Stochastics Meeting Lunteren

II. Metastable dynamics of stochastic PDEs

Nils Berglund

Institut Denis Poisson, Université d'Orléans, France

Lunteren, November 12, 2018

Joint works with Giacomo Di Gesù (Vienna), Bastien Fernandez (Paris),
Barbara Gentz (Bielefeld) and Hendrik Weber (Warwick/Bath)




Contents

1. Reversible metastable Markov chains

$$\mathbb{E}^{\nu_{AB}}[\tau_B] = \frac{1}{\text{cap}(A,B)} \sum_{x \in B^c} \pi(x) \mathbb{P}^x\{\tau_A < \tau_B\}$$

Reversible Markov chains

[Bovier, den Hollander 2015], [Slowik, PhD thesis, TU Berlin 2012]

$\mathcal{X}$ countable, $\{X_n\}_{n\in\mathbb{N}_0}$ irreducible positive rec M.C., matrix P, $\pi P = \pi$

- Reversible: $\pi(x)P(x,y) = \pi(y)P(y,x) \ \ \forall x,y \in \mathcal{X}$ (detailed balance)

  $\Rightarrow$ P self-adjoint in $L^2(\mathcal{X},\pi)$, $\langle f,g\rangle = \sum_{x\in\mathcal{X}} \pi(x)f(x)g(x)$

Reversible Markov chains

[Bovier, den Hollander 2015], [Slowik, PhD thesis, TU Berlin 2012]

$\mathcal{X}$ countable, $\{X_n\}_{n\in\mathbb{N}_0}$ irreducible positive rec M.C., matrix P, $\pi P = \pi$

- Reversible: $\pi(x)P(x,y) = \pi(y)P(y,x) \ \ \forall x,y \in \mathcal{X}$ (detailed balance)

  $\Rightarrow$ P self-adjoint in $L^2(\mathcal{X},\pi)$, $\langle f,g\rangle = \sum_{x\in\mathcal{X}} \pi(x)f(x)g(x)$

- Generator: $(Lf)(x) := \mathbb{E}^x[f(X_1)] - f(x) = \sum_{y\in\mathcal{X}} P(x,y)[f(y)-f(x)]$

Reversible Markov chains

[Bovier, den Hollander 2015], [Slowik, PhD thesis, TU Berlin 2012]

$\mathcal{X}$ countable, $\{X_n\}_{n\in\mathbb{N}_0}$ irreducible positive rec M.C., matrix P, $\pi P = \pi$

- Reversible: $\pi(x)P(x,y) = \pi(y)P(y,x) \ \ \forall x,y \in \mathcal{X}$ (detailed balance)

  $\Rightarrow$ P self-adjoint in $L^2(\mathcal{X},\pi)$, $\langle f,g\rangle = \sum_{x\in\mathcal{X}} \pi(x)f(x)g(x)$

- Generator: $(Lf)(x) := \mathbb{E}^x[f(X_1)] - f(x) = \sum_{y\in\mathcal{X}} P(x,y)[f(y) - f(x)]$

- Dirichlet form: $\mathcal{E}(f) := \langle f, -Lf\rangle = \frac{1}{2} \sum_{x,y\in\mathcal{X}} \pi(x)P(x,y)[f(x) - f(y)]^2$

  $\mathcal{E}(f,g) := \langle f, -Lg\rangle$ (polarization)

Reversible Markov chains

[Bovier, den Hollander 2015], [Slowik, PhD thesis, TU Berlin 2012]

$\mathcal{X}$ countable, $\{X_n\}_{n\in\mathbb{N}_0}$ irreducible positive rec M.C., matrix P, $\pi P = \pi$

- Reversible: $\pi(x)P(x,y) = \pi(y)P(y,x) \;\; \forall x,y \in \mathcal{X}$ (detailed balance)

  $\Rightarrow$ P self-adjoint in $L^2(\mathcal{X},\pi)$, $\langle f,g\rangle = \sum_{x\in\mathcal{X}} \pi(x)f(x)g(x)$

- Generator: $(Lf)(x) := \mathbb{E}^x[f(X_1)] - f(x) = \sum_{y\in\mathcal{X}} P(x,y)[f(y) - f(x)]$

- Dirichlet form: $\mathcal{E}(f) := \langle f, -Lf\rangle = \dfrac{1}{2}\sum_{x,y\in\mathcal{X}} \pi(x)P(x,y)[f(x) - f(y)]^2$

  $\mathcal{E}(f,g) := \langle f, -Lg\rangle$ (polarization)

- Green function (fundamental matrix): For $\varnothing \neq A \subset \mathcal{X}$, $G_A := L_{A^c}^{-1}$

  $\pi(x)G_A(x,y) = \pi(y)G_A(y,x) \;\; \forall x,y \in A^c$

Expected hitting times and committors

- Given $\varnothing \neq A \subset \mathcal{X}$

$$w_A(x) = \mathbb{E}^x[\tau_A] \qquad \text{satisfies} \qquad \begin{cases} (Lw_A)(x) = -1 & x \in A^c \\ w_A(x) = 0 & x \in A \end{cases}$$

$$\Rightarrow w_A(x) = -\sum_{y \in A^c} G_A(x,y)$$

Expected hitting times and committors

- Given $\varnothing \neq A \subset \mathcal{X}$

$$w_A(x) = \mathbb{E}^x[\tau_A] \qquad \text{satisfies} \qquad \begin{cases} (Lw_A)(x) = -1 & x \in A^c \\ w_A(x) = 0 & x \in A \end{cases}$$

$$\Rightarrow w_A(x) = -\sum_{y \in A^c} G_A(x,y)$$

- Committor: Given $\varnothing \neq A, B \subset \mathcal{X}$, $A \cap B = \varnothing$

$$h_{AB}(x) = \mathbb{P}^x\{\tau_A < \tau_B\} \quad \text{satisfies} \quad \begin{cases} (Lh_{AB})(x) = 0 & x \in (A \cup B)^c \\ h_{AB}(x) = 1 & x \in A \\ h_{AB}(x) = 0 & x \in B \end{cases}$$

Expected hitting times and committors

- Given $\emptyset \neq A \subset \mathcal{X}$

$$w_A(x) = \mathbb{E}^x[\tau_A] \qquad \text{satisfies} \qquad \begin{cases} (Lw_A)(x) = -1 & x \in A^c \\ w_A(x) = 0 & x \in A \end{cases}$$

$$\Rightarrow w_A(x) = -\sum_{y \in A^c} G_A(x,y)$$

- Committor: Given $\emptyset \neq A, B \subset \mathcal{X}$, $A \cap B = \emptyset$

$$h_{AB}(x) = \mathbb{P}^x\{\tau_A < \tau_B\} \quad \text{satisfies} \quad \begin{cases} (Lh_{AB})(x) = 0 & x \in (A \cup B)^c \\ h_{AB}(x) = 1 & x \in A \\ h_{AB}(x) = 0 & x \in B \end{cases}$$

Lemma: Let $e_{AB}(x) := -(Lh_{AB})(x) \quad \forall x \in A$ (equilibrium measure)

Then $h_{AB}(x) = -\sum_{y \in A} G_B(x,y) e_{AB}(y) \quad \forall x \in (A \cup B)^c$

Expected hitting times and committors

- Given $\varnothing \neq A \subset \mathcal{X}$

$$w_A(x) = \mathbb{E}^x[\tau_A] \qquad \text{satisfies} \qquad \begin{cases} (Lw_A)(x) = -1 & x \in A^c \\ w_A(x) = 0 & x \in A \end{cases}$$

$$\Rightarrow w_A(x) = -\sum_{y \in A^c} G_A(x,y)$$

- Committor: Given $\varnothing \neq A, B \subset \mathcal{X}$, $A \cap B = \varnothing$

$$h_{AB}(x) = \mathbb{P}^x\{\tau_A < \tau_B\} \quad \text{satisfies} \quad \begin{cases} (Lh_{AB})(x) = 0 & x \in (A \cup B)^c \\ h_{AB}(x) = 1 & x \in A \\ h_{AB}(x) = 0 & x \in B \end{cases}$$

Lemma: Let $e_{AB}(x) := -(Lh_{AB})(x) \quad \forall x \in A$ (equilibrium measure)

Then $h_{AB}(x) = -\sum_{y \in A} G_B(x,y) e_{AB}(y) \quad \forall x \in (A \cup B)^c$

Proof: $L_{B^c}\begin{pmatrix} h_{AB} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -e_{AB} \end{pmatrix} \Rightarrow \begin{pmatrix} h_{AB} \\ 1 \end{pmatrix} = G_B\begin{pmatrix} 0 \\ -e_{AB} \end{pmatrix}$ □

Potential theory

Capacity: Given A, B as before,

$$\text{cap}(A, B) := \sum_{x \in A} \pi(x) e_{AB}(x)$$

Potential theory

Capacity: Given A, B as before,

$$\mathrm{cap}(A,B) := \sum_{x\in A} \pi(x) e_{AB}(x) = \mathcal{E}(h_{AB})$$

(since $e_{AB}(x) = (-Lh_{AB})(x)$, $h_{AB} = 1\!\!1_A$ in $A \cup B$, $Lh_{AB} = 0$ in $(A\cup B)^c$)

$\Rightarrow \nu_{AB}(x) = \dfrac{\pi(x)e_{AB}(x)}{\mathrm{cap}(A,B)}$ is probability on A

Potential theory

Capacity: Given A, B as before,

$$\operatorname{cap}(A,B) := \sum_{x \in A} \pi(x) e_{AB}(x) = \mathcal{E}(h_{AB})$$

(since $e_{AB}(x) = (-Lh_{AB})(x)$, $h_{AB} = 1\!\!1_A$ in $A \cup B$, $Lh_{AB} = 0$ in $(A \cup B)^c$)

$\Rightarrow \nu_{AB}(x) = \dfrac{\pi(x) e_{AB}(x)}{\operatorname{cap}(A,B)}$ is probability on A

Theorem ("Magic" formula):

$$\mathbb{E}^{\nu_{AB}}[\tau_B] := \sum_{x \in A} \nu_{AB}(x) w_B(x) = \frac{1}{\operatorname{cap}(A,B)} \sum_{x \in B^c} \pi(x) h_{AB}(x)$$

Potential theory

Capacity: Given A, B as before,

$$\operatorname{cap}(A,B) := \sum_{x\in A} \pi(x) e_{AB}(x) = \mathcal{E}(h_{AB})$$

(since $e_{AB}(x) = (-Lh_{AB})(x)$, $h_{AB} = \mathbb{1}_A$ in $A \cup B$, $Lh_{AB} = 0$ in $(A \cup B)^c$)

$\Rightarrow \nu_{AB}(x) = \dfrac{\pi(x) e_{AB}(x)}{\operatorname{cap}(A,B)}$ is probability on A

Theorem ("Magic" formula):

$$\mathbb{E}^{\nu_{AB}}[\tau_B] := \sum_{x\in A} \nu_{AB}(x) w_B(x) = \frac{1}{\operatorname{cap}(A,B)} \sum_{x\in B^c} \pi(x) h_{AB}(x)$$

Proof: Divide by $\operatorname{cap}(A,B)$ the identity

$$\begin{aligned}\sum_{x\in A} \pi(x) e_{AB}(x) w_B(x) &= -\sum_{x\in A}\sum_{y\in B^c} \pi(x) G_B(x,y) e_{AB}(x) \\ &= -\sum_{y\in B^c}\sum_{x\in A} \pi(y) G_B(y,x) e_{AB}(x) = \sum_{y\in B^c} \pi(y) h_{AB}(y)\end{aligned}$$

□

Upper bound on capacity: Dirichlet principle

Theorem: Dirichlet principle

Let $\mathcal{H}_{AB} = \{h : \mathcal{X} \to [0,1] : h|_A = 1, h|_B = 0\}$. Then

$$\text{cap}(A,B) = \inf_{h \in \mathcal{H}_{AB}} \mathcal{E}(h) = \mathcal{E}(h_{AB})$$

Upper bound on capacity: Dirichlet principle

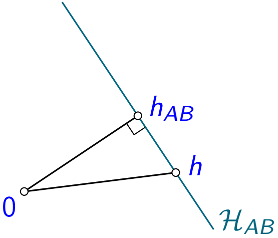
Theorem: Dirichlet principle

Let $\mathcal{H}_{AB} = \{h : \mathcal{X} \to [0,1] : h|_A = 1, h|_B = 0\}$. Then

$$\text{cap}(A,B) = \inf_{h \in \mathcal{H}_{AB}} \mathcal{E}(h) = \mathcal{E}(h_{AB})$$

Proof: $h \in \mathcal{H}_{AB} : \mathcal{E}(h, h_{AB}) = \langle h, -Lh_{AB} \rangle = \sum_{x \in A} \pi(x) e_{AB}(x) = \text{cap}(A,B)$

Cauchy–Schwarz $\Rightarrow$ $\text{cap}(A,B)^2 \leqslant \mathcal{E}(h)\mathcal{E}(h_{AB}) = \mathcal{E}(h)\,\text{cap}(A,B)$ $\square$



Lower bound on capacity: Thomson principle

- Flow: $f : \mathcal{X} \times \mathcal{X}$, $f(x,y) = -f(y,x) \ \ \forall x, y \in \mathcal{X}$
- Divergence: $\operatorname{div} f(x) := \sum_y f(x,y)$
- Harmonic unit flow:

$$f_{AB}(x,y) = \frac{\pi(x)P(x,y)}{\operatorname{cap}(A,B)}\left[h_{AB}(x) - h_{AB}(y)\right] \qquad \operatorname{div} f_{AB}\big|_{(A\cup B)^c} = 0$$

- For flows f, g, define

$$\mathcal{D}(f,g) = \frac{1}{2}\sum_{x,y\in\mathcal{X}} \frac{f(x,y)g(x,y)}{\pi(x)P(x,y)}, \qquad \mathcal{D}(f) = \mathcal{D}(f,f)$$

Lower bound on capacity: Thomson principle

- Flow: $f : \mathcal{X} \times \mathcal{X}$, $f(x,y) = -f(y,x)$ $\forall x, y \in \mathcal{X}$
- Divergence: $\operatorname{div} f(x) := \sum_y f(x,y)$
- Harmonic unit flow:

$$f_{AB}(x,y) = \frac{\pi(x)P(x,y)}{\operatorname{cap}(A,B)}\left[h_{AB}(x) - h_{AB}(y)\right] \qquad \operatorname{div} f_{AB}|_{(A\cup B)^c} = 0$$

- For flows f, g, define

$$\mathcal{D}(f,g) = \frac{1}{2}\sum_{x,y\in\mathcal{X}} \frac{f(x,y)g(x,y)}{\pi(x)P(x,y)}, \qquad \mathcal{D}(f) = \mathcal{D}(f,f)$$

Theorem: Thomson principle

Let $\mathcal{U}_{AB} = \{f : \operatorname{div} f|_{(A\cup B)^c} = 0, \sum_{x\in A} \operatorname{div} f(x) = 1\}$. Then

$$\operatorname{cap}(A,B) = \sup_{f\in\mathcal{U}_{AB}} \frac{1}{\mathcal{D}(f)} = \frac{1}{\mathcal{D}(f_{AB})}$$

Lower bound on capacity: Thomson principle

- Flow: $f : \mathcal{X} \times \mathcal{X}$, $f(x,y) = -f(y,x) \quad \forall x,y \in \mathcal{X}$
- Divergence: $\operatorname{div} f(x) := \sum_y f(x,y)$
- Harmonic unit flow:

$$f_{AB}(x,y) = \frac{\pi(x)P(x,y)}{\operatorname{cap}(A,B)}\left[h_{AB}(x) - h_{AB}(y)\right] \qquad \operatorname{div} f_{AB}\big|_{(A\cup B)^c} = 0$$

- For flows f, g, define

$$\mathcal{D}(f,g) = \frac{1}{2}\sum_{x,y\in\mathcal{X}} \frac{f(x,y)g(x,y)}{\pi(x)P(x,y)}, \qquad \mathcal{D}(f) = \mathcal{D}(f,f)$$

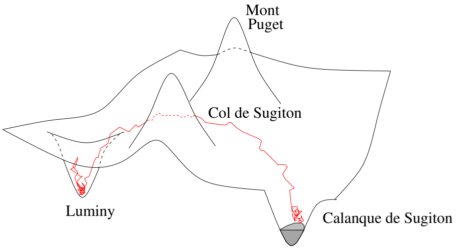
Theorem: Thomson principle

Let $\mathcal{U}_{AB} = \{f \colon \operatorname{div} f|_{(A\cup B)^c} = 0, \sum_{x\in A} \operatorname{div} f(x) = 1\}$. Then

$$\operatorname{cap}(A,B) = \sup_{f\in\mathcal{U}_{AB}} \frac{1}{\mathcal{D}(f)} = \frac{1}{\mathcal{D}(f_{AB})}$$

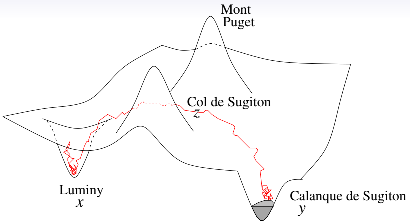
Proof: Check that $\mathcal{D}(f_{AB}) = \operatorname{cap}(A,B)^{-1}$, that $\mathcal{D}(f_{AB},f) = \operatorname{cap}(A,B)^{-1}$ for all $f \in \mathcal{U}_{AB}$, and apply Cauchy–Schwarz. □

2. Reversible diffusions



Reversible diffusion in a double-well

$$\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$$

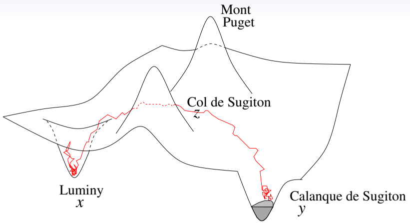


$V : \mathbb{R}^d \to \mathbb{R}$ confining potential

$\tau_y^x = \inf\{t > 0 \colon x_t \in \mathcal{B}_\varepsilon(y)\}$
first-hitting time of small ball $\mathcal{B}_\varepsilon(y)$, when starting in x

Reversible diffusion in a double-well

$$\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$$



$V : \mathbb{R}^d \to \mathbb{R}$ confining potential

$\tau_y^x = \inf\{t > 0 \colon x_t \in \mathcal{B}_\varepsilon(y)\}$
first-hitting time of small ball $\mathcal{B}_\varepsilon(y)$, when starting in x

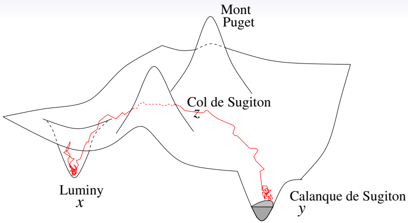
Arrhenius' law (1889): $\mathbb{E}[\tau_y^x] \simeq \mathrm{e}^{[V(z)-V(x)]/\varepsilon}$

Eyring–Kramers law (1935, 1940):
Eigenvalues of Hessian of V at minimum x: $0 < \nu_1 \leqslant \nu_2 \leqslant \dots \leqslant \nu_d$
Eigenvalues of Hessian of V at saddle z: $\lambda_1 < 0 < \lambda_2 \leqslant \dots \leqslant \lambda_d$

$$\mathbb{E}[\tau_y^x] = 2\pi \sqrt{\frac{\lambda_2 \dots \lambda_d}{|\lambda_1|\nu_1 \dots \nu_d}}\, \mathrm{e}^{[V(z)-V(x)]/\varepsilon}\big[1 + \mathcal{O}_\varepsilon(1)\big]$$

Reversible diffusion in a double-well

$$dx_t = -\nabla V(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t$$



$V : \mathbb{R}^d \to \mathbb{R}$ confining potential

$\tau_y^x = \inf\{t > 0 \colon x_t \in \mathcal{B}_\varepsilon(y)\}$
first-hitting time of small ball $\mathcal{B}_\varepsilon(y)$, when starting in x

Arrhenius' law (1889): $\mathbb{E}[\tau_y^x] \simeq e^{[V(z)-V(x)]/\varepsilon}$

Eyring–Kramers law (1935, 1940):

Eigenvalues of Hessian of V at minimum x: $0 < \nu_1 \leqslant \nu_2 \leqslant \dots \leqslant \nu_d$

Eigenvalues of Hessian of V at saddle z: $\lambda_1 < 0 < \lambda_2 \leqslant \dots \leqslant \lambda_d$

$$\mathbb{E}[\tau_y^x] = 2\pi \sqrt{\frac{\lambda_2 \dots \lambda_d}{|\lambda_1| \nu_1 \dots \nu_d}}\, e^{[V(z)-V(x)]/\varepsilon} \big[1 + \mathcal{O}_\varepsilon(1)\big]$$

Arrhenius' law: proved by [Freidlin, Wentzell, 1979] using large deviations

Eyring–Kramers law: [Bovier, Eckhoff, Gayrard, Klein, 2004] using potential theory, [Helffer, Klein, Nier, 2004] using Witten Laplacian, . . .

Potential-theoretic proof

$$\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$$

- Generator: $\mathcal{L} = \varepsilon\Delta - \nabla V \cdot \nabla = \varepsilon\,\mathrm{e}^{V/\varepsilon}\,\nabla \cdot \mathrm{e}^{-V/\varepsilon}\,\nabla$
- Invariant probability: $\pi(\mathrm{d}x) = \frac{1}{Z}\,\mathrm{e}^{-V(x)/\varepsilon}\,\mathrm{d}x \quad \Rightarrow \quad \mathcal{L}^\dagger\pi = 0$
- Reversible: $\langle f, \mathcal{L}g\rangle = \langle \mathcal{L}f, g\rangle$ for $\langle f, g\rangle = \int_{\mathbb{R}^d} \mathrm{e}^{-V(x)/\varepsilon} f(x)g(x)\,\mathrm{d}x$

Potential-theoretic proof

$$\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$$

- Generator: $\mathcal{L} = \varepsilon\Delta - \nabla V \cdot \nabla = \varepsilon\,\mathrm{e}^{V/\varepsilon}\,\nabla\cdot\mathrm{e}^{-V/\varepsilon}\,\nabla$
- Invariant probability: $\pi(\mathrm{d}x) = \frac{1}{Z}\,\mathrm{e}^{-V(x)/\varepsilon}\,\mathrm{d}x \quad \Rightarrow \quad \mathcal{L}^\dagger\pi = 0$
- Reversible: $\langle f, \mathcal{L}g\rangle = \langle \mathcal{L}f, g\rangle$ for $\langle f, g\rangle = \int_{\mathbb{R}^d} \mathrm{e}^{-V(x)/\varepsilon} f(x)g(x)\,\mathrm{d}x$
- Dirichlet form: $\mathcal{E}(f) = \langle f, -\mathcal{L}f\rangle = \varepsilon\int_{\mathbb{R}^d} \mathrm{e}^{-V(x)/\varepsilon}|\nabla f(x)|^2\,\mathrm{d}x$
  $\mathcal{E}(f,g) = \langle f, -\mathcal{L}g\rangle$

Potential-theoretic proof

$$dx_t = -\nabla V(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t$$

- ▷ Generator: $\mathcal{L} = \varepsilon\Delta - \nabla V \cdot \nabla = \varepsilon\,e^{V/\varepsilon}\,\nabla \cdot e^{-V/\varepsilon}\,\nabla$
- ▷ Invariant probability: $\pi(dx) = \frac{1}{Z}\,e^{-V(x)/\varepsilon}\,dx \quad \Rightarrow \quad \mathcal{L}^\dagger \pi = 0$
- ▷ Reversible: $\langle f, \mathcal{L}g\rangle = \langle \mathcal{L}f, g\rangle$ for $\langle f, g\rangle = \int_{\mathbb{R}^d} e^{-V(x)/\varepsilon} f(x)g(x)\,dx$
- ▷ Dirichlet form: $\mathcal{E}(f) = \langle f, -\mathcal{L}f\rangle = \varepsilon \int_{\mathbb{R}^d} e^{-V(x)/\varepsilon} |\nabla f(x)|^2\,dx$
  $\mathcal{E}(f,g) = \langle f, -\mathcal{L}g\rangle$
- ▷ Committor:

$$h_{AB}(x) = \mathbb{P}^x\{\tau_A < \tau_B\} \quad \text{satisfies} \quad \begin{cases} (\mathcal{L}h_{AB})(x) = 0 & x \in (A\cup B)^c \\ h_{AB}(x) = 1 & x \in A \\ h_{AB}(x) = 0 & x \in B \end{cases}$$

Potential-theoretic proof

$$\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$$

- Generator: $\mathcal{L} = \varepsilon\Delta - \nabla V \cdot \nabla = \varepsilon\,\mathrm{e}^{V/\varepsilon}\,\nabla \cdot \mathrm{e}^{-V/\varepsilon}\,\nabla$
- Invariant probability: $\pi(\mathrm{d}x) = \frac{1}{Z}\,\mathrm{e}^{-V(x)/\varepsilon}\,\mathrm{d}x \quad \Rightarrow \quad \mathcal{L}^\dagger\pi = 0$
- Reversible: $\langle f, \mathcal{L}g\rangle = \langle \mathcal{L}f, g\rangle$ for $\langle f, g\rangle = \int_{\mathbb{R}^d} \mathrm{e}^{-V(x)/\varepsilon} f(x)g(x)\,\mathrm{d}x$
- Dirichlet form: $\mathcal{E}(f) = \langle f, -\mathcal{L}f\rangle = \varepsilon \int_{\mathbb{R}^d} \mathrm{e}^{-V(x)/\varepsilon}|\nabla f(x)|^2\,\mathrm{d}x$
  $\mathcal{E}(f,g) = \langle f, -\mathcal{L}g\rangle$
- Committor:

$$h_{AB}(x) = \mathbb{P}^x\{\tau_A < \tau_B\} \quad \text{satisfies} \quad \begin{cases} (\mathcal{L}h_{AB})(x) = 0 & x \in (A\cup B)^c \\ h_{AB}(x) = 1 & x \in A \\ h_{AB}(x) = 0 & x \in B \end{cases}$$

- Equilibrium measure: $e_{AB}(\mathrm{d}x) = (-\mathcal{L}h_{AB})(\mathrm{d}x)$ measure on $x \in \partial A$

$$\Rightarrow \qquad h_{AB}(x) = \int_A G_B(x,y)e_{AB}(\mathrm{d}y) \qquad G_B = \mathcal{L}_{B^c}^{-1}$$

Capacity

Capacity: $\mathrm{cap}(A,B) = \int_{\partial A} \mathrm{e}^{-V(x)/\varepsilon}\, e_{AB}(\mathrm{d}x) = \mathcal{E}(h_{AB})$

$\Rightarrow \nu_{AB}(\mathrm{d}x) = \frac{1}{\mathrm{cap}(A,B)}\, \mathrm{e}^{-V(x)/\varepsilon}\, e_{AB}(\mathrm{d}x)$ is a probability measure on ∂A

Capacity

Capacity: $\mathrm{cap}(A,B) = \int_{\partial A} e^{-V(x)/\varepsilon}\, e_{AB}(dx) = \mathcal{E}(h_{AB})$

$\Rightarrow \nu_{AB}(dx) = \frac{1}{\mathrm{cap}(A,B)} e^{-V(x)/\varepsilon}\, e_{AB}(dx)$ is a probability measure on ∂A

Theorem ("Magic" formula):

$$\mathbb{E}^{\nu_{AB}}[\tau_B] := \int_{\partial A} \mathbb{E}^x[\tau_B]\nu_{AB}(dx) = \frac{1}{\mathrm{cap}(A,B)} \int_{B^c} e^{-V(x)/\varepsilon}\, h_{AB}(x)\, dx$$

Capacity

Capacity: $\text{cap}(A,B) = \int_{\partial A} e^{-V(x)/\varepsilon} \, e_{AB}(dx) = \mathcal{E}(h_{AB})$

$\Rightarrow \nu_{AB}(dx) = \frac{1}{\text{cap}(A,B)} e^{-V(x)/\varepsilon} \, e_{AB}(dx)$ is a probability measure on ∂A

Theorem ("Magic" formula):

$$\mathbb{E}^{\nu_{AB}}[\tau_B] := \int_{\partial A} \mathbb{E}^x[\tau_B] \nu_{AB}(dx) = \frac{1}{\text{cap}(A,B)} \int_{B^c} e^{-V(x)/\varepsilon} \, h_{AB}(x) \, dx$$

Theorem: Dirichlet principle

Let $\mathcal{H}_{AB} = \{h : \mathbb{R}^d \to [0,1] \colon h|_A = 1, h|_B = 0\}$. Then

$$\text{cap}(A,B) = \inf_{h \in \mathcal{H}_{AB}} \mathcal{E}(h) = \mathcal{E}(h_{AB})$$

Capacity

Capacity: $\mathrm{cap}(A,B) = \int_{\partial A} \mathrm{e}^{-V(x)/\varepsilon}\, e_{AB}(\mathrm{d}x) = \mathcal{E}(h_{AB})$

$\Rightarrow \nu_{AB}(\mathrm{d}x) = \frac{1}{\mathrm{cap}(A,B)} \mathrm{e}^{-V(x)/\varepsilon}\, e_{AB}(\mathrm{d}x)$ is a probability measure on ∂A

Theorem ("Magic" formula):

$$\mathbb{E}^{\nu_{AB}}[\tau_B] := \int_{\partial A} \mathbb{E}^x[\tau_B]\nu_{AB}(\mathrm{d}x) = \frac{1}{\mathrm{cap}(A,B)} \int_{B^c} \mathrm{e}^{-V(x)/\varepsilon}\, h_{AB}(x)\,\mathrm{d}x$$

Theorem: Dirichlet principle

Let $\mathcal{H}_{AB} = \{h : \mathbb{R}^d \to [0,1] \colon h|_A = 1, h|_B = 0\}$. Then

$$\mathrm{cap}(A,B) = \inf_{h\in\mathcal{H}_{AB}} \mathcal{E}(h) = \mathcal{E}(h_{AB})$$

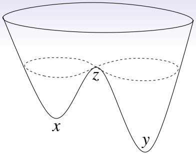
Theorem: Thomson principle [Landim, Mariani, Seo 2018]

Let $\mathcal{U}_{AB} = \{f \colon \nabla\cdot f|_{(A\cup B)^c} = 0, \int_{\partial A} f(x)\cdot n_A(x)\sigma(\mathrm{d}x) = 1\}$. Then

$$\mathrm{cap}(A,B) = \sup_{f\in\mathcal{U}_{AB}} \frac{1}{\mathcal{D}(f)} = \frac{1}{\mathcal{D}(f_{AB})} \qquad \mathcal{D}(f) = \frac{1}{\varepsilon}\int \mathrm{e}^{V(x)/\varepsilon}|f(x)|^2\,\mathrm{d}x$$

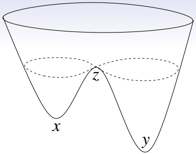
Proof of Eyring–Kramers law

$$dx_t = -\nabla V(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t$$



▷ A, B small balls around x, y

Proof of Eyring–Kramers law

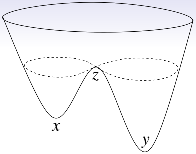


$$\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$$

- ▷ A, B small balls around x, y
- ▷ Laplace asymptotics: $\displaystyle\int_{B^c} h_{A,B}(y)\,\mathrm{e}^{-V(y)/\varepsilon}\,\mathrm{d}y \simeq \sqrt{\frac{(2\pi\varepsilon)^d}{\nu_1\dots\nu_d}}\,\mathrm{e}^{-V(x)/\varepsilon}$

Proof of Eyring–Kramers law

$$dx_t = -\nabla V(x_t)\,dt + \sqrt{2\varepsilon}\,dW_t$$



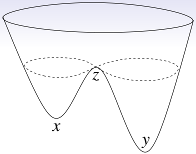
- A, B small balls around x, y
- Laplace asymptotics: $\displaystyle\int_{B^c} h_{A,B}(y)\,e^{-V(y)/\varepsilon}\,dy \simeq \sqrt{\frac{(2\pi\varepsilon)^d}{\nu_1\dots\nu_d}}\,e^{-V(x)/\varepsilon}$
- Variational arguments: $\displaystyle\operatorname{cap}(A,B) \simeq \varepsilon\sqrt{\frac{|\lambda_1|}{2\pi\varepsilon}}\sqrt{\frac{(2\pi\varepsilon)^{d-1}}{\lambda_2\dots\lambda_d}}\,e^{-V(z)/\varepsilon}$

  Test function for Dirichlet principle: $1d$ committor

$$h(x_1) = \int_{x_1}^{\delta} e^{V(\xi,0)/\varepsilon}\,d\xi\left[\int_{-\delta}^{\delta} e^{V(\xi,0)/\varepsilon}\,d\xi\right]^{-1} \qquad \delta = \mathcal{O}([\varepsilon\log(\varepsilon^{-1})]^{1/2})$$

Proof of Eyring–Kramers law

$$\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$$

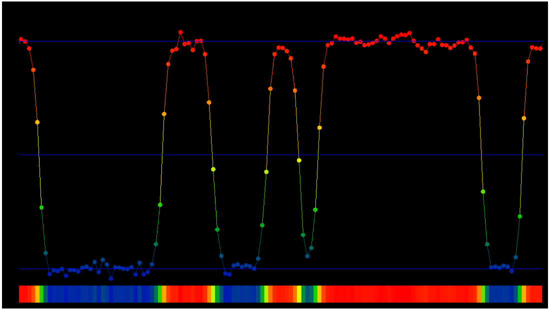


- A, B small balls around x, y
- Laplace asymptotics: $\displaystyle\int_{B^c} h_{A,B}(y)\,\mathrm{e}^{-V(y)/\varepsilon}\,\mathrm{d}y \simeq \sqrt{\frac{(2\pi\varepsilon)^d}{\nu_1\dots\nu_d}}\,\mathrm{e}^{-V(x)/\varepsilon}$
- Variational arguments: $\operatorname{cap}(A,B) \simeq \varepsilon\sqrt{\frac{|\lambda_1|}{2\pi\varepsilon}}\sqrt{\frac{(2\pi\varepsilon)^{d-1}}{\lambda_2\dots\lambda_d}}\,\mathrm{e}^{-V(z)/\varepsilon}$
  Test function for Dirichlet principle: $1d$ committor
  $$h(x_1) = \int_{x_1}^{\delta} \mathrm{e}^{V(\xi,0)/\varepsilon}\,\mathrm{d}\xi \left[\int_{-\delta}^{\delta} \mathrm{e}^{V(\xi,0)/\varepsilon}\,\mathrm{d}\xi\right]^{-1} \qquad \delta = \mathcal{O}([\varepsilon\log(\varepsilon^{-1})]^{1/2})$$
- Use Harnack inequalities to show that $\mathbb{E}^{\nu_{A,B}}[\tau_B] \simeq \mathbb{E}^{x}[\tau_B]$
  Alternative: coupling argument by [Martinelli, Olivieri & Scoppola]

Eyring–Kramers formula:

$$\mathbb{E}^x[\tau_B] = 2\pi\sqrt{\frac{\lambda_2\dots\lambda_d}{|\lambda_1|\nu_1\dots\nu_d}}\,\mathrm{e}^{[V(z)-V(x)]/\varepsilon}\left[1 + \mathcal{O}(\varepsilon^{1/2}\log(\varepsilon^{-1})^{3/2})\right]$$

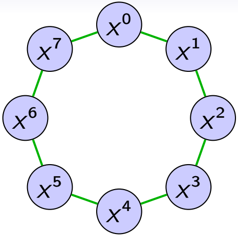
3. Allen–Cahn SPDE in dimension 1



A particle system

[B, Fernandez, Gentz, Nonlinearity 2007]

- N particles on a circle $\mathbb{Z}/N\mathbb{Z}$
- Bistable local dynamics
- Ferromagnetic nearest neighbour coupling
- Independent noise on each site

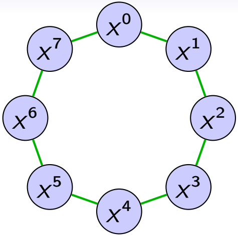


$$\mathrm{d}x_t^i = \left[x_t^i - (x_t^i)^3\right]\mathrm{d}t + \frac{\gamma}{2}\left[x_t^{i+1} - 2x_t^i + x_t^{i-1}\right]\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t^i$$

A particle system

[B, Fernandez, Gentz, Nonlinearity 2007]

- N particles on a circle $\mathbb{Z}/N\mathbb{Z}$
- Bistable local dynamics
- Ferromagnetic nearest neighbour coupling
- Independent noise on each site



$$\mathrm{d}x_t^i = \left[x_t^i - (x_t^i)^3\right]\mathrm{d}t + \frac{\gamma}{2}\left[x_t^{i+1} - 2x_t^i + x_t^{i-1}\right]\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t^i$$

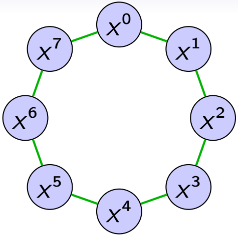
Gradient system $\quad \mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$

potential $V(x) = \sum_i U(x^i) + \frac{\gamma}{4}\sum_i (x^{i+1} - x^i)^2 \qquad U(\xi) = \frac{1}{4}\xi^4 - \frac{1}{2}\xi^2$

A particle system

[B, Fernandez, Gentz, Nonlinearity 2007]

- N particles on a circle $\mathbb{Z}/N\mathbb{Z}$
- Bistable local dynamics
- Ferromagnetic nearest neighbour coupling
- Independent noise on each site



$$\mathrm{d}x_t^i = \left[x_t^i - (x_t^i)^3\right]\mathrm{d}t + \frac{\gamma}{2}\left[x_t^{i+1} - 2x_t^i + x_t^{i-1}\right]\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t^i$$

Gradient system $\qquad \mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t$

potential $V(x) = \sum_i U(x^i) + \frac{\gamma}{4}\sum_i (x^{i+1} - x^i)^2 \qquad U(\xi) = \frac{1}{4}\xi^4 - \frac{1}{2}\xi^2$

Scaling limit $\gamma \sim N^2$, $N \to \infty$: expects convergence to Allen–Cahn SPDE

$$\partial_t u(t,x) = u(t,x) - u(t,x)^3 + \frac{\gamma}{2N^2}\Delta u(t,x) + \sqrt{2\varepsilon}\xi(t,x)$$

Space-time white noise: $\langle \xi, \varphi \rangle \sim \mathcal{N}(0, \|\varphi\|^2)$, $\mathbb{E}[\langle \xi, \varphi_1 \rangle \langle \xi, \varphi_2 \rangle] = \langle \varphi_1, \varphi_2 \rangle$

Coarsening dynamics with noise

(Link to simulation)

Deterministic Allen–Cahn PDE

[Chafee & Infante 74, Allen & Cahn 75]

$$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x))$$

- $x \in [0, L]$, L: bifurcation parameter
- $u(t,x) \in \mathbb{R}$
- Either periodic or zero-flux Neumann boundary conditions
- In this talk: $f(u) = u - u^3$ (results more general)

Deterministic Allen–Cahn PDE

[Chafee & Infante 74, Allen & Cahn 75]

$$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x))$$

- $x \in [0,L]$, L: bifurcation parameter
- $u(t,x) \in \mathbb{R}$
- Either periodic or zero-flux Neumann boundary conditions
- In this talk: $f(u) = u - u^3$ (results more general)

Energy function:

$$V[u] = \int_0^L \Big[\frac{1}{2}u'(x)^2 - \frac{1}{2}u(x)^2 + \frac{1}{4}u(x)^4\Big]\,\mathrm{d}x \qquad \to \quad \text{min}$$

Scaling limit of particle system with $\gamma = 2\frac{N^2}{L^2}$

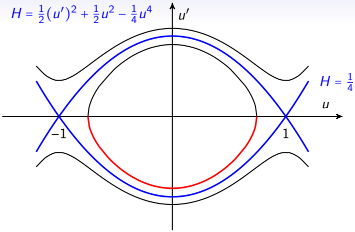
Stationary solutions: $u_0''(x) = -u_0(x) + u_0(x)^3$ critical points of V

Stability: Sturm–Liouville problem $\partial_t v_t(x) = v_t''(x) + [1 - 3u_0(x)^2]v_t(x)$

Stationary solutions

$u_0''(x) = -f(u_0(x)) = -u_0(x) + u_0(x)^3$

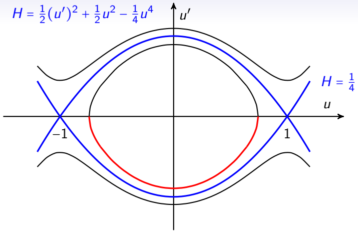
- $u_\pm(x) \equiv \pm 1$
- $u_0(x) \equiv 0$
- Nonconstant solutions satisfying b.c.
  (expressible in terms of Jacobi elliptic fcts)



Stationary solutions

$u_0''(x) = -f(u_0(x)) = -u_0(x) + u_0(x)^3$

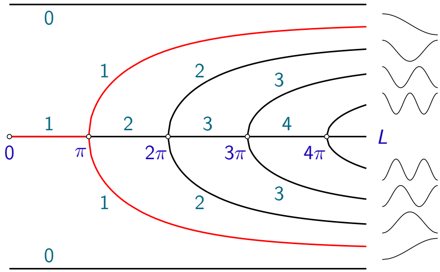
- $u_\pm(x) \equiv \pm 1$
- $u_0(x) \equiv 0$
- Nonconstant solutions satisfying b.c.
  (expressible in terms of Jacobi elliptic fcts)



- Neumann b.c: $2k$ nonconstant solutions when $L > k\pi$

Number of positive eigenvalues (= unstable directions)

Transition state



- Periodic b.c: k families when $L > 2k\pi$

Eyring–Kramers law for 1D SPDEs: heuristics

$$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x)) \quad + \sqrt{2\varepsilon}\,\xi(t,x) \qquad (f(u) = u - u^3)$$

Initial condition: u_{in} near $u_- \equiv -1$ with eigenvalues $\nu_k = (\frac{\beta k\pi}{L})^2 + 2$

Target: $u_+ \equiv 1$, $\tau_+ = \inf\{t > 0 \colon \|u_t - u_+\|_{L^\infty} < \rho\}$

Transition state: ($\beta = 1$ for Neumann b.c., $\beta = 2$ for periodic b.c.)

$$u_{\text{ts}}(x) = \begin{cases} u_0(x) \equiv 0 & \text{if } L \leqslant \beta\pi \quad \text{with ev } \lambda_k = (\frac{\beta k\pi}{L})^2 - 1 \\ u_1(x) \text{ } \beta\text{-kink stationary sol.} & \text{if } L > \beta\pi \quad \text{with ev } \lambda'_k \end{cases}$$

Eyring–Kramers law for 1D SPDEs: heuristics

$$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x)) \quad + \sqrt{2\varepsilon}\,\xi(t,x) \qquad (f(u) = u - u^3)$$

Initial condition: $u_{\rm in}$ near $u_- \equiv -1$ with eigenvalues $\nu_k = (\frac{\beta k\pi}{L})^2 + 2$

Target: $u_+ \equiv 1$, $\tau_+ = \inf\{t > 0\colon \|u_t - u_+\|_{L^\infty} < \rho\}$

Transition state: ($\beta = 1$ for Neumann b.c., $\beta = 2$ for periodic b.c.)

$$u_{\rm ts}(x) = \begin{cases} u_0(x) \equiv 0 & \text{if } L \leqslant \beta\pi \quad \text{with ev } \lambda_k = (\frac{\beta k\pi}{L})^2 - 1 \\ u_1(x) \ \beta\text{-kink stationary sol.} & \text{if } L > \beta\pi \quad \text{with ev } \lambda'_k \end{cases}$$

[Faris & Jona-Lasinio 82]: large-deviation principle

$\Rightarrow$ Arrhenius law: $\mathbb{E}^{u_{\rm in}}[\tau_+] \simeq e^{(V[u_{\rm ts}] - V[u_-])/\varepsilon}$

[Maier & Stein 01]: formal computation; for Neumann b.c.

$\Rightarrow \mathbb{E}^{u_{\rm in}}[\tau_+] \simeq 2\pi\sqrt{\frac{1}{|\lambda_0|\nu_0}\prod_{k=1}^{\infty}\frac{\lambda_k}{\nu_k}}\, e^{(V[u_{\rm ts}] - V[u_-])/\varepsilon}$

Eyring–Kramers law for 1D SPDEs: main result

Theorem: Neumann b.c. [B & Gentz, 2013]

- If $L < \pi - c$ with $c > 0$, then

$$\mathbb{E}^{u_{\text{in}}}[\tau_+] = 2\pi \sqrt{\frac{1}{|\lambda_0|\nu_0} \prod_{k=1}^{\infty} \frac{\lambda_k}{\nu_k}}\, \mathrm{e}^{(V[u_{\text{ts}}]-V[u_-])/\varepsilon} \Big[1 + \underbrace{\mathcal{O}(\varepsilon^{1/2}|\log\varepsilon|^{3/2})}_{\text{error not optimal}}\Big]$$

- If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles) and λ'_k instead of λ_k

Eyring–Kramers law for 1D SPDEs: main result

Theorem: Neumann b.c. [B & Gentz, 2013]

- If $L < \pi - c$ with $c > 0$, then

$$\mathbb{E}^{u_{\text{in}}}[\tau_+] = 2\pi\sqrt{\frac{1}{|\lambda_0|\nu_0}\prod_{k=1}^{\infty}\frac{\lambda_k}{\nu_k}}\, e^{(V[u_{\text{ts}}]-V[u_-])/\varepsilon}\Big[1 + \underbrace{\mathcal{O}(\varepsilon^{1/2}|\log\varepsilon|^{3/2})}_{\text{error not optimal}}\Big]$$

- If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles) and λ_k' instead of λ_k

- Results also for L near π and periodic b.c.

Eyring–Kramers law for 1D SPDEs: main result

Theorem: Neumann b.c. [B & Gentz, 2013]

- If $L < \pi - c$ with $c > 0$, then

$$\mathbb{E}^{u_{\rm in}}[\tau_+] = 2\pi\sqrt{\frac{1}{|\lambda_0|\nu_0}\prod_{k=1}^{\infty}\frac{\lambda_k}{\nu_k}}\, e^{(V[u_{\rm ts}]-V[u_-])/\varepsilon}\Big[1 + \underbrace{\mathcal{O}(\varepsilon^{1/2}|\log\varepsilon|^{3/2})}_{\text{error not optimal}}\Big]$$

- If $L > \pi + c$, then same formula with extra factor $\frac{1}{2}$ (since 2 saddles) and λ'_k instead of λ_k

- Results also for L near π and periodic b.c.
- Prefactor involves a Fredholm determinant:
  $\Delta_\perp$ Laplacian acting on mean zero functions

$$\prod_{k=1}^{\infty}\frac{\lambda_k}{\nu_k} = \det\Big[(-\Delta_\perp - 1)(-\Delta_\perp + 2)^{-1}\Big] = \det\Big[\mathbb{1} - 3(-\Delta_\perp + 2)^{-1}\Big]$$

  converges because $\log\det = \operatorname{Tr}\log$ and $(-\Delta_\perp + 2)^{-1}$ is trace class
  (limit $= \frac{\sqrt{2}\sin(L)}{\sinh(\sqrt{2}L)}$)

Ideas of the proof ($L < \pi$)

▷ Spectral Galerkin approximation: $u(t,x) = \sum_{|k| \leqslant N} z_k(t) e_k(x)$ (Fourier)

Ideas of the proof ($L < \pi$)

- Spectral Galerkin approximation: $u(t,x) = \sum_{|k| \leqslant N} z_k(t) e_k(x)$ (Fourier)
- Change of variables $u(x) = u_0 + \sqrt{\varepsilon} u_\perp(x)$ where $\int_0^L u_\perp(x)\,\mathrm{d}x = 0$

$$\Rightarrow \frac{1}{\varepsilon} V[u_0 + \sqrt{\varepsilon} u_\perp] = \frac{1}{\varepsilon} \underbrace{\left(\frac{1}{4} u_0^4 - \frac{1}{2} u_0^2\right)}_{V_0(u_0)} + Q_{u_0}[u_\perp] + \sqrt{\varepsilon} R_{u_0}[u_\perp]$$

where

$$\begin{cases} Q_{u_0}[u_\perp] = \dfrac{1}{2} \displaystyle\int_0^L \left[u_\perp'(x)^2 - (1 - 3u_0^2) u_\perp(x)^2\right] \mathrm{d}x = \dfrac{1}{2} \langle u_\perp, [-\Delta - (1 - 3u_0^2)] u_\perp \rangle \\ R_{u_0}[u_\perp] = u_0 \displaystyle\int_0^L u_\perp(x)^3 \,\mathrm{d}x + \sqrt{\varepsilon} \int_0^L u_\perp(x)^4 \,\mathrm{d}x \qquad \text{(remainder)} \end{cases}$$

Ideas of the proof ($L < \pi$)

- Spectral Galerkin approximation: $u(t,x) = \sum_{|k|\leqslant N} z_k(t) e_k(x)$ (Fourier)
- Change of variables $u(x) = u_0 + \sqrt{\varepsilon} u_\perp(x)$ where $\int_0^L u_\perp(x)\,dx = 0$

$$\Rightarrow \frac{1}{\varepsilon} V[u_0 + \sqrt{\varepsilon} u_\perp] = \frac{1}{\varepsilon}\underbrace{\left(\frac{1}{4}u_0^4 - \frac{1}{2}u_0^2\right)}_{V_0(u_0)} + Q_{u_0}[u_\perp] + \sqrt{\varepsilon} R_{u_0}[u_\perp]$$

where

$$\begin{cases} Q_{u_0}[u_\perp] = \dfrac{1}{2}\displaystyle\int_0^L \left[u_\perp'(x)^2 - (1-3u_0^2)u_\perp(x)^2\right] dx = \dfrac{1}{2}\langle u_\perp, [-\Delta - (1-3u_0^2)]u_\perp\rangle \\ R_{u_0}[u_\perp] = u_0 \displaystyle\int_0^L u_\perp(x)^3\,dx + \sqrt{\varepsilon}\int_0^L u_\perp(x)^4\,dx \qquad \text{(remainder)} \end{cases}$$

- Dirichlet principle with $h = h(u_0)$ s.t. $h'(u_0) = -\frac{1}{c}\,e^{V_0(u_0)/\varepsilon}$, $c \simeq \sqrt{\frac{2\pi\varepsilon}{|\lambda_0|}}$

$$\operatorname{cap}(A,B) \leqslant \mathcal{E}(h) = \frac{\varepsilon^{1+\frac{N}{2}}}{c^2}\int_{-1}^{1} e^{V_0(u_0)/\varepsilon} \underbrace{\int e^{-Q_{u_0}[u_\perp]}\, e^{-\sqrt{\varepsilon}R_{u_0}[u_\perp]}\, du_\perp}_{=\sqrt{\frac{(2\pi)^N}{\det[-\Delta_\perp - (1-3u_0^2)]}}\,\mathbb{E}^\gamma[e^{-\sqrt{\varepsilon}R_{u_0}}]}\, du_0$$

Ideas of the proof ($L < \pi$)

- Thomson principle with divergence-free unit flow $f = K^{-1}\,\mathrm{e}^{-Q_0[u_\perp]}\,e_{u_0}$

  Normalisation $K = \varepsilon^{\frac{N}{2}} \int \mathrm{e}^{-Q_0[u_\perp]}\,\mathrm{d}u_\perp = \sqrt{\frac{(2\pi\varepsilon)^N}{\det[-\Delta_\perp - 1]}}$

Ideas of the proof ($L < \pi$)

- ▷ Thomson principle with divergence-free unit flow $f = K^{-1}\,\mathrm{e}^{-Q_0[u_\perp]}\,e_{u_0}$

  Normalisation $K = \varepsilon^{\frac{N}{2}} \int \mathrm{e}^{-Q_0[u_\perp]}\,\mathrm{d}u_\perp = \sqrt{\dfrac{(2\pi\varepsilon)^N}{\det[-\Delta_\perp - 1]}}$

$$\operatorname{cap}(A,B)^{-1} \leqslant \mathcal{D}(h) = \frac{1}{\varepsilon K^2} \int_{-1}^{1} \mathrm{e}^{V_0(u_0)/\varepsilon} \underbrace{\int \mathrm{e}^{-Q_0}\,\mathrm{e}^{Q_{u_0} - Q_0 - \sqrt{\varepsilon} R_{u_0}}\,\mathrm{d}u_\perp}_{= K \mathbb{E}^{\gamma}[\mathrm{e}^{Q_{u_0} - Q_0 - \sqrt{\varepsilon} R_{u_0}}]}\,\mathrm{d}u_0$$

Ideas of the proof ($L < \pi$)

- Thomson principle with divergence-free unit flow $f = K^{-1}\,\mathrm{e}^{-Q_0[u_\perp]}\,e_{u_0}$

  Normalisation $K = \varepsilon^{\frac{N}{2}} \int \mathrm{e}^{-Q_0[u_\perp]}\,\mathrm{d}u_\perp = \sqrt{\frac{(2\pi\varepsilon)^N}{\det[-\Delta_\perp - 1]}}$

$$\operatorname{cap}(A,B)^{-1} \leqslant \mathcal{D}(h) = \frac{1}{\varepsilon K^2} \int_{-1}^{1} \mathrm{e}^{V_0(u_0)/\varepsilon} \underbrace{\int \mathrm{e}^{-Q_0}\,\mathrm{e}^{Q_{u_0} - Q_0 - \sqrt{\varepsilon} R_{u_0}}\,\mathrm{d}u_\perp}_{= K \mathbb{E}^\gamma[\mathrm{e}^{Q_{u_0} - Q_0 - \sqrt{\varepsilon} R_{u_0}}]}\,\mathrm{d}u_0$$

- Conclusion: $\operatorname{cap}(A,B) = \varepsilon \sqrt{\frac{|\lambda_0|}{2\pi\varepsilon}} \sqrt{\frac{(2\pi\varepsilon)^N}{\det[-\Delta_\perp - 1]}} \big[1 + \mathcal{O}(\varepsilon)\big]$

Ideas of the proof ($L < \pi$)

- Thomson principle with divergence-free unit flow $f = K^{-1}\,\mathrm{e}^{-Q_0[u_\perp]}\,e_{u_0}$

  Normalisation $K = \varepsilon^{\frac{N}{2}} \displaystyle\int \mathrm{e}^{-Q_0[u_\perp]}\,\mathrm{d}u_\perp = \sqrt{\frac{(2\pi\varepsilon)^N}{\det[-\Delta_\perp - 1]}}$

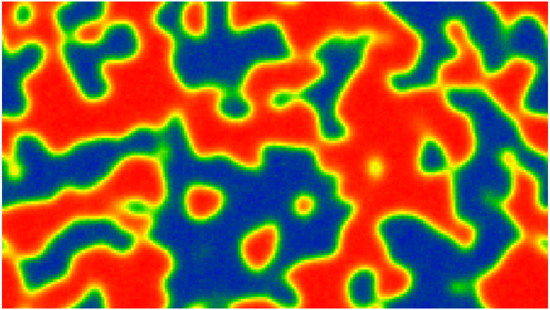
$$\operatorname{cap}(A,B)^{-1} \leqslant \mathcal{D}(h) = \frac{1}{\varepsilon K^2}\int_{-1}^{1} \mathrm{e}^{V_0(u_0)/\varepsilon} \underbrace{\int \mathrm{e}^{-Q_0}\,\mathrm{e}^{Q_{u_0} - Q_0 - \sqrt{\varepsilon}R_{u_0}}\,\mathrm{d}u_\perp}_{= K\mathbb{E}^{\gamma}[\mathrm{e}^{Q_{u_0} - Q_0 - \sqrt{\varepsilon}R_{u_0}}]}\,\mathrm{d}u_0$$

- Conclusion: $\operatorname{cap}(A,B) = \varepsilon\sqrt{\frac{|\lambda_0|}{2\pi\varepsilon}}\sqrt{\frac{(2\pi\varepsilon)^N}{\det[-\Delta_\perp - 1]}}\big[1 + \mathcal{O}(\varepsilon)\big]$

Other elements of the proof:

- A priori bounds on h_{AB}: large deviations (or symmetry argument)
- Convergence of hitting times as $N \to \infty$: a priori estimate for $\mathbb{E}[\tau_B^2]$
- Coupling argument for start in u_{in} [Martinelli, Olivieri & Scoppola]
- Bifurcation at $L = \beta\pi$

4. Allen–Cahn SPDE in dimension 2



The two-dimensional case

$\partial_t u = \Delta u + u - u^3 + \sqrt{2\varepsilon}\xi$ (Link to simulation)

The two-dimensional case

- Large-deviation principle: [Hairer & Weber, 2015]
- Naive computation of prefactor fails:

$$\log \prod_{k\in(\mathbb{N}^2)^*} \frac{1-\left(\frac{L}{|k|\pi}\right)^2}{1+2\left(\frac{L}{|k|\pi}\right)^2} \simeq \sum_{k\in(\mathbb{N}^2)^*} \log\left(1-\frac{3L^2}{|k|^2\pi^2}\right)$$

$$\simeq -\sum_{k\in(\mathbb{N}^2)^*} \frac{3L^2}{|k|^2\pi^2} \simeq -\frac{3L^2}{\pi^2}\int_1^\infty \frac{r\,\mathrm{d}r}{r^2} = -\infty$$

The two-dimensional case

- Large-deviation principle: [Hairer & Weber, 2015]
- Naive computation of prefactor fails:

$$\log \prod_{k\in(\mathbb{N}^2)^*} \frac{1-\left(\frac{L}{|k|\pi}\right)^2}{1+2\left(\frac{L}{|k|\pi}\right)^2} \simeq \sum_{k\in(\mathbb{N}^2)^*} \log\left(1-\frac{3L^2}{|k|^2\pi^2}\right)$$

$$\simeq -\sum_{k\in(\mathbb{N}^2)^*} \frac{3L^2}{|k|^2\pi^2} \simeq -\frac{3L^2}{\pi^2}\int_1^\infty \frac{r\,\mathrm{d}r}{r^2} = -\infty$$

- In fact, the equation needs to be renormalised

Theorem: [Da Prato & Debussche 2003]
Let ξ^δ be a mollification on scale δ of white noise. Then

$$\partial_t u = \Delta u + \big[1 + 3\varepsilon C(\delta)\big]u - u^3 + \sqrt{2\varepsilon}\xi^\delta$$

with $C(\delta) \simeq \log(\delta^{-1})$ admits local solution converging as $\delta \to 0$
(Global version: [Mourrat & Weber 2015])
[Mourrat & Weber 2014]: Renormalised eq = scaling limit of Ising–Kac model

Main result in dimension 2

Theorem: [B, Di Gesù, Weber, 2017]

For $L < 2\pi$, appropriate $A \ni u_-$, $B \ni u_+$, $\exists \mu_N$ probability measures on ∂A:

$$\limsup_{N\to\infty} \mathbb{E}^{\mu_N}\big[\tau_B\big] \leqslant \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k}\, \mathrm{e}^{\frac{\nu_k-\lambda_k}{|\lambda_k|}}}\, \mathrm{e}^{(V[u_{\mathrm{ts}}]-V[u_-])/\varepsilon}\big[1 + c_+\sqrt{\varepsilon}\,\big]$$

$$\liminf_{N\to\infty} \mathbb{E}^{\mu_N}\big[\tau_B\big] \geqslant \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k}\, \mathrm{e}^{\frac{\nu_k-\lambda_k}{|\lambda_k|}}}\, \mathrm{e}^{(V[u_{\mathrm{ts}}]-V[u_-])/\varepsilon}\big[1 - c_-\varepsilon\big]$$

Main result in dimension 2

Theorem: [B, Di Gesù, Weber, 2017]

For $L < 2\pi$, appropriate $A \ni u_-$, $B \ni u_+$, $\exists \mu_N$ probability measures on ∂A:

$$\limsup_{N\to\infty} \mathbb{E}^{\mu_N}\big[\tau_B\big] \leqslant \frac{2\pi}{|\lambda_0|} \sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k} \mathrm{e}^{\frac{\nu_k-\lambda_k}{|\lambda_k|}}} \, \mathrm{e}^{(V[u_{\text{ts}}]-V[u_-])/\varepsilon}\big[1 + c_+\sqrt{\varepsilon}\,\big]$$

$$\liminf_{N\to\infty} \mathbb{E}^{\mu_N}\big[\tau_B\big] \geqslant \frac{2\pi}{|\lambda_0|} \sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k} \mathrm{e}^{\frac{\nu_k-\lambda_k}{|\lambda_k|}}} \, \mathrm{e}^{(V[u_{\text{ts}}]-V[u_-])/\varepsilon}\big[1 - c_-\varepsilon\big]$$

▷ Inverse of prefactor involves Carleman–Fredholm determinant:

$$\det{}_2(\mathbb{1} + T) = \det(\mathbb{1} + T)\mathrm{e}^{-\operatorname{Tr} T}$$

with $T = 3(-\Delta_\perp - 1)^{-1}$

$\det_2$ defined whenever T is only Hilbert–Schmidt (true for $d \leqslant 3$)

Main result in dimension 2

Theorem: [B, Di Gesù, Weber, 2017]

For $L < 2\pi$, appropriate $A \ni u_-$, $B \ni u_+$, $\exists \mu_N$ probability measures on ∂A:

$$\limsup_{N\to\infty} \mathbb{E}^{\mu_N}\big[\tau_B\big] \leqslant \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k}\, e^{\frac{\nu_k-\lambda_k}{|\lambda_k|}}}\; e^{(V[u_{\text{ts}}]-V[u_-])/\varepsilon}\big[1 + c_+\sqrt{\varepsilon}\,\big]$$

$$\liminf_{N\to\infty} \mathbb{E}^{\mu_N}\big[\tau_B\big] \geqslant \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k}\, e^{\frac{\nu_k-\lambda_k}{|\lambda_k|}}}\; e^{(V[u_{\text{ts}}]-V[u_-])/\varepsilon}\big[1 - c_-\varepsilon\big]$$

- ▷ Inverse of prefactor involves Carleman–Fredholm determinant:

$$\det{}_2(\mathbb{1} + T) = \det(\mathbb{1} + T)e^{-\operatorname{Tr} T}$$

  with $T = 3(-\Delta_\perp - 1)^{-1}$
  $\det_2$ defined whenever T is only Hilbert–Schmidt (true for $d \leqslant 3$)
- ▷ [Tsatsoulis & Weber 2018]: Same result for $\mathbb{E}^{u_0}\big[\tau_B\big]$

Renormalisation

Problem: Stoch. convolution $w_t(x) = \int_0^t e^{\Delta(t-s)}\xi(s,x)\,ds$ is distribution

- δ-mollification should be equivalent to Galerkin approx. $|k| \leqslant N = \delta^{-1}$:

$$w_N(x,t) = \sum_{|k|\leqslant N} a_k(t)\frac{1}{L}e^{i\Omega k\cdot x} \qquad a_k = \int_0^t e^{-\mu_k(t-s)}\,dW_s^{(k)}$$

$$\mu_k = (\Omega|k|)^2 \qquad \Omega = \beta\pi/L$$

Renormalisation

Problem: Stoch. convolution $w_t(x) = \int_0^t e^{\Delta(t-s)} \xi(s,x)\,ds$ is distribution

- δ-mollification should be equivalent to Galerkin approx. $|k| \leqslant N = \delta^{-1}$:

$$w_N(x,t) = \sum_{|k|\leqslant N} a_k(t) \frac{1}{L} e^{i\Omega k \cdot x} \qquad a_k = \int_0^t e^{-\mu_k(t-s)}\,dW_s^{(k)}$$

$$\mu_k = (\Omega|k|)^2 \qquad \Omega = \beta\pi/L$$

- $\lim_{t\to\infty} \int_0^t e^{(\Delta-1)(t-s)} \xi_N(s,x)\,ds = \phi_N$ is a Gaussian free field, s.t.

$$L^2 C_N := L^2 \mathbb{E}\phi_N^2 = \mathbb{E}\|\phi_N\|_{L^2}^2 = \sum_{|k|\leqslant N} \frac{1}{2(\mu_k+1)} = \frac{\operatorname{Tr}(P_N[-\Delta+\mathbb{1}]^{-1})}{2} \simeq \log(N)$$

Renormalisation

Problem: Stoch. convolution $w_t(x) = \int_0^t \mathrm{e}^{\Delta(t-s)} \xi(s,x) \, \mathrm{d}s$ is distribution

- δ-mollification should be equivalent to Galerkin approx. $|k| \leqslant N = \delta^{-1}$:

$$w_N(x,t) = \sum_{|k| \leqslant N} a_k(t) \frac{1}{L} \mathrm{e}^{\mathrm{i}\,\Omega k \cdot x} \qquad a_k = \int_0^t \mathrm{e}^{-\mu_k(t-s)} \, \mathrm{d}W_s^{(k)}$$

$$\mu_k = (\Omega |k|)^2 \qquad \Omega = \beta\pi/L$$

- $\lim_{t\to\infty} \int_0^t \mathrm{e}^{(\Delta-1)(t-s)} \xi_N(s,x) \, \mathrm{d}s = \phi_N$ is a Gaussian free field, s.t.

$$L^2 C_N := L^2 \mathbb{E}\phi_N^2 = \mathbb{E}\|\phi_N\|_{L^2}^2 = \sum_{|k| \leqslant N} \frac{1}{2(\mu_k + 1)} = \frac{\operatorname{Tr}(P_N[-\Delta + \mathbb{1}]^{-1})}{2} \simeq \log(N)$$

- Wick powers

$$:\phi_N^2: = \phi_N^2 - C_N$$
$$:\phi_N^3: = \phi_N^3 - 3C_N\phi_N$$
$$:\phi_N^4: = \phi_N^4 - 6C_N\phi_N^2 + 3C_N^2$$

have zero mean and uniformly bounded variance (when integrated)

Nelson's estimate

Lemma: For X random variable in n^{th} inhomogeneous Wiener chaos

$$\mathbb{E}[X^{2p}]^{\frac{1}{2p}} \leqslant C_n(2p-1)^{\frac{n}{2}}\mathbb{E}[X^2]^{\frac{1}{2}}$$

Nelson's estimate

Lemma: For X random variable in n^{th} inhomogeneous Wiener chaos

$$\mathbb{E}[X^{2p}]^{\frac{1}{2p}} \leqslant C_n(2p-1)^{\frac{n}{2}}\mathbb{E}[X^2]^{\frac{1}{2}}$$

$$\sup_N \mathbb{E}\Big[\Big(\int_{\mathbb{T}^2} :\phi_N^n(x):\mathrm{d}x\Big)^{2p}\Big] < \infty$$

$$\forall M > N : \ \mathbb{E}\Big[\Big(\int_{\mathbb{T}^2} :\phi_M^n(x):\mathrm{d}x - \int_{\mathbb{T}^2} :\phi_N^n(x):\mathrm{d}x\Big)^{2p}\Big]^{\frac{1}{p}} \leqslant C_n'(2p-1)\frac{(\log N)^{n-2}}{N}$$

Nelson's estimate

Lemma: For X random variable in n^{th} inhomogeneous Wiener chaos

$$\mathbb{E}[X^{2p}]^{\frac{1}{2p}} \leqslant C_n(2p-1)^{\frac{n}{2}}\mathbb{E}[X^2]^{\frac{1}{2}}$$

$$\sup_N \mathbb{E}\Big[\Big(\int_{\mathbb{T}^2} :\phi_N^n(x): \mathrm{d}x\Big)^{2p}\Big] < \infty$$

$$\forall M > N: \ \mathbb{E}\Big[\Big(\int_{\mathbb{T}^2} :\phi_M^n(x): \mathrm{d}x - \int_{\mathbb{T}^2} :\phi_N^n(x): \mathrm{d}x\Big)^{2p}\Big]^{\frac{1}{p}} \leqslant C_n'(2p-1)\frac{(\log N)^{n-2}}{N}$$

Corollary:

$$\mathbb{E}\Big[\exp\Big\{-\frac{\varepsilon}{4}\int_{\mathbb{T}^2} :\phi_N^4(x): \mathrm{d}x\Big\}\Big] \leqslant 1 + \mathcal{O}(\varepsilon)$$

- ▷ Integrand is bounded below by term of order $-C_N^2$
- ▷ Use Markov's ineq to bound tails of integral and integrate by parts

Nelson's estimate

Lemma: For X random variable in n^{th} inhomogeneous Wiener chaos

$$\mathbb{E}[X^{2p}]^{\frac{1}{2p}} \leqslant C_n (2p-1)^{\frac{n}{2}} \mathbb{E}[X^2]^{\frac{1}{2}}$$

$$\sup_N \mathbb{E}\Big[\Big(\int_{\mathbb{T}^2} :\phi_N^n(x): \mathrm{d}x\Big)^{2p}\Big] < \infty$$

$$\forall M > N: \ \mathbb{E}\Big[\Big(\int_{\mathbb{T}^2} :\phi_M^n(x): \mathrm{d}x - \int_{\mathbb{T}^2} :\phi_N^n(x): \mathrm{d}x\Big)^{2p}\Big]^{\frac{1}{p}} \leqslant C_n'(2p-1)\frac{(\log N)^{n-2}}{N}$$

Corollary:

$$\mathbb{E}\Big[\exp\Big\{-\frac{\varepsilon}{4}\int_{\mathbb{T}^2} :\phi_N^4(x): \mathrm{d}x\Big\}\Big] \leqslant 1 + \mathcal{O}(\varepsilon)$$

- Integrand is bounded below by term of order $-C_N^2$
- Use Markov's ineq to bound tails of integral and integrate by parts

Useful as $\operatorname{cap}(A,B) \leqslant \sqrt{\frac{|\lambda_0|\varepsilon}{2\pi}} \prod_{0<|k|\leqslant N} \sqrt{\frac{2\pi\varepsilon}{\lambda_k}} \mathbb{E}\Big[\exp\Big\{-\frac{\varepsilon}{4}\int_{\mathbb{T}^2} :u_N^4(x): \mathrm{d}x\Big\}\Big]$

Computation of the prefactor

- ▷ Consider for simplicity $L < \beta\pi \Rightarrow$ transition state in 0
- ▷ Galerkin-truncated renormalised potential

$$V_N = \frac{1}{2}\int_{\mathbb{T}^2}\Big[\|\nabla u_N(x)\|^2 - u_N(x)^2\Big]\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{T}^2} :u_N(x)^4:\,\mathrm{d}x$$

Computation of the prefactor

- ▷ Consider for simplicity $L < \beta\pi \Rightarrow$ transition state in 0
- ▷ Galerkin-truncated renormalised potential

$$V_N = \frac{1}{2}\int_{\mathbb{T}^2}\left[\|\nabla u_N(x)\|^2 - u_N(x)^2\right]\mathrm{d}x + \frac{1}{4}\int_{\mathbb{T}^2} :u_N(x)^4: \mathrm{d}x$$

- ▷ Using Nelson estimate: $\operatorname{cap}(A,B) \simeq \sqrt{\frac{|\lambda_0|\varepsilon}{2\pi}} \prod_{0<|k|\leqslant N} \sqrt{\frac{2\pi\varepsilon}{\lambda_k}}$

Computation of the prefactor

- ▷ Consider for simplicity $L < \beta\pi \Rightarrow$ transition state in 0
- ▷ Galerkin-truncated renormalised potential

$$V_N = \frac{1}{2}\int_{\mathbb{T}^2}\left[\|\nabla u_N(x)\|^2 - u_N(x)^2\right]\mathrm{d}x + \frac{1}{4}\int_{\mathbb{T}^2} {:}u_N(x)^4{:}\,\mathrm{d}x$$

- ▷ Using Nelson estimate: $\operatorname{cap}(A,B) \simeq \sqrt{\frac{|\lambda_0|\varepsilon}{2\pi}} \prod_{0<|k|\leqslant N} \sqrt{\frac{2\pi\varepsilon}{\lambda_k}}$
- ▷ Symmetry argument:

$$\int_{B^c} h_{A,B}(z)\,\mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z = \frac{1}{2}\int \mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z = \frac{1}{2}\mathcal{Z}_N(\varepsilon)$$

- ▷ $\mathcal{Z}_N(\varepsilon) \simeq 2\prod_{|k|\leqslant N}\sqrt{\frac{2\pi\varepsilon}{\nu_k}}\,\mathrm{e}^{-V_N(L,0)/\varepsilon}$ where $-V_N(L,0) = \frac{1}{4}L^2 + \frac{3}{2}L^2 C_N\varepsilon$

Computation of the prefactor

- Consider for simplicity $L < \beta\pi \Rightarrow$ transition state in 0
- Galerkin-truncated renormalised potential

$$V_N = \frac{1}{2}\int_{\mathbb{T}^2}\left[\|\nabla u_N(x)\|^2 - u_N(x)^2\right]\mathrm{d}x + \frac{1}{4}\int_{\mathbb{T}^2} :u_N(x)^4:\mathrm{d}x$$

- Using Nelson estimate: $\operatorname{cap}(A,B) \simeq \sqrt{\frac{|\lambda_0|\varepsilon}{2\pi}}\prod_{0<|k|\leqslant N}\sqrt{\frac{2\pi\varepsilon}{\lambda_k}}$
- Symmetry argument:

$$\int_{B^c} h_{A,B}(z)\,\mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z = \frac{1}{2}\int \mathrm{e}^{-V_N(z)/\varepsilon}\,\mathrm{d}z = \frac{1}{2}\mathcal{Z}_N(\varepsilon)$$

- $\mathcal{Z}_N(\varepsilon) \simeq 2\prod_{|k|\leqslant N}\sqrt{\frac{2\pi\varepsilon}{\nu_k}}\,\mathrm{e}^{-V_N(L,0)/\varepsilon}$ where $-V_N(L,0) = \frac{1}{4}L^2 + \frac{3}{2}L^2 C_N\varepsilon$
- Prefactor proportional to (since $\nu_k = \lambda_k + 3$)

$$\prod_{0<|k|\leqslant N}\frac{\lambda_k}{\lambda_k+3}\mathrm{e}^{3/\lambda_k} \qquad \text{converges since} \qquad \log\left[\frac{\lambda_k}{\lambda_k+3}\mathrm{e}^{3/\lambda_k}\right] = \mathcal{O}\left(\frac{1}{|k|^4}\right)$$

Outlook

- ▷ Case $d = 3$: 2 renormalisation constants needed [Hairer 2014]
  Nelson argument does not work (Gibbs measure singular wrt GFF)
  Proved: lower bound on $\mathbb{E}[\tau_B]$, full E–K law in dimension $3 - \delta$

References:

- ▷ N. B., Bastien Fernandez & Barbara Gentz, *Metastability in interacting nonlinear stochastic differential equations I: From weak coupling to synchronisation* & *II: Large-N behaviour*, Nonlinearity **20**, 2551–2581; 2583–2614 (2007)
- ▷ N. B. & Barbara Gentz, *Sharp estimates for metastable lifetimes in parabolic SPDEs: Kramers' law and beyond*, Electronic J. Probability **18**, (24):1–58 (2013)
- ▷ N. B., Giacomo Di Gesù & Hendrik Weber, *An Eyring–Kramers law for the stochastic Allen–Cahn equation in dimension two*, Electronic J. Probability **22**, 1–27 (2017)

Related:

- ▷ N. B. & Barbara Gentz, , *The Eyring–Kramers law for potentials with nonquadratic saddles*, Markov Processes Relat. Fields **16**, 549–598 (2010)
- ▷ N. B., *Kramers' law: Validity, derivations and generalisations*, Markov Processes Relat. Fields **19**, 459–490 (2013)
- ▷ N. B. & Sébastien Dutercq, *The Eyring–Kramers law for Markovian jump processes with symmetries*, J. Theoretical Probability, **29**, 1240–1279 (2016)