ENS Lyon – Groupe de travail "MathsInFluids"

Modèles stochastiques et dynamique du climat

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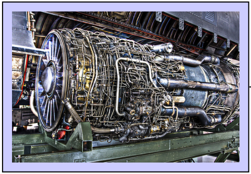
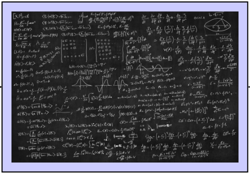
18 mars 2022

Basé sur des travaux avec Barbara Gentz et Rita Nader

Mathematical modeling,
or “let S be a spherical cow”





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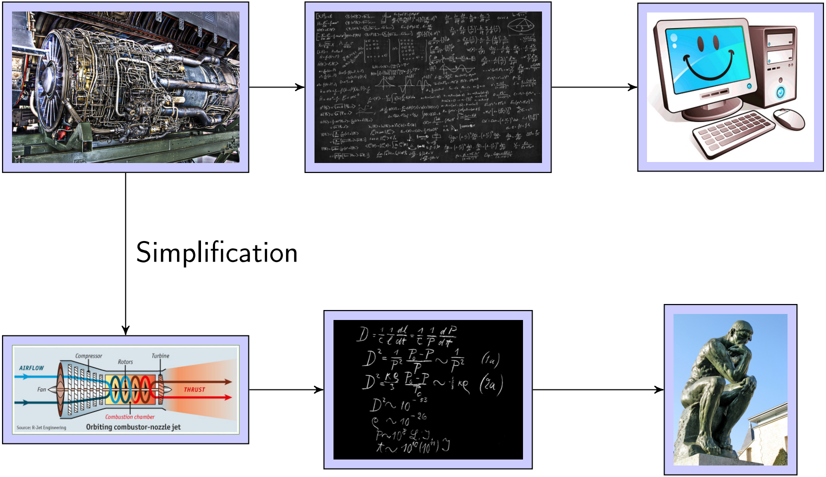

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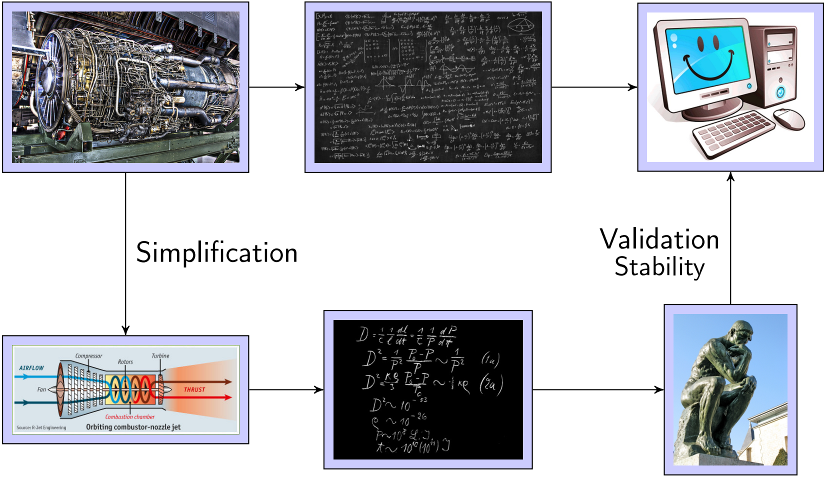
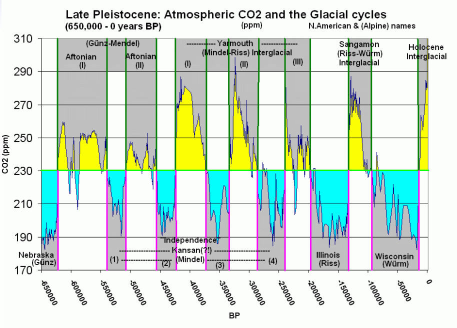


Climate in the past



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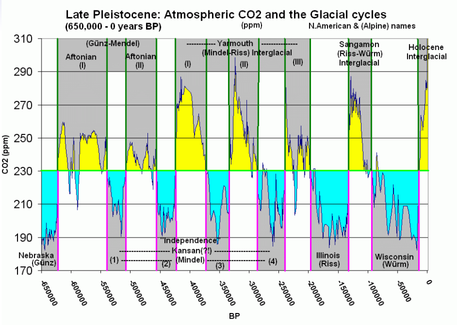


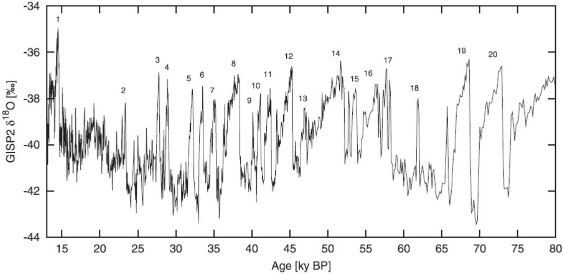
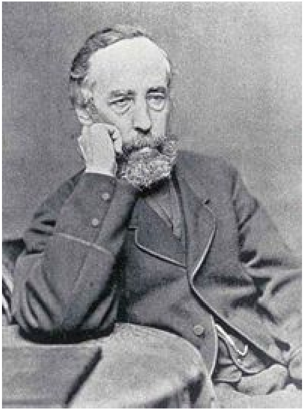
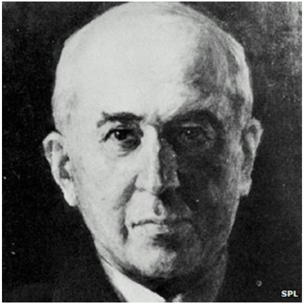


Figure 1. Oxygen isotope ($\delta^{18}O$) record from Greenland (GISP2 ice core [*Grootes and Stuiver*, 1997]). Numerals above $\delta^{18}O$ maxima denote the "classical" Dansgaard-Oeschger interstadial events [*Johnsen et al.*, 1992; *Dansgaard et al.*, 1993].

Ice Ages – Milanković cycles



James Croll (1821–1890)



Milutin Milanković (1879–1958)

Theory: Ice Ages are caused by (quasi-)periodic variations in the Earth's orbital parameters (eccentricity, semi-axis, inclination)

Climate models by decreasing complexity

- General Circulation Models (GCMs)
  Atmosphere + Oceans + Land masses/Ice sheets + . . .

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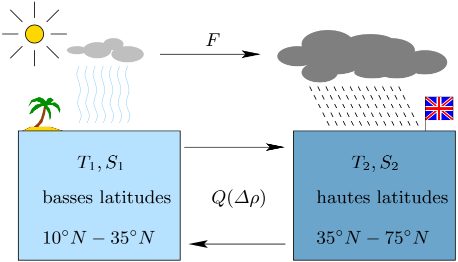
- Box Models
  Simple compartmental models for the evolution of averaged quantities



The simplest box model: One single box

$$c\frac{\mathrm{d}T}{\mathrm{d}t} = R_{\mathrm{in}}(t) - R_{\mathrm{out}}(T, t)$$

- ▷ T: average temperature on the Earth
- ▷ c: specific heat
- ▷ $R_{\mathrm{in}}(t) = Q(1 + K\cos(\omega t))$: incoming solar radiation, $K \simeq 5 \cdot 10^{-4}$

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  $\alpha(T)$: albedo, complicated dependence on T

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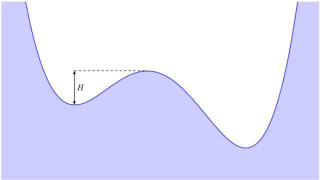
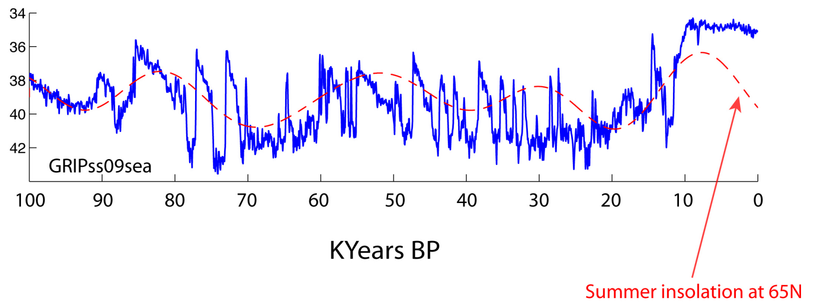
$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{E_0}{c}\Big[\gamma(T)(1 + K\cos(\omega t)) + K\cos(\omega t)\Big]$$

$\gamma(T) = \frac{Q}{E_0}(1 - \alpha(T)) - 1$

derives from double-well potential



Dansgaard–Oeschger events



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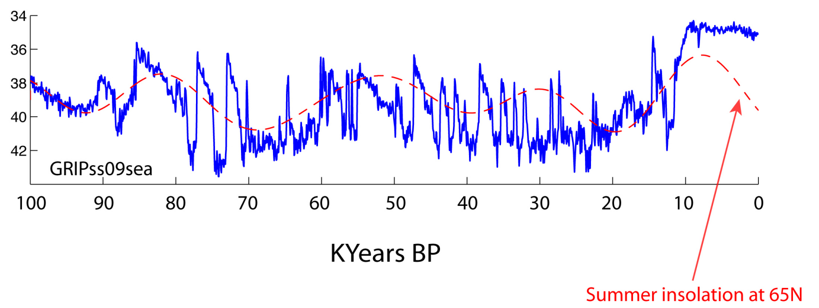
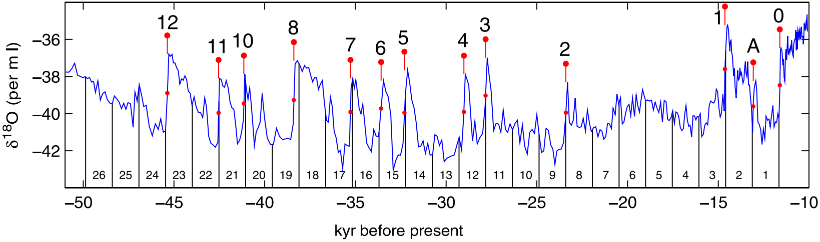
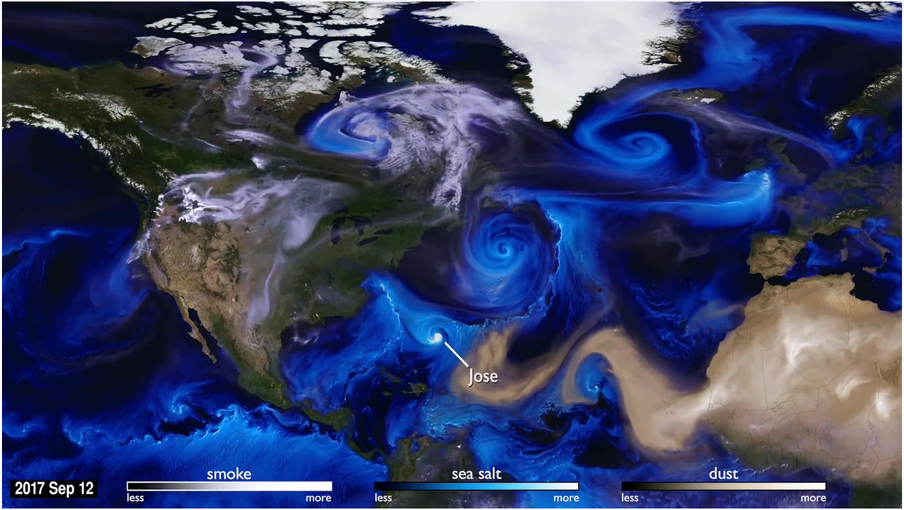


Is there a periodic forcing involved?

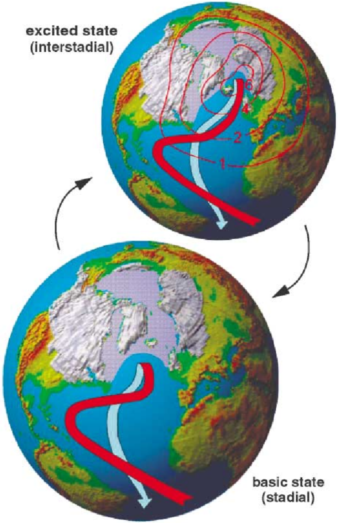


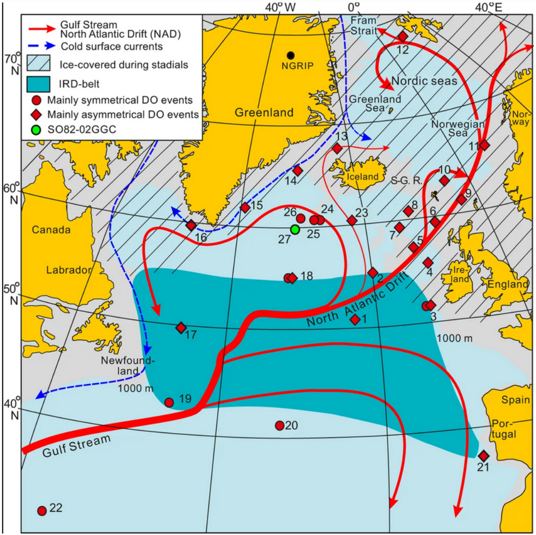
Weather over the North Atlantic in Fall 2017



Source: NASA, https://youtu.be/h1eRp0EG0mE

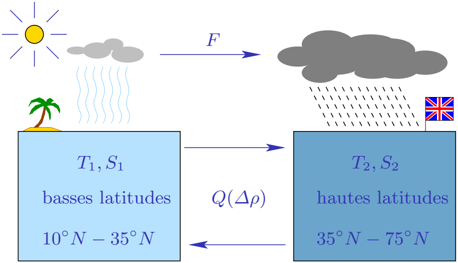
Bistability: Stadial vs interstadial regime





Stommel's box model

- T_i: temperatures
- S_i: salinities
- F: freshwater flux
- $Q(\Delta\rho)$: mass exchange
- $\Delta\rho = \alpha_S \Delta S - \alpha_T \Delta T$
- $\Delta T = T_1 - T_2$
- $\Delta S = S_1 - S_2$



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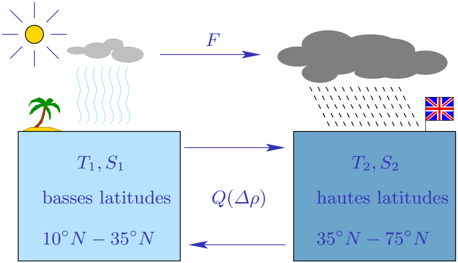

$$\frac{d}{dt}\Delta T = -\frac{1}{\tau_r}(\Delta T - \theta) - Q(\Delta\rho)\Delta T$$
$$\frac{d}{dt}\Delta S = \frac{S_0}{H}F - Q(\Delta\rho)\Delta S$$

Model for Q [Cessi]: $Q(\Delta\rho) = \frac{1}{\tau_d} + \frac{q}{V}\Delta\rho^2$.

Stommel's box model

Scaling: $x = \frac{\Delta T}{\theta}$, $y = \frac{\Delta S \alpha_S}{\alpha_T \theta}$, $t \mapsto \tau_d t$

Separation of time scales: $\tau_r \ll \tau_d$, $\varepsilon = \tau_r / \tau_d \ll 1$

$$\begin{aligned} \varepsilon \dot{x} &= -(x-1) - \varepsilon x \left[1 + \eta^2 (x-y)^2\right] \\ \dot{y} &= \mu - y\left[1 + \eta^2 (x-y)^2\right] \end{aligned}$$

where $\eta^2 = \tau_d (\alpha_T \theta)^2 \frac{q}{V}$, $\mu = \frac{\alpha_S S_0 \tau_d}{\alpha_T \theta H} F$

Slow manifold [Fenichel '79]: $x = 1 + \mathcal{O}(\varepsilon) \Rightarrow \varepsilon \dot{x} = 0$.

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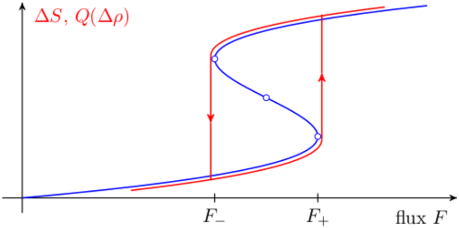
Reduced equation on slow manifold:

$$\dot{y} = \mu - y\left[1 + \eta^2 (1-y)^2 + \mathcal{O}(\varepsilon)\right]$$

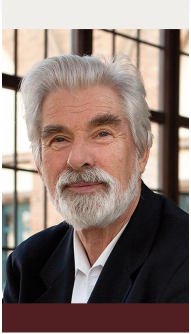
One or two stable equilibria, depending on μ (and η).



Improving the models – adding noise

[Klaus Hasselmann. Stochastic climate models. Part I. Theory. Tellus, **28**:473–485, 1976.]

- ▷ Separation of length/time scales
- ▷ Galerkin truncation to long/slow modes
- ▷ Unresolved modes: typically represented by parameterization (as in BBGKY hierarchy)
- ▷ Alternative: represent unresolved modes by stochastic process (random noise)



Picture by Daniel Reinhardt

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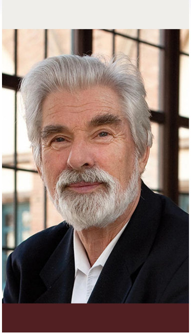

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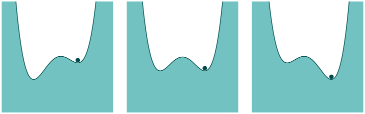
Justification of effective description by stochastic differential equations:

- ▷ System coupled to infinitely many harmonic oscillators
  [Ford, Kac, Mazur '65], [Lebowitz, Spohn '77],
  [Eckmann, Pillet, Rey-Bellet '99], [Rey-Bellet, Thomas '00, '02]
- ▷ Stochastic averaging for slow–fast systems
  [Khasminski '66], [Hasselmann '76], [Kifer '03]

Stochastic resonance

[Benzi/Sutera/Vulpiani '81, Nicolis/Nicolis '81]

$$\mathrm{d}x_t = \underbrace{\left[-x^3 + x + A\cos\varepsilon t\right]}_{=-\frac{\partial}{\partial x}\left[\frac{1}{4}x^4 - \frac{1}{2}x^2 - Ax\cos\varepsilon t\right]} \mathrm{d}t + \sigma\,\mathrm{d}W_t$$



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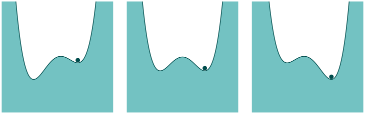
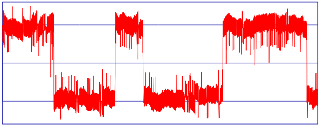
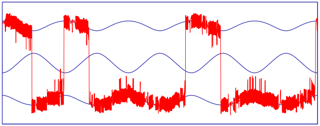


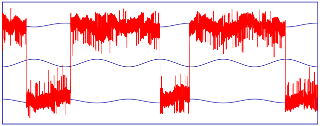
Sample paths $\{x_t\}_t$ for $\varepsilon = 0.001$:



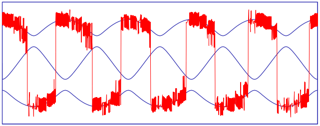
$A = 0,\ \sigma = 0.3$



$A = 0.24,\ \sigma = 0.2$

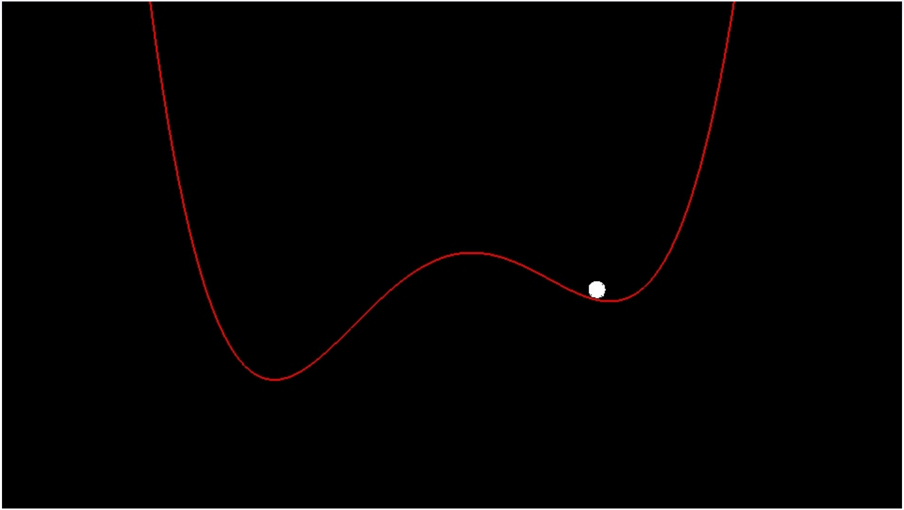


$A = 0.1,\ \sigma = 0.27$



$A = 0.35,\ \sigma = 0.2$

Stochastic resonance



Simulation available at https://youtu.be/h1eRp0EGOmE

Descriptions of stochastic resonance

- Fokker–Planck equation: [Caroli, Caroli, Roulet & Saint-James '81]
- Two-state Markov chain: [Eckmann & Thomas '82], [Imkeller & Pavljukevich '02], [Herrmann & Imkeller '02]
- Signal-to-noise ratio: [Gammaitoni, Menichella-Saetta & . . . '89], [Fox '89], [Jung& Hänggi '89], [McNamara & Wiesenfeld '89]
- Slow forcing: [Jung & Hänggi '91], [Talkner '99], [Talkner & Łuczka '04]
- Large deviations: [Freidlin '00, Freidlin '01]
- Residence-time distributions: [Zhou, Moss & Jung '90], [Choi, Fox & Jung '98], . . .
- Overview articles: [Moss, Pierson & O'Gorman '94], [Wiesenfeld & Moss '95], [McNamara & Wiesenfeld '95], [Wiesenfeld & Jaramillo '98], [Gammaitoni, Hänggi, Jung & Marchesoni '98], [Hänggi '02], [Wellens, Shatokhin & Buchleitner '04], . . .
- Monograph: [Herrmann, Imkeller, Pavlyukevich & Peithmann '14]

Slow–fast systems with noise

On the slow time scale εt:

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t)$$

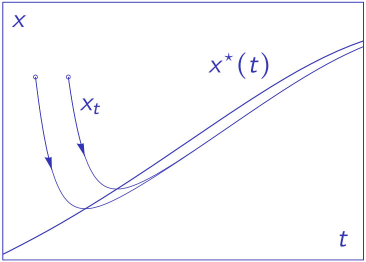
- ▷ Equilibrium branch: $\{x = x^\star(t)\}$ where $f(x^\star(t),t) = 0$ for all t
- ▷ Stable if $a^\star(t) = \partial_x f(x^\star(t),t) \leqslant -a_0 < 0$ for all t

Then [Tikhonov '52, Fenichel '79]:

- ▷ There exists particular solution

$$\bar{x}(t) = x^\star(t) + \mathcal{O}(\varepsilon)$$

- ▷ $\bar{x}$ attracts nearby orbits exp. fast
- ▷ $\bar{x}$ admits asymptotic series in ε



Slow–fast systems with noise

Stochastic perturbation:

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, t)\, \mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\, \mathrm{d}W_t$$

Write $x_t = \bar{x}(t) + \xi_t$ and Taylor-expand:

$$\mathrm{d}\xi_t = \frac{1}{\varepsilon}\Big[\bar{a}(t)\xi_t + \underbrace{b(\xi_t, t)}_{=\mathcal{O}(\xi_t^2)}\Big]\, \mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\, \mathrm{d}W_t$$

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Variations of constants (Duhamel formula), if $\xi_0 = 0$:

$$\xi_t = \underbrace{\frac{\sigma}{\sqrt{\varepsilon}} \int_0^t \mathrm{e}^{\bar{\alpha}(t,s)/\varepsilon}\,\mathrm{d}W_s}_{\xi_t^0\text{: sol of linearised system}} + \underbrace{\frac{1}{\varepsilon} \int_0^t \mathrm{e}^{\bar{\alpha}(t,s)/\varepsilon}\, b(\xi_s, s)\,\mathrm{d}s}_{\text{treat as a perturbation}}$$

where $\bar{\alpha}(t,s) = \int_s^t \bar{a}(u)\,\mathrm{d}u$

Slow–fast systems with noise

Properties of $\xi_t^0 = \dfrac{\sigma}{\sqrt{\varepsilon}} \displaystyle\int_0^t e^{\bar{\alpha}(t,s)/\varepsilon} \, dW_s$:

- Gaussian process, $\mathbb{E}[\xi_t^0] = 0$, $\mathrm{Var}(\xi_t^0) = \frac{\sigma^2}{\varepsilon} \int_0^t e^{2\bar{\alpha}(t,s)/\varepsilon} \, ds$
- Confidence interval: $\mathbb{P}\Bigl\{|\xi_t^0| > \frac{h}{\sigma}\sqrt{\mathrm{Var}(\xi_t^0)}\Bigr\} = \mathcal{O}(e^{-h^2/2\sigma^2})$
- $\sigma^{-2}\,\mathrm{Var}(\xi_t^0)$ satisfies ODE $\varepsilon \dot{v} = 2\bar{a}(t)v + 1$

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Lemma [B & Gentz, Proba. Theory Relat. Fields 2002]

$\bar{v}(t)$ solution of ODE bounded away from 0: $\bar{v}(t) = \frac{1}{-2\bar{a}(t)} + \mathcal{O}(\varepsilon)$

$$\mathbb{P}\bigg\{\sup_{0\leqslant s\leqslant t} \frac{|\xi_s^0|}{\sqrt{\bar{v}(s)}} > h\bigg\} = C_0(t,\varepsilon)\, e^{-h^2/2\sigma^2}$$

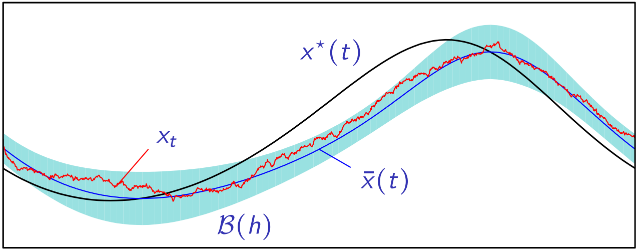
where $C_0(t,\varepsilon) = \sqrt{\frac{2}{\pi}}\frac{1}{\varepsilon}\frac{h}{\sigma}\Big|\displaystyle\int_0^t \bar{a}(s)\,ds\Big|\Big[1 + \mathcal{O}\big(\varepsilon + \frac{t}{\varepsilon}e^{-h^2/\sigma^2}\big)\Big]$

Proof based on Doob's submartingale inequality and partition of $[0,t]$

Slow–fast systems with noise

Nonlinear equation: $\mathrm{d}\xi_t = \frac{1}{\varepsilon}\big[\bar{a}(t)\xi_t + b(\xi_t, t)\big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$

Confidence strip: $\mathcal{B}(h) = \Big\{|\xi| \leqslant h\sqrt{\bar{v}(t)}\ \forall t\Big\} = \Big\{|x - \bar{x}(t)| \leqslant h\sqrt{\bar{v}(t)}\ \forall t\Big\}$



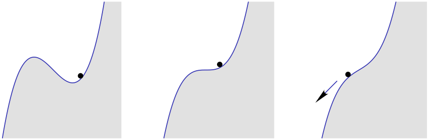
Theorem [B & Gentz, Proba. Theory Relat. Fields 2002]

$$C(t,\varepsilon)\,\mathrm{e}^{-\kappa_- h^2/2\sigma^2} \leqslant \mathbb{P}\Big\{\text{leaving } \mathcal{B}(h) \text{ before time } t\Big\} \leqslant C(t,\varepsilon)\,\mathrm{e}^{-\kappa_+ h^2/2\sigma^2}$$

where $\kappa_\pm = 1 \mp \mathcal{O}(h)$ and $C(t,\varepsilon) = C_0(t,\varepsilon)\big[1 + \mathcal{O}(h)\big]$ (requires $h \leqslant h_0$)

Rising variance near tipping points

Saddle–node (fold) bifurcation: $\mathrm{d}x_t = \frac{1}{\varepsilon}\Big[-x_t^2 - t\Big]\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\mathrm{d}W_t$



- Fluctuations grow like $\frac{\sigma}{|\bar{a}(t)|^{1/2}} \asymp \frac{\sigma}{\max\{(-t)^{1/4},\varepsilon^{1/6}\}}$
- Early transitions occur if $\sigma \gg \varepsilon^{1/2}$ at time $\asymp -\sigma^{4/3}$

Rising variance near tipping points

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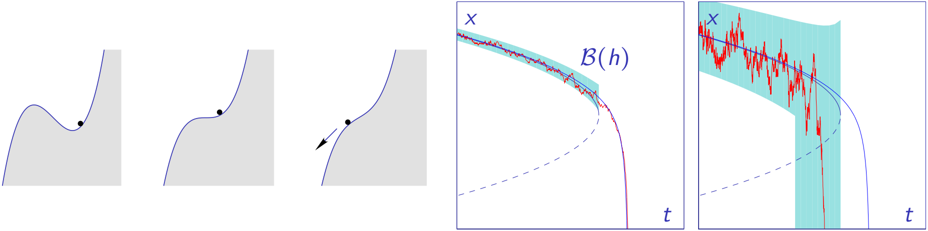

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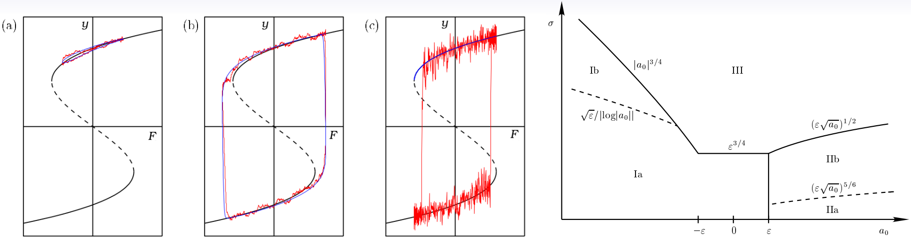
Theorem [B & Gentz, Nonlinearity 2002]

- ▷ $\sigma < \varepsilon^{1/2}$: transition probability before fold $\leqslant \mathrm{e}^{-c\varepsilon/\sigma^2}$
- ▷ $\sigma > \varepsilon^{1/2}$: probability of early transition $\geqslant 1 - \mathrm{e}^{-c\sigma^2/(\varepsilon|\log\sigma|)}$

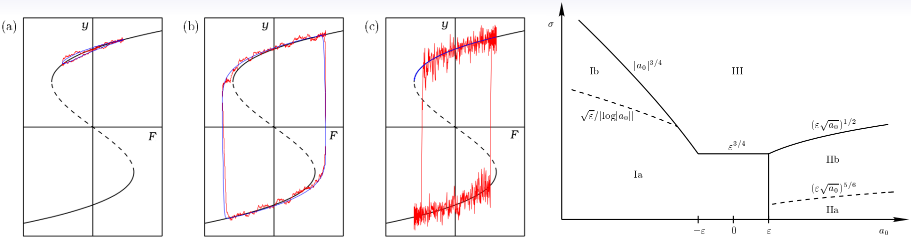
[Carpenter, Brock, *Rising variance: a leading indicator of ecological transition*, Ecology Letters **9**, 311–318 (2006)]

Hysteresis



[B & Gentz, Nonlinearity 2002]

Hysteresis



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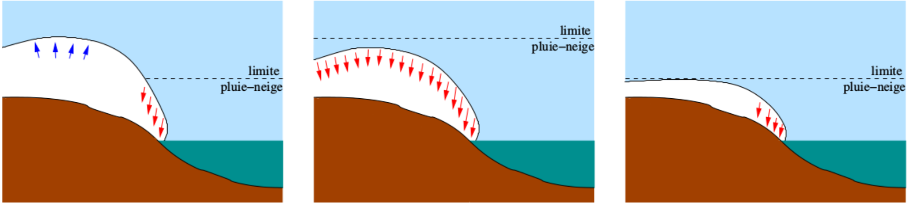
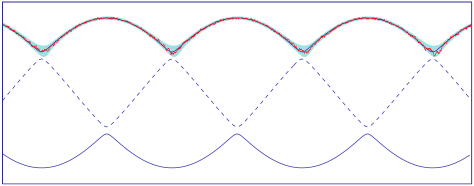


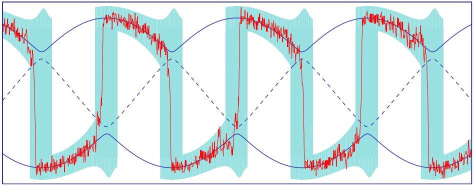
Stochastic resonance

Critical noise intensity: $\sigma_c = \max\{\delta, \varepsilon\}^{3/4}$, $\delta = A_c - A$, $A_c = \frac{2}{3\sqrt{3}}$

$\sigma \ll \sigma_c$:
transitions unlikely



$\sigma \gg \sigma_c$:
synchronisation



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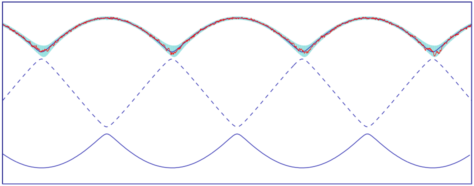

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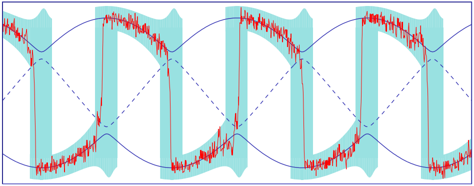

Theorem [B & Gentz, Annals App. Proba 2002]

- $\sigma < \sigma_c$: transition probability per period $\leqslant e^{-\sigma_c^2/\sigma^2}$
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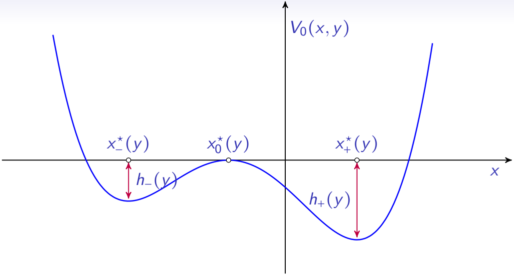
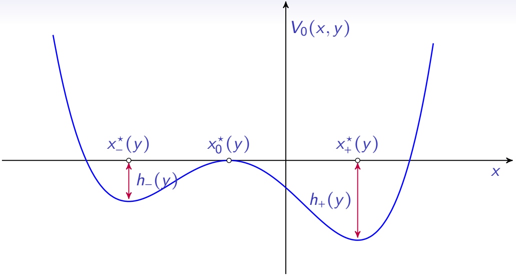
Residence-time distribution



$$dx_t = -\partial_x V_0(x_t, y_t)\,dt + \sigma\,dW_t^x$$
$$dy_t = \varepsilon\,dt + \sigma\sqrt{\varepsilon}\varrho\,dW_t^y$$

Residence-time distribution



$$\begin{aligned} dx_t &= -\partial_x V_0(x_t, y_t)\, dt + \sigma\, dW_t^x \\ dy_t &= \varepsilon\, dt + \sigma\sqrt{\varepsilon}\varrho\, dW_t^y \end{aligned}$$

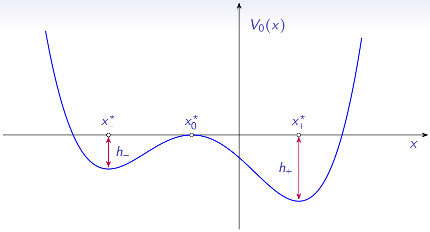
- ▷ $x \mapsto V_0(x,y)$ confining double-well potential, class $\mathcal{C}^4$
  $V_0(x, y+1) = V_0(x,y)$
- ▷ $0 \leqslant \varepsilon, \sigma \ll 1$, $\varrho > 0$
- ▷ W_t^x, W_t^y independent standard Wiener processes

Question: describe law of $\tau_+ = \inf\{t > 0 \colon x_t = x_+^*(y_t) | (x_0 = x_-^*(y_0), y_0)\}$

Static case

$$dx_t = -V_0'(x_t)\,dt + \sigma\,dW_t$$

$$\omega_\pm = \sqrt{V_0''(x_\pm^*)} \quad \omega_0 = \sqrt{-V_0''(x_0^*)}$$



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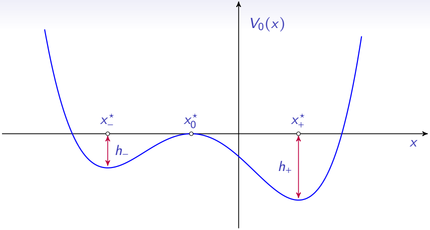

- By Dynkin's equation, $\forall x < x_+^*$,

$$\mathbb{E}^x[\tau_+] = \frac{2}{\sigma^2}\int_x^{x_+^*}\int_{-\infty}^{x_2} e^{2[V_0(x_2)-V_0(x_1)]/\sigma^2}\,dx_1\,dx_2$$

$\Rightarrow$ Eyring–Kramers law: $\mathbb{E}^{x_-^*}[\tau_+] = \dfrac{2\pi}{\omega_0\omega_-}\,e^{2h_-/\sigma^2}\,[1+\mathcal{O}(\sigma^2)]$

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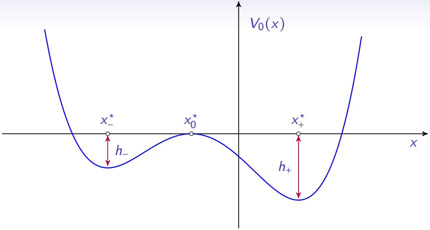

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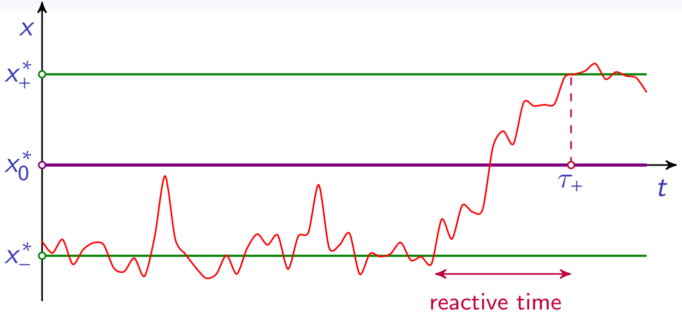
▷ [Day '83]: $\forall s \geqslant 0$, $\lim_{\sigma\to 0}\mathbb{P}^{x_-^*}\Big\{\tau_+ > s\,\mathbb{E}^{x_-^*}[\tau_+]\Big\} = e^{-s}$

(Convergence to exponential law $\mathscr{E}(1)$)

Static case: reactive time



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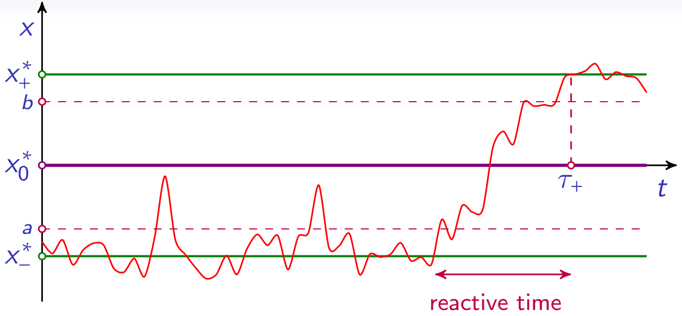

▷ [Cérou, Guyader, Lelièvre, Malrieu '13]: $x_-^* < a < x_0 < x_0^* < b < x_+^*$

$$\lim_{\sigma\to 0} \text{Law}\Big(\omega_0^2 \tau_b - 2\log(\sigma^{-1}) \;\Big|\; \tau_b < \tau_a\Big) = \text{Law}\Big(\underbrace{\mathcal{G}}_{\text{Gumbel}} + \underbrace{T(x_0, b)}_{\text{deterministic}} \Big)$$

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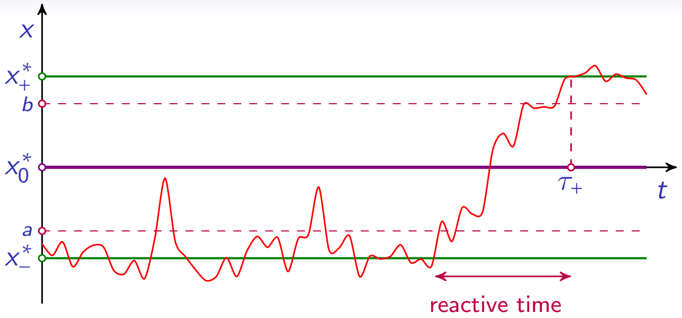

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Gumbel law: $\mathbb{P}\{\mathcal{G} < t\} = e^{-e^{-t}} \quad \forall t \in \mathbb{R}$

(max-stable distribution from extreme value theory, cf. [Bakhtin '15])

$\Rightarrow$ reactive time $\simeq \omega_0^{-2}[2\log(\sigma^{-1}) + \mathcal{G} + T(x_0, b)]$

Law of first-passage times

$$\mathrm{d}x_t = -\frac{1}{\varepsilon}\partial_x V_0(x_t, y_t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t^x$$

$$\mathrm{d}y_t = \mathrm{d}t + \sigma\varrho\,\mathrm{d}W_t^y$$

(time scaled by ε)

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- Det. eq. $\varepsilon\dot{x} = -\partial_x V_0(x,t)$: $\exists!\ 3$ periodic orbits $\bar{x}_i(t) = x_i^*(t) + \mathcal{O}(\varepsilon)$
- τ_0 hitting time of $\bar{x}_0(y)$

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$$
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Theorem: [B & Gentz, SIAM J Math Analysis 2014]

$$\lim_{\sigma\to 0} \mathrm{Law}\Big(\theta(y_{\tau_0}) - \log(\sigma^{-1}) - \frac{\lambda_+}{\varepsilon} Y^\sigma\Big) = \mathrm{Law}\Big(\frac{\mathcal{G}}{2} - \frac{\log 2}{2}\Big)$$

- $\theta(y)$: explicit parametrisation of $\bar{x}_0(y)$, $\theta(y+1) = \theta(y) + \frac{\lambda_+}{\varepsilon}$
- λ_+: Lyapunov exponent of $\bar{x}_0(y)$ $(\lambda_+ = \int_0^1 \omega_0(y)^2\,\mathrm{d}y + \mathcal{O}(\varepsilon))$
- $Y^\sigma \in \mathbb{N}$: asymptotically geometric $\mathbb{N}$-valued r.v:

$$\lim_{n\to\infty} \mathbb{P}\{Y^\sigma = n+1 | Y^\sigma > n\} = p(\sigma)$$

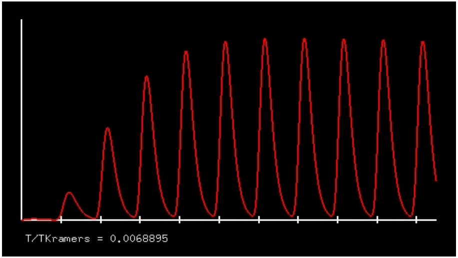
$p(\sigma) \simeq \mathrm{e}^{-\mathcal{I}/\sigma^2}$, $\mathcal{I}$ Freidlin–Wentzell quasipotential, $\mathbb{E}[\tau_0] \simeq p(\sigma)^{-1}$

First-passage and residence time distributions

First-passage time distribution

$$f(t) = f_{\text{trans}}(t)\frac{e^{-t/T_{\mathrm{K}}}}{T_{\mathrm{K}}} \times \sum_{k=-\infty}^{\infty} A(\lambda_+ T(k+\theta) - |\log\sigma|)$$

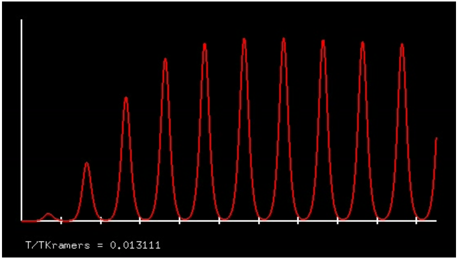
$$A(x) = e^{-x-e^{-x}}$$



Simulation available at https://youtu.be/XDM5aCmQ_HI

Residence-time distribution

$$q(t) = f_{\text{trans}}(t)\frac{e^{-t/T_{\mathrm{K}}}}{T_{\mathrm{K}}} \times \sum_{k=-\infty}^{\infty} \frac{1}{\cosh^2(\lambda_+(t+\frac{T}{2}-kT))}$$



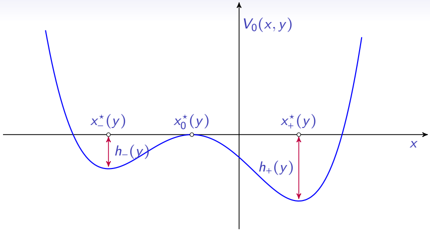
Simulation available at https://youtu.be/v-PZIedBQO0

Eyring–Kramers-type law for $\mathbb{E}[\tau_+]$

$$\omega_\pm(y) = \sqrt{\partial_{xx} V_0(x^*_\pm(y), y)}$$

$$\omega_0(y) = \sqrt{-\partial_{xx}(x^*_0(y), y)}$$

$$r_\pm(y) = \frac{\omega_\pm(y)\omega_0(y)}{2\pi}\, \mathrm{e}^{-2h_\pm(y)/\sigma^2}$$



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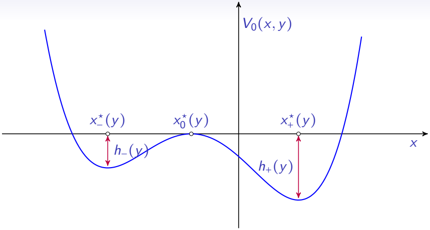

- Leading eigenvalue of $-\mathscr{L}_x = -\frac{\sigma^2}{2}\partial_{xx} + \partial_x V_0 \partial_x$:

$$\lambda_1(y) = [r_+(y) + r_-(y)][1 + \mathcal{O}(\sigma^2)]$$

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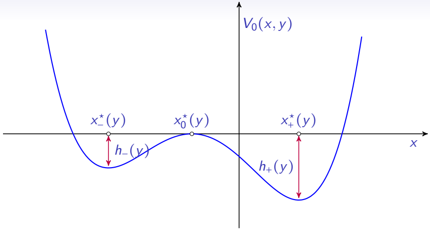

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$$\lambda_1(y) = [r_+(y) + r_-(y)][1 + \mathcal{O}(\sigma^2)] \qquad \langle \lambda_1 \rangle = \int_0^1 \lambda_1(y)\,\mathrm{d}y$$

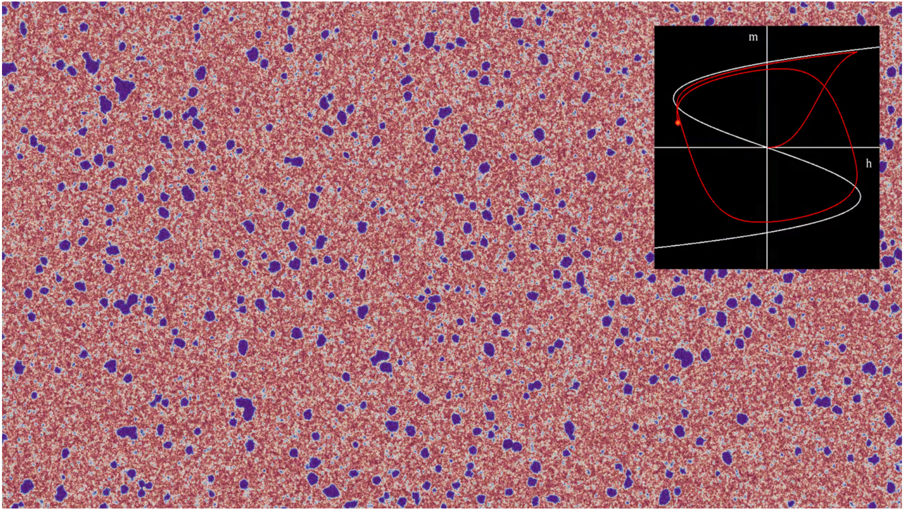
Theorem: [B, Proba & Math Phys 2021]

$$\mathbb{E}^{(x^*_-(y_0),y_0)}[\tau_+] = \frac{2\pi\varepsilon[1 + R(\varepsilon,\sigma)]}{\displaystyle\int_0^1 \omega_0(y)\omega_-(y)\,\mathrm{e}^{-2h_-(y)/\sigma^2}\,\mathrm{d}y}$$

where $R(\varepsilon,\sigma)$ complicated but small if $\langle\lambda_1\rangle \ll \varepsilon \ll \langle\lambda_1\rangle^{1/4}$

Stochastic resonance in stochastic PDEs

$$d\phi(t,x) = \left[\Delta\phi(t,x) + \phi(t,x) - \phi(t,x)^3 + A\cos(\varepsilon t)\right]dt + \sigma\, dW(t,x)$$



Simulation available at `https://youtu.be/eN3NWiEjBK8`

Stochastic resonance in SPDEs

$$\mathrm{d}\phi(t,x) = \left[\Delta\phi(t,x) + f(\varepsilon t, \phi(t,x))\right]\mathrm{d}t + \sigma\,\mathrm{d}W(t,x)$$

- $\phi = \phi(t,x) \in \mathbb{R}$, $\varepsilon t \in [0, T]$ or f is T-periodic, $x \in \mathbb{T} = \mathbb{R}/L\mathbb{Z}$, $L > 0$
- $\phi \mapsto f(s,\phi)$ bistable, e.g. $f(s,\phi) = \phi - \phi^3 + A\cos(s)$
- $\mathrm{d}W(t,x)$ space-time white noise on $\mathbb{R}_+ \times \mathbb{T}$
- $0 < \varepsilon, \sigma \ll 1$
- δ measures closeness to bifurcation (e.g. $A_c - A$)

Stochastic resonance in SPDEs

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- $0 < \varepsilon, \sigma \ll 1$
- δ measures closeness to bifurcation (e.g. $A_\mathrm{c} - A$)

Theorem [B & Nader, SPDEs: Analysis & Computations, 2022]

- Away from bifurcations, solutions are concentrated around deterministic solutions in Sobolev H^s-norm for any $s < \frac{1}{2}$
- At bifurcations, $\phi_\perp = \phi - \int_{\mathbb{T}^2} \phi\,\mathrm{d}x$ remains small in H^s-norm
- $\sigma < \sigma_\mathrm{c} = (\delta \vee \varepsilon)^{3/4}$: transition probability per period $\leqslant \mathrm{e}^{-\sigma_\mathrm{c}^2/\sigma^2}$
- $\sigma > \sigma_\mathrm{c}$: transition probability per period $\geqslant 1 - \mathrm{e}^{-c\sigma^{4/3}/(\varepsilon|\log\sigma|)}$

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Thanks for your attention!