P} = \begin{pmatrix} {}_AP & 0 \\ P_{A^cA} & P_{A^c} \end{pmatrix}$

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Norm: $\|Q\| = \sup_{\|\varphi\|_\infty = 1} \|Q\varphi\|_\infty = \sup_{\|\mu\|_1 = 1} \|\mu Q\|_1 = \sup_{x \in \mathcal{X}} \sum_{y \in \mathcal{X}} |Q(x,y)|$

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Fact from spectral theory (using complex analysis, Riesz projector):
$\hat{\lambda}$ simple eigenvalue of $\widehat{P}$ at distance $> \|P - \widehat{P}\|$ from remaining spectrum
$\Rightarrow$ P has unique eigenvalue at distance $\mathcal{O}(\|P - \widehat{P}\|)$ from $\hat{\lambda}$

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Consequence: If $A^c = \{x\}$ then $p_{\text{in}}(A) = 1 - P(x,x) = 1 - \hat{\lambda}$
$\Rightarrow 1 - \lambda = 1 - \hat{\lambda} + \mathcal{O}(p_{\text{out}}(A)) = (1 - \hat{\lambda})\Big[1 + \mathcal{O}\Big(\frac{p_{\text{out}}(A)}{p_{\text{in}}(A)}\Big)\Big]$

Example: $\hat{\lambda}_2 = 1 - \varepsilon$ perturbs to $\lambda_2 = 1 - \varepsilon + \mathcal{O}(\varepsilon^2)$
The argument does not suffice to compare spectra of P_A and ${}_AP$

Laplace transforms

$u \in \mathbb{C} \quad \Rightarrow \quad \mathbb{E}^x[e^{u\tau_A^+}]$ exists for $|e^{-u}| > 1 - p_{\text{in}}(A) \quad (\star)$

Follows from $\mathbb{P}^y\{\tau_A^+ > n\} \leqslant (1 - p_{\text{in}}(A))^n \ \forall y \in A^c$

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Proposition [Feynman–Kac type relation]

Under $(\star)$,

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Uniqueness: Apply Fredholm alternative to difference of two solutions □

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Corollary [Reduction to eigenvalue problem on A]

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Proof of $\Rightarrow$: For $x \in A$

$$\begin{aligned}
e^{-u}\phi(x) = (P\phi)(x) &= \mathbb{E}^x[\phi(X_1)] \\
&= \mathbb{E}^x\left[\phi(X_1)\mathbb{1}_{\{X_1\in A\}}\right] + \mathbb{E}^x\left[\phi(X_1)\mathbb{1}_{\{X_1\in A^c\}}\right] \\
&= \mathbb{E}^x\left[\phi(X_{\tau_A^+})\mathbb{1}_{\{\tau_A^+=1\}}\right] + \mathbb{E}^x\left[\mathbb{E}^{X_1}\left[e^{u\tau_A^+}\phi(X_{\tau_A})\right]\mathbb{1}_{\{\tau_A^+>1\}}\right] \\
&= \mathbb{E}^x\left[\phi(X_{\tau_A^+})\mathbb{1}_{\{\tau_A^+=1\}}\right] + \mathbb{E}^x\left[e^{u(\tau_A^+-1)}\phi(X_{\tau_A^+})\mathbb{1}_{\{\tau_A^+>1\}}\right] = ({}_AP^u\phi)(x)
\end{aligned}$$

Laplace transforms

Corollary [Reduction to eigenvalue problem on A]

Under $(\star)$, $P\phi = \mathrm{e}^{-u}\phi$ in $\mathcal{X}$ $\Leftrightarrow$ ${}_AP^u\phi = \mathrm{e}^{-u}\phi$ in A
where ${}_AP^u(x,y) = \mathbb{E}^x\Big[\mathrm{e}^{u(\tau_A^+-1)}\,\mathbb{1}_{\{X_{\tau_A^+}=y\}}\Big]$ is such that ${}_AP^0 = {}_AP$

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\end{aligned}$$

⚠ ${}_AP^u$ depends on $u \Rightarrow$ solve system $({}_AP^u\phi = \lambda\phi, \mathrm{e}^{-u} = \lambda)$

Proposition

$$\|{}_AP^u - {}_AP^0\| \leqslant \frac{|1-\mathrm{e}^{-u}|\sup_{x\in A}\mathbb{E}^x[\tau_A^+ - 1]}{1 - |1-\mathrm{e}^{-u}|\sup_{x\in A^c}\mathbb{E}^x[\tau_A^+]} \leqslant \frac{|1-\mathrm{e}^{-u}|p_{\text{out}}(A)}{p_{\text{in}}(A) - |1-\mathrm{e}^{-u}|}$$

Main result – nondegenerate case

Algorithm in nondegenerate case:

- Assume $\exists x \in \mathcal{X}$ such that $1 - P(x,x) \gg 1 - P(y,y)\ \forall y \neq x$
- Take $A = \mathcal{X} \setminus \{x\}$ (A is a good set)
- Then $\mathbb{1} - P$ has ev $1 - \lambda = P(x,x)\big[1 + \mathcal{O}(p_{\text{in}}(A)/p_{\text{out}}(A))\big]$ $\quad \in \mathbb{R}$
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Theorem [Baudel & B, 2017]

- Non-degenerate case: $\exists A_1 \subset A_2 \subset \dots \subset A_n = \mathcal{X}$ s.t. $\#(A_{k+1} \setminus A_k) = 1$, each A_k good set for ${}_{A_{k+1}}P$
  Renumber states s.t. $A_k = \{1,\dots,k\}$. Then
- $\lambda_0 = 1,\ \lambda_k = 1 - \mathbb{P}^{k+1}\{\tau^+_{A_k} < \tau^+_{k+1}\}\Big[1 + \mathcal{O}\Big(\frac{p_{\text{out}}(A_k|A_{k+1})}{p_{\text{in}}(A_k|A_{k+1})}\Big)\Big] \quad \in \mathbb{R}$

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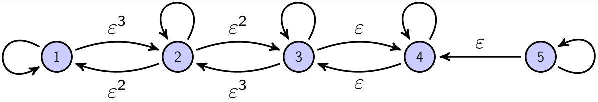
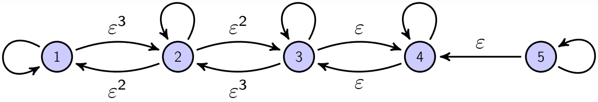
- kth left eigenvector π_k close to quasistationary distribution (QSD) of P_{A_k} (left eigenvect of P_{A_k} for Perron–Frobenius principal eigenval)

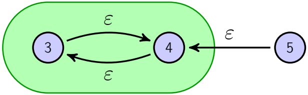
Algorithm in degenerate case



Algorithm in degenerate case



Degenerate part, leading order:

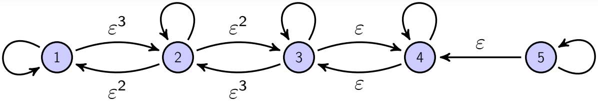


Eigenvalues: 1

$1 - \varepsilon$

$1 - 2\varepsilon$

Algorithm in degenerate case



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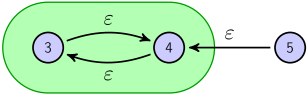

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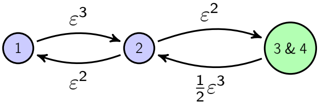
Effective trace process:



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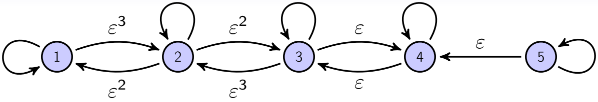

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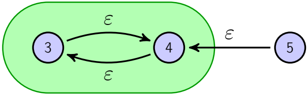

Eigenvalues: 1

$1-\varepsilon$

$1-2\varepsilon$

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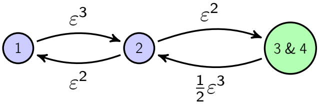
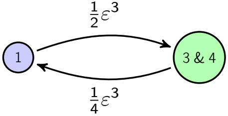


Eigenvalues: $1-2\varepsilon^2$

Trace on $\{1, 3\&4\}$:



$1-\frac{3}{4}\varepsilon^3$

1

2. Continuous-space Markov chains

$$\mathbb{P}\{X_{n+1} \in A | X_n = x\} = \int_A k_\sigma(x, y)\,\mathrm{d}y$$

Continuous-space Markov chains

$(X_n)_{n\in\mathbb{N}_0}$ Markov chain in $\mathcal{X} \subset \mathbb{R}^d$ with kernel K_σ:

$$\mathbb{P}\{X_{n+1} \in A | X_n = x\} = K_\sigma(x, A) = \int_A K_\sigma(x, \mathrm{d}y)$$

- $K_0(x, A) = 1\!\!1_{\{\Pi(x)\in A\}}$ defined by deterministic map $\Pi : \mathcal{X} \to \mathcal{X}$
- For $\sigma > 0$, K_σ admits continuous density k_σ

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Example 1: Randomly perturbed map

$$X_{n+1} = \Pi(X_n) + \sigma\xi_{n+1}$$

$(\xi_n)_{n\geqslant 1}$ i.i.d. r.v. with density (e.g. $\sigma\xi_n$ Gaussian of variance σ^2)

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Example 2: Random Poincaré map

SDE

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sigma g(x_t)\,\mathrm{d}W_t$$

X_n suitably defined location of nth return to surface of section $\Sigma \subset \mathcal{X}$

Assumptions

Assumption 1: Deterministic dynamics

$\Pi : \mathcal{X} \to \mathcal{X}$ admits positively invariant compact set $\mathcal{X}_0 \subset \mathcal{X}$, finitely many limit sets in $\mathcal{X}_0$, all hyperbolic fixed points, N of which are stable

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Assumption 2: Large-deviation principle

K_σ satisfies LDP with good rate function I ($K_\sigma(x, A) \sim e^{-\inf_A I(x,\cdot)/\sigma^2}$)
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Assumption 4: Uniform positivity (Doeblin-type condition)

$\forall x_i^\star$ stable fixed point, $\exists B_i$ nbh of $x_i^\star$ s.t. $k_i = {}_{B_1 \cup \dots \cup B_i} k_{B_i}$ satisfies

$$\sup_{x \in B_i} k_i^n(x, y) \leqslant L \inf_{x \in B_i} k_i^n(x, y) \quad \forall y \in B_i \qquad \text{for some } L \in (1, 2),\ n(\sigma) \in \mathbb{N}$$

Main result

Theorem [Baudel & B, 2017]

- Non-degenerate case ($x_1^\star, \dots, x_N^\star$ in metastable order)
  - Eigenvalues of K_σ:

$$
\begin{aligned}
\lambda_0 &= 1 \\
\lambda_k &= 1 - \mathbb{P}^{\mathring{\pi}_0^{k+1}} \{ \tau^+_{B_1 \cup \dots \cup B_k} < \tau^+_{B_{k+1}} \} \big[1 + \mathcal{O}(\mathrm{e}^{-\theta/\sigma^2}) \big] \in \mathbb{R} & 1 \leqslant k < N \\
|\lambda_k| &< \varrho = 1 - \frac{c}{\log(\sigma^{-1})} & k \geqslant N
\end{aligned}
$$

  where $\mathring{\pi}_0^{k+1}$ is a certain QSD on B_{k+1} and $c, \theta > 0$
  - kth right eigenfunction ϕ_k close to $\mathbb{P}^x\{\tau_{B_{k+1}} < \tau_{B_1 \cup \dots \cup B_k}\}$
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- Degenerate case: similar to finite chain...

Approximation result

Theorem: Approximation by a finite Markov chain [Baudel & B, 2017]

$\exists m(\sigma)$, (signed) measures μ_i s.t. $\mu_i(B_j) = \delta_{ij}$:

$$\mathbb{P}^{\mu_i}\{X_{\tau^{+,nm}_{B_1\cup\dots\cup B_N}} \in B_j\} = \mathbb{P}^i\{Y_n = j\} + \underbrace{\mathcal{O}(\mathrm{e}^{-\theta/\sigma^2})}_{\text{uniform in } n}$$

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where $(Y_n)_{n\in\mathbb{N}_0}$ Markov chain with matrix

$$P_{ij} = \mathbb{P}^{\mathring{\pi}_0^{B_i}}\{X_{\tau^{+,nm}_{B_1\cup\dots\cup B_N}} \in B_j\}[1 + \mathcal{O}(\mathrm{e}^{-\theta/\sigma^2})]$$

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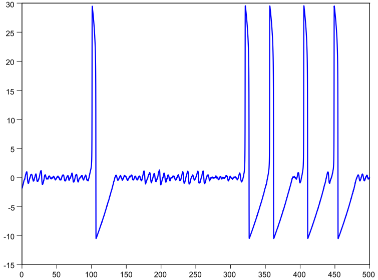
Truncated spectral decomposition of ${}_{B_1\cup\dots\cup B_N}K$:

$$K^0_{\mathrm{tr}} = \Pi^0({}_{B_1\cup\dots\cup B_N}K) \qquad \Pi^0(x,\mathrm{d}y) = \sum_{k=0}^{N-1} \phi^0_k(x)\pi^0_k(\mathrm{d}y)$$

$P_{ij} = \mu_i(K^0_{\mathrm{tr}})^m\psi_j$ where $\mu_i = \mathring{\pi}_0^{B_i}\Pi^0$, $\psi_j = \Pi^0 1\!\!1_{B_j}$, $\|\psi_j - 1\!\!1_{B_j}\|_\infty \leqslant \mathrm{e}^{-\theta/\sigma^2}$

3. FitzHugh–Nagumo equations



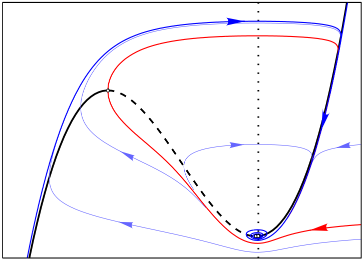
Stochastic FitzHugh–Nagumo equation

$$dx_t = \frac{1}{\varepsilon}\left[x_t - x_t^3 + y_t\right] dt$$ neuron membrane potential

$$dy_t = \left[a - x_t - by_t\right] dt$$ open ion channels

▷ $b = 0$ for simplicity in this talk, bifurcation parameter $\delta := \frac{3a^2-1}{2}$

$\varepsilon = 0.1$
$\delta = 0.02$



Stochastic FitzHugh–Nagumo equation

$$dx_t = \frac{1}{\varepsilon}\left[x_t - x_t^3 + y_t\right] dt + \frac{\sigma_1}{\sqrt{\varepsilon}}\, dW_t^{(1)} \qquad \text{neuron membrane potential}$$

$$dy_t = \left[a - x_t - b y_t\right] dt + \sigma_2\, dW_t^{(2)} \qquad \text{open ion channels}$$

- $b = 0$ for simplicity in this talk, bifurcation parameter $\delta := \frac{3a^2-1}{2}$
- $W_t^{(1)}, W_t^{(2)}$: independent Wiener processes
- $0 < \sigma_1, \sigma_2 \ll 1$, $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$

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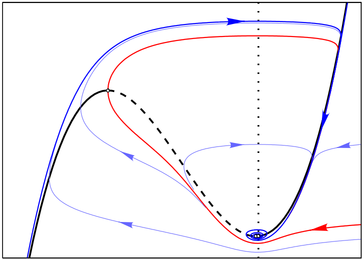

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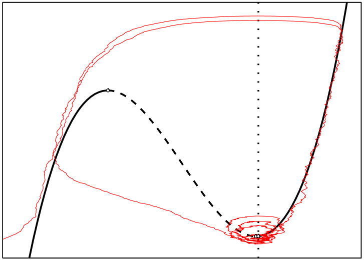
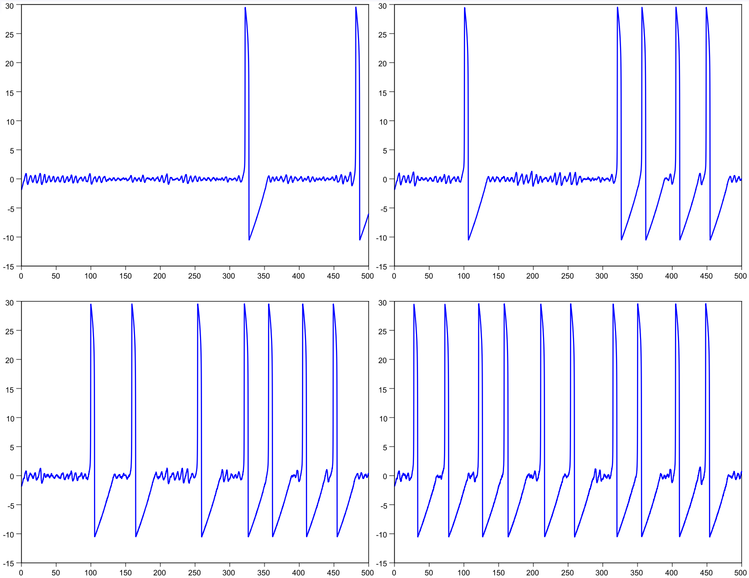
$\delta = 0.02$
$\sigma_1 = \sigma_2 = 0.03$

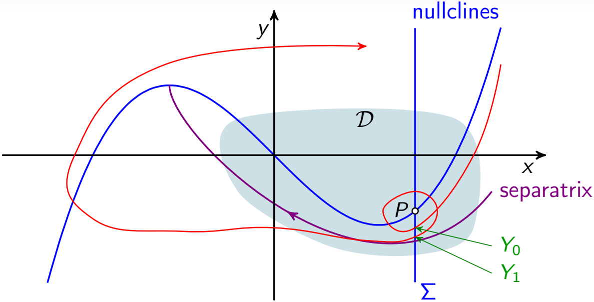


Mixed-mode oscillations (MMOs)



Time series $t \mapsto -x_t$ for $\varepsilon = 0.01$, $\delta = 3 \cdot 10^{-3}$, $\sigma = 1.46 \cdot 10^{-4}, \dots, 3.65 \cdot 10^{-4}$

Random Poincaré map



$Y_0, Y_1, \dots$ substochastic Markov chain describing process killed on $\partial\mathcal{D}$
Number of small oscillations: $N = \inf\{n \geqslant 1 \colon Y_n \notin \Sigma\}$

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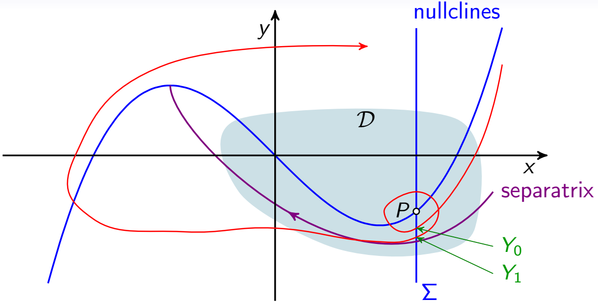

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Theorem 1 [B & Landon, 2012]

N is asymptotically geometric: $\lim_{n\to\infty} \mathbb{P}\{N = n+1 | N > n\} = 1 - \lambda_0$

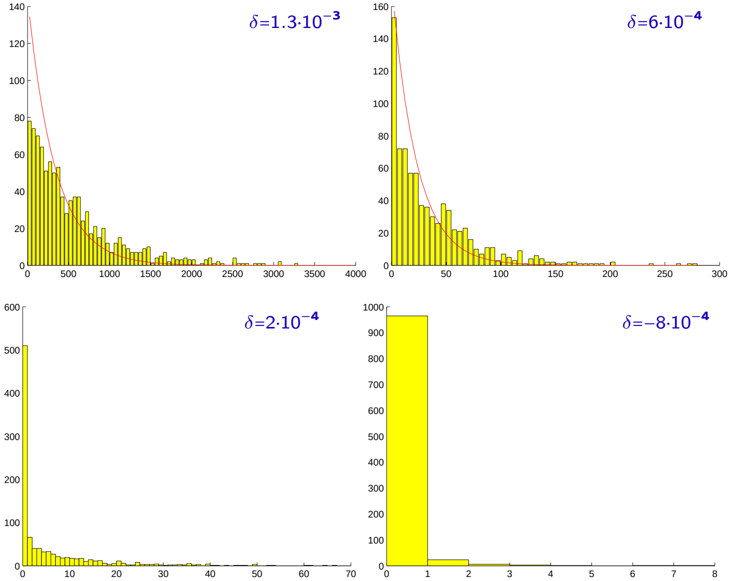
where $\lambda_0 \in \mathbb{R}_+$: principal eigenvalue of the kernel K, $\lambda_0 < 1$ if $\sigma > 0$

Proof: follows from existence of spectral gap

▸▸ Details

Histograms of distribution of N (1000 spikes)

$\sigma = \varepsilon = 10^{-4}$



Weak-noise regime

Theorem 2 [B & Landon, 2012]

Assume ε and $\delta/\sqrt{\varepsilon}$ sufficiently small
There exists $\kappa > 0$ s.t. for $\sigma^2 \leqslant (\varepsilon^{1/4}\delta)^2/\log(\sqrt{\varepsilon}/\delta)$

- Principal eigenvalue:

$$1 - \lambda_0 \leqslant \exp\left\{-\kappa \frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$$

- Expected number of small oscillations:

$$\mathbb{E}^{\mu_0}[N] \geqslant C(\mu_0)\exp\left\{\kappa \frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$$

where $C(\mu_0)$ = probability of starting on Σ above separatrix

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There exists $\kappa > 0$ s.t. for $\sigma^2 \leqslant (\varepsilon^{1/4}\delta)^2/\log(\sqrt{\varepsilon}/\delta)$

- Principal eigenvalue:

$$1 - \lambda_0 \leqslant \exp\left\{-\kappa \frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$$

- Expected number of small oscillations:

$$\mathbb{E}^{\mu_0}[N] \geqslant C(\mu_0) \exp\left\{\kappa \frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$$

where $C(\mu_0)$ = probability of starting on Σ above separatrix

Proof: Let $A \subset \Sigma$ have positive Lebesgue measure

$$\lambda_0\pi_0(A) = \int_\Sigma \pi_0(\mathrm{d}x)K(x,A) \geqslant \int_A \pi_0(\mathrm{d}x)K(x,A) \quad \Rightarrow \quad \lambda_0 \geqslant \inf_{x\in A} K(x,A)$$

$\Rightarrow$ construct A such that $K(x,A)$ exponentially close to 1 for all $x \in A$

Dynamics near the separatrix

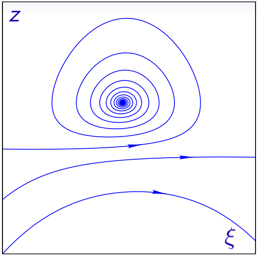
Change of variables:

- ▷ Translate to Hopf bif. point
- ▷ Scale space and time
- ▷ Straighten nullcline $\dot{x} = 0$

$\Rightarrow$ variables (ξ, z) where nullcline: $\{z = \frac{1}{2}\}$

$$d\xi_t = \left(\frac{1}{2} - z_t - \frac{\sqrt{\varepsilon}}{3}\xi_t^3\right)dt$$

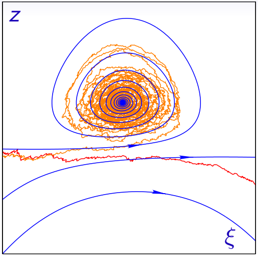
$$dz_t = \left(\mu + 2\xi_t z_t + \frac{2\sqrt{\varepsilon}}{3}\xi_t^4\right)dt$$



where

$$\mu = \frac{\delta}{\sqrt{\varepsilon}}$$

Dynamics near the separatrix



Change of variables:

▷ Translate to Hopf bif. point

▷ Scale space and time

▷ Straighten nullcline $\dot{x} = 0$

$\Rightarrow$ variables (ξ, z) where nullcline: $\{z = \frac{1}{2}\}$

$$\mathrm{d}\xi_t = \left(\frac{1}{2} - z_t - \frac{\sqrt{\varepsilon}}{3}\xi_t^3\right)\mathrm{d}t + \tilde{\sigma}_1\,\mathrm{d}W_t^{(1)}$$

$$\mathrm{d}z_t = \left(\tilde{\mu} + 2\xi_t z_t + \frac{2\sqrt{\varepsilon}}{3}\xi_t^4\right)\mathrm{d}t - 2\tilde{\sigma}_1\xi_t\,\mathrm{d}W_t^{(1)} + \tilde{\sigma}_2\,\mathrm{d}W_t^{(2)}$$

where

$$\tilde{\mu} = \frac{\delta}{\sqrt{\varepsilon}} - \tilde{\sigma}_1^2 \qquad \tilde{\sigma}_1 = -\sqrt{3}\frac{\sigma_1}{\varepsilon^{3/4}} \qquad \tilde{\sigma}_2 = \sqrt{3}\frac{\sigma_2}{\varepsilon^{3/4}}$$

Upward drift dominates if $\tilde{\mu}^2 \gg \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 \Rightarrow (\varepsilon^{1/4}\delta)^2 \gg \sigma_1^2 + \sigma_2^2$

Rotation around P: use that $2z\,\mathrm{e}^{-2z-2\xi^2+1}$ is constant for $\tilde{\mu} = \varepsilon = 0$

Take $A = \{z > \tilde{\mu}^{1-\gamma}\}$ with $0 < \gamma < \frac{1}{4}$ □

From below to above threshold

Linear approximation:

$$\mathrm{d}z_t^0 = \left(\tilde{\mu} + t z_t^0\right) \mathrm{d}t - \tilde{\sigma}_1 t \, \mathrm{d}W_t^{(1)} + \tilde{\sigma}_2 \, \mathrm{d}W_t^{(2)}$$

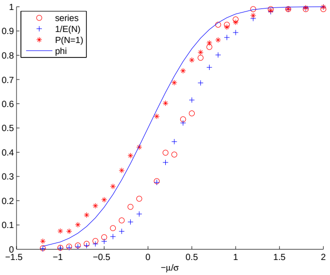
$$\Rightarrow \quad \mathbb{P}\{\text{no small osc}\} \simeq \Phi\Big(-\pi^{1/4} \frac{\tilde{\mu}}{\sqrt{\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2}}\Big) \qquad \Phi(x) = \int_{-\infty}^{x} \frac{\mathrm{e}^{-y^2/2}}{\sqrt{2\pi}} \, \mathrm{d}y$$

From below to above threshold

Linear approximation:

$$\mathrm{d}z_t^0 = \left(\tilde\mu + t z_t^0\right)\mathrm{d}t - \tilde\sigma_1\, t\,\mathrm{d}W_t^{(1)} + \tilde\sigma_2\,\mathrm{d}W_t^{(2)}$$

$$\Rightarrow \quad \mathbb{P}\{\text{no small osc}\} \simeq \Phi\biggl(-\pi^{1/4}\frac{\tilde\mu}{\sqrt{\tilde\sigma_1^2+\tilde\sigma_2^2}}\biggr) \qquad \Phi(x) = \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,\mathrm{d}y$$



$\ast$: $\mathbb{P}\{\text{no small osc}\}$

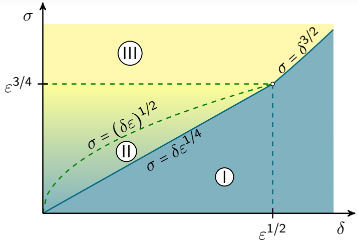
$+$: $1/\mathbb{E}[N]$

$\circ$: $1-\lambda_0$

curve: $x \mapsto \Phi(\pi^{1/4}x)$

$$x = -\frac{\tilde\mu}{\sqrt{\tilde\sigma_1^2+\tilde\sigma_2^2}} = -\frac{\varepsilon^{1/4}(\delta-\sigma_1^2/\varepsilon)}{\sqrt{\sigma_1^2+\sigma_2^2}}$$

Summary: Parameter regimes



$\sigma_1 = \sigma_2$:

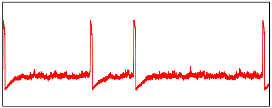
$\mathbb{P}\{N = 1\} \simeq \Phi\Big(-\frac{(\pi\varepsilon)^{1/4}(\delta - \sigma^2/\varepsilon)}{\sigma}\Big)$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\,dy$$

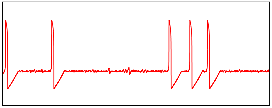
see also

[Muratov & Vanden Eijnden '08]

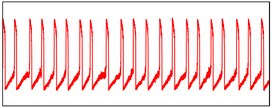
Regime I: rare isolated spikes
Theorem 2 applies ($\delta \ll \varepsilon^{1/2}$)
Interspike interval $\simeq$ exponential



Regime II: clusters of spikes
interspike osc asympt geometric
$\sigma = (\delta\varepsilon)^{1/2}$: geom(1/2)



Regime III: repeated spikes
$\mathbb{P}\{N = 1\} \simeq 1$
Interspike interval $\simeq$ constant



Outlook

- ▷ Finite $\mathcal{X}$ case: simple algorithm to obtain eigenvalues and vectors (complexity $\mathcal{O}(n^2)$, $n = \#(\mathcal{X})$)
- ▷ Continuous-space Markov chains: eigen-elements in terms of committors and QSDs
- ▷ Needed: better ways to approximate QSDs and committors

References:

- ▷ N. B. & Damien Landon, *Mixed-mode oscillations and interspike interval statistics in the stochastic FitzHugh–Nagumo model*, Nonlinearity **25**, 2303-2335 (2012)
- ▷ Manon Baudel & N. B., *Spectral theory for random Poincaré maps*, SIAM J. Math. Analysis **49**, 4319–4375 (2017)

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- ▷ N. B., Barbara Gentz & Christian Kuehn, *From random Poincaré maps to stochastic mixed-mode-oscillation patterns*, J. Dynam. Diff. Eq. **27**, 83–136 (2015)
- ▷ N. B. & Barbara Gentz, *On the noise-induced passage through an unstable periodic orbit II: General case*, SIAM J. Math. Analysis **46**, 310–352 (2014)

Proof of asymptotically geometric distribution

Theorem 1 [B & Landon, 2012]

N is asymptotically geometric: $\lim_{n\to\infty} \mathbb{P}\{N = n+1 | N > n\} = 1 - \lambda_0$

where $\lambda_0 \in (0,1)$ if $\sigma > 0$ is principal eigenvalue of the kernel K

Proof:

Markov chain on Σ, kernel K with density k [Ben Arous, Kusuoka, Stroock '84]

- $\lambda_0 \leqslant \sup_{x\in\Sigma} K(x,\Sigma) < 1$ by ellipticity (k bounded below)
- $$\begin{aligned}\mathbb{P}^{\mu_0}\{N > n\} = \mathbb{P}^{\mu_0}\{X_n \in \Sigma\} &= \int_\Sigma \mu_0(\mathrm{d}x) K^n(x,\Sigma) \\ &= \int_\Sigma \mu_0(\mathrm{d}x) \lambda_0^n \phi_0(x) [1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)] \\ &= \lambda_0^n \langle \mu_0, \phi_0 \rangle [1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)]\end{aligned}$$
- $$\begin{aligned}\mathbb{P}^{\mu_0}\{N = n+1\} &= \int_\Sigma \int_\Sigma \mu_0(\mathrm{d}x) K^n(x,\mathrm{d}y)[1 - K(y,\Sigma)] \\ &= \lambda_0^n (1-\lambda_0) \langle \mu_0, \phi_0 \rangle [1 + \mathcal{O}((|\lambda_1|/\lambda_0)^n)]\end{aligned}$$
- Existence of spectral gap follows from positivity condition [Birkhoff '57]

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